TIGHTER SETTLING-TIME BOUNDS FOR FIXED-TIME STABLE SYSTEMS

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This paper addresses the problem of refining settling-time upper-bound estimates for fixed-time stable systems under a three-term Lyapunov characterization. Without resorting to inequality-based techniques or special functions, a closed-form analytical upper bound for the settling-time function is derived using only elementary functions. Comparative analyses demonstrate that the proposed bound is tighter than existing results. The theoretical findings are further validated through the design of a fixed-time controller for a memristive neural network, with numerical simulations confirming the accuracy of the estimated settling time and the independence of the control scheme from initial conditions.

Keywords: fixed-time stability, settling-time estimation, Lyapunov inequality, nonlinear systems, generalized homogeneity

Classification: 93C10, 93D40, 93D05

1. INTRODUCTION

Fixed-time stability (FXTS) [20] addresses a critical limitation of finite-time stability [2] by removing the dependence of the settling time on initial conditions. This feature ensures a pre-computable, uniform upper bound on the convergence time that is invariant with respect to the initial states, thereby enabling reliable deployment across a wide range of applications, including mechanical systems [16, 22, 26], power systems [7, 9, 17], image encryption [24, 28], and neural networks [11, 23, 25].

The Lyapunov-based characterization of FXTS has evolved significantly since Polyakov's seminal work, which established a foundational framework using the condition $\dot{V}(x) \leq -\big(aV^{\alpha}(x) + bV^{\beta}(x)\big)^{\chi}$ with positive parameters satisfying $\alpha\chi < 1$ and $\beta\chi > 1$ [20]. Subsequent studies have primarily aimed at tightening the associated settling-time estimates. Parsegov et al. introduced an alternative characterization by selecting the special parameterization $\chi = 1$, $\alpha = 1 - 1/u$, and $\beta = 1 + 1/u$ with $u > 1$, which leads to a closed-form and less conservative bound [19]. Zuo and Tie developed distributed robust fixed-time (globally finite-time) consensus protocols for multi-agent systems subject to external disturbances, guaranteeing that the settling time is uniformly bounded by a constant depending solely on design parameters and independent of initial conditions [29]. More recently, Aldana-López et al. derived the least upper bound of the

DOI: 10.14736/kyb-2026-3-0546

settling-time function based on Polyakov's characterization [1]. By circumventing the inequality-based estimation techniques employed in earlier works, their approach resolved a long-standing issue and reduced the conservatism inherent in previous bounds. Guo et al. further advanced the two-term fixed-time framework to high-order uncertain nonlinear systems under output-feedback control, and established a novel settling-time solution method together with an observer-based controller synthesis [8].

To further improve convergence performance beyond the classical two-term Lyapunov characterization, Chen et al. introduced an additional linear term into the Lyapunov derivative, leading to $\dot{V}(x) \leq -aV^{\alpha}(x) - bV^{\beta}(x) - kV(x)$ with $k > 0$, $0 < \alpha < 1$, and $\beta > 1$ [4]. By combining definite integrals, scaling inequalities, and variable substitutions, they obtained improved upper bounds for the settling time. Hu et al. subsequently relaxed the parameter conditions to $k \geq 0$ and $0 \leq \alpha < 1$. Through a more refined integral-based analysis, they established tighter and more unified settling-time estimates, and further applied the generalized framework to fixed-time synchronization of complex networks [12]. Cai et al. developed a novel Lyapunov framework with an indefinite derivative for the fixed-time stability analysis of discontinuous systems, and applied it to the fixed-time synchronization of fuzzy neural networks under a switching control protocol [3]. Nevertheless, the continued reliance on inequality-based scaling arguments in these works still tends to introduce conservatism into the settling-time analysis.

Fixed-time estimation has been revisited from the perspective of special functions, leading to highly accurate settling-time formulas for a class of fixed-time stable systems [10]. While such special-function-based expressions enhance estimation accuracy, they often reduce transparency and complicate direct evaluation as well as parameter tuning in practical applications. More recently, Mapui et al. investigated the optimal upper bound of the settling time for a class of fixed-time stable systems and obtained estimates expressible in elementary functions [15]. However, the analysis still relied on inequality-based techniques. Consequently, deriving less conservative, or even optimal, upper bounds on the settling time associated with three-term Lyapunov inequalities remains a challenging problem.

