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EQUILIBRIUM STRATEGIES IN TIME-INCONSISTENT STOCHASTIC CONTROL PROBLEMS WITH DELAYED FEEDBACK

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This paper develops a game-theoretic framework for analyzing stochastic differential delayed equations (SDDEs) in time-inconsistent control problems. By extending the Bellman equation to a system of nonlinear equations, the framework identifies subgame-perfect Nash equilibrium strategies for delayed processes with functional objectives. The approach accounts for the challenges introduced by delays and time inconsistency, providing a robust method for deriving equilibrium strategies. To illustrate its applicability, the framework is applied to a mean-variance portfolio selection problem with state-dependent risk aversion and delay, demonstrating how past decisions influence current outcomes. This work advances the theoretical understanding of SDDEs and offers practical insights for applications in finance and related fields.

Keywords: time inconsistency, equilibrium strategy, extended HJB equations, mean-variance criterion, investment problem with delay

Classification: 93E20, 60H30, 93E99, 60H10

1. INTRODUCTION

Over the past two decades, significant attention has been devoted to stochastic control problems with delays, highlighting their applications in fields such as life sciences, engineering, and financial mathematics. Researchers including [7, 8, 9, 12, 16, 17, 18, 22] have made notable contributions in this area. Stochastic models with delays, commonly referred to as stochastic differential delay equations (SDDEs), account for phenomena dependent on historical states, where the system's current behavior at time t depends on both the present and a finite portion of its past. For instance, Chang et al. [6], considered a portfolio management problem of Merton's type, incorporating a risky asset return linked to the return history. Employing the dynamic programming approach, they derived an explicit solution for the CRRA utility case. Shi [22] extended this work to a recursive utility framework.

The exploration of the mean-variance portfolio problem with delays remains relatively scarce in existing research. In [8], David was among the first to examine the optimal investment problem within a jump-diffusion delayed system, focusing on a single-objective

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mean-variance framework. By applying a sufficient maximum principle to the quadratic loss minimization problem inherent to this framework, an optimal investment strategy was derived in closed-loop form. Later, Shen et al. [23] formulated two distinct sufficient maximum principles for stochastic optimal control problems featuring delay and mean-field terms. Through the application of their second principle to the mean-variance portfolio problem with delay, they were able to identify efficient portfolios and compute efficient frontiers, relying on solutions to two linear ordinary differential equation systems. In a related study, Shen and Zeng [24] addressed an optimal investment and reinsurance problem for insurers under the mean-variance criterion while accounting for delay.

To address time inconsistency in control problems, the game-theoretic perspective, which focuses on Nash equilibrium strategies, has been widely utilized. This approach becomes particularly relevant when the discount function diverges from the exponential form, leading to a loss of time-consistency in utility models. Such models no longer satisfy Bellman's optimality principle, rendering the conventional dynamic programming method unsuitable. In response to this challenge, two principal methods have been proposed to manage time inconsistency in utility models with non-exponential discounting. The first method considers agents referred to as "naive," who make decisions without anticipating how their preferences might change in the future. At a given time $t \in [0, T]$, the naive agent treats the problem as a standard optimal control problem with the initial condition $X(t) = x$. When this agent solves the problem at $t = 0$, the solution is known as the pre-commitment strategy, which remains optimal provided the agent can fully commit to the planned strategy at $t = 0$.

The second approach involves formulating a time-inconsistent decision problem as a non-cooperative game among different instances of the decision maker at various points in time. Nash equilibrium strategies are then considered to define a new concept of solution for the original problem. Strotz, as referenced in [26], was the first to propose a game-theoretic formulation to address dynamic time-inconsistent optimal decision problems, specifically focusing on the deterministic Ramsey problem, as mentioned in [21]. By introducing the concept of non-commitment and allowing for an infinitesimally small commitment period, Strotz provided a primitive notion of Nash equilibrium strategy. Subsequent research along this line, in both continuous and discrete time, has been conducted by Pollak [20], Phelps and Pollak [19].

Continuing with the game-theoretic approach, Ekland and Lazrak [10] and Marin-Solano and Navas [27] addressed the optimal consumption problem in a deterministic framework where the utility function incorporates a non-exponential discount function. They characterized equilibrium strategies using a value function that must satisfy an "extended HJB equation," a nonlinear differential equation with a non-local term dependent on the global behavior of the solution. In this situation, each decision at time t is made by a $t-$agent, representing the controller's incarnation at that time, referred to as a "sophisticated $t-$agent" in [27]. Bjork and Murgoci, as referenced in [3], extend this idea to the stochastic setting, where the controlled dynamics are driven by a general class of Markov processes and a general objective function. Yong, in [29], studied a class of time-inconsistent deterministic linear quadratic models by discretizing time and deriving equilibrium controls via a class of Riccati–Volterra equations. In [30], Yong investigated a general time-inconsistent stochastic optimal control problem with discounting, also by discretizing time, and characterized a feedback time-consistent Nash equilibrium control using the "equilibrium HJB equation".

Numerical methods for solving stochastic control problems with delays have been a topic of extensive research due to their complexity and broad applicability in finance, engineering, and other disciplines. The presence of delays introduces significant challenges, as the system dynamics depend not only on the current state but also on historical states. Classical approaches, such as those detailed in Kushner [15], leverage numerical schemes specifically designed for controlled stochastic delay systems, providing a robust foundation for addressing such problems. These methods often extend finite difference schemes and dynamic programming principles to account for the delayed dynamics, albeit with increased computational demands in higher dimensions. Recent advancements include the application of machine learning-based techniques; in particular [14] shows how recurrent neural networks such as LSTM offer a more effective alternative to classical numerical schemes for handling delayed dynamics. Applied to examples like portfolio optimization, this approach outperforms traditional methods by naturally capturing past dependence and improving efficiency.

