ON THE ADVANTAGES OF USING VIRTUAL OUTPUTS TO DESIGN NONLINEAR UNKNOWN-INPUT MIMO OBSERVERS: A NOVEL LMI-BASED SOLUTION

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Sufficient conditions in the form of linear matrix inequalities for design of a novel nonlinear unknown-input MIMO observer are given in this paper. The proposed scheme uses knowledge of real and virtual outputs in order to express the unknown input dynamics, based on which an extended observer-error system is obtained. Asymptotic stability of this system is ensured by means of exact factorization of the error signal, convex rewriting of expressions in a region of interest, finite-time estimation of a set of time derivatives of the system outputs, and the direct Lyapunov method. As examples show, the proposal is able to successfully reconstruct states, inputs, faults, and disturbances, where former methodologies fail.

Keywords: nonlinear observer, linear matrix inequality, virtual outputs, direct Lyapunov method

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1. INTRODUCTION

Motivation: Besides their main goal of estimating states and inputs of a given system based on its mathematical model, unknown-input observers (UIO) are relevant to reconstruct other signals acting on the system such as disturbances and sensor/actuator faults [2, 34]. Such estimation is the first step for diagnosis [16, 25], isolation [29, 42], reconstruction [13, 40], and fault-tolerant control schemes [1, 21]; needless to say, the effectiveness of the latter tasks critically depend on the quality of the former; hence, the motivation of this work.

Background: Standard observers use the information of the input and output to reconstruct the states of a system under the assumption that the model is known and observability/detectability holds, at least locally [17]; their design usually involves the construction of an error system which is driven to 0 by means of a set of gains in order to make the observer states match those of the system. Both in the linear case as well as in some restricted classes of nonlinear systems, the input plays no role as it is cancelled out in the resulting error system. In contrast, UIO design faces a variety of challenges: inputs are unavailable and cannot be fed into the observer [4], input dynamic

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equations are not known [12], conditions for existence of UIOs depend on notions such as distinguishability [12], and observability/detectability concepts require a significant redefinition [23].

Methodologies for UIO design can have one or several of the following characteristics: being based on an algebraic computation of the unknown input [22], requiring a linear nominal model to satisfy a decoupling condition [15], requiring nonlinearities to satisfy a Lipschitz bound [24], being based on the assumption of a finite number of non-zero derivatives of the input [6], adding virtual outputs to solve the unknown signals from an extended set of algebraic relationships [37], using linear matrix inequalities (LMIs) [14], embedding nonlinearities as quasi-linear-parameter-varying (quasi-LPV) terms [7], and implementing a dynamical system for reconstruction of the unknown input [28]. This work involves virtual outputs, unknown input dynamics, quasi-LPV embedding, and LMI design conditions.

Indeed, nonlinear systems are increasingly treated by means of convex techniques inherited from the LPV framework [32]: they handle nonlinearities by rewriting them in a convex form within a region of interest, which allows using the direct Lyapunov method to find design conditions in terms of LMIs [3]. Such formulations are advantageous over tuning approaches due to its numerical efficiency and systematicness [5]; they also deal with multiple-input multiple-output (MIMO) systems in a straightforward manner [35]. As a result, several UIO schemes have been developed following this direction [18], e. g., [14] where nonlinear systems are embedded in quasi-LPV structures and [11] where Takagi-Sugeno models are employed; some others obtain the estimation error system by means of the factorization in [27], e. g., in [7] unknown inputs are algebraically reconstructed from a convex observer and in [10] fault signals are estimated employing a sliding-mode-based convex observer.

Problem statement: UIO design faces several limitations: in the sliding mode framework they require linear nominal systems subject to matching conditions; in the LPV context they assume scheduling variables and their derivatives are measurable; in the quasi-LPV framework they lack a systematic way of writing the observation error system for Lyapunov treatment. The first two approaches do not manage to exploit nonlinearities; the efficiency of the third one is hindered by the missing input dynamics, an obstacle that is poorly circumvented. In [28] a solution to the latter problem was proposed; it is, nevertheless, limited to SISO systems, do not use virtual outputs [7], and lacks a thorough analysis of observability issues [23]. This work addresses these problems.

Methodology: This work employs three methodologies whose details can be found in the referred literature: convex treatment of nonlinear expressions in order to obtain LMI conditions via the direct Lyapunov method [3]; any sort of robust differentiator which converges after a finite process of transient, e. g. [19, 20, 33], and exact factorization of error signals in the observation error system [27]. The numerical solver used throughout this work is the LMI Toolbox of MATLAB [9].

Contribution: This work provides sufficient conditions for LMI-based design of nonlinear unknown-input MIMO observers. The proposal overcomes former problems such as rank conditions, handling of MIMO characteristics, systematic use of real and virtual outputs; and numerical implementability of solutions, among others. These contributions come at the price of solving the unknown input dynamics from successive time derivatives of real and/or virtual outputs via any robust differentiator, convex rewriting of nonlinearities, and exact factorization of error signals.

Organization: Section 2 summarizes the methodologies required to develop our proposal (basics on observability, convex rewriting of expressions, robust differentiators, and factorization of error signals), section 3 develops the main results of this work (sufficient conditions for LMI-based design of nonlinear unknown-input MIMO observers), section 4 provides detailed examples to put at test the proposal and compare it against former methodologies, and finally, section 5 presents some concluding remarks and discusses future work.

2. PRELIMINARIES

Formal basics on convex rewriting of expressions, factorization of error signals, robust differentiators, and observability, are concisely presented in this section. The reason for this is as follows: we will differentiate a set of real and virtual outputs in order to solve equations describing the input dynamics $\dot{u}$ (hence the need of robust differentiators); we will propose the nonlinear UIO based on the system and input dynamics in order to express the resulting error system with the error signal $e = [e_x^T \; e_u^T]^T$, $e_x = \hat{x} - x$, $e_u = \hat{u} - u$, factorized as $\dot{e} = F(\cdot)e$ (hence the need of exact factorization); we will rewrite the error system in a convex form (hence the need of convex rewriting); finally, we will use the direct Lyapunov method to obtain sufficient conditions for asymptotic stability of the origin of the resulting error system (hence the need of observability/detectability issues to be subsumed in the LMI conditions).

2.1. Convex rewriting of nonlinear expressions

Consider a function $z(\cdot) : \mathbb{R}^n \to \mathbb{R}$, which is well-defined $\forall x \in \mathcal{C} \subset \mathbb{R}^n$; thus $z(x) \in [z^0, z^1]$, where $z^0 = \inf_{x\in\mathcal{C}} z(x)$ and $z^1 = \sup_{x\in\mathcal{C}} z(x)$. Let us define

$$w_0(x) \equiv \frac{z^1 - z(x)}{z^1 - z^0}, \; w_1(x) \equiv 1 - w_0(x),$$

then, it is easy to see that

$$z(x) = w_0(x)z^0 + w_1(x)z^1 = \sum_{i=0}^{1} w_i(x)z^i, \tag{1}$$

is an identity that holds $\forall x \in \mathbb{R}^n$. On the other hand, it can be verified that $w_0(x) \in [0, 1]$ and $w_1(x) \in [0, 1]$ are only guaranteed $\forall x \in \mathcal{C}$; it is worth noticing that the last term of (1) is a convex sum of constant terms.

If a collection of terms $z_1(x)$, $z_2(x)$, …, $z_r(x)$, is well-defined $\forall x \in \mathcal{C} \subset \mathbb{R}^n$, then the rewriting above can be easily applied to each of them. To illustrate it, let us introduce the following notation: define $z_i^0 = \inf_{x\in\mathcal{C}} z_i(x)$, $z_i^1 = \sup_{x\in\mathcal{C}} z_i(x)$, $i \in \{1, 2, \dots, r\}$, $\forall x \in \mathcal{C}$; then, the possible scalar, vector or matricial expression $f(z_1, z_2, \dots, z_r)$ can be rewritten as a convex sum:

$$f(z_1, z_2, \dots, z_r) = \sum_{\mathbf{i}\in\mathbb{B}^r} \mathbf{w_i}(x) f_{\mathbf{i}}, \quad w_0^i(x) \equiv \frac{z_i^1 - z_i(x)}{z_i^1 - z_i^0}, \; w_1^i(x) \equiv 1 - w_0^i(x),$$

where $\mathbb{B} = \{0,1\}$, $\mathbf{i} = (i_1, i_2, \ldots, i_r)$, $\mathbf{w_i}(x) = w^1_{i_1}(x)\ w^2_{i_2}(x) \cdots w^r_{i_r}(x)$, and $f_\mathbf{i} = f(z_1, z_2, \ldots, z_r)|_{\mathbf{w_i}(x)=\mathbf{1}}$.

In order to briefly illustrate the convex rewriting just described, consider the expression

$$f(x_1, x_2) = \begin{bmatrix} 3\cos x_1 - 10 & x_2^2 \cos x_1 \\ 15 & -6\cos x_1 \end{bmatrix},$$

in a region of interest $\mathcal{C} = \{|x_2| < 1\}$. Defining $z_1(x) = \cos x_1$ and $z_2(x) = x_2^2$ we can define $z_1^0 = -1$, $z_1^1 = 1$, $z_2^0 = 0$, $z_2^1 = 1$, in order to rewrite the expression above as:

$$\begin{aligned} f(z_1, z_2) &= \begin{bmatrix} 3z_1(x) - 10 & z_1(x)z_2(x) \\ 15 & -6z_1(x) \end{bmatrix} \\ &= \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} w^1_{i_1}(x) w^2_{i_2}(x) \begin{bmatrix} 3z_1^{i_1} - 10 & z_1^{i_1} z_2^{i_2} \\ 15 & -6z_1^{i_1} \end{bmatrix} = \sum_{\mathbf{i}\in\mathbb{B}^2} \mathbf{w_i}(x) f_\mathbf{i} \end{aligned}$$

with $\mathbf{i} = (i_1, i_2)$, $\mathbf{w_i}(x) = w^1_{i_1}(x) w^2_{i_2}(x)$, $w_0^i(x) = (z_i^1 - z_i(x))/(z_i^1 - z_i^0)$, $w_1^i(x) = 1 - w_0^i(x)$, $f_\mathbf{i} = \begin{bmatrix} 3z_1^{i_1} - 10 & z_1^{i_1} z_2^{i_2} \\ 15 & -6z_1^{i_1} \end{bmatrix}$.