The analytical difficulty arises from the intractable integral $\int (av^{\alpha} + bv^{\beta} + kv)^{-1}\mathrm{d}v$ for $0 < \alpha < 1$ and $\beta > 1$, which generally admits no closed-form expression without resorting to inequality-based approximations. To address this limitation, the constraint $\alpha + \beta = 2$ is imposed. From the viewpoint of homogeneity [21], this choice gives the fixed-time mechanism a symmetric bi-limit structure around the linear term: the low-order term $V^{\alpha}(x)$ dominates near the origin, while the high-order term $V^{\beta}(x)$ dominates for large states. It also allows the local nonlinear convergence strength and the global restoring effect to be coordinated by a single parameter. This structural symmetry is reflected analytically in the associated integral transformation, which makes the settling-time bound explicitly computable in elementary functions.

Motivated by these developments and inspired by the emphasis of homogeneity-based fixed-time analysis on explicit settling-time characterization, this paper develops an analytical approach for deriving a tighter settling-time upper bound under a three-term Lyapunov characterization. By relying solely on elementary functions, a closed-form analytical upper bound for the settling-time function is derived, which is shown to be tighter than existing estimates.

The remainder of this paper is organized as follows. Section 2 introduces the preliminaries, while Section 3 presents the main theoretical results. Numerical results are provided in Section 4, and concluding remarks are given in Section 5.

2. PRELIMINARIES

This section establishes the foundational concepts for the analysis of fixed-time stable systems. Consider the following nonlinear control system:

$$\dot{x} = f(x,t) + u, \quad x(0) = x_0, \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{R}^n$ denotes the state vector, $u = (u_1, u_2, \ldots, u_n)^\top \in \mathbb{R}^n$ denotes the control input, and $f(x,t) : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ is a nonlinear vector field with components $f(x,t) = (f_1(x,t), f_2(x,t), \ldots, f_n(x,t))^\top$. If the function $f(x,t)$ or the controller is discontinuous, the solutions of the corresponding system are understood in the Filippov sense.

Definition 2.1. (Bhat and Bernstein [2]) The origin of system (1) is said to be globally finite-time stable if:

1. It is globally asymptotically stable,
2. Every solution $x(t)$ reaches the equilibrium in finite time, that is, there exists a settling-time function $T(x_0) : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ such that $x(t) = 0$ for all $t \geq T(x_0)$.

Definition 2.2. (Polyakov [20]) The origin is defined as fixed-time stable if it is globally finite-time stable and there exists a constant $T_{\max} > 0$, independent of the initial condition x_0, such that the settling-time function satisfies $T(x_0) \leq T_{\max}$ for all $x_0 \in \mathbb{R}^n$.

Lemma 2.3. For the integral of the reciprocal of a quadratic function: $I = \int \frac{\mathrm{d}v}{av^2+bv+c}$, the result is given in the following form:

$$I = \begin{cases} \dfrac{2}{\sqrt{4ac-b^2}} \arctan \dfrac{2av+b}{\sqrt{4ac-b^2}} + C, & b^2 < 4ac, \\ -\dfrac{1}{av + \dfrac{b}{2}} + C, & b^2 = 4ac, \\ \dfrac{1}{\sqrt{b^2-4ac}} \ln \left| \dfrac{2av+b-\sqrt{b^2-4ac}}{2av+b+\sqrt{b^2-4ac}} \right| + C, & b^2 > 4ac. \end{cases}$$

Lemma 2.4. (Li et al. [14]) For nonnegative real numbers $z_1, z_2, \ldots, z_n$, the following inequalities hold:

$$\Big(\sum_{i=1}^n z_i\Big)^p \leq \sum_{i=1}^n z_i^p, \quad \forall\, 0 < p \leq 1,$$

$$n^{1-p}\Big(\sum_{i=1}^n z_i\Big)^p \leq \sum_{i=1}^n z_i^p, \quad \forall\, p > 1.$$

3. MAIN RESULTS

This section establishes sufficient conditions for FXTS and derives tighter settling-time bounds.

3.1. Fixed-time convergence

Theorem 3.1. Let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be a C-regular function for system (1), and suppose there exist constants $\mathcal{A}, \mathcal{B} > 0$, $\mathcal{K} > -2\sqrt{\mathcal{A}\mathcal{B}}$, and parameters $0 \le \alpha < 1$, $\beta > 1$ satisfying $\alpha + \beta = 2$ such that

$$\dot{V}(x) \le -\mathcal{A}V^{\alpha}(x) - \mathcal{B}V^{\beta}(x) - \mathcal{K}V(x), \tag{2}$$

then the origin of system (1) is fixed-time stable, and the settling time $T_s(x_0)$ satisfies $T_s(x_0) \le T_{\max} = T_1$, where