In contrast to Bjork et al. [4], we extend the state-dependent risk aversion mean-variance optimization problem to account for delays, where the system state is governed by a stochastic delay differential equation. Our primary focus is the terminal state, $X(T) + \varpi Y(T)$, which includes state-dependent risk aversion. Using stochastic control theory with delays, we derive extended Hamilton–Jacobi–Bellman (HJB) equations.

However, solving the mean-variance optimization problem with state-dependent risk aversion explicitly proves challenging. This requires constructing an exponential martingale process related to wealth evolution. Since the wealth process is governed by a stochastic delay differential equation, the approach in Bjork et al. [4] is not directly applicable. As a solution, we transform the wealth process $X(t)$ into a combinatorial wealth dynamic $X(t) + \varpi Y(t)$, which is described by a stochastic differential equation. We then construct the exponential martingale over $X(t) + \varpi Y(t)$ and seek the optimal strategy based on historical performance.

To our knowledge, no existing literature addresses the optimal time-consistent (Nash equilibrium) problem with delay under a general utility function. This paper fills that gap by exploring time-consistent solutions for systems with delay and state-dependent risk aversion. The framework we introduce builds on the work of Bjork et al. [3], making it applicable to a variety of practical situations. By formulating the problem within a game-theoretic framework, we establish sufficient conditions for Nash equilibrium strategies in the context of the extended HJB equation. In Part 2 of this paper, we present the mean-variance investment problem with delays and state-dependent risk aversion, solve the extended HJB equations with delay, and derive the explicit expression for the optimal time-consistent investment strategy, along with the corresponding equilibrium value functions.

Notations In this work, we use the following notations:

- $\mathcal{C}([t,T];\mathbb{R})$: the space of continuously functions $f:[t,T]\to\mathbb{R}$.

- $\mathcal{C}_p^{1,2,1}([0,T]\times\mathbb{R}^2)$: the space of continuously differentiable functions $f:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ such that the partial derivatives $f_t,\ f_x,\ f_{xx},\ f_y$ satisfy a polynomial growth condition, meaning that there exist constants $C>0$ and $m\in\mathbb{N}$ such that

$$|f_t(t,x,y)|+|f_x(t,x,y)|+|f_{xx}(t,x,y)|+|f_y(t,x,y)|\le C(1+|x|^m+|y|^m),$$

  for all $(t,x,y)\in[0,T]\times\mathbb{R}^2$.

- $\mathcal{C}_p^{1,2,1,1,2,1}([0,T]\times\mathbb{R}^2\times[0,T]\times\mathbb{R}^2)$: the space of continuously differentiable functions $f:[0,T]\times\mathbb{R}^2\times[0,T]\times\mathbb{R}^2\to\mathbb{R}$ such that the partial derivatives $f_t,\ f_x,\ f_{xx},\ f_y,\ f_s,\ f_z,\ f_{zz},\ f_w$ satisfy a polynomial growth condition, i. e., there exist constants $C>0$ and $m\in\mathbb{N}$ such that, for $\chi=f_t,f_s,\ f_x,\ f_{xx},\ f_y,\ f_z,\ f_{zz}$ and f_w we have

$$|\chi(t,x,y,s,z,w)|\le C(1+|x|^m+|y|^m+|z|^m+|w|^m),$$

  for all $(t,s,x,y,z,w)\in[0,T]^2\times\mathbb{R}^4$.

2. THE MODEL AND PROBLEM FORMULATIONS

We consider a filtered, complete probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$ supporting a one-dimensional standard Brownian motion $\mathcal{W}$, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural filtration of $\mathcal{W}$, augmented by all P-null sets. The terminal filtration satisfies $\mathcal{F}_T=\mathcal{F}$, and the time horizon is finite with $T>0$. This provides the probabilistic setting in which all processes are defined.

Given a closed subset $U\subset\mathbb{R}$, let $b:[0,T]\times\mathbb{R}^3\to\mathbb{R}$, $\widetilde{b}:[0,T]\times\mathbb{R}^2\times U\to\mathbb{R}$ and $\sigma:[0,T]\times\mathbb{R}^2\times U\to\mathbb{R}$ be deterministic functions. These functions specify the drift and volatility of the system under control.

We study the controlled stochastic delay differential equation (SDDE)

$$\begin{aligned} dX^\xi(s)&=\Big(b(s,X^\xi(s),Y^\xi(s),Z^\xi(s))+\widetilde{b}(s,X^\xi(s),Y^\xi(s),\pi(s))\Big)ds\\ &\quad+\sigma(s,X^\xi(s),Y^\xi(s),\pi(s))d\mathcal{W}(s), \end{aligned} \tag{2.1}$$

for $s\in[0,T]$, with initial condition $X^\xi(s)=\xi(s)$ for $s\in[-\delta,0]$. Here, $\pi:[0,T]\times\Omega\to U$ is the control process, ξ is the initial path, $Y^\xi(s)=\int_{-\delta}^0 e^{\lambda\tau}X^\xi(s+\tau)\,\mathrm{d}\tau$, and $Z^\xi(s)=X^\xi(s-\delta)$ are functionals of the path segment $\{X^\xi(s+\tau)\}_{\tau\in[-\delta,0]}$. Thus, the dynamics of the state variable X^ξ depend not only on its current value but also on past values, reflecting the presence of delay. The parameter $\lambda\in\mathbb{R}$ is an averaging parameter, and $\delta>0$ represents a fixed delay.