Distinguishing measurable and unmeasurable signals is critical in any observer design. When applying convex rewriting to expressions in this work, different measurable signals will be grouped as $z(\cdot) = [z_1(\cdot)\ z_2(\cdot)\ \cdots\ z_r(\cdot)]^T$, while different unmeasurable signals will be grouped as $\zeta(\cdot) = [\zeta_1(\cdot)\ \zeta_2(\cdot)\ \cdots \zeta_\rho(\cdot)]^T$, assuming all of them are well-defined in some compact set, which implies that therein $z_i(\cdot) \in [z_i^0, z_i^1]$, $\zeta_j \in [\zeta_j^0, \zeta_j^1]$, $j \in \{1, 2, \ldots, \rho\}$. $i \in \{1, 2, \ldots, r\}$, $\forall x \in \mathcal{C}$. Thus, an expression $f(z_1, z_2, \ldots, z_r, \zeta_1, \zeta_2, \ldots, \zeta_\rho)$ can be rewritten as:

$$f(z_1, z_2, \ldots, z_r, \zeta_1, \zeta_2, \ldots, \zeta_\rho) = \sum_{\mathbf{i}\in\mathbb{B}^r} \sum_{\mathbf{j}\in\mathbb{B}^\rho} \mathbf{w_i}(\cdot) \boldsymbol{\omega}_\mathbf{j}(\cdot) f_\mathbf{ij},$$

with $\mathbb{B} = \{0,1\}$, $\mathbf{i} = (i_1, i_2, \ldots, i_r)$, $\mathbf{j} = (j_1, j_2, \ldots, j_\rho)$, $\mathbf{w_i}(\cdot) = w^1_{i_1}(\cdot) w^2_{i_2}(\cdot) \cdots w^r_{i_r}(\cdot)$, $w_0^i(\cdot) \equiv (z_i^1 - z_i(\cdot))/(z_i^1 - z_i^0)$, $w_1^i(\cdot) \equiv 1 - w_0^i(\cdot)$, $\boldsymbol{\omega}_\mathbf{j}(\cdot) = \omega^1_{j_1}(\cdot)\ \omega^2_{j_2}(\cdot) \cdots \omega^\rho_{j_\rho}(\cdot)$, $\omega_0^j(\cdot) \equiv (\zeta_j^1 - \zeta_j(\cdot))/(\zeta_j^1 - \zeta_j^0)$, $\omega_1^j(\cdot) \equiv 1 - \omega_0^j(\cdot)$, and $f_\mathbf{ij} = f(z_1^{i_1}, z_2^{i_2}, \ldots, z_r^{i_r}, \zeta_1^{j_1}, \zeta_2^{j_2}, \ldots, \zeta_\rho^{j_\rho})$.

The interested reader can find details on convex modelling in [3, 30].

2.2. Factorization of error signals

Lyapunov-based observer design usually faces the challenge of factorizing the estimation error $e = \hat{x} - x$ from expressions of the form $f(\hat{x}) - f(x)$. In [27], a methodology to explicitly obtain $F(x, \hat{x})$ such that $f(\hat{x}) - f(x) = F(x, \hat{x})(\hat{x} - x)$ holds, has been developed. Indeed, defining $x_{i,0} = \hat{x}_i$, $x_{i,1} = x_i$, $\mathbf{j} = (j_1, j_2, \ldots, j_n)$, $j_i \in \{0, 1\}$, $\mathbf{q} = (q_1, q_2, \ldots, q_n)$, $q_i \in \{\mathbb{N} \cup 0\}$, $\mathbf{x_j^q} = x_{1,j_1}^{q_1} x_{2,j_2}^{q_2} \cdots x_{n,j_n}^{q_n}$, we have that:

Lemma 2.1. A multivariate expression of the form $\mathbf{x_0^q} - \mathbf{x_1^q}$ with $\deg \mathbf{x_0} \geq 2$ can be written as a convex sum whose terms have the form $\mathbf{x_0^{r_1}}(\mathbf{x_0^{r_2}} - \mathbf{x_1^{r_2}}) + \mathbf{x_1^{r_2}}(\mathbf{x_0^{r_1}} - \mathbf{x_1^{r_1}})$, with $\deg \mathbf{x_0^{r_i}} < \deg \mathbf{x_0^q}$, $i \in \{1, 2\}$.

The repeated application of the previous lemma produces expressions whose corresponding differences $\mathbf{x_0^{r_2}} - \mathbf{x_1^{r_2}}$ and $\mathbf{x_0^{r_1}} - \mathbf{x_1^{r_1}}$ have a lower degree than the preceding ones; the last of them are the desired error signals which can now be directly factorized. Hence, given a multivariate polynomial $p(x)$, $x \in \mathbb{R}^n$, the expression $p(\hat{x}) - p(x)$ can be written as $q(x, \hat{x})e$ with $e = \hat{x} - x$ and $q(x, \hat{x})$ any polynomial resulting from the repeated application of Lemma 2.1. Non-polynomial expressions holding the differential mean value theorem (i. e., with convergent Taylor series on a compact set), can be approximated with any degree of accuracy by polynomial expressions which can then be treated as before. The lemma above guarantees an infinite number of choices for the sought factorization.

The following development illustrates the factorization just described for $f(\hat{x}) - f(x)$ where $f(\hat{x}) = \hat{x}_1\hat{x}_2\hat{x}_3$ and $f(x) = x_1x_2x_3$:

$$\begin{aligned}
&\hat{x}_1\hat{x}_2\hat{x}_3 - x_1x_2x_3 = \hat{x}_1\hat{x}_2\hat{x}_3 - \hat{x}_1x_2x_3 + \hat{x}_1x_2x_3 - x_1x_2x_3 \\
&= \hat{x}_1(\hat{x}_2\hat{x}_3 - x_2x_3) + x_2x_3(\hat{x}_1 - x_1) = \hat{x}_1(\hat{x}_2\hat{x}_3 - \hat{x}_2x_3 + \hat{x}_2x_3 - x_2x_3) + x_2x_3e_1 \\
&= \hat{x}_1\left(\hat{x}_2\left(\hat{x}_3 - x_3\right) + x_3\left(\hat{x}_2 - x_2\right)\right) + x_2x_3e_1 = \hat{x}_1\hat{x}_2e_3 + \hat{x}_1x_3e_2 + x_2x_3e_1 \\
&= [x_2x_3 \; \hat{x}_1x_3 \; \hat{x}_1\hat{x}_2][e_1 \; e_2 \; e_3]^T,
\end{aligned}$$

which can also be rewritten as:

$$\hat{x}_1\hat{x}_2\hat{x}_3 - x_1x_2x_3 = [\hat{x}_2\hat{x}_3 \; x_1x_3 \; x_1\hat{x}_2][e_1 \; e_2 \; e_3]^T.$$

The interested reader is referred to [27, 31] for details.

2.3. Robust differentiators

As stated, our proposal requires a number of time derivatives of real and virtual outputs in order to reconstruct the input dynamics. A variety of robust differentiators is available in the literature that, in noiseless environments, ensure finite-time convergence. In this work, we will employ the Levant's robust differentiator which has the following structure [19]:

$$\begin{aligned}
\dot{v}_0 &= -\lambda_0 \left|v_0 - y(t)\right|^{\frac{s}{s+1}} \operatorname{sign}\left(v_0 - y(t)\right) + v_1 \\
\dot{v}_1 &= -\lambda_1 \left|v_1 - v_0\right|^{\frac{s-1}{s}} \operatorname{sign}\left(v_1 - v_0\right) + v_2 \\
&\;\;\vdots \\
\dot{v}_{s-1} &= -\lambda_{s-1} \left|v_{s-1} - v_{s-2}\right|^{\frac{1}{2}} \operatorname{sign}\left(v_{s-1} - v_{s-2}\right) + v_s \\
\dot{v}_s &= -\lambda_s \operatorname{sign}\left(v_s - v_{s-1}\right),
\end{aligned} \tag{2}$$

where parameters $\lambda_i > 0$, $i \in \{0, 1, \ldots, s\}$, are designed taking the Lipschitz constants of the successive time derivatives of $y(t)$. In the free noise case, $v_i = y^{(i)}(t)$, $i \in \{0, 1, \ldots, s\}$, in finite time. The higher the order s of the differentiator, the better the accuracy and convergence time of the estimation.

The interested reader is referred to [19, 20, 33] for details on different differentiation options.

2.4. Observability and detectability

In [38] it is defined that, for a nonlinear system $\dot{x} = f(x) + \sum_{i=1}^{m} u_i g(x)$, whose output is given by $y = h(x)$, two states x_0 and x_1 are *distinguishable* if there exists an input u such that $y(\cdot, x_0, u) \neq y(\cdot, x_i, u)$. Moreover, the system is *locally observable* at x_0 if there exists a neighbourhood N of x_0 such that for every $x \in N$, $x \neq x_0$, x is distinguishable from x_0. However, when an LMI test, obtained by the direct Lyapunov method, is employed to obtain sufficient conditions that ensure asymptotic stability of the origin of the estimation error system (as in this work), these definitions are implied [31].