$$T_1 = \begin{cases} \dfrac{2}{(\beta-1)\sqrt{-\Delta}}\left(\dfrac{\pi}{2} - \arctan\dfrac{\mathcal{K}}{\sqrt{-\Delta}}\right), & \Delta < 0, \\ \dfrac{2}{(\beta-1)\mathcal{K}}, & \Delta = 0, \\ \dfrac{1}{(\beta-1)\sqrt{\Delta}}\ln\left|\dfrac{\mathcal{K}+\sqrt{\Delta}}{\mathcal{K}-\sqrt{\Delta}}\right|, & \Delta > 0, \end{cases} \tag{3}$$

with $\Delta = \mathcal{K}^2 - 4\mathcal{A}\mathcal{B}$.

Proof. It follows from inequality (2) that the settling time is bounded by

$$\begin{aligned} T_s(x_0) &= \int_0^{T_s(x_0)} \mathrm{d}t \\ &\le \int_0^{+\infty} \frac{\mathrm{d}V(x)}{\mathcal{A}V^{\alpha}(x) + \mathcal{B}V^{\beta}(x) + \mathcal{K}V(x)}. \end{aligned}$$

Substituting $\alpha = 2 - \beta$ gives

$$\begin{aligned} T_s(x_0) &\le \int_0^{+\infty} \frac{\mathrm{d}V(x)}{\mathcal{A}V^{2-\beta}(x) + \mathcal{B}V^{\beta}(x) + \mathcal{K}V(x)} \\ &= \frac{1}{\beta-1}\int_0^{+\infty} \frac{\mathrm{d}(V^{\beta-1}(x))}{\mathcal{B}(V^{\beta-1}(x))^2 + \mathcal{K}V^{\beta-1}(x) + \mathcal{A}}. \end{aligned} \tag{4}$$

By Lemma 2.3, the integral can be evaluated for the three cases of $\Delta = \mathcal{K}^2 - 4\mathcal{A}\mathcal{B}$.

Case 1: $\Delta < 0$

Eq. (4) becomes

$$T_s(x_0) \le \frac{1}{\beta-1}\left[\frac{2}{\sqrt{4\mathcal{A}\mathcal{B} - \mathcal{K}^2}} \arctan \frac{2\mathcal{B}V^{\beta-1}(x) + \mathcal{K}}{\sqrt{4\mathcal{A}\mathcal{B} - \mathcal{K}^2}}\Bigg|_0^{+\infty}\right]$$

$$= \frac{1}{\beta - 1}\left[\frac{2}{\sqrt{4\mathcal{A}\mathcal{B} - \mathcal{K}^2}}\left(\frac{\pi}{2} - \arctan\frac{\mathcal{K}}{\sqrt{4\mathcal{A}\mathcal{B} - \mathcal{K}^2}}\right)\right]$$

$$= \frac{2}{(\beta - 1)\sqrt{-\Delta}}\left(\frac{\pi}{2} - \arctan\frac{\mathcal{K}}{\sqrt{-\Delta}}\right).$$

Case 2: $\Delta = 0$

$$T_s(x_0) \leq \frac{1}{\beta - 1}\left[-\frac{1}{(\mathcal{B}V^{\beta-1}(x) + \frac{\mathcal{K}}{2})}\Bigg|_0^{+\infty}\right]$$

$$= \frac{2}{(\beta - 1)\mathcal{K}}.$$

Case 3: $\Delta > 0$

$$T_s(x_0) \leq \frac{1}{\beta - 1}\left[\frac{1}{\sqrt{\mathcal{K}^2 - 4\mathcal{A}\mathcal{B}}}\ln\left|\frac{2\mathcal{B}V^{\beta-1}(x) + \mathcal{K} - \sqrt{\mathcal{K}^2 - 4\mathcal{A}\mathcal{B}}}{2\mathcal{B}V^{\beta-1}(x) + \mathcal{K} + \sqrt{\mathcal{K}^2 - 4\mathcal{A}\mathcal{B}}}\right|\Bigg|_0^{+\infty}\right]$$

$$= \frac{1}{(\beta - 1)}\left[\frac{1}{\sqrt{\mathcal{K}^2 - 4\mathcal{A}\mathcal{B}}}\left(\ln|1| - \ln\left|\frac{\mathcal{K} - \sqrt{\mathcal{K}^2 - 4\mathcal{A}\mathcal{B}}}{\mathcal{K} + \sqrt{\mathcal{K}^2 - 4\mathcal{A}\mathcal{B}}}\right|\right)\right]$$

$$= \frac{1}{(\beta - 1)\sqrt{\Delta}}\ln\left|\frac{\mathcal{K} + \sqrt{\Delta}}{\mathcal{K} - \sqrt{\Delta}}\right|.$$