To evaluate the performance of a control process π, we introduce the payoff functional:

$$\begin{aligned} J(t,\xi,\pi)&=\mathbb{E}_{t,\xi}\left[\int_t^T C(t,s,X^\xi(s),Y^\xi(s),\pi(s))\,\mathrm{d}s+L(t,\xi,X^\xi(T)+\varpi Y^\xi(T))\right]\\ &\quad+\Psi(t,\xi,\mathbb{E}_{t,\xi}[X^\xi(T)+\varpi Y^\xi(T)]), \end{aligned} \tag{2.2}$$

where $\mathbb{E}_{t,\xi}[\cdot]$ denote conditional expectation, given the initial path ξ. The deterministic functions $C:[0,T]^2\times\mathbb{R}^2\times U\to\mathbb{R}$, $L:[0,T]\times\mathbb{R}^2\to\mathbb{R}$, and $\Psi:[0,T]\times\mathbb{R}^2\to\mathbb{R}$ capture the economic objectives of the system. The parameter $\varpi \in \mathbb{R}$ balances the contributions of $X^{\xi}(T)$ and $Y^{\xi}(T)$ to the payoff. Hence, the performance criterion combines both the terminal wealth and its delayed component.

All the terms in the payoff functional (2.2) are unconventional. Specifically, the first two terms, are initial time-dependent and can be motivated by the non-exponential discounted utility functions commonly used in economics. However, the last term, Ψ, can be motivated by the variance term in a mean–variance portfolio problem with a delay model. This structure illustrates how delay and time-dependent preferences are jointly embedded in the optimization problem.

With the choice of this functional, time inconsistency arises at two distinct points:

1. The present time t appears explicitly in the local utility function C, as well as in the functions L and Ψ. Consequently, the utility function evolves with time. At time $t \in [0,T]$, the utility function is given by $L(t,\xi,X^{\xi}(T)+\varpi Y^{\xi}(T))$, which is maximized as a function of $X^{\xi}(T)+\varpi Y^{\xi}(T)$. However, at a later time $t+h$ $(h>0)$, the utility function becomes $L(t+h,\xi(t+h),X^{\xi}(T)+\varpi Y^{\xi}(T))$. This dynamic evolution of the utility function leads to time inconsistency.

2. The term $\Psi(t,\xi,\mathbb{E}_{t,\xi}[X^{\xi}(T)+\varpi Y^{\xi}(T)])$ introduces a nonlinear dependence on the conditional expectation. This results in the failure of the iterated-expectations property. Consequently, the Bellman optimality principle does not hold, further reinforcing the time inconsistency of the problem.

The dual appearance of time inconsistency in this framework, through the evolving utility function and the nonlinear conditional expectation term, highlights the challenges inherent in addressing such problems within the standard stochastic control framework.

Definition 2.1. (Admissible Control) An admissible control π over $[t,T]$ is a U-valued measurable $(\mathcal{F}_s)_{s\in[t,T]}$ adapted process such that: For each initial state (t,ξ) the SDDE (2.1) admits unique strong solution, with

$$\mathbb{E}_{t,\xi}\left[\int_t^T C\left(t,s,X^{\xi}(s),Y^{\xi}(s),\pi(s)\right)\,\mathrm{d}s + L\left(t,\xi,X^{\xi}(T)+\varpi Y^{\xi}(T)\right)\right] < \infty, \quad (2.3)$$

$$\Psi\left(t,\xi,\mathbb{E}_{t,\xi}\left[X^{\xi}(T)+\varpi Y^{\xi}(T)\right]\right) < \infty.$$

This ensures that the performance functional is well-defined and finite for every admissible strategy. In the rest of this paper we denote by $\mathcal{U}[t,T]$ the set of all admissible control over $[t,T]$.

Let $(t,\xi) \in [0,T]\times C([-\delta,0];\mathbb{R})$ be a given initial pair, where ξ represents the initial path of the state process over the delay interval $[-\delta,0]$. Following the approach in [16], we assume that the objective functional depends on ξ only through the following two quantities: $x=\xi(0), \quad y=\int_{-\delta}^{0} e^{\lambda\tau}\xi(\tau)\,\mathrm{d}\tau$. Here, x represents the current state, and y is a weighted integral of the delayed states. This reduction from the entire path ξ to two sufficient statistics (x,y) transforms the problem into a finite-dimensional one, making it mathematically more tractable.

For any admissible control $\pi \in U[t,T]$, we define

$$\begin{aligned}\bar{J}(t,\xi,\pi) &:= J(t,x,y,\pi)\\ &= \mathbb{E}_{t,x,y}\left[\int_t^T C\left(t,s,X^{x,y}(s),Y^{x,y}(s),\pi(s)\right)\,\mathrm{d}s\ + L\left(t,x,y,X^{x,y}(T)+\varpi Y^{x,y}(T)\right)\right]\\ &\quad + \Psi\left(t,x,y,\mathbb{E}_{t,x,y}\left(X^{x,y}(T)+\varpi Y^{x,y}(T)\right)\right).\end{aligned} \tag{2.4}$$

where $\mathbb{E}_{t,x,y}[\cdot] := \mathbb{E}[\cdot \mid X(t)=x, Y(t)=y]$. Thus, the performance functional can now be expressed entirely in terms of the reduced state variables (x,y), conditional on their initial values.

We can now formulate the stochastic optimal control problem as follows

Problem (N). Given $(t,\xi) \in [0,T]\times\mathbb{R}^2$, find $\hat{\pi} \in \mathcal{U}[t,T]$ such that

$$\bar{J}(t,\xi,\hat{\pi}) = \min_{\pi \in U[t,T]} \bar{J}(t,\xi,\pi).$$

This states that the goal is to select an admissible control strategy $\hat{\pi}$ that minimizes the objective functional (2.3), given the initial condition $(t,\xi) \in [0,T]\times\mathbb{R}^2$.