In order to illustrate this fact, consider a linear system $\dot{x} = Ax + Bu$ whose output is given by $y = Cx$. Given an observer structure such as the Luenberger one $\dot{\hat{x}} = A\hat{x} + Bu + L(\hat{y} - y)$ leads to an error system of the form $\dot{e} = (A + LC)e$, where $e = \hat{x} - x$. Such system can be stabilized by means of L if the direct Lyapunov method is employed. Indeed, it is clear that if $V(e) = e^T Pe$, $P = P^T > 0$, then $\dot{V} = e^T(PA + PLC + A^T P + C^T L^T P)e < 0$, $e \neq 0$, if there exists $P > 0$ and N of adequate dimensions such that $PA + NC + A^T P + C^T N^T < 0$ with $L = P^{-1}N$. Is there any chance of these LMI conditions being feasible for an undetectable system? As a matter of fact, no. If an undetectable linear system is put in its canonical observer decomposition

$$\begin{bmatrix} \dot{\bar{x}}_o \\ \dot{\bar{x}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} u, \; y = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix},$$

it will be obliged to have an unstable matrix $\bar{A}_{\bar{o}}$. If the LMIs hold then

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}\begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}\begin{bmatrix} L_1\bar{C}_o & 0 \\ L_2\bar{C}_o & 0 \end{bmatrix} + \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} + \begin{bmatrix} L_1\bar{C}_o & 0 \\ L_2\bar{C}_o & 0 \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}$$

$$= \begin{bmatrix} P_1(\bar{A}_o + L_1\bar{C}_o) + P_2(\bar{A}_{21} + L_2\bar{C}_o) + (*) & (*) \\ P_2^T(\bar{A}_o + L_1\bar{C}_o) + P_3(\bar{A}_{21} + L_2\bar{C}_o) + \bar{A}_{\bar{o}}^T P_2^T & P_3\bar{A}_{\bar{o}} + \bar{A}_{\bar{o}}^T P_3 \end{bmatrix} < 0 \Rightarrow P_3\bar{A}_{\bar{o}} + \bar{A}_{\bar{o}}^T P_3 < 0,$$

but this is impossible since $P_3 > 0$ and the last inequality implies $\bar{A}_{\bar{o}}$ is stable, which is a contradiction.

Similarly, since the compact sets considered for the estimation error systems in this paper include the origin $e = 0$, the arguments above about sufficient LMI conditions implying detectability, necessarily hold. Moreover, since the UIO extension of detectability is *distinguishability* of input signals from system outputs, sufficient LMI conditions based on extended estimation error systems that include the input dynamics must necessarily imply distinguishability; this coincides with global conditions for distinguishability in nonlinear systems as described in [17].

3. MAIN RESULTS

Consider a MIMO nonlinear system of the form

$$\dot{x}(t) = f(x) + g(x)u(t), \; y(t) = h(x), \tag{3}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ the input vector, and $y \in \mathbb{R}^p$ the output vector; $f(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^n$, $g(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^{n \times m}$, and $h(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^p$ are sufficiently smooth vector fields.

Assumption 1. Both the state x and the input u are assumed unknown; the output y is the only available information.

Assumption 2. The vector fields $f(x)$, $g(x)$, and $h(x)$ are well-defined in $\mathcal{C}_x \subset \mathbb{R}^n$.

Assumption 3. In the absence of noise, based on the output signals gathered in $y(t)$ and any robust differentiator scheme, $\dot{y}$, $\ddot{y}$, ..., $y^{(s)}$, are available for fixed s.

Assumption 4. There exists a differentiation order s allowing $\dot{u}_1$, $\dot{u}_2$, ..., $\dot{u}_m$, to be solved from the explicit equations of $\ddot{y}$, ..., $y^{(s)}$, in terms of states x, inputs u, and output derivatives $\ddot{y}$, ..., $y^{(s)}$ only.

Remark 3.1. Assumptions 1 and 2 are standard for UIO design and nonlinear models representing physical plants, respectively; assumption 3 is readily fulfilled by any robust differentiator of order $\bar{s} \geq s$, e. g., Levant's [19]; assumption 4 means that the system is unknown-input observable.

Some consequences of these assumptions follow:

1. Explicit state- and input-dependencies of the derivatives can be inferred from the output vector field $h(x)$ in (3).

2. Failing to fulfill assumption 4 implies indistinguishability of inputs from outputs and their derivatives.

3. Solving $\dot{u}_1$, $\dot{u}_2$, ..., $\dot{u}_m$, may require solving for higher-order time derivatives of u too.

4. A minimum of 2nd-order time derivatives of y is required for the input dynamics to appear.

5. Once the sth-order time derivative of the output is obtained, only a selection of the $(s-1)p$ equations might be required to solve the desired dynamics.

Since assumption 4 implies $\dot{u}$ can be solved in terms of x, u, y, $\ddot{y}$, ..., $y^{(s)}$, let call these dynamics $q(\cdot)$, i. e.:

$$\dot{u}(t) = q(\ddot{y}, \dots, y^{(s)}, x, u). \tag{4}$$

Based on the system dynamics (3) and the input dynamics (4), we propose the following observer:

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{u}} \end{bmatrix} = \begin{bmatrix} f(\hat{x}) + g(\hat{x})\hat{u} \\ q\left(\ddot{y}, \dots, y^{(s)}, \hat{x}, \hat{u}\right) \end{bmatrix} + \begin{bmatrix} L_1(\hat{x}, \hat{u}, y, \dot{y}, \dots, y^{(s)}) \\ L_2(\hat{x}, \hat{u}, y, \dot{y}, \dots, y^{(s)}) \end{bmatrix} (\hat{y} - y), \ \hat{y} = h(\hat{x}), \tag{5}$$

where $\hat{x} \in \mathbb{R}^n$ is the observer state, $\hat{u} \in \mathbb{R}^m$ is the observer input, $\hat{y} \in \mathbb{R}^p$ is the observer output, and $L_1(\hat{x}, \hat{u}, y, \dot{y}, \dots, y^{(s)}) \in \mathbb{R}^{n \times p}$, $L_2(\hat{x}, \hat{u}, y, \dot{y}, \dots, y^{(s)}) \in \mathbb{R}^{m \times p}$ are possibly nonlinear observer gains to be found. Notice that, according to assumptions 1 and 3, these gains depend only on available information.

Defining the observer errors $e_x \equiv \hat{x} - x$ and $e_u \equiv \hat{u} - u$, we have that the error dynamics can be expressed as follows:

$$\begin{aligned}\begin{bmatrix}\dot{e}_x\\ \dot{e}_u\end{bmatrix} &= \begin{bmatrix} f(\hat{x}) + g(\hat{x})\hat{u} - f(x) - g(x)u \\ q\left(\ddot{y},\ldots,y^{(s)},\hat{x},\hat{u}\right) - q\left(\ddot{y},\ldots,y^{(s)},x,u\right)\end{bmatrix} + \begin{bmatrix} L_1(\ddot{y},\ldots,y^{(s)},\hat{x},\hat{u}) \\ L_2(\ddot{y},\ldots,y^{(s)},\hat{x},\hat{u})\end{bmatrix}(h(\hat{x})-h(x))\\ &= \begin{bmatrix} F_1(x,\hat{x},u,\hat{u}) & F_2(x,\hat{x},u,\hat{u}) \\ Q_1(\ddot{y},\ldots,y^{(s)},x,\hat{x},u,\hat{u}) & Q_2(\ddot{y},\ldots,y^{(s)},x,\hat{x},u,\hat{u})\end{bmatrix}\begin{bmatrix}e_x\\ e_u\end{bmatrix}\\ &\quad + \begin{bmatrix} L_1(\ddot{y},\ldots,y^{(s)},\hat{x},\hat{u}) \\ L_2(\ddot{y},\ldots,y^{(s)},\hat{x},\hat{u})\end{bmatrix}\begin{bmatrix} H(x,\hat{x}) & 0\end{bmatrix}\begin{bmatrix}e_x\\ e_u\end{bmatrix},\end{aligned} \tag{6}$$

where $F_1(x,\hat{x},u,\hat{u}) \in \mathbb{R}^{n\times n}$, $F_2(x,\hat{x},u,\hat{u}) \in \mathbb{R}^{n\times m}$, $Q_1(\ddot{y},\ldots,y^{(s)},x,\hat{x},u,\hat{u}) \in \mathbb{R}^{m\times n}$, $Q_2(\ddot{y},\ldots,y^{(s)},x,\hat{x},u,\hat{u}) \in \mathbb{R}^{m\times m}$, and $H(x,\hat{x}) \in \mathbb{R}^{p\times n}$, must satisfy

$$\begin{aligned}&F_1(x,\hat{x},u,\hat{u})e_x + F_2(x,\hat{x},u,\hat{u})e_u = f(\hat{x}) + g(\hat{x})\hat{u} - f(x) - g(x)u,\\ &H(x,\hat{x})e_x = h(\hat{x}) - h(x),\\ &Q_1(\ddot{y},\ldots,y^{(s)},x,\hat{x},u,\hat{u})e_x + Q_2(\ddot{y},\ldots,y^{(s)},x,\hat{x},u,\hat{u})e_u\\ &\ = q(\ddot{y},\ldots,y^{(s)},\hat{x},\hat{u}) - q(\ddot{y},\ldots,y^{(s)},x,u).\end{aligned}$$

Such requirement can be fulfilled by means of the factorization in [27] briefly described in Section 2.2.

Let $\mathcal{C}_x \subset \mathbb{R}^n$, $\mathcal{C}_u \subset \mathbb{R}^m$, and $\mathcal{C}_{y^{(i)}}$, be state, input, and ith-order-output-time-derivative domains of interest (design regions) for $i \in \{0,1,\ldots,s\}$, respectively, on which nonlinear expressions $F_1(\cdot)$, $F_2(\cdot)$, $Q_1(\cdot)$, $Q_2(\cdot)$, and $H(\cdot)$ are well defined due to assumption 2; let us assume these nonlinear expressions depend on a total of r measurable signals $z_1(\hat{x},\hat{u},y,\dot{y},\ldots,y^{(s)}) \in [z_1^0,z_1^1]$, $z_2(\hat{x},\hat{u},y,\dot{y},\ldots,y^{(s)}) \in [z_2^0,z_2^1]$, …, $z_r(\hat{x},\hat{u},y,\dot{y},\ldots,y^{(s)}) \in [z_r^0,z_r^1]$ and ρ remaining ones $\zeta_1(\cdot) \in [\zeta_1^0,\zeta_1^1]$, $\zeta_2(\cdot) \in [\zeta_2^0,\zeta_2^1]$, … $\zeta_\rho(\cdot) \in [\zeta_\rho^0,\zeta_\rho^1]$, all of them bounded in the design regions as a consequence of being well-defined therein. Thus, following the methodology in Section 2.1, we have:

$$F_l(x,\hat{x},u,\hat{u}) = \sum_{\mathbf{i}\in\mathbb{B}^r}\sum_{\mathbf{j}\in\mathbb{B}^\rho}\mathbf{w}_\mathbf{i}(\hat{x},\hat{u},y,\dot{y},\ldots,y^{(s)})\boldsymbol{\omega}_\mathbf{j}(\cdot)F_l^{\mathbf{ij}}, \tag{7}$$

