KYBERNETIKA — VOLUME *28* (1992), NUMBER 3, PAGES 227-233

EVERY CONTINUOUS FIRST ORDER AUTOREGRESSIVE STOCHASTIC PROCESS IS A GAUSSIAN PROCESS

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The notion of a process $X(t)$ with independent increments is generalized. It is required that for $0 = t_0 < t_1 < \cdots < t_n \leq T$ the r.v. $X(t_0), X(t_1) - \alpha(t_0,t_1)X(t_0), \ldots, X(t_n) - \alpha(t_{n-1},t_n)X(t_{n-1})$ are independent with some suitable function $\alpha(s,t)$. This class consists of Markov processes with a special structure of transition kernels and includes both the processes with independent increments and the regular Markov processes introduced by Vajda. The main result is that under some mild additional conditions every continuous process from this class is a Gauß-Markov-process.

1. FIRST ORDER AUTOREGRESSIVE PROCESSES

A discrete time stochastic process $X_0, X_1, \ldots$ is commonly called a *first order autoregressive* process if there are constants $\alpha_0, \alpha_1, \ldots$ so that the random variables X_0, $X_1 - \alpha_0 X_0$, $X_2 - \alpha_1 X_1, \ldots$ are independent. Analogously we call a continuous time stochastic process $X(t)$, $0 \leq t \leq T$, *a first order autoregressive* process if there exists a function $\alpha(s,t)$, $0 \leq s \leq t \leq T$, so that for every $0 = t_0 < t_1 < \cdots < t_n \leq T$ the random variables $X(t_0), X(t_1) - \alpha(t_0,t_1)X(t_0), \ldots, X(t_n) - \alpha(t_{n-1},t_n)X(t_{n-1})$ are independent. Obviously every process with independent increments is a first order autoregressive process with $\alpha = 1$ and every first order autoregressive process is a Markov process. But these Markov processes have a special structure.

Proposition. A real valued process $X(t)$, $0 \leq t \leq T$, is a first order autoregressive process iff there are distributions $Q_{s,t}$ and a function $\alpha(s,t)$ for $0 \leq s < t \leq T$ so that for every Borel set B of the real line

$$P(X(t) \in B \mid X(s)) = Q_{s,t}(B - \alpha(s,t)X(s)) \quad \text{a.s.} \tag{1}$$

Suppose now $X(t)$, $0 \leq t \leq T$, is a Gauß-Markov-process and denote by $a(t) = \mathsf{E}X(t)$ and $R(s,t) = \mathsf{E}X(s)X(t) - a(s)a(t)$ the expectation and covariance function, respectively. Using the convention $ab^+ = a/b$ if $b \neq 0$ and $ab^+ = 0$ if $b = 0$ the conditional distribution of $X(t)$ given $X(s) = x$ is known to be normal distribution with expectation $a(t) - a(s)R(s,t)\,(R(s,s))^{+} + xR(s,t)\,(R(s,s))^{+}$ and variance $\sigma^2(s,t) = R(t,t)\,(1 - R^2(s,t)\,(R(s,s)R(t,t))^{+})$, respectively. (see [1]). In this connection a normal distribution with mean μ and variance zero is understood to be a δ-distribution which is concentrated at the point μ . If we denote by $Q_{s,t}$ a normal distribution with mean $a(t) - a(s)R(s,t)(R(s,s))^{+}$ and variance $\sigma^2(s,t)$ and introduce α by $\alpha(s,t) = R(s,t)(R(s,s))^{+}$ then the representation (1) holds which implies that X(t) is a first order autoregressive process. Furthermore, if we suppose $\sigma(s,t) > 0$ for every $0 < s < t < T$ then with φ as standard normal density

$$P(X(t) \in B \mid X(s) = x) = \int_B \frac{1}{\sigma(s,t)} \varphi\left(\frac{y - \alpha(s,t)x - \xi(s,t)}{\sigma(s,t)}\right) \mathrm{d}y \tag{2}$$

where $\xi(s,t) = a(t) - \alpha(s,t)a(s)$.

Vajda [4] calls every Markov process X for which the representation (2) with some density holds which is not necessarily the density of a normal distribution *a regular Markov process.* Consequently by the Proposition every regular Markov process is a first order autoregressive process.

The meaning of regular Markov processes is due to the fact that for two regular Markov processes which differ only in the mean value function the *Renyi distance* of the corresponding distributions may be explicitly calculated. This distance is important for determining the asymptotic behaviour of error probabilities in the problem of testing statistical hypotheses. For details we refer to [2] and [4].