Remark 2.2. For a given initial state (t,ξ), where ξ is the general full initial path in $C([-\delta,0];\mathbb{R})$, any admissible strategy $\hat{\pi}$ satisfying (2.4) is called a pre-commitment optimal solution to Problem (N) at (t,ξ). Without the reduction to (x,y), the problem would be infinite-dimensional, since the objective functional could depend on the entire initial path ξ in a complicated way. The above assumption makes the problem finite-dimensional and therefore more tractable.

The above assumption therefore plays a crucial role in simplifying the model: it reduces the dimensionality of the problem while retaining the key effects of delay.

3. TIME INCONSISTENCY AND FEEDBACK EQUILIBRIUMS

The dynamic optimization problem (2.4) demonstrates time inconsistency due to the non-linear dependence of the objective functional J on the combined terminal wealth and average performance. Given the importance of time consistency in rational decision-making, this study aims to characterize the optimal time-consistent solution, referred to as the equilibrium to Problem (N). To achieve this, we adopt an approach inspired by the extended Hamilton–Jacobi–Bellman (HJB) framework introduced by Bjork et al. [3]. While their work considers state variables governed by general stochastic differential equations without delay, our problem incorporates past dependence, necessitating an adaptation of their equilibrium definition and extended HJB equations. A key step in this process involves defining feedback equilibriums by introducing the class of admissible feedback controls, also known as control laws in [3].

Definition 3.1. (Feedback Strategy) An admissible feedback strategy is a map $\pi : [0,T] \times \mathbb{R}^2 \to U$ such that, for any $(t,\xi) \in [0,T] \times \mathcal{C}([-\delta,0];\mathbb{R})$, the SDDE

$$\begin{cases} dX^{\xi}(s) & = \left\{ b\left(s, X^{\xi}(s), Y^{\xi}(s), Z^{\xi}(s)\right) \right. \\ & \quad \left. + \widetilde{b}\left(s, X^{\xi}(s), Y^{\xi}(s), \pi\left(s, X^{\xi}(s), Y^{\xi}(s)\right)\right) \right\} ds \\ & \quad + \sigma\left(s, X^{\xi}(s), Y^{\xi}(s), \pi\left(s, X^{\xi}(s), Y^{\xi}(s)\right)\right) d\mathcal{W}(s),\ s \in [0,T], \\ X(0) & = \xi(0) = x,\ Y(0) = y = \int_{-\delta}^{0} e^{\lambda\tau} \xi(\tau)\, \mathrm{d}\tau, \end{cases} \tag{3.1}$$

has a unique strong solution denoted by X^{ξ}.

We denote by $\mathcal{U}[0,T]$ the set of all admissible feedback control. In addition, we will sometimes use the notations $\pi(t)$ instead of $\pi(t,x,y)$ and X instead of X^{ξ}.

Remark 3.2. It's crucial to note that our assumption entails that the feedback controls are independent of z. This assumption can be broadly understood through the representation (2.3), which stipulates that the objective function J is solely dependent on x and y. This simplification reflects the fact that the reduced state representation (x,y) already captures the relevant impact of delay.

We refer the readers to [3] and [11] for the the intuition behind the following definition.

Definition 3.3. (Feedback Equilibrium) An admissible feedback control $\hat{\pi} \in \mathcal{U}[0,T]$ is an equilibrium control if the following condition holds

$$\lim_{\epsilon \downarrow 0} \inf \frac{1}{\epsilon} \left\{ J(t,x,y;\hat{\pi}) - J(t,x,y;\pi^{\epsilon}) \right\} \geq 0, \tag{3.2}$$

where for any $\epsilon \in [0, T-t]$,

$$\pi^{\epsilon}(s,x,y) = \begin{cases} \pi(s,x,y) & \text{for } (s,x,y) \in [t,t+\epsilon] \times \mathbb{R}^2, \\ \hat{\pi}(s,x,y) & \text{for } (s,x,y) \in]0,t[\cup]t+\epsilon, T[\times \mathbb{R}^2. \end{cases} . \tag{3.3}$$

The deterministic function $W : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$,

$$W(t,x,y) = J(t,x,y,\hat{\pi}) \tag{3.4}$$

is called the equilibrium value function of the Problem (N).

Before presenting the extended HJB equations and the corresponding verification theorem for equilibriums, we introduce the infinitesimal generator associated to our model, see, for instance [12]. For any feedback control $\pi \in \mathcal{U}[0,T]$ the operator $\mathcal{A}^{\pi}$ is defined for any $\phi \in \mathcal{C}^{1,2,1}\left([0,T] \times \mathbb{R}^2\right)$ as follows

$$\begin{aligned} \mathcal{A}^{\pi}\phi(t,x,y) = & \frac{\partial \phi}{\partial t}(t,x,y) + \frac{\partial \phi}{\partial x}(t,x,y) \left\{ b(t,x,y,z) + \widetilde{b}(t,x,y,\pi) \right\} \\ & + \frac{\partial \phi}{\partial y}(t,x,y) \left\{ x - e^{-\delta\lambda} z - \lambda y \right\} \\ & + \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2}(t,x,y)\, \sigma^2(t,x,y,z,\pi). \end{aligned} \tag{3.5}$$