$$Q_l(\ddot{y},\ldots,y^{(s)},x,\hat{x},u,\hat{u}) = \sum_{\mathbf{i}\in\mathbb{B}^r}\sum_{\mathbf{j}\in\mathbb{B}^\rho}\mathbf{w}_\mathbf{i}(\hat{x},\hat{u},y,\dot{y},\ldots,y^{(s)})\boldsymbol{\omega}_\mathbf{j}(\cdot)Q_l^{\mathbf{ij}}, \tag{8}$$

$$H(x,\hat{x}) = \sum_{\mathbf{i}\in\mathbb{B}^r}\sum_{\mathbf{j}\in\mathbb{B}^\rho}\mathbf{w}_\mathbf{i}(\hat{x},\hat{u},y,\dot{y},\ldots,y^{(s)})\boldsymbol{\omega}_\mathbf{j}(\cdot)H^{\mathbf{ij}}, \tag{9}$$

where functions $\mathbf{w}_\mathbf{i}(\hat{x},\hat{u},y,\dot{y},\ldots,y^{(s)})$ and $\boldsymbol{\omega}_\mathbf{j}(\cdot)$, defined as

$$\begin{aligned}&\mathbf{w}_\mathbf{i}(\hat{x},\hat{u},y,\dot{y},\ldots,y^{(s)}) = w_{i_1}^1(\hat{x},\hat{u},y,\dot{y},\ldots,y^{(s)})\cdots w_{i_r}^r(\hat{x},\hat{u},y,\dot{y},\ldots,y^{(s)}),\\ &w_0^i(\hat{x},\hat{u},y,\dot{y},\ldots,y^{(s)}) \equiv (z_i^1 - z_i(\hat{x},\hat{u},y,\dot{y},\ldots,y^{(s)}))/(z_i^1 - z_i^0),\\ &w_1^i(\hat{x},\hat{u},y,\dot{y},\ldots,y^{(s)}) \equiv 1 - w_0^i(\hat{x},\hat{u},y,\dot{y},\ldots,y^{(s)}),\\ &\boldsymbol{\omega}_\mathbf{j}(\cdot) = \omega_{j_1}^1(\cdot)\omega_{j_2}^2(\cdot)\cdots\omega_{j_\rho}^\rho(\cdot),\ \omega_0^j(\cdot) \equiv (\zeta_j^1 - \zeta_j(\cdot))/(\zeta_j^1 - \zeta_j^0),\ \omega_1^j(\cdot) \equiv 1 - \omega_0^j(\cdot),\end{aligned}$$

hold the convex sum property $\sum_{\mathbf{i}\in\mathbb{B}^r}\mathbf{w}(\cdot)=1$, $\mathbf{w}(\cdot)\in[0,1]$ and $\sum_{\mathbf{j}\in\mathbb{B}^\rho}\boldsymbol{\omega}(\cdot)=1$, $\boldsymbol{\omega}(\cdot)\in[0,1]$, $\forall x\in\mathcal{C}_x$, $\forall u\in\mathcal{C}_u$, $\forall\hat{x}\in\mathcal{C}_x$, $\forall\hat{u}\in\mathcal{C}_u$, $\forall y^{(i)}\in\mathcal{C}_{y^{(i)}}$, $i\in\{0,1,\dots,s\}$.

In the sequel, nonlinear observer gains exploiting measurable signals will be found; this means they will have dependencies $L_l(\hat{x},\hat{u},y,\dot{y},\dots,y^{(s)})$, $l\in\{1,2\}$. This is why in the developments above the dependency of functions $\mathbf{w_i}(\hat{x},\hat{u},y,\dot{y},\dots,y^{(s)})$ has been stressed; for the same reason, it is not relevant to explicitly show the arguments of the remaining terms gathered in functions $\boldsymbol{\omega}_\mathbf{j}(\cdot)$.

Theorem 3.2. The origin of the nonlinear error system (6) is asymptotically stable if there exists matrices $P_1\in\mathbb{R}^{n\times n}$, $P_2\in\mathbb{R}^{n\times m}$, $P_3\in\mathbb{R}^{m\times m}$, $N_1^\mathbf{k}\in\mathbb{R}^{n\times p}$, $N_2^\mathbf{k}\in\mathbb{R}^{m\times p}$, such that the following LMIs hold $\forall\mathbf{i}\in\mathbb{B}^r$, $\forall\mathbf{k}\in\mathbb{B}^r$ and $\forall\mathbf{j}\in\mathbb{B}^\rho$:

$$\begin{bmatrix}P_1 & P_2\\ P_2^T & P_3\end{bmatrix}>0,\ \mathrm{He}\left(\begin{bmatrix}P_1 & P_2\\ P_2^T & P_3\end{bmatrix}\begin{bmatrix}F_1^\mathbf{ij} & F_2^\mathbf{ij}\\ Q_1^\mathbf{ij} & Q_2^\mathbf{ij}\end{bmatrix}+\begin{bmatrix}N_1^\mathbf{k}\\ N_2^\mathbf{k}\end{bmatrix}\begin{bmatrix}H^\mathbf{ij} & 0\end{bmatrix}\right)<0, \tag{10}$$

where $\mathrm{He}(M)=M+M^T$. Once solved, the nonlinear observer gains in (5) are given by

$$L_l(\hat{x},\hat{u},y,\dot{y},\dots,y^{(s)})=\sum_{\mathbf{k}\in\mathbb{B}^r}\mathbf{w_k}(\hat{x},\hat{u},y,\dot{y},\dots,y^{(s)})L_l^\mathbf{k},\ l\in\{1,2\}, \tag{11}$$

$$\text{where}\ \begin{bmatrix}L_1^\mathbf{k}\\ L_2^\mathbf{k}\end{bmatrix}=\begin{bmatrix}P_1 & P_2\\ P_2^T & P_3\end{bmatrix}^{-1}\begin{bmatrix}N_1^\mathbf{k}\\ N_2^\mathbf{k}\end{bmatrix}.$$

Moreover, any trajectory within any level set

$$\Omega_k=\left\{\begin{bmatrix}e_x\\ e_u\end{bmatrix}^T\begin{bmatrix}P_1 & P_2\\ P_2^T & P_3\end{bmatrix}\begin{bmatrix}e_x\\ e_u\end{bmatrix}\le k\right\}\subset\mathcal{C}_e\subset\mathbb{R}^{n+m},$$

with $k>0$ and $\mathcal{C}_e$ being the region induced in e by $\mathcal{C}_x$ and $\mathcal{C}_u$, goes asymptotically to 0, provided that $\forall y^{(i)}\in\mathcal{C}_{y^{(i)}}$, $i\in\{0,1,\dots,s\}$.

Proof. The following implication is guaranteed by the first LMI in (10):

$$\begin{bmatrix}P_1 & P_2\\ P_2^T & P_3\end{bmatrix}>0\Rightarrow V(e)=\begin{bmatrix}e_x\\ e_u\end{bmatrix}^T\begin{bmatrix}P_1 & P_2\\ P_2^T & P_3\end{bmatrix}\begin{bmatrix}e_x\\ e_u\end{bmatrix};$$

it means that $V(e)$ is a Lyapunov function candidate for the error dynamics in (6).

Due to the convex sum properties of functions $\mathbf{w_i}(\cdot)$ and $\boldsymbol{\omega}_\mathbf{j}(\cdot)$, $\forall x\in\mathcal{C}_x$, $\forall u\in\mathcal{C}_u$, $\forall\hat{x}\in\mathcal{C}_x$, $\forall\hat{u}\in\mathcal{C}_u$, $\forall y^{(i)}\in\mathcal{C}_{y^{(i)}}$, $i\in\{0,1,\dots,s\}$, it follows that the second LMI expression in (10) validates the following inequality

$$\begin{bmatrix}e_x\\ e_u\end{bmatrix}^T\Bigg(\sum_{\mathbf{i},\mathbf{k}\in\mathbb{B}^r}\sum_{\mathbf{j}\in\mathbb{B}^s}\mathbf{w_i}(\hat{x},\hat{u},y,\dot{y},\dots,y^{(s)})\boldsymbol{\omega}_\mathbf{j}(\cdot)\mathbf{w_k}(\hat{x},\hat{u},y,\dot{y},\dots,y^{(s)})$$

$$\times\mathrm{He}\left(\begin{bmatrix}P_1 & P_2\\ P_2^T & P_3\end{bmatrix}\begin{bmatrix}F_1^\mathbf{ij} & F_2^\mathbf{ij}\\ Q_1^\mathbf{ij} & Q_2^\mathbf{ij}\end{bmatrix}+\begin{bmatrix}N_1^\mathbf{k}\\ N_2^\mathbf{k}\end{bmatrix}\begin{bmatrix}H^\mathbf{ij} & 0\end{bmatrix}\right)\Bigg)\begin{bmatrix}e_x\\ e_u\end{bmatrix}<0,$$

which, using convex sum properties, can be rewritten as

$$\begin{bmatrix} e_x \\ e_u \end{bmatrix}^T \left(\mathrm{He} \left(\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \begin{bmatrix} \sum_{\mathbf{i}\in\mathbb{B}^r} \sum_{\mathbf{j}\in\mathbb{B}^s} \mathbf{w_i}\boldsymbol{\omega}_\mathbf{j} F_1^{\mathbf{ij}} & \sum_{\mathbf{i}\in\mathbb{B}^r} \sum_{\mathbf{j}\in\mathbb{B}^s} \mathbf{w_i}\boldsymbol{\omega}_\mathbf{j} F_2^{\mathbf{ij}} \\ \sum_{\mathbf{i}\in\mathbb{B}^r} \sum_{\mathbf{j}\in\mathbb{B}^s} \mathbf{w_i}\boldsymbol{\omega}_\mathbf{j} Q_1^{\mathbf{ij}} & \sum_{\mathbf{i}\in\mathbb{B}^r} \sum_{\mathbf{j}\in\mathbb{B}^s} \mathbf{w_i}\boldsymbol{\omega}_\mathbf{j} Q_2^{\mathbf{ij}} \end{bmatrix} \right.\right.$$
$$\left.\left. + \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \begin{bmatrix} \sum_{\mathbf{k}\in\mathbb{B}^r} \mathbf{w_k} L_1^{\mathbf{k}} \\ \sum_{\mathbf{k}\in\mathbb{B}^r} \mathbf{w_k} L_2^{\mathbf{k}} \end{bmatrix} \begin{bmatrix} \sum_{\mathbf{i}\in\mathbb{B}^r} \sum_{\mathbf{j}\in\mathbb{B}^s} \mathbf{w_i}\boldsymbol{\omega}_\mathbf{j} H^{\mathbf{ij}} & 0 \end{bmatrix} \right)\right) \begin{bmatrix} e_x \\ e_u \end{bmatrix} < 0.$$