By Definition 2.2, the origin of system (1) is fixed-time stable, with the upper bound of the settling time given by Eq. (3). □

Corollary 3.2. If $V(x) : \mathbb{R}^n \to \mathbb{R}$ is C-regular for system (1), and there exist constants $\mathcal{A}, \mathcal{B} > 0$, $\mathcal{K} > -2\sqrt{\mathcal{A}\mathcal{B}}$, and $0 < \eta \leq 1$ such that

$$\dot{V}(x) \leq -\mathcal{A}V^{1-\eta}(x) - \mathcal{B}V^{1+\eta}(x) - \mathcal{K}V(x),$$

then the origin of system (1) is fixed-time stable, with the settling-time bound T_1^* given by

$$T_1^* = \begin{cases} \dfrac{2}{\eta\sqrt{-\Delta}}\left(\dfrac{\pi}{2} - \arctan\dfrac{\mathcal{K}}{\sqrt{-\Delta}}\right), & \Delta < 0, \\ \dfrac{2}{\eta\mathcal{K}}, & \Delta = 0, \\ \dfrac{1}{\eta\sqrt{\Delta}}\ln\left|\dfrac{\mathcal{K} + \sqrt{\Delta}}{\mathcal{K} - \sqrt{\Delta}}\right|, & \Delta > 0, \end{cases}$$

with $\Delta = \mathcal{K}^2 - 4\mathcal{A}\mathcal{B}$.

P r o o f. The corollary follows directly from Lemma 2.3 and Theorem 3.1. □

Remark 3.3. By parameterizing the exponents as $\alpha = 1 - \eta$ and $\beta = 1 + \eta$ with $\eta \in (0, 1]$, the low-order term $V^{\alpha}(x)$ and the high-order term $V^{\beta}(x)$ exhibit homogeneity degrees that are exactly symmetric about the zero-degree linear term $V(x)$ [21]. This gives rise to a symmetric bi-limit structure: the low-order term $V^{\alpha}(x)$ contributes the negative-degree mechanism that ensures finite-time compression near the origin, while the high-order term $V^{\beta}(x)$ contributes the positive-degree mechanism that enhances the restoring effect for large states. Hence, the local convergence behavior and the global attractive behavior are unified through the single parameter η, which makes the exponent structure more interpretable and easier to tune. Furthermore, this symmetry is also reflected in the settling-time integral, which becomes exactly computable after a suitable change of variables. Therefore, the proposed analysis produces a closed-form settling-time upper bound in elementary functions, preserves the explicit role of the linear term $V(x)$, and reveals a clearer quantitative relationship between the design parameters and the convergence-time bound.

3.2. Analysis and Comparison of Settling-Time Bounds

From Eq. (3), it is evident that T_1 decreases monotonically with respect to both $\mathcal{K}$ and β within the domain of definition. Notably, as $\mathcal{K}$ approaches $-2\sqrt{\mathcal{AB}}$, T_1 tends to infinity, reflecting the singularity in the denominator of the time-estimation formula. For the specific case $\mathcal{A} = \mathcal{B} = 1$, the dependence of T_1 with respect to $\mathcal{K}$ and β is visualized in Figure 1, which includes a 3D surface plot and its 2D projections. Among the parameters influencing T_1, $\mathcal{K}$ plays a dominant role in reducing the upper bound of the settling time.

The optimization of fixed-time convergence bounds has attracted considerable attention. In [4], the Lyapunov condition (2) leads to the following upper bound:

$$T_2 = \frac{1}{\mathcal{K}(1-\alpha)} \ln\left(1 + \frac{\mathcal{K}}{\mathcal{A}}\right) + \frac{1}{\mathcal{K}(\beta-1)} \ln\left(1 + \frac{\mathcal{K}}{\mathcal{B}}\right).$$

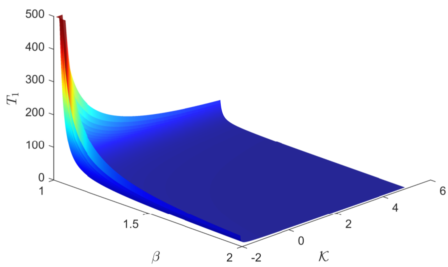
In [12], the Lyapunov condition (2) yields an alternative upper bound:

$$T_3 = \frac{\pi}{(\beta-\alpha)\mathcal{B}} \left(\frac{\mathcal{B}}{\mathcal{A}}\right)^{\frac{1-\alpha}{\beta-\alpha}} \csc\left(\frac{1-\alpha}{\beta-\alpha}\pi\right).$$