The operator $\mathcal{A}^{\pi}$ plays the role of the infinitesimal generator of the controlled process, and will be central in deriving the extended HJB system. Inspired from [3], we formulate the extended HJB equations as follows, $\forall\,(t,x,y)\in[0,T]\times\mathbb{R}^2$, we have

$$\begin{aligned}\sup_{\pi\in\mathcal{U}[0,T]}\ &\{\mathcal{A}^{\pi}W(t,x,y)+C(t,t,x,y,\pi)-\mathcal{A}^{\pi}l(t,x,y,t,x,y)+\mathcal{A}^{\pi}l^{t,x,y}(t,x,y)\\ &-\mathcal{A}^{\pi}\Psi\diamond\vartheta(t,x,y)+\mathcal{H}^{\pi}\vartheta(t,x,y)\}=0,\end{aligned}\tag{3.6}$$

with the boundary condition

$$W(T,x,y)=L(T,x,y,x+\varpi y)+\Psi(T,x,y,x+\varpi y).$$

Here, W represents the equilibrium value function, while the auxiliary functions ϑ, $l^{t,x,y}$and l are introduced to handle time inconsistency in the problem. Note that W, ϑ, $l^{t,x,y}\in\mathcal{C}_p^{1,2,1}\left([0,T]\times\mathbb{R}^2\right)$ and $l\in\mathcal{C}_p^{1,2,1,1,2,1}([0,T]\times\mathbb{R}^2\times[0,T]\times\mathbb{R}^2)$ are deterministic functions, with $\hat{\pi}$ denotes the feedback control that attains the supremum in the W-equation i. e. $\forall\,(t,x,y)\in[0,T]\times\mathbb{R}^2$

$$\begin{aligned}\hat{\pi}(t)=\quad\arg\sup_{\pi\in\mathcal{U}[0,T]}\ &\{\mathcal{A}^{\pi}W(t,x,y)+C(t,t,x,y,\pi(t))-\mathcal{A}^{\pi}l(t,x,y,t,x,y)\\ &+\mathcal{A}^{\pi}l^{t,x,y}(t,x,y)-\mathcal{A}^{\pi}\Psi\diamond\vartheta(t,x,y)+\mathcal{H}^{\pi}\vartheta(t,x,y)\}.\end{aligned}\tag{3.7}$$

Thus, the equilibrium control $\hat{\pi}$ is defined as the maximizer of the extended HJB expression, ensuring consistency of the strategy over time. We then have the following auxiliary characterizations

1. For any fixed s, x_1 and y_1 the function l^{s,x_1,y_1} is defined as the solution of

$$\begin{aligned}&\mathcal{A}^{\hat{\pi}}l^{s,x_1,y_1}(t,x,y)+C(s,t,x,y,\hat{\pi})=0,\ 0\le t<T,\\ &l^{s,x_1,y_1}(T,x,y)=L(s,x_1,y_1,x+\varpi y).\end{aligned}\tag{3.8}$$

   This auxiliary function l^{s,x_1,y_1} encodes the dependence of future costs on the initial state (s,x_1,y_1).

2. The function ϑ is defined as the solution of

$$\begin{aligned}&\mathcal{A}^{\hat{\pi}}\vartheta(t,x,y)=0,\ 0\le t\le T,\\ &\vartheta(T,x,y)=x+\varpi y.\end{aligned}\tag{3.9}$$

   Note that ϑ acts as a propagator for the terminal state, carrying information about future values back to earlier times.

3. We have used the notation

$$\begin{cases}l(t,x,y,s,x_1,y_1)=l^{s,x_1,y_1}(t,x,y),\ \forall\,(t,s,x,y,x_1,y_1)\in[0,T]^2\times\mathbb{R}^4,\\ \Psi\diamond\vartheta(t,x,y)=\Psi(t,x,y,\vartheta(t,x,y)),\ \forall\,(t,x,y)\in[0,T]\times\mathbb{R}^2,\\ \mathcal{H}^{\pi}\vartheta(t,x,y)=\dfrac{\partial\Psi}{\partial\vartheta}(t,x,y,\vartheta(t,x,y))\,\mathcal{A}^{\pi}\vartheta(t,x,y),\ \forall\,(t,x,y)\in[0,T]\times\mathbb{R}^2,\\ \Psi_{\vartheta}(t,x,\vartheta(t,x,y))=\dfrac{\partial\Psi}{\partial\vartheta}(t,x,y,\vartheta(t,x,y)),\ \forall\,(t,x,y)\in[0,T]\times\mathbb{R}^2.\end{cases}\tag{3.10}$$

4. EXTENDED HJB EQUATIONS AND VERIFICATION THEOREM

Theorem 4.1. Assume that there exists the functions W, l^{s,x_1,y_1}, l and ϑ which have the followig properties

1. W, l^{s,x_1,y_1}, l and ϑ do not depend on z.
2. W, l^{s,x_1,y_1} and ϑ solve the extended HJB system (3.6) – (3.9).
3. W, l^{s,x_1,y_1}, $\vartheta \in \mathcal{C}_p^{1,2,1}\left([0,T]\times\mathbb{R}^2\right)$ and $l \in \mathcal{C}_p^{1,2,1,1,2,1}([0,T]\times\mathbb{R}^2\times[0,T]\times\mathbb{R}^2)$.
   In addition, let $\hat{\pi}$ be an admissible feedback control, i. e. $\hat{\pi}\in\mathcal{U}[0,T]$, which realizes the supremum in the W-equation. Then, $\hat{\pi}$ is a feedback equilibrium control and W is the corresponding equilibrium value function, i. e.

$$\begin{aligned} &W(t,x,y)\\ &= \mathbb{E}_{t,x,y}\left[\int_t^T C\left(t,s,\hat{X}(s),\hat{Y}(s),\hat{\pi}(s)\right)\mathrm{d}s + L\left(t,x,y,\hat{X}(T)+\varpi\hat{Y}(T)\right)\right]\\ &\qquad + \Psi\left(t,x,y,\mathbb{E}_{t,x,y}\left[\hat{X}(T)+\varpi\hat{Y}(T)\right]\right). \end{aligned} \tag{4.1}$$

Furthermore $l^{s,x_1,y_1}$and ϑ has the following probabilistic representations for $0\le t\le T$

$$\begin{aligned} l^{s,x_1,y_1}(t,x,y) &= \mathbb{E}_{t,x,y}\left[\int_s^T C\left(s,r,\hat{X}(r),\hat{Y}(r),\pi\left(\hat{X}(r),\hat{Y}(r)\right)\right)\mathrm{d}r\right.\\ &\qquad\left. + L\left(s,x_1,y_1,\hat{X}(T)+\varpi\hat{Y}(T)\right)\right],\\ \vartheta(t,x,y) &= \mathbb{E}_{t,x,y}\left[\hat{X}(T)+\varpi\hat{Y}(T)\right]. \end{aligned} \tag{4.2}$$