Now, recalling equivalences (7), (8), (9), and (11), the inequality above can be rewritten as

$$\begin{bmatrix} e_x \\ e_u \end{bmatrix}^T \left(\mathrm{He} \left(\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \begin{bmatrix} F_1(x,\hat{x},u,\hat{u}) & F_2(x,\hat{x},u,\hat{u}) \\ Q_1(\ddot{y},\ldots,y^{(s)},x,\hat{x},u,\hat{u}) & Q_2(\ddot{y},\ldots,y^{(s)},x,\hat{x},u,\hat{u}) \end{bmatrix} \right.\right.$$
$$\left.\left. + \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \begin{bmatrix} L_1(\hat{x},\hat{u},y,\dot{y},\ldots,y^{(s)}) \\ L_2(\hat{x},\hat{u},y,\dot{y},\ldots,y^{(s)}) \end{bmatrix} \begin{bmatrix} H(x,\hat{x}) & 0 \end{bmatrix} \right)\right) \begin{bmatrix} e_x \\ e_u \end{bmatrix} = \dot{V}(e) < 0.$$

Hence, we have just proved that $V(e)$ is a valid Lyapunov function establishing asymptotic stability of the origin of the nonlinear error system (6). Since the previous developments ensure positive-definiteness of $V(e)$ and $-\dot{V}(e)$ provided functions $\mathbf{w_i}(\cdot)$ and $\boldsymbol{\omega}_\mathbf{j}(\cdot)$ hold the convex sum property, it follows that trajectories belonging to any level set Ω_k as defined above go asymptotically to 0, thus concluding the proof. □

Remark 3.3. The LMI formulation of conditions in Theorem 3.2 allows a straightforward inclusion of decay rate specifications as well as $\mathscr{H}_\infty$ disturbance attenuation: the first one may speed up the rate of convergence of the error signals to 0; the second one may help dealing with additive square-integrable disturbance signals. The interested reader may refer to [3].

Remark 3.4. Numerical complexity of LMIs in Theorem 3.2 amounts to $\log_{10}(n_d^3 n_l)$ [9], where $n_d = 0.5(n+m)(n+m+1) + 2^r(n+m)p$ is the number of scalar decision variables and $n_l = (2^r+1)(n+m)$ is the number of LMI rows. Notice that a higher order of output derivatives may lead to an increase of nonlinearities r.

Assumption 3 implies that the time derivatives of $y(t)$ up to order s are available; their corresponding analytic expressions are known too. This fact can be exploited to add *virtual outputs* to the UIO design, which may favourably influence the observability/distinguishability properties of the whole scheme by adding degrees of freedom to the resulting LMI problems. A *virtual output* of the system (3) is a possibly nonlinear function of the time derivatives of $y(t)$ up to order s, i. e.:

$$y_v(t) = h_v(y,\dot{y},\ddot{y},\ldots,y^{(s)}), \tag{12}$$

where $y_v \in \mathbb{R}^{p_v}$, $h_v : \mathbb{R}^{ps} \mapsto \mathbb{R}^{p_v}$ is a sufficiently smooth map $\forall y^{(i)} \in \mathcal{C}_{y^{(i)}}$, $i \in \{0,1,\ldots,s\}$, such that expressions taken from $y = h(x)$, $\dot{y} = (\partial h/\partial x)\dot{x}$, …, $y^{(s)} = (\partial y^{(s-1)}/\partial x)\dot{x}$, depend exclusively on x and u, and are well-defined $\forall x \in \mathcal{C}_x$ and $\forall u \in \mathcal{C}_u$.

Since the system resulting from (3) altogether with (4) is the one for which the UIO (5) is designed, the incorporation of virtual outputs must be guided by the objective of guaranteeing the following system is observable:

$$\dot{\bar{x}}(t) = \bar{f}(\bar{x}),\ \bar{y}(t) = \bar{h}(\bar{x}), \tag{13}$$

with

$$\bar{x} = \begin{bmatrix} x \\ u \end{bmatrix},\ \bar{f}(\bar{x}) = \begin{bmatrix} f(x) + g(x)u \\ q(x,u) \end{bmatrix},\ \bar{y} = \begin{bmatrix} y \\ y_v \end{bmatrix} = \begin{bmatrix} h(x) \\ h_v(x,u) \end{bmatrix},$$

where the arguments in $q(\ddot{y}, \ldots, y^{(s)}, x, u)$ and $h_v(y, \dot{y}, \ddot{y}, \ldots, y^{(s)})$ have been properly rewritten in terms of x and u.

Naturally, if virtual outputs are added, system (13) should be considered instead of the original one both for the observer structure (5) and the resulting error system (6), with the corresponding changes when invoking Theorem 3.2.

Figure 1 summarizes the methodology just developed. There is an off-line process (on the left) and an on-line implementation (on the right). Notice that no LMI is solved during the on-line process; all the structures being already "fixed" and the numerical values in them readily available; the computational burden lies on the off-line process, more specifically on finding a feasible solution to a set of LMIs. Algorithm 1 also summarizes our proposal, adding some additional conditions for termination. The first *while* loop is obliged to end as a consequence of assumption 4; virtual outputs are optional, but as line 34 suggests, their inclusion may influence the feasibility outcome; so it does the size of the design regions $\mathcal{C}_x$, $\mathcal{C}_u$, and $\mathcal{C}_{y^{(i)}}$ (second *while* loop).

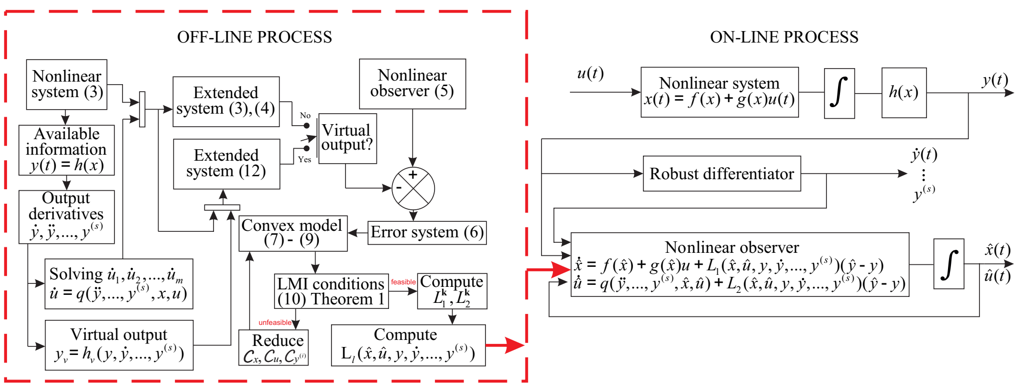


Fig. 1. Block diagram of the proposed methodology.

Remark 3.5. Assumption 3 implies a noiseless environment. However, if a Levant's robust differentiator is employed, sensitivity to measurement noise is precisely measured by means of the formula $\left| v_i - f_0^{(i+1)}(t) \right| \leq \nu_i \epsilon^{(n-i)/(n+1)}$, where $|y(t) - y_0(t)| \leq \epsilon$ is the noise inequality and ν_i are positive constants depending exclusively on the parameters of the differentiator (see [19, Theorem 6]).

Algorithm 1 Design of an LMI-based nonlinear unknown-input MIMO observer using virtual outputs for nonlinear system (3)

1: Fix $s = 2$, $solvable = 0$
2: **while** $solvable = 0$ **do**
3: Calculate $\dot{y}$, $\ddot{y}$, …, $y^{(s)}$
4: **if** $\dot{u}(t)$ can be solved as (4) using the nonlinear system (3) and the s derivatives of $y(t)$ **then**
5: $solvable = 1$
6: **else**
7: $s = s + 1$
8: **end if**
9: **end while**
10: Fix $tolerancia = 0$ and $max\_tolerancia \geq 1$ at some integer value
11: **if** virtual outputs are desired **then**
12: Provide a virtual output $y_v(t) = h(y, \dot{y}, \ldots, y^{(s)})$
13: Extend the output as in (13)
14: **end if**
15: Consider an UIO observer of the form (5) where gains $L_i(\cdot)$, $i \in \{1, 2\}$, are to be found later
16: Obtain the nonlinear observation error system (6) by means of a suitable factorization
17: Provide design regions $\mathcal{C}_x$, $\mathcal{C}_u$, $\mathcal{C}_{y^{(i)}}$
18: Fix $feasible = 0$, $reducciones = 0$ and $max\_reducciones \geq 1$ at some integer value
19: **while** $feasible = 0$ and $reducciones \leq max\_reducciones$ **do**
20: Define a set of measurable signals $z_i(\cdot)$, $i \in \{1, 2, \ldots, r\}$, to rewrite matrices in (6) as (7)-(9)
21: **if** LMI conditions in Theorem (3.2) are feasible **then**
22: $feasible = 1$
23: **else**
24: $reducciones = reducciones + 1$
25: Reduce the design regions $\mathcal{C}_x$, $\mathcal{C}_u$, $\mathcal{C}_{y^{(i)}}$
26: **end if**
27: **end while**
28: **if** $feasible = 1$ **then**
29: Compute $L_1^{\mathbf{k}}$ and $L_2^{\mathbf{k}}$ and construct $L_l(\hat{x}, u, y, \dot{y}, \ldots, y^{(s)})$
30: Design a robust differentiator of your choice to obtain signals $\dot{y}$, $\ddot{y}$, …, $y^{(s)}$ to implement the UIO (5)
31: **else**
32: **if** $tolerancia \leq max\_tolerancia$ **then**
33: $tolerancia = tolerancia + 1$
34: Go to step 11 to redefine virtual outputs
35: **else**
36: Claim the approach fails
37: **end if**
38: **end if**

4. EXAMPLES

Three examples are given in this section: example 4.1 illustrates the influence of the design region as well as the incorporation of virtual outputs in a 2-input plant; example 4.2 has recently appeared in the UIO literature, is fully nonlinear, and is subjected to sudden changes to test the proposal performance; finally, example 4.3 compares the real-time estimations provided by two former methodologies against ours in a plant with a sufficiently simple model for former methodologies to be applicable, namely, the inertia wheel pendulum.