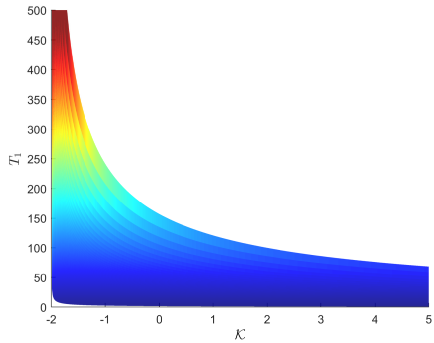
Notably, T_3 is independent of the coefficient $\mathcal{K}$. This arises because the linear term $\mathcal{K}V(x)$ is omitted through inequality scaling during the settling-time estimation to simplify integration. As a result, the upper bound T_3 is insensitive to $\mathcal{K}$, implying that $\dot{V}(x) \leq -\mathcal{A}V^{\alpha}(x) - \mathcal{B}V^{\beta}(x) - \mathcal{K}V(x)$ and $\dot{V}(x) \leq -\mathcal{A}V^{\alpha}(x) - \mathcal{B}V^{\beta}(x)$ produce the same estimate. However, as noted in Remark 3.3, the three-term Lyapunov framework explicitly retains the linear term, yielding a tighter bound $T_1 < T_3$ whenever $\mathcal{K} > 0$.

To illustrate this, consider the case $\alpha + \beta = 2$. When $\mathcal{K} = 0$ (so that $\Delta < 0$), the expression for T_1 is given by

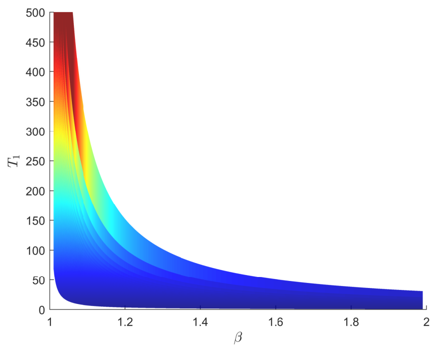
$$T_1 = \frac{\pi}{2(\beta-1)\sqrt{\mathcal{AB}}}.$$



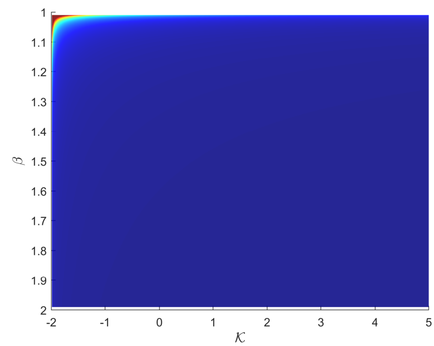
(a) Three-dimensional surface of $T_1(\beta, \mathcal{K})$



(b) Projection onto the $\mathcal{K}$-T_1 plane



(c) Projection onto the β-T_1 plane



(d) Projection onto the $\mathcal{K}$-β plane

Fig. 1. Surface and projections of the settling-time bound $T_1(\beta, \mathcal{K})$ for $\mathcal{A} = \mathcal{B} = 1$.

For the settling time T_3, substituting $\alpha = 2 - \beta$ into its expression gives

$$\begin{aligned} T_3 &= \frac{\pi}{(2\beta - 2)\mathcal{B}} \left(\frac{\mathcal{B}}{\mathcal{A}}\right)^{\frac{1}{2}} \csc\left(\frac{\beta - 1}{2\beta - 2}\pi\right) \\ &= \frac{\pi}{2(\beta - 1)\mathcal{B}} \sqrt{\frac{\mathcal{B}}{\mathcal{A}}} \cdot 1 \\ &= \frac{\pi}{2(\beta - 1)\sqrt{\mathcal{A}\mathcal{B}}} = T_1. \end{aligned}$$

This confirms that $T_1 = T_3$ when $\alpha + \beta = 2$ and $\mathcal{K} = 0$.

However, for $\mathcal{K} > 0$, T_1 decreases as $\mathcal{K}$ increases, resulting in $T_1 < T_3$. This behavior is confirmed numerically in Table 1, which shows that $T_1 \leq T_3$ across all tested parameter sets, while T_3 remains invariant to $\mathcal{K}$. Therefore, when $\alpha + \beta = 2$, the proposed method provides a tighter upper bound for the fixed-time convergence. For cases where $\alpha + \beta \neq 2$, alternative strategies such as those in [3–5, 12, 13, 18, 27] offer practical and effective approaches.