From this, it follows that

$$W(t,x,y) = l(t,x,y,t,x,y) + \Psi(t,x,y,\vartheta(t,x,y)). \tag{4.3}$$

Proof. Following the approach in [3], we first demonstrate that W, l, l^{s,x_1,y_1} and ϑ satisfy the Feynman-Kac representation and that W corresponds to the equilibrium value function associated with the strategy $\hat{\pi}$, i. e., $W(t,x,y) = J(t,x,y;\hat{\pi})$. Subsequently, we prove that $\hat{\pi}$ constitutes a feedback equilibrium control. To establish the interpretation of ϑ given in (4.2), we apply Itô's formula (see e. g. [12]) to the process $\kappa\mapsto\vartheta(\kappa,\hat{X}(\kappa),\hat{Y}(\kappa))$ we get

$$\begin{aligned} &d\vartheta\left(\kappa,\hat{X}(\kappa),\hat{Y}(\kappa)\right)\\ &= \mathcal{A}^{\hat{\pi}}\vartheta\left(\kappa,\hat{X}(\kappa),\hat{Y}(\kappa)\right)d\kappa + \frac{\partial\vartheta}{\partial x}\left(\kappa,\hat{X}(\kappa),\hat{Y}(\kappa)\right)\sigma\left(\kappa,\hat{X}(\kappa),\hat{Y}(\kappa),\hat{\pi}(\kappa)\right)d\mathcal{W}(\kappa). \end{aligned}$$

From (3.9), it follows that the process $\vartheta\left(\kappa,\hat{X}(\kappa),\hat{Y}(\kappa)\right)$ is a martingale. Using the boundary conditions for ϑ, we deduce that

$$\vartheta(t,x,y) = \mathbb{E}_{t,x,y}\left[\hat{X}(T)+\varpi\hat{Y}(T)\right]. \tag{4.4}$$

Now applying Itô formula to $\kappa \to l^{s,x_1,y_1}\left(\kappa, \hat{X}(\kappa), \hat{Y}(\kappa)\right)$, we obtain that

$$\begin{aligned} &dl^{s,x_1,y_1}\left(\kappa, \hat{X}(\kappa), \hat{Y}(\kappa)\right) \\ &= \mathcal{A}^{\hat{\pi}} l^{s,x_1,y_1}\left(\kappa, \hat{X}(\kappa), \hat{Y}(\kappa)\right) d\kappa \\ &+ \frac{\partial l^{s,x_1,y_1}}{\partial x}\left(\kappa, \hat{X}(\kappa), \hat{X}(\kappa)(\kappa)\right) \sigma\left(\kappa, \hat{X}(\kappa), \hat{Y}(\kappa), \hat{\pi}(\kappa)\right) d\mathcal{W}(\kappa). \end{aligned}$$

Using (3.8) it follows that

$$\begin{aligned} dl^{s,x_1,y_1}\left(\kappa, \hat{X}(\kappa), \hat{Y}(\kappa)\right) &= -C\left(s, \kappa, \hat{X}(\kappa), \hat{Y}(\kappa), \hat{Z}(\kappa), \hat{\pi}(\kappa)\right) d\kappa \\ + \frac{\partial l^{s,x_1,y_1}}{\partial x}\left(\kappa, \hat{X}(\kappa), \hat{Y}(\kappa)\right) &\sigma\left(\kappa, \hat{X}(\kappa), \hat{Y}(\kappa), \hat{\pi}(\kappa)\right) d\mathcal{W}(\kappa). \end{aligned}$$

From the boundary conditions for l^{s,x_1,y_1} we obtain the following representation

$$\begin{aligned} &l^{s,x_1,y_1}(t,x,y) \\ &= \mathbb{E}_{t,x,y}\left[\int_s^T C\left(s, \kappa, \hat{X}(\kappa), \hat{Y}(\kappa), \hat{Z}(\kappa), \hat{\pi}(\kappa)\right) \mathrm{d}\kappa + L\left(s, x_1, y_1, \hat{X}(T) + \varpi \hat{Y}(T)\right)\right]. \end{aligned}$$

To show that $W(t,x,y) = J(t,x,y;\hat{\pi})$, we use the equation (3.6) to obtain

$$\begin{aligned} &\mathcal{A}^{\hat{\pi}} W(t,x,y) + C(t,t,x,y,\hat{\pi}) - \mathcal{A}^{\hat{\pi}} l(t,x,y,t,x,y) + \mathcal{A}^{\hat{\pi}} l^{t,x,y}(t,x,y) \\ &-\mathcal{A}^{\hat{\pi}}(\Psi \diamond \vartheta)(t,x,y) + \mathcal{H}^{\hat{\pi}} \vartheta(t,x,y) = 0, \end{aligned} \tag{4.5}$$

By using (3.8) and (3.9) the equation (4.5) takes the form

$$\mathcal{A}^{\hat{\pi}} W(t,x,y) - \mathcal{A}^{\hat{\pi}} l(t,x,y,t,x,y) - \mathcal{A}^{\hat{\pi}}(\Psi \diamond \vartheta)(t,x,y) = 0. \tag{4.6}$$

We now apply Itô's formula to the process $W\left(\kappa, \hat{X}(\kappa), \hat{Y}(\kappa)\right)$. Integrating and taking expectations we obtain

$$\mathbb{E}_{t,x,y}\left[W\left(T, \hat{X}(T), \hat{Y}(T)\right)\right] = W(t,x,y) + \mathbb{E}_{t,x,y}\left[\int_t^T \mathcal{A}^{\hat{\pi}} W\left(\kappa, \hat{X}(\kappa), \hat{Y}(\kappa)\right) \mathrm{d}\kappa\right]. \tag{4.7}$$