Example 4.1. A model of a single-link flexible joint robot is given in [8]; a disturbed version of it is given in [41]; it can be put in the form (3), i. e., $\dot{x}(t) = f(x) + g(x)u(t)$, $y = h(x)$, with state vector $x = [x_1\ x_2\ x_3\ x_4]^T \in \mathbb{R}^4$, where x_1 and x_2 are the motor position and its velocity, respectively; x_3 and x_4 are the link position and its velocity, respectively; with vector $u = [u_1\ u_2]^T \in \mathbb{R}^2$, where u_1 is the system input and u_2 is an additive disturbance acting on the system; with output vector $y = [x_1\ x_3]^T \in \mathbb{R}^2$ rendering only the motor and link position as measurable signals; finally, with vector fields

$$f(x)=\begin{bmatrix} x_2 \\ \frac{1}{J_m}\left(k_1(x_3-x_1)+k_2(x_3-x_1)^3\right)-\frac{B_v}{J_m}x_2 \\ x_4 \\ -\frac{1}{J_l}\left(k_1(x_3-x_1)+k_2(x_3-x_1)^3\right)-\frac{m_l g h}{J_l}\sin x_3 \end{bmatrix},\ g(x)=\begin{bmatrix} 0 & 0 \\ \frac{K_\tau}{J_m} & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},\ h(x)=\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}^T,$$

where the plant parameters are as follows: the motor inertia $J_m = 3.7\times10^{-3}\text{kgm}^2$, the link inertia $J_l = 9.3\times10^{-3}\text{kgm}^2$, the length of the link $2h = 3\times10^{-1}\text{m}$, the mass of the link $m_l = 0.21\text{kg}$, the viscous friction $B_v = 4.6\times10^{-2}\text{NmV}^{-1}$, the torsional spring constant $k_1 = k_2 = 1.8\times10^{-1}\text{Nm rad}^{-1}$, and the amplifier gain $K_\tau = 8\times10^{-2}\text{NmV}^{-1}$. This example contains that in [39, Section 4.1] as a particular case.

Taking the time derivative of each entry in y up to order 3 and solving for $\dot{u}$, the input dynamics can be solved as in (4), i. e., $\dot{u}(t) = q(\ddot{y}, y^{(3)}, x, u) = [\dot{u}_1\ \dot{u}_2]^T$, where

$$\begin{aligned}\dot{u}_1 =& \frac{k_1}{K_\tau}(x_2-x_4)+\frac{B_v}{J_m}u_1+\frac{3k_2}{K_\tau}(x_3-x_1)^2(x_2-x_4)\\ &-\frac{B_v}{J_m K_\tau}\Big(B_v x_2-k_2(x_3-x_1)^3-k_1(x_3-x_1)\Big)+\frac{J_m}{K_\tau}y_1^{(3)}\\ \dot{u}_2 =& y_2^{(3)}-\frac{k_1}{J_l}(x_2-x_4)-\frac{3k_2}{J_l}(x_3-x_1)^2(x_2-x_4)+\frac{m_l g h}{J_l}x_4\cos x_3.\end{aligned}$$

Taking into account the system and input dynamics above, a nonlinear observer of the form (5) can be proposed, where $\hat{x} \in \mathbb{R}^2$ is the observer state vector, $\hat{u} \in \mathbb{R}^2$ is the observer input vector, $\hat{y} \in \mathbb{R}^2$ is the observer output vector, the nonlinear gains to be determined are $L_1(\hat{x}, \hat{u}, y, \dot{y}, \ddot{y}, \dddot{y}) \in \mathbb{R}^{4\times2}$ and $L_2(\hat{x}, \hat{u}, y, \dot{y}, \ddot{y}, \dddot{y}) \in \mathbb{R}^{2\times2}$.

Using the factorization method given in section 2.2, the nonlinear error system (6) is obtained with:

$$F_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{B_v}{J_m} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, F_2 = \begin{bmatrix} 0 & 0 \\ \frac{K_\tau}{J_m} & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, Q_2 = \begin{bmatrix} \frac{B_v}{J_m} & 0 \\ 0 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$Q_1(y) = \begin{bmatrix} 0 & \frac{1}{K_\tau}(k_1 + 3k_2 z_1 - \frac{B_v^2}{J_m}) & 0 & -\frac{1}{K_\tau}(k_1 + 3k_2 z_1) \\ 0 & -\frac{1}{J_l}(3k_2 z_1 + k_1) & 0 & \frac{1}{J_l}(k_1 + 3k_2 z_1 + m_l g h z_2) \end{bmatrix},$$

where the nonlinearities in $Q_1(y)$ are $z_1 = (y_2 - y_1)^2 = (x_3 - x_1)^2$ and $z_2 = \cos x_3 = \cos y_2$.

Design region 1 (original output): Consider $z_1 \in [0,1]$ and $z_2 \in [0.85, 1]$; the convex modelling in section 2.1 can be employed with $w_0^1(z_1) = 1 - z_1$, $w_1^1(z_1) = z_1$, $w_0^2(z_2) = 6.67 - 6.67z_2$, and $w_1^1(z_2) = -5.67 + 6.67z_2$ to rewrite $Q_1(y)$ as in (8). Once this is made, Theorem 3.2 can be invoked; it renders LMI conditions (10) feasible with a decay rate of $\alpha = 0.25$. Some observer gains $L_1^{\mathbf{k}}$, $L_2^{\mathbf{k}}$, $\mathbf{k} \in \mathbb{B}^2$ are

$$L_1^{00} = \begin{bmatrix} -86.83 & -7.04 \\ -4755 & -1031 \\ -269.71 & -136.57 \\ -505.90 & -714.65 \end{bmatrix}, L_1^{01} = \begin{bmatrix} -86.83 & -6.98 \\ -4753 & -1002 \\ -269.42 & -131.59 \\ -504.02 & -682.64 \end{bmatrix}, L_1^{10} = \begin{bmatrix} -73.31 & -18.17 \\ -4022 & -1633 \\ -237.71 & -162.71 \\ -510.74 & -709.60 \end{bmatrix},$$

$$L_2^{01} = \begin{bmatrix} -4624.54 & -660.45 \\ 5808 & -4626 \end{bmatrix}, L_2^{10} = \begin{bmatrix} -3876 & -1293 \\ 4131 & -3479 \end{bmatrix}, L_2^{11} = \begin{bmatrix} -3875 & -1275 \\ 4143 & -3246 \end{bmatrix}.$$

Design region 2 (extended output): Consider the design specifications given in [41], i.e., $z_1 = (y_2 - y_1)^2 \in [0, 7.84]$, and $z_2 = \cos y_2 \in [-1, 1]$. Again, using the convex functions $w_0^1(z_1) = 1 - 0.128z_1$, $w_1^1(z_1) = 0.128z_1$, $w_0^2(z_2) = 0.5 - 0.5z_2$, and $w_1^1(z_2) = 0.5 + 0.5z_2$ we can rewrite $Q_1(y)$ as in (8). The new vertices render LMI conditions (10) infeasible, which makes sense as design region 2 is bigger than design region 1. Nevertheless, virtual outputs come at hand as they might increase the observer design flexibility. To this end, consider the extended output vector $\bar{y} = [x_1 \; x_3 \; y_2 + \dot{y}_2]^T$, which includes the virtual output $y_3 = y_2 + \dot{y}_2$ (measurable) which corresponds to the signal $y_3 = x_3 + x_4$ (analytically); this idea is pursued in [7]. The new output only affects matrix H when $\hat{\bar{y}} - \bar{y} = He(t)$ is computed; all the other expressions remain the same; thus:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Now LMI conditions (10) in Theorem 3.2 are feasible with decay rate $\alpha = 0.25$. Some observer gains $L_1^{\mathbf{k}}$, $L_2^{\mathbf{k}}$, $\mathbf{k}\in\mathbb{B}^2$ are

$$L_1^{00}=\begin{bmatrix}-4197 & -385.4 & 944.9\\ -18423 & -2135 & 4751\\ -0.8 & -79 & -40.18\\ 1288 & 203 & -416.65\end{bmatrix},\ L_1^{01}=\begin{bmatrix}-4220 & -161.2 & 748.6\\ -18532 & -1155 & 3892\\ -1.33 & -79 & -40.5\\ 1297 & 135.4 & -357\end{bmatrix},$$

$$L_2^{00}=\begin{bmatrix}-14085 & -2218 & 4355\\ -5205 & 5105 & -5778\end{bmatrix},\quad L_2^{01}=\begin{bmatrix}-14174 & -1479 & 3705\\ -5177 & 5534 & -6143\end{bmatrix},$$

$$L_2^{10}=\begin{bmatrix}-19919 & 1079 & 1171\\ -4893 & 5702 & -6371\end{bmatrix},\quad L_2^{11}=\begin{bmatrix}-20021 & 2236 & 198\\ -4869 & 6177 & -6773\end{bmatrix}.$$

Note that gains $L_1^{\mathbf{k}}$ and $L_2^{\mathbf{k}}$ have an additional column when compared with the set of observer gains in the design region 1; this is due to the fact that an additional virtual output was considered.