Now we restrict to continuous processes X, i.e. every realization of the process X is a continuous real valued function on $[0,T]$. If X is a process with independent increments and consequently an autoregressive process with $\alpha = 1$ then X is known to be a Gaussian process with independent increments. The main goal of this paper is to give an answer to the question whether a similar statement continues to hold in the more general class of first order autoregressive processes.

2. RESULTS

To formulate the results we need some special classes of functions. Denote by $\mathcal{C}$ the set of all real continuous functions $\alpha(s,t)$, $0 \le s \le t \le T$ with $\alpha(s,s) = 1$ and by $\mho$ the set of all real functions so that

$$d(T) = \sup \prod_{i=1}^{n} |\alpha(t_{i-1}, t_i)| < \infty \tag{3}$$

where the supremum is taken over all $0 \le t_0 \le \cdots \le t_n \le T$.

Theorem 1. If $X(t)$, $0 \le t \le T$, is a continuous first order autoregressive stochastic process with $\alpha \in \mathcal{C} \cap \mho$ and $X(0)$ is normally distributed then $X(t)$ is a Gauß-Markov process.

If we impose moment conditions to the process X then the conditions required to α may be weakened.

Theorem 2. If $X(t)$, $0 \le t \le T$, is a continuous first order autoregressive stochastic process with $\alpha \in \mathcal{C}$, $\mathsf{E}X^2(t) < \infty$, $0 \le t \le T$, and if $X(0)$ is normally distributed then $X(t)$ is a Gauß-Markov process.

Corollary. Let $X(t)$, $0 \le t \le T$, be a continuous regular Markov process and assume $X(0)$ is normally distributed. If either α fulfils $\alpha \in \mathcal{C} \cap \mathcal{U}$ or both $\alpha \in \mathcal{C}$ and $\mathsf{E}X^2(t) < \infty$, $0 \le t \le T$, are satisfied then $X(t)$ is a Gauß-Markov process.

3. PROOFS

At first we proof the Proposition.

Let f be any non-negative Borel measurable function on R^{n+1} and assume $X(t)$ is first order autoregressive. If $Q_{s,t}$ denotes the distribution of $X(t) - \alpha(s,t)X(s)$ then for $0 \le t_1 \le \cdots \le t_n \le T$, $\alpha_k = \alpha(t_{k-1}, t_k)$, $Y_k = X(t_k) - \alpha_k X(t_{k-1})$ if $k \ge 1$, $Y_0 = X(0)$, $Q_k = Q_{t_{k-1},t_k}$

$$\begin{aligned} & \mathsf{E}\, f\,(X(0), X(t_1), \ldots, X(t_n)) = \\ & \mathsf{E}\, f\,(Y_0, Y_1 + \alpha_1 Y_0, \ldots, Y_n + \alpha_n Y_{n-1} + \cdots + \alpha_n \ldots \alpha_1 Y_0) = \\ = \; & \int \cdots \int f\,(y_0, y_1 + \alpha_1 y_0, \ldots, y_n + \alpha_n y_{n-1} + \cdots + \alpha_n \cdots \alpha_1 y_0) \\ & \qquad Q_n(\mathrm{d}y_n) \ldots Q_1(\mathrm{d}y_1)\, P_{X(0)}(\mathrm{d}y_0) \\ = \; & \int \cdots \int f(x_0, x_1, \ldots, x_n)\, Q_n\,(\mathrm{d}x_n - \alpha_n x_{n-1}) \cdots Q_1\,(\mathrm{d}x_1 - \alpha_1 x_0)\, P_{X(0)}(\mathrm{d}x_0). \end{aligned}$$

Consequently $X(t)$ is a Markov process with transition kernels

$$P(X(t) \in B \mid X(s) = x) = Q_{s,t}(B - \alpha(s,t)x)\,.$$

If conversely $X(t)$ is a Markov process with these transition kernels then the above calculation shows that the random variables $X(0), X(t_1) - \alpha(0,t_1)X(0), \ldots, X(t_n) - \alpha(t_{n-1}, t_n)X(t_{n-1})$ are independent which means that $X(t)$ is a first order autoregressive process. □

The proofs of the Theorems 1 and 2 are based on a series of lemmas which will be now established.

The first lemma is a well-known characterization of continuous stochastic processes with independent increments. For a proof we refer to [3].