Using equation (4.6), we thus obtain

$$\begin{aligned} &\mathbb{E}_{t,x,y}\left[W\left(T, \hat{X}(T), \hat{Y}(T)\right)\right] - W(t,x,y) \\ &= \mathbb{E}_{t,x,y}\left[\int_t^T \left(\mathcal{A}^{\hat{\pi}} l\left(\kappa, \hat{X}(\kappa), \hat{Y}(\kappa), \kappa, \hat{X}(\kappa), \hat{Y}(\kappa)\right)\right.\right. \\ &\left.\left. + \mathcal{A}^{\hat{\pi}}(\Psi \diamond \vartheta)\left(\kappa, \hat{X}(\kappa), \hat{Y}(\kappa)\right)\right) \mathrm{d}\kappa\right]. \end{aligned} \tag{4.8}$$

In the same way we obtain

$$\begin{aligned} &\mathbb{E}_{t,x,y}\left[l\left(T, \hat{X}(T), \hat{Y}(T), T, \hat{X}(T), \hat{Y}(T)\right)\right] - l(t,x,y,t,x,y) \\ &= \mathbb{E}_{t,x,y}\left[\int_t^T \mathcal{A}^{\hat{\pi}} l\left(\kappa, \hat{X}(\kappa), \hat{Y}(\kappa), \kappa, \hat{X}(\kappa), \hat{Y}(\kappa)\right) \mathrm{d}\kappa\right] \end{aligned} \tag{4.9}$$

and

$$\begin{aligned}&\mathbb{E}_{t,x,y}\left[(\Psi\diamond\vartheta)\left(T,\hat{X}\left(T\right),\hat{Y}\left(T\right)\right)\right]-(\Psi\diamond\vartheta)\left(t,x,y\right)\\&=\mathbb{E}_{t,x,y}\left[\int_t^T\mathcal{A}^{\hat{\pi}}\left(\Psi\diamond\vartheta\right)\left(\kappa,\hat{X}\left(\kappa\right),\hat{Y}\left(\kappa\right)\right)\,\mathrm{d}\kappa\right],\end{aligned}\tag{4.10}$$

By utilizing the two equalities above and the boundary conditions for W, l and ϑ we derive

$$W\left(t,x,y\right)=L\left(t,x,y,t,x,y\right)+\Psi\left(t,x,y,\vartheta\left(t,x,y\right)\right).\tag{4.11}$$

The second part of the proof aims to highlight that $\hat{\pi}$ is indeed an equilibrium strategy. For any admissible strategy π, we define l^{π} and ϑ^{π} by

$$\begin{aligned}&l^{\pi}(t,x,y,t,x_1,y_1)=\mathbb{E}_{t,x,y}[L\left(t,x_1,y_1,X(T)+\varpi Y\left(T\right)\right)],\\&\vartheta^{\pi}(T,x,y)=\mathbb{E}_{t,x,y}[X(T)+\varpi Y\left(T\right)].\end{aligned}\tag{4.12}$$

Note that $l=l^{\hat{\pi}}$ and $\vartheta=\vartheta^{\hat{\pi}}$ for $\pi=\hat{\pi}$. For any $\epsilon>0$ and for any admissible strategy, we proceed to construct an admissible strategy as given in the relevant definition. From Lemma 3.3 in [3] applied to the points t and $t+\epsilon$, we get

$$\begin{aligned}&J(t,x,y,\pi^{\epsilon})\\&=\mathbb{E}_{t,x,y}\left[J\left(t+\epsilon,X^{\epsilon}(t+\epsilon),Y^{\epsilon}(t+\epsilon),\pi^{\epsilon}\right)\right]\\&-\left(\mathbb{E}_{t,x,y}\left[l^{\pi^{\epsilon}}\left(t+\epsilon,X^{\epsilon}(t+\epsilon),Y^{\epsilon}(t+\epsilon),t+\epsilon,X^{\epsilon}(t+\epsilon),Y^{\epsilon}(t+\epsilon)\right)\right]\right.\\&\left.-\mathbb{E}_{t,x,y}\left[l^{\pi^{\epsilon}}(t+\epsilon,X^{\epsilon}(t+\epsilon),Y^{\epsilon}(t+\epsilon),t,x,y)\right]\right)\\&-\left(\mathbb{E}_{t,x,y}\left[\Psi(t+\epsilon,X^{\epsilon}(t+\epsilon),Y^{\epsilon}(t+\epsilon),\vartheta^{\pi^{\epsilon}}(t+\epsilon,X^{\epsilon}(t+\epsilon),Y^{\epsilon}(t+\epsilon))\right]\right.\\&\left.-\Psi(t+\epsilon,x,y,\mathbb{E}_{t,x,y}\left[\vartheta^{\pi^{\epsilon}}(t+\epsilon,X^{\epsilon}(t+\epsilon),Y^{\epsilon}(t+\epsilon)\right]\right).\end{aligned}$$