T_1 (Proposed)	T_2 ([4])	T_3 ([12])	$(\mathcal{A}, \mathcal{B}, \mathcal{K}, \alpha, \beta)$
2.4184	2.7726	3.1416	$(1, 1, 1, 0.5, 1.5)$
2.0153	2.3105	2.6180	$(1, 1, 1, 0.4, 1.6)$
1.2092	1.3863	1.5708	$(1, 1, 1, 0.0, 2.0)$
2.0000	2.1972	3.1416	$(1, 1, 2, 0.5, 1.5)$
1.6667	1.8310	2.6180	$(1, 1, 2, 0.4, 1.6)$
1.0000	1.0986	1.5708	$(1, 1, 2, 0.0, 2.0)$
1.7216	1.8484	3.1416	$(1, 1, 3, 0.5, 1.5)$
1.4347	1.5403	2.6180	$(1, 1, 3, 0.4, 1.6)$
0.8608	0.9242	1.5708	$(1, 1, 3, 0.0, 2.0)$

Tab. 1. Comparison of T_1, T_2, and T_3 for various parameter sets $(\mathcal{A}, \mathcal{B}, \mathcal{K}, \alpha, \beta)$

Remark 3.4. Although both T_2 and T_3 include the case $\alpha + \beta = 2$, Table 1 reveals that T_1 is not a special case of either T_2 or T_3 when $\mathcal{K} > 0$. This underscores the advantage of the proposed method in utilizing the full three-term structure to obtain tighter bounds.

3.3. Fixed-time control

In this section, the stabilization problem of system (1) is investigated. To this end, the following controller is designed, that is,

$$u_i = -f_i(x,t) - \mathcal{A}\,\mathrm{sign}(x_i)|x_i|^{\alpha} - \mathcal{B}n^{\frac{\beta-1}{2}}\,\mathrm{sign}(x_i)|x_i|^{\beta} - \mathcal{K}x_i, \quad i = 1, 2, \dots, n, \tag{5}$$

where n is the system dimension, $\mathcal{A}, \mathcal{B} > 0$, $\mathcal{K} > -2\sqrt{\mathcal{A}\mathcal{B}}$, and $\alpha \in [0, 1)$, $\beta > 1$ satisfying $\alpha + \beta = 2$.

Theorem 3.5. Under the control law (5), system (1) achieves FXTS with the settling time bounded by T_1.

P r o o f. Define the Lyapunov function candidate as

$$V(x) = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}.$$

Differentiating $V(x)$ along the system trajectories yields

$$\begin{aligned} \dot{V}(x) &= \frac{1}{V(x)}\sum_{i=1}^{n}(x_i\dot{x}_i) \\ &= \frac{1}{V(x)}\sum_{i=1}^{n}\left(-\mathcal{A}|x_i|^{\alpha+1} - \mathcal{B}n^{\frac{\beta-1}{2}}|x_i|^{\beta+1} - \mathcal{K}x_i^2\right). \end{aligned}$$

By applying Lemma 2.4, the following result is obtained:

$$\begin{aligned}\dot{V}(x) &= -\frac{1}{V(x)}\Big[\mathcal{A}\sum_{i=1}^{n}\Big(|x_i|^2\Big)^{\frac{\alpha+1}{2}} + \mathcal{B}n^{\frac{\beta-1}{2}}\sum_{i=1}^{n}\Big(|x_i|^2\Big)^{\frac{\beta+1}{2}} + \sum_{i=1}^{n}\mathcal{K}x_i^2\Big]\\ &\le -\frac{1}{V(x)}\Big(\mathcal{A}V^{\alpha+1}(x) + \mathcal{B}V^{\beta+1}(x) + \mathcal{K}V^2(x)\Big)\\ &= -\mathcal{A}V^{\alpha}(x) - \mathcal{B}V^{\beta}(x) - \mathcal{K}V(x).\end{aligned}$$

According to Theorem 3.1, the origin of system (1) achieves FXTS with the settling time T_1. □

Remark 3.6. From $\dot{V}(x) \le -\mathcal{A}V^{\alpha}(x) - \mathcal{B}V^{\beta}(x) - \mathcal{K}V(x)$, let $\Xi(x) = V^{1-\alpha}(x)$ and $\delta = \frac{\beta-\alpha}{1-\alpha} > 1$. Substituting these expressions into the above inequality yields:

$$\dot{\Xi}(x) \le -\mathcal{A}(1-\alpha) - \mathcal{B}(1-\alpha)\Xi^{\delta}(x) - \mathcal{K}(1-\alpha)\Xi(x). \tag{6}$$

When $\alpha + \beta = 2$, it follows that $\delta = 2$, and the inequality reduces to a quadratic form:

$$\dot{\Xi}(x) \le -\mathcal{B}(1-\alpha)\Xi^{2}(x) - \mathcal{K}(1-\alpha)\Xi(x) - \mathcal{A}(1-\alpha). \tag{7}$$

By exploiting inequalities (6) and (7), one can construct appropriate fixed-time controllers [12].