Lemma 1. If $W(t), 0 \leq t \leq T$, is a continuous stochastic process with independent increments and $W(0) = 0$ then there exist a continuous function $a(t)$ and a non-decreasing continuous function $b(t)$, $0 \leq t \leq T$, so that $W(t) - W(s)$ is normally distributed with mean $a(t) - a(s)$ and variance $b(t) - b(s)$ for every $0 \leq s \leq t \leq T$, i.e. W is a Gaussian process with expectation function $a(t)$ and covariance function $R(s,t) = \min(b(s), b(t))$.

Lemma 2. Let $Z_1, \ldots, Z_n$ be independent random variables with $P(|Z_i| > 1) = 0$, $i = 1, \ldots, n$. If $P(|Z_1 + \cdots + Z_n| \geq a) \leq \frac{1}{8e}$ where e is the basis of the natural logarithm then there exist constants L_m depending on m only so that

$$\mathsf{E}|Z_1 + \cdots + Z_n|^m \leq L_m(1+a)^m \tag{4}$$

The proofs of Lemma 1 and 2 may be found in [3].

Lemma 3. Let $X(t)$, $0 \leq t \leq T$, be a continuous first order autoregressive stochastic process with $\alpha \in \mathcal{C} \cap \mathcal{U}$. If $\mathsf{E}|X(0)|^m < \infty$ for every $m > 0$ then

$$\mathsf{E}|X(t)|^m < \infty \tag{5}$$

for every $0 \leq t \leq T$, $m > 0$.

P r o o f. Let $0 \leq t \leq T$ be fixed and define $t_{k,n} = \frac{kt}{n}$, $k = 0, \ldots, n$. Put

$$Y_{k,n} = X(t_{k,n}) - \alpha(t_{k-1,n}, t_{k,n}) X(t_{k-1,n})$$

for $k = 1, \ldots, n$ and set $Y_{0,n} = X(0)$. Using the abbreviation

$$\beta_{k,n} = \prod_{l=k}^{n} \alpha(t_{l-1,n}, t_{l,n})$$

we get

$$X(t) = Y_{n,n} + \beta_{n,n} Y_{n-1,n} + \cdots + \beta_{2,n} Y_{1,n} + \beta_{1,n} Y_{0,n}$$

Introduce the event $A_{k,n}$ by

$$A_{k,n} = \{|\beta_{k,n} Y_{k-1,n}| \leq 1\}.$$

Let $I(A)$ denote the indicator function of the event A. To approximate $X(t)$ we introduce $X_n(t)$ by

$$X_n(t) = \sum_{k=2}^{n} \beta_{k,n} Y_{k-1,n} I(A_{k,n}).$$

Then

$$P(|X(t)-\beta_{1,n}X(0)-X_n(t)|>\varepsilon)\le P\left(\max_{1\le k\le n-1}|Y_{k,n}|>\frac{1}{d(t)}\right)$$
$$\le P\left(\max_{1\le k\le n-1}|X(t_{k,n})-X(t_{k-1,n})|>\frac{1}{2d(t)}\right)+$$
$$+P\left(\left(\max_{1\le k\le n-1}|1-\alpha(t_{k-1,n},t_{k,n})|\right)\sup_{0\le s\le t}|X(s)|>\frac{1}{2d(t)}\right)$$

The continuity of the process X ensures that the first term on the right hand side tends to zero as $n\to\infty$. By the continuity of X again we get $\sup_{0\le s\le t}|X(s)|<\infty$ a.s. As $\alpha(s,t)$ is uniformly continuous for $0\le s\le t\le T$ the second term also tends to zero as $n\to\infty$. Hence

$$\lim_{n\to\infty}P(|X(t)-\beta_{1,n}X(0)-X_n(t)|>\varepsilon)=0 \tag{6}$$

Note also that

$$P(|X_n(t)|>a)\le P(|X(t)-\beta_{1,n}X(0)-X_n(t)|>\varepsilon)+P(|X(t)-\beta_{1,n}X(0)|>a-\varepsilon)$$
$$\le P\left(|X(t)|>\frac{a-\varepsilon}{2}\right)+P\left(|X(0)|>\frac{a-\varepsilon}{2d(t)}\right)+P(|X(t)-\beta_{1,n}X(0)-X_n(t)|>\varepsilon)$$

Hence for all sufficiently large a and n the sum on the right hand side does not exceed $\frac{1}{8e}$. Thus we may apply Lemma 2 and obtain

$$\mathsf{E}|X_n(t)|^m\le L_m(1+a)^m.$$

Hence by the Minkowski inequality for $m>1$

$$(\mathsf{E}|\beta_{1,n}X(0)+X_n(t)|^m)^{1/m}\le d(t)(\mathsf{E}|X(0)|^m)^{1/m}+(L_m(1+a)^m)^{1/m} \tag{7}$$