Comparisons: For simulation purposes the unknown input

$$u_1(t)=\begin{cases}0.5-0.1t, & 0<t\le 10\\ 0.9\sin 10t+0.2\sin t+0.1\cos 5t, & 10<t\le 17\\ 1, & 17<t\le 25\\ 0.25\sin 10t-0.2\sin 15t+0.15\cos 20t, & 25<t\le 32\\ 0.5-0.1(t-42), & 32<t\le 42\\ 0.5\cos 5t, & 42<t\le 50\end{cases}$$

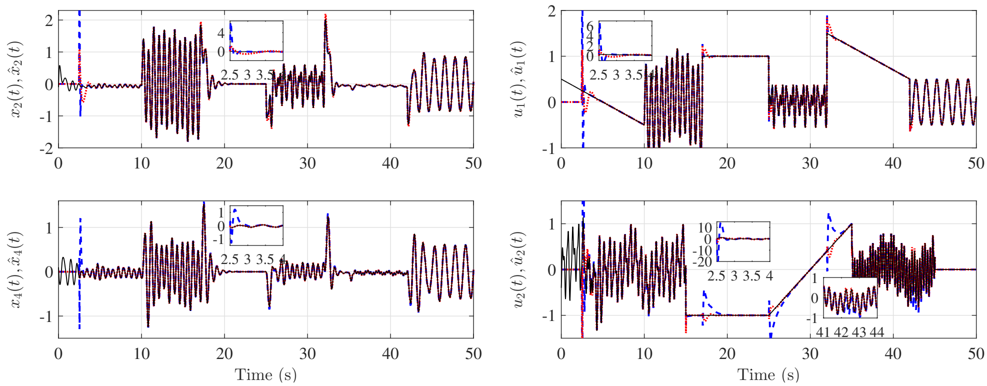
and the disturbance signal

$$u_2(t)=\begin{cases}0.3\sin t-0.1\sin 2t-0.2\cos 25t+0.5\sin 10t\\ \quad +0.1\cos 15t-0.15\sin 12t-0.2\cos 13t-0.1\cos 6t, & 0<t\le 5\\ 0.2\sin t-0.2\sin 2t-0.2\cos 25t+0.5\sin 10t+0.1\cos 15t, & 5<t\le 15\\ -1, & 15<t\le 25\\ -1+0.2(t-25), & 25<t\le 35\\ 0.2\sin t-0.15\sin 5t-0.5\cos 20t, & 35<t\le 45\\ 0, & 45<t\le 50\end{cases}$$

were considered.

Figures 2(a) − 2(b) depict the time evolution of the unmeasurable signals (solid black line) and their estimates, both for the case with the original output $y=[x_1\ x_3]^T$ (dashed blue line) and for the case with the extended output $\bar{y}=[x_1\ x_3\ y_2+\dot{y}_2]^T$ (dotted red line). Clearly, observation of the state and the unknown inputs is adequately performed.

Due to the structure of the system, it can be verified that it does not hold the condition $\text{rank}(CB)=\text{rank}(B)$, i.e., $\text{rank}(CB)=0$ whilst $\text{rank}(B)=2$, where B is the distribution matrix of the unknown input. This fact leaves works [14, 22, 41] unable to deal with the observation problem; works [6, 36] can be applied, but they do not guarantee asymptotic convergence of $e(t)$ to 0 because of the uncoupled unknown input ($\text{rank}(CB)=0$), which can only be dealt with by means of $\mathcal{L}_2$-norm attenuation; on the other hand, notice that [11] cannot be used because the unknown input does not vanish at some fixed oth-order time derivative; finally, [28] cannot be employed because it only addresses SISO systems. □



(a) Estimation of x_2 and x_4 (black) by $\hat{x}_2$ and $\hat{x}_4$. (b) Estimation of the unknown input u (black) by $\hat{u}$.

Fig. 2. Time evolution of the unmeasurable signals and their estimations using $y(t)$ (blue) and $\bar{y}(t)$ (red) in Example 4.1.

Example 4.2. Consider the following nonlinear system, given in [7]:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_2(3+2\sin x_1) - x_4 \\ x_4 \\ -x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 3+2\sin x_1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \tag{14}$$

Time derivatives $\dot{u}_1$ and $\dot{u}_2$ appear for the first time in $\ddot{y}_2$ and $y_2^{(3)}$, respectively. Nevertheless, they cannot be solved from these equations as the latter involves $\ddot{u}_1$ too. In order to solve them, a 4th-order time derivative of the output y should be taken as to form the following system:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 4x_2\cos x_1 + 2\cos 2x_1 - 12\sin x_1 - 11 & 3+2\sin x_1 & -1 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \ddot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} \ddot{y}_2 - u_2 + x_3 \\ y_2^{(3)} + u_1 + x_4 \\ \alpha\left(x, u, y_1^{(4)}\right) \end{bmatrix}, \tag{15}$$

where $\alpha\left(x, u, y_1^{(4)}\right) = y_1^{(4)} - 46u_1 - 3u_2 + 45x_2 + 3x_3 + 10x_4 + 18u_1\cos 2x_1 - 6u_1^2\cos x_1 - 18x_2\cos 2x_1 - 2x_4\cos 2x_1 - 24x_2^2\cos x_1 + 2u_1\sin 3x_1 - 2x_2\sin 3x_1 - 2x_2^3\sin x_1 - 2u_1^2\sin 2x_1 - 8x_2^2\sin 2x_1 - 60u_1\sin x_1 - 2u_2\sin x_1 + 60x_2\sin x_1 + 2x_3\sin x_1 + 12x_4\sin x_1 + 30u_1x_2\cos x_1 + 2u_1x_4\cos x_1 - 6x_2x_4\cos x_1 + 10u_1x_2\sin 2x_1 + 2u_1x_2^2\sin x_1$. Since the matrix on the left side is full rank, the equations can be solved yielding the inputs dynamics.

Consider a virtual output $y_v = \ddot{y}_1 + (2\sin y_1 + 3)(\dot{y}_1 - \dot{y}_2)$ (measurable), which is analytically equivalent to $y_v = (3+2\sin x_1)(u_1 - x_2) - x_4 + (2\sin y_1 + 3)(x_2 - x_4 - u_1) = -x_4(2\sin y_1 + 4)$, for inclusion in an extended output vector $\bar{y} = [x_1\ x_3\ y_v]^T$. Based on system (14), input dynamics solved from (15), and extended output $\bar{y}$, an observer of the form (5) can be constructed with $\hat{x} \in \mathbb{R}^4$, $\hat{u} \in \mathbb{R}^2$, $L_1(\cdot) \in \mathbb{R}^{4\times 3}$, $L_2(\cdot) \in \mathbb{R}^{2\times 3}$, and

$\hat{y} = [\hat{x}_1 \; \hat{x}_3 \;\; -\hat{x}_4(2\sin y_1{+}4)]^T$ being the observer output, where we took advantage of the fact that y_1 is measurable to avoid a Taylor representation of $\sin y_1$.

Employing the factorization in section 2.2, the error dynamics (6) can be obtained, where

$$F_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -2z_2-3 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},\; F_2 = \begin{bmatrix} 0 & 0 \\ 2z_2+3 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},\; H(z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -(2z_2+4) \end{bmatrix},$$

$$Q_1(z,\zeta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & z_1\zeta_1 & 0 & z_1\zeta_2 \end{bmatrix},\; Q_2 = \begin{bmatrix} 0 & -1 \\ z_1\zeta_3 & z_1\zeta_4 \end{bmatrix}.$$

with $z_1 = 1/(2\sin y_1{+}4)$ and $z_2 = \sin y_1$ being measurable terms; $\zeta_1 = -2\sin y_1(18\sin y_1 + 4\sin^2 y_1 + u_1x_2 + u_1\hat{x}_2 - x_2\hat{x}_2 - x_2^2 - \hat{x}_2^2 + 27) + 4\cos y_1\sin y_1(4x_2 - 5u_1 + 4\hat{x}_2) + 2\cos y_1(12x_2 - 2u_2 - 15u_1 + 3x_4 + 12\hat{x}_2 + 2y_2) + 4\ddot{y}_2\cos y_1 - 27$, $\zeta_2 = 2\cos y_1(3\hat{x}_2 - u_1 + 2\cos y_1) - 10\sin y_1 - 9$, $\zeta_3 = -2\sin y_1(\hat{x}_2^2 - 31) - 18\cos 2y_1 - 2\sin 3y_1 + 2\sin 2y_1(u_1 + \hat{u}_1) - 10\hat{x}_2\sin 2y_1 + 2\cos y_1(3u_1 + 3\hat{u}_1 - 15\hat{x}_2 - \hat{x}_4) + 49$, and $\zeta_4 = 14\sin y_1 - 4\cos y_1(\hat{x}_2 + \cos y_1) + 16$, being the remaining (possibly unmeasurable) terms.

Let us consider the design regions $\mathcal{C}_x = \{x : |x_1| \le 2, |x_2| \le 1.2, |x_3| \le 4, |x_4| \le 3\}$, $\mathcal{C}_u = \{u : |u_1| \le 2, |u_2| \le 12\}$, and $\mathcal{C}_{y_2^{(2)}} = \{y_2^{(2)} : |y_2^{(2)}| \le 40\}$; they translate into the design intervals $z_1 \in [0.16 \; 0.5]$, $z_2 \in [-1 \; 1]$, $\zeta_1 \in [-407 \; 335]$, $\zeta_2 \in [-22.31 \; 8.71]$, $\zeta_3 \in [-35.7 \; 170]$, and $\zeta_4 \in [0.45 \; 30.52]$, which are used to perform the convex modelling described in section 2.1, based on which the LMI conditions in (10) can be tested: they were feasible with observer gains $L_1^{\mathbf{k}}$ and $L_2^{\mathbf{k}}$ as follows:

$$L_1^{00} = L_1^{01} = \begin{bmatrix} -363.3 & -432 & -89 \\ -787 & -105 & -228 \\ -369 & -567 & -128 \\ 126 & 171 & 74 \end{bmatrix},\; L_1^{10} = L_1^{11} = \begin{bmatrix} -362 & -435 & -81 \\ -786 & -1060 & -209 \\ -368 & -571 & -118 \\ 123 & 183 & 66 \end{bmatrix},$$

$$L_2^{00} = L_2 01 = \begin{bmatrix} -991 & -1407 & -347 \\ 165 & -1288 & 1209 \end{bmatrix},\; L_2^{10} = L_2^{11} = \begin{bmatrix} -987 & -1427 & -318 \\ 61 & -658 & 877 \end{bmatrix}.$$