4. NUMERICAL SIMULATIONS

To validate the proposed control scheme (5), a two-dimensional memristive neural network [6] governed by the following equations is considered:

$$\dot{x}_m(t) = -\,d_m\big(x_m(t)\big)x_m(t) + \sum_{s=1}^{2} a_{ms}\big(x_m(t)\big)f(x_s(t)) + u_m(t), \quad m = 1, 2, \tag{8}$$

where $x_m(t)$ are state variables, $f(x_s) = \tanh(x_s)$ are activation functions, and $u_m(t)$ are control inputs.

The memristor-based parameters exhibit state-dependent switching behavior as follows:

$$d_1(x_1) = \begin{cases}1.2, & x_1 < 1,\\ 1.0, & x_1 \ge 1\end{cases} \quad d_2(x_2) = \begin{cases}1.0, & x_2 < 1,\\ 1.2, & x_2 \ge 1,\end{cases}$$

$$a_{11}(x_1) = \begin{cases}3.0, & x_1 < 1,\\ 2.9, & x_1 \ge 1,\end{cases} \quad a_{12}(x_1) = \begin{cases}4.9, & x_1 < 1,\\ 4.8, & x_1 \ge 1,\end{cases}$$

$$a_{21}(x_2) = \begin{cases}0.1, & x_2 < 1,\\ 0.08, & x_2 \ge 1,\end{cases} \quad a_{22}(x_2) = \begin{cases}2.0, & x_2 < 1,\\ 1.9, & x_2 \ge 1.\end{cases}$$

The initial conditions are set as $(x_1(0), x_2(0)) = (1.6, -1)$, and the controllers $u_m(t)$ are designed according to Eq. (5). Two simulation scenarios are conducted to evaluate the effects of the parameters.

- Case A: $(\mathcal{A}, \mathcal{B}, \mathcal{K}, \alpha, \beta) = (1, 1, 1, 0.5, 1.5)$, resulting in $T_1 = 2.4184$.
- Case B: $(\mathcal{A}, \mathcal{B}, \mathcal{K}, \alpha, \beta) = (1, 1, 3, 0, 2)$, resulting in $T_1 = 0.8608$.

The simulation results are shown in Figures 2, 3, and 4. Figure 2 shows the state trajectories of the closed-loop system under two parameter sets, where the theoretical settling-time bounds T_1, T_2, and T_3 are explicitly marked for visual comparison. It can be observed that the system states converge to zero within these theoretical bounds in both cases, and the comparison also provides a more intuitive illustration of the relative tightness of different settling-time estimates. The corresponding control inputs are displayed in Figure 3. To illustrate that the proposed fixed-time control strategy is independent of the system's initial states, Figure 4 depicts state trajectories for 20 randomly selected initial values uniformly distributed over the interval $[-100, 100]$ for Case A. These numerical results align with the theoretical bounds listed in Table 1, confirming the effectiveness of the proposed controller.

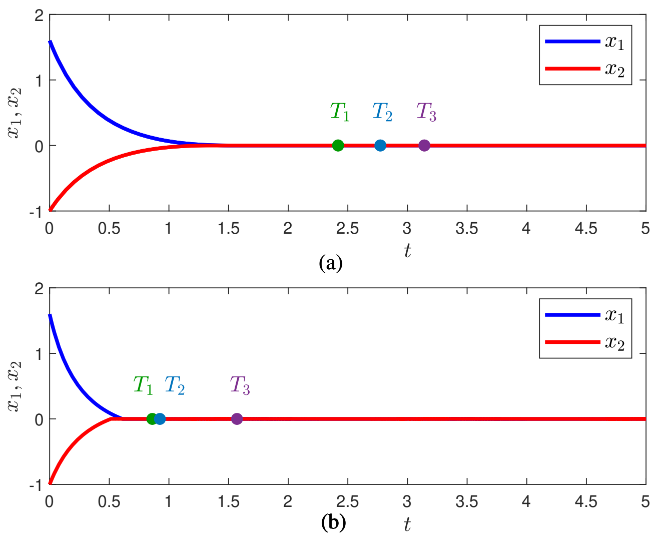


Fig. 2. State trajectories of system (8) under controller (5). (a) x_1, x_2 for Case A, (b) x_1, x_2 for Case B.