Using this inequality with $m+\delta$, $\delta>0$, instead of m we see that the sequence $|\beta_{1,n}X(0)+X_n(t)|^m$ is uniformly integrable, $n=1,2,\ldots$. The rest follows from (6) and (7). □

Proof of Theorem 2. As $\mathsf{E}X^2(t)<\infty$ the expectation $a(t)=\mathsf{E}X(t)$ also exists and it is easy to see that the process $X-a$ is autoregressive of first order iff the process X has the same property. Hence we may assume $\mathsf{E}X(t)=0$, $0\le t\le T$, without loss of generality in the sequel.

Put $R(s,t)=\mathsf{E}X(s)X(t)$. For $0<s<t<T$ the random variables $X(0), X(s)-\alpha(0,s)X(0)$, $X(t)-\alpha(s,t)X(s)$ are independent. Hence $X(s)$ and $X(t)-\alpha(s,t)X(s)$ are also independent and we get;

$$R(s,t)=\alpha(s,t)R(s,s) \tag{8}$$

If $R(t_0,t_0)=0$ for some $0<t_0\le T$ then $R(s,t_0)=0$ for all s for which t_0-s is sufficiently small as the assumption $\alpha\in\mathcal{C}$ implies $\alpha(s,t_0)\ne 0$. But then (8) yields $R(s,s)=0$. Hence there are only two cases

1. $R(s,s) > 0$ for every $0 < s \le T$.

2. There exists $0 < t_0 \le T$ with $R(t_0, t_0) = 0$ and $R(s,s) = 0$ which means $X(s) = 0$ for every $0 \le s \le t_0$, where $X(0) = 0$ follows from the continuity of the process X.

In the second case we shift the origin to t_0 in order to get a process $\tilde{X}$ for which $\tilde{R}(t,t) = \mathsf{E}\tilde{X}^2(t) > 0$ for every $0 < t \le \tilde{T} = T - t_0$ The above consideration shows that we may assume $R(s,s) > 0$ for every $0 < s \le T$ without loss of generality.

Assume now $0 < s \le t \le u \le T$. Then by the independence of $X(t) - \alpha(s,t)X(s)$ and $X(u) - \alpha(t,u)X(t)$ and (8)

$$0 = \mathsf{E}(X(t) - \alpha(s,t)X(s))(X(u) - \alpha(t,u)X(t)) = R(s,t)(\alpha(s,u) - \alpha(s,t)\alpha(t,u))$$

$R(s,s) > 0$ and (8) imply for every $0 < s \le t \le u \le T$

$$\alpha(s,t)(\alpha(s,u) - \alpha(s,t)\alpha(t,u)) = 0 \tag{9}$$

The continuity of α shows that this relation also holds for $0 \le s \le t \le u \le T$. Hence

$$\alpha(0,t)(\alpha(0,u) - \alpha(0,t)\alpha(t,u)) = 0 .$$

If $\alpha(0,u_0) = 0$ for some $0 < u_0 \le T$ then $\alpha \in \mathcal{C}$ implies $\alpha(0,t) = 0$ for all t for which $u_0 - t$ is sufficiently small. Hence under the assumption $\alpha(0,u_0) = 0$ we have $\inf\{u : \alpha(0,u) = 0\} = 0$ in contradiction to $\alpha(0,0) = 1$. This means $\alpha(0,u) \ne 0$ for every $0 \le u \le T$. Introduce the process W by

$$W(t) = (\alpha(0,t))^{-1} X(t) - X(0) .$$

W is a continuous stochastic process with $W(0) = 0$. Furthermore we get from (9) the relation $\alpha(0,s)\alpha(s,t) = \alpha(0,t)$. Hence $W(t) - W(s) = \frac{1}{\alpha(0,t)}(X(t) - \alpha(s,t)X(s))$. As X is a first order autoregressive process we see that W is a continuous stochastic process with independent increments. Lemma 1 yields that W is Gaussian process. The definition of an autoregressive process implies that $X(0)$ is independent of the process W. Since $X(0)$ was supposed to be normally distributed and

$$X(t) = \alpha(0,t)(W(t) + X(0))$$

the process X is again a Gaussian process and Theorem 2 is established. □

The proof of Theorem 1 is a consequence of Lemma 3 and Theorem 2. The Corollary follows from the Proposition and Theorem 1 and 2.

(Received May 6, 1991.)

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