It is easy to remark that for any $\epsilon\in[0,T-t]$

$$\pi^{\epsilon}(s,x,y)=\begin{cases}\pi(s,x,y)&\text{for }(s,x,y)\in[t,t+\epsilon]\times\mathbb{R}^2,\\\hat{\pi}(s,x,y)&\text{for }(s,x,y)\in[t+\epsilon,T]\times\mathbb{R}^2,\end{cases}$$

and by continuity, we have $X^{\epsilon}(t+\epsilon)=X(t+\epsilon)$ and $Y^{\epsilon}(t+\epsilon)=Y(t+\epsilon)$. Then, we obtain that

$$J\left(t+\epsilon,X^{\epsilon}(t+\epsilon),Y^{\epsilon}(t+\epsilon),\pi^{\epsilon}\right)=W\left(t+\epsilon,X(t+\epsilon),Y(t+\epsilon)\right),\tag{4.13}$$

and

$$\begin{aligned}&l^{\pi^{\epsilon}}\left(t+\epsilon,X^{\epsilon}(t+\epsilon),Y^{\epsilon}(t+\epsilon),t+\epsilon,X^{\epsilon}(t+\epsilon),Y^{\epsilon}(t+\epsilon)\right)\\&=l\left(t+\epsilon,X(t+\epsilon),Y(t+\epsilon),t+\epsilon,X(t+\epsilon),Y(t+\epsilon)\right),\\&l^{\pi^{\epsilon}}\left(t+\epsilon,X^{\epsilon}(t+\epsilon),Y^{\epsilon}(t+\epsilon),t,x,y\right)=l\left(t+\epsilon,X(t+\epsilon),Y(t+\epsilon),t,x,y\right),\end{aligned}$$

with

$$\vartheta^{\pi^{\epsilon}}(t+\epsilon,X^{\epsilon}(t+\epsilon),Y^{\epsilon}(t+\epsilon))=\vartheta(t+\epsilon,X(t+\epsilon),Y(t+\epsilon)).$$

Consequently,

$$\begin{aligned}J(t,x,y,\pi^{\epsilon}) &= \mathbb{E}_{t,x,y}\left[W\left(t+\epsilon, X(t+\epsilon), Y(t+\epsilon)\right)\right] \\ &- \left(\mathbb{E}_{t,x,y}\left[l^{\pi}\left(t+\epsilon, X(t+\epsilon), Y(t+\epsilon), t+\epsilon, X(t+\epsilon), Y\left(t+\epsilon\right)\right)\right]\right. \\ &\left.-\mathbb{E}_{t,x,y}\left[l^{\pi}\left(t+\epsilon, X(t+\epsilon), Y\left(t+\epsilon\right), t, x, y\right)\right]\right) \\ &- \left(\mathbb{E}_{t,x,y}\left[\Psi(t+\epsilon, X(t+\epsilon), Y\left(t+\epsilon\right), \vartheta^{\pi}(t+\epsilon, X(t+\epsilon), Y\left(t+\epsilon\right))\right]\right. \\ &\left.-\Psi\left(t,x,y,\mathbb{E}_{t,x,y}\left[\vartheta^{\pi}\left(t+\epsilon, X(t+\epsilon), Y\left(t+\epsilon\right)\right)\right]\right)\right).\end{aligned}$$

Furthermore, from the extended HJB equation, we have that

$$\begin{gathered}\mathcal{A}^{\pi}W(t,x,y) + C\left(t,t,x,y,\pi\right) - \mathcal{A}^{\pi}l(t,x,y,t,x,y) + \mathcal{A}^{\pi}l^{t,x,y}(t,x,y) \\ -\mathcal{A}^{\pi}(\Psi\diamond\vartheta)(t,x,y) + \mathcal{H}^{\pi}\vartheta(t,x,y) \leq 0,\end{gathered} \tag{4.14}$$

which implies that

$$\begin{aligned}&\mathbb{E}_{t,x,y}\left[W\left(t+\epsilon, X(t+\epsilon), Y(t+\epsilon)\right)\right] - W(t,x,y) \\ &- \left(\mathbb{E}_{t,x,y}\left[l^{\pi}\left(t+\epsilon, X(t+\epsilon), Y(t+\epsilon), t+\epsilon, X(t+\epsilon), Y(t+\epsilon)\right)\right]\right. \\ &\left.-l(t,x,y,t,x,y)\right) + \mathbb{E}_{t,x,y}\left[l^{\pi}\left(t+\epsilon, X(t+\epsilon), Y(t+\epsilon), t, x, y\right)\right] \\ &- l(t,x,y,t,x,y) \\ &- \left(\mathbb{E}_{t,x,y}\left[\Psi\left(t+\epsilon, X(t+\epsilon), Y(t+\epsilon), \vartheta^{\pi}\left(t+\epsilon, X(t+\epsilon), Y(t+\epsilon)\right)\right)\right]\right. \\ &\left.-\Psi\left(t,x,y,\vartheta(t,x,y)\right)\right) + \Psi\left(t,x,y,\mathbb{E}_{t,x,y}\left[\vartheta^{\pi}\left(t+\epsilon, X(t+\epsilon), Y(t+\epsilon)\right)\right]\right) \\ &- \Psi(t,x,y,\vartheta(t,x,y)) \\ &\leq o(\epsilon).\end{aligned}$$

After performing several simplifications, we arrive at

$$\begin{aligned}W(t,x,y) &\geq \mathbb{E}_{t,x,y}\left[W\left(t+\epsilon, X(t+\epsilon), Y(t+\epsilon)\right)\right] \\ &- \mathbb{E}_{t,x,y}\left[l^{\pi}\left(t+\epsilon, X(t+\epsilon), Y(t+\epsilon), X(t+\epsilon), Y(t+\epsilon)\right)\right] \\ &+ \mathbb{E}_{t,x,y}\left[l^{\pi}\left(t+\epsilon, X(t+\epsilon), Y(t+\epsilon), t, x, y\right)\right] \\ &- \mathbb{E}_{t,x,y}\left[\Psi\left(t+\epsilon, X(t+\epsilon), Y(t+\epsilon), \vartheta^{\pi}\left(t+\epsilon, X(t+\epsilon), Y(t+\epsilon)\right)\right)\right] \\ &+ \Psi\left(t,x,y,\mathbb{E}_{t,x,y}\left[\vartheta^{\pi}\left(t+\epsilon