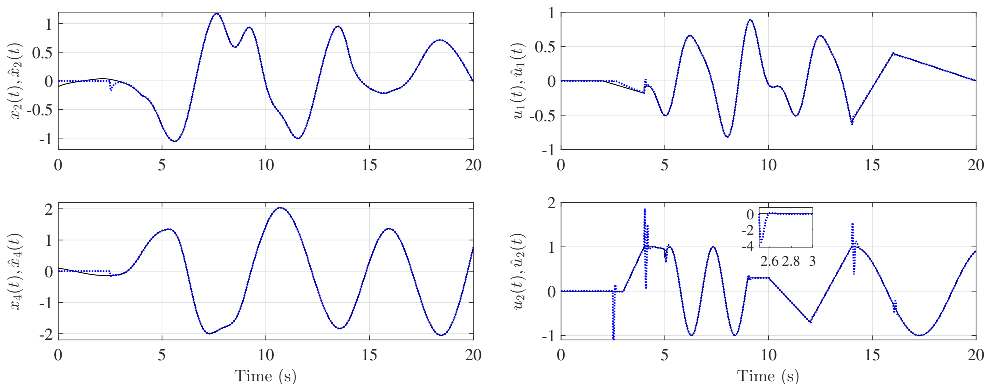
For simulation purposes the unknown inputs

$$u_1(t) = \begin{cases} 0 & t \le 2 \\ -0.09(t-2), & 2 < t \le 4 \\ 0.6\cos 2t + 0.05 + 0.2\sin 3t - 0.2\sin 4t & 4 < t \le 14 \\ 0.5(t-14) - 0.6, & 14 < t \le 16 \\ -0.1(t-16) + 0.4, & 16 < t \end{cases}$$

and

$$u_2(t) = \begin{cases} 0 & t \leq 3 \\ (t-3) & 3 < t \leq 4 \\ 1 & 4 < t \leq 4.5 \\ 1 - 0.1(t-4.5) & 4.5 < t \leq 5 \\ -\cos 3t, & 5 < t \leq 9 \\ 0.3, & 9 < t \leq 10 \\ 0.3 - 0.5(t-10), & 10 < t \leq 12 \\ -0.7 + 0.85(t-12), & 12 < t \leq 14 \\ \sin t, & 14 < t \end{cases}$$

were considered.



(a) Estimation of x_2 and x_4 by $\hat{x}_2$ and $\hat{x}_4$. (b) Estimation of the unknown input u by $\hat{u}$.

Fig. 3. Time evolution of the unmeasurable signals (black) and their estimations (blue) in Example 4.2.

Comparisons: Figures 3(a) – 3(b) depict the time evolution of the unmeasurable signals (solid black lines) and their estimates (dotted blue lines). Observation takes place as expected, both for the states and the unknown inputs; sudden changes of the references sometimes produce transient peaks in the estimation signals. As in the previous example, it can be verified that system (14) does not hold the condition $\text{rank}(CB) = \text{rank}(B)$, i. e., $\text{rank}(CB) = 1$ whilst $\text{rank}(B) = 2$, where B is the distribution matrix of the unknown input; similarly, the methodologies in [14, 22, 41] are unable to deal with the observation problem just presented. Proposals in [6, 36] can be used but, as before, they do not guarantee asymptotic convergence of $e(t)$ to 0 because of the uncoupled unknown input ($\text{rank}(CB) = 1$). Finally, the fact that the inputs do not vanish after a finite number of time derivatives hinders [11] from being applied. □

Example 4.3. Consider the inertia wheel pendulum whose model is given by [26]:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} x_2 \\ -\dfrac{gp_3}{p_1}\sin x_1 - \dfrac{1}{p_1}u \\ x_4 \\ \dfrac{gp_3}{p_1}\sin x_1 + \dfrac{p_1+p_2}{p_1p_2}u \end{bmatrix}, \; y = \begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}x_1\\ x_3\end{bmatrix}, \tag{16}$$

where $g = 9.804$, $p_1 = 0.9245$, $p_2 = 0.005$, and $p_3 = 7.3173$. By successive differentiation of output y_1 or y_2, dynamics of $\dot{u}$ can be inferred. Using y_1 and after differentiating up to the 3rd order we obtain:

$$\dot{u} = -gp_3x_2\cos x_1 - p_1y_1^{(3)}, \tag{17}$$

where $y_1^{(3)}$ is assumed to be finite-time available by means of a robust differentiator. The proposed UIO, based on (16) and (17), and taking into account that x_1 and x_3 are measurable, has the form:

$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\\ \dot{\hat{x}}_3\\ \dot{\hat{x}}_4\\ \dot{\hat{u}}\end{bmatrix} = \begin{bmatrix} \hat{x}_2 \\ -\dfrac{gp_3}{p_1}\sin y_1 - \dfrac{1}{p_1}\hat{u} \\ \hat{x}_4 \\ \dfrac{gp_3}{p_1}\sin y_1 + \dfrac{p_1+p_2}{p_1p_2}\hat{u} \\ -gp_3\hat{x}_2\cos y_1 - p_1y_1^{(3)} \end{bmatrix} + L(z)(\hat{y}-y), \; \hat{y} = \begin{bmatrix}\hat{y}_1\\ \hat{y}_2\end{bmatrix} = \begin{bmatrix}\hat{x}_1\\ \hat{x}_3\end{bmatrix}, \tag{18}$$

from which the observation error system can be written as

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\\ \dot{e}_4\\ \dot{e}_u\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{1}{p_1} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{p_1+p_2}{p_1p_2} \\ 0 & -gp_3\cos y_1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\\ e_u\end{bmatrix} + L(z)\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\\ e_u\end{bmatrix}, \tag{19}$$

which is already in the form (6) with

$$F_1=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}, \; F_2=\begin{bmatrix}0\\ -\dfrac{1}{p_1}\\ 0\\ \dfrac{p_1+p_2}{p_1p_2}\end{bmatrix}, \quad Q_1(y_1)=\begin{bmatrix}0 & -gp_3\cos y_1 & 0 & 0\end{bmatrix}, \; Q_2=0,$$
$$H=\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix}, \; L(z)=\begin{bmatrix}L_1(z)\\ L_2(z)\end{bmatrix},$$

where $L_1(z) \in \mathbb{R}^{4\times 2}$ and $L_2(z) \in \mathbb{R}^{1\times 2}$. This error system has only one non-constant term to be modelled, namely, $z = \cos y_1$, which is measurable and therefore can be used in $L(z)$ by means of convex modelling.

LMIs (10) in Theorem 3.2 are slightly modified for real-time implementation purposes by including a decay rate α such that $\dot{V}(e) \leq -2\alpha V(e)$, i.e.:

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} > 0, \ \mathrm{He}\left(\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}\begin{bmatrix} F_1 & F_2 \\ Q_1^{\mathbf{i}} & Q_2 \end{bmatrix} + \begin{bmatrix} N_1^{\mathbf{i}} \\ N_2^{\mathbf{i}} \end{bmatrix}\begin{bmatrix} H & 0 \end{bmatrix}\right) + 2\alpha \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} < 0.$$

where $Q_1(y_1) = w_i^0(y_1)Q_1^0 + w_i^1(y_1)Q_1^1$, with $Q_1^0 = [0 \ -53.8041 \ 0 \ 0]$, $Q_1^1 = [0 \ - \ 71.7388 \ 0 \ 0]$, $w_1^0(y_1) = 4 - 4\cos y_1$, $w_1^1 = 4\cos y_1 - 3$, $z = \cos y_1 \in [0.75 \ 1]$.

These LMIs are feasible for $\alpha = 35$; based on them it was found that

$$L_1^0 = \begin{bmatrix} -53.24 & -49.06 \\ -1959 & 3421 \\ 53.77 & -226.15 \\ 11882 & -34956 \end{bmatrix}, \ L_1^1 = \begin{bmatrix} -60.42 & -36.14 \\ -2246 & 3937 \\ 57.58 & -233 \\ 12973 & -36918 \end{bmatrix}, \quad \begin{array}{l} L_2^0 = \begin{bmatrix} 4552 & -10014 \end{bmatrix}, \\ L_2^1 = \begin{bmatrix} 5072 & -10950 \end{bmatrix}. \end{array}$$

Comparisons: The results of real-time implementing observer (18) for this plant are given in Figure 4 in red solid line. For the sake of comparison, the estimations provided by the approach in [7] (blue solid line) and that in [11] (black solid line) are also included. The PI estimation in [11] is smooth but far from the right values: it has a mean square error (MSE) of 0.0319; that in [7] is smoother than our proposal but has a MSE of 0.0153 against 0.0061 of ours. □

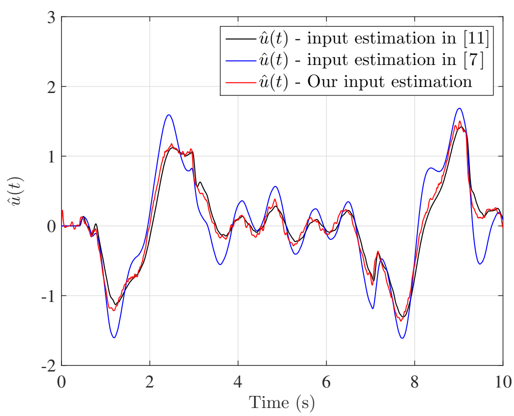


Fig. 4. Real-time estimations of the unknown input in Example 4.3.

5. CONCLUSIONS

A novel unknown input observer for MIMO nonlinear systems has been presented, whose advantages can be summarized as follows:

1. It has been shown that the input dynamics can be characterized by a set of equations solved from a succession of time derivatives of the outputs, properly tuned by a robust differentiator.

2. A recently appeared factorization has been exploited to obtain the observation error model from the input/system dynamics and its nonlinear observer counterpart, based on which convex modelling of nonlinearities and the direct Lyapunov method have been employed to deduce sufficient design conditions in the form of linear matrix inequalities, which are efficiently solved by commercially-available software.

3. The proposal has been enhanced by means of virtual outputs which might turn an unobservable model into an observable one.

4. Asymptotic state, input, and fault estimation have been successfully accomplished, even simultaneously, in examples that most former methodologies cannot solve.

Future work is under course to address the following issues:

1. Systematic selection of virtual outputs and their maximum time-derivative order, choice of factorization of the error signal and non-constant terms in the factorized matrices, in order to achieve feasibility of the resulting LMIs.

2. Systematic design to guarantee a variety of performance specifications under noisy environments or parametric uncertainties.

3. Immediate applications of the proposal for, say, fault-tolerant estimation and control.

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