Remark 4.1. The simulation results indicate that proper parameter tuning can effectively shorten the settling time. However, such an improvement is usually accompanied by larger control inputs and potentially more severe chattering. In practical implementations, it is therefore necessary to strike a balance among convergence speed, available control effort, and signal smoothness. To reduce oscillations, the discontinuous sign function may be replaced by a smooth approximation, for example, $\tanh(\cdot/\varepsilon)$ with a small $\varepsilon > 0$.

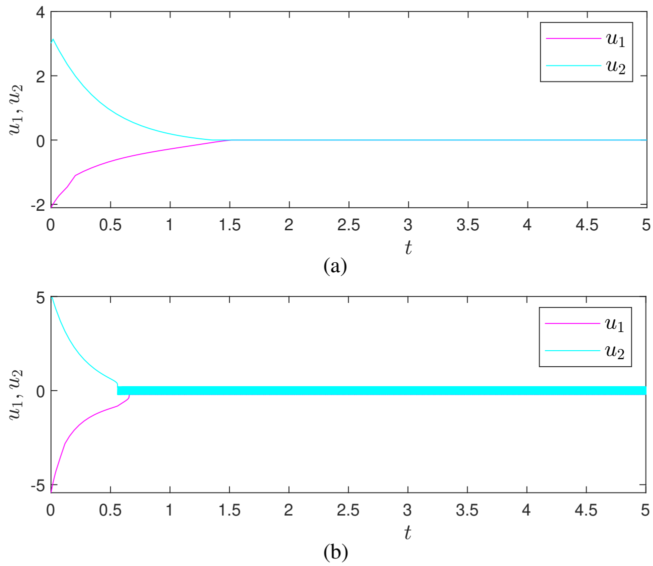


Fig. 3. Control input trajectories. (a) u_1, u_2 for Case A, (b) u_1, u_2 for Case B.

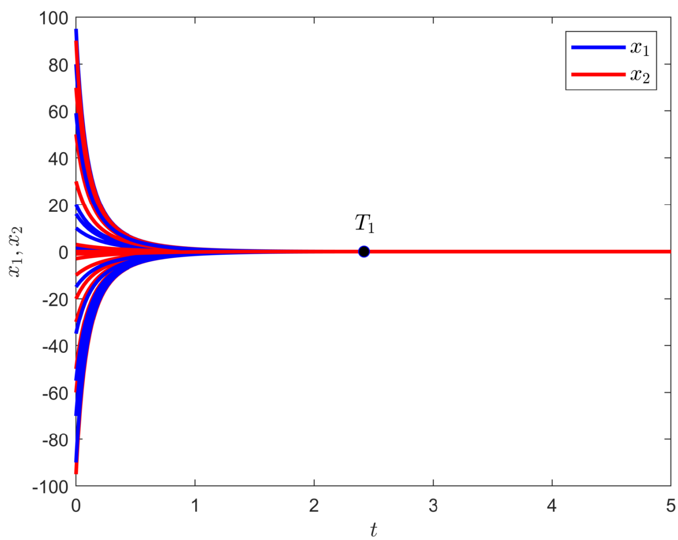


Fig. 4. State trajectories of system (8) under controller (5) and 20 randomly selected initial values in $[-100, 100]$ for Case A.

5. CONCLUSION

This study investigates the problem of reducing conservatism in settling-time upper-bound estimation for fixed-time stable systems under a three-term Lyapunov inequality framework. By transforming the associated settling-time integral into a reciprocal-quadratic form, a closed-form analytical bound is derived using only elementary functions, without resorting to auxiliary inequality techniques or special functions. In addition, from the viewpoint of generalized homogeneity, the adopted Lyapunov structure can be interpreted as possessing a symmetric bi-limit feature, which also helps explain the tractability and reduced conservatism of the resulting estimate. Rigorous theoretical comparisons show that the proposed estimate is tighter than the existing bounds reported in the literature. Numerical examples further validate the effectiveness of the fixed-time controller, confirm the improved accuracy of the proposed settling-time estimate, and verify that the guaranteed settling-time bound is independent of the initial condition.

ACKNOWLEDGEMENT

This work was supported by the Funds of the Natural Science Foundation of Hangzhou (No. 2025SZRJJ1777).

(Received February 8, 2026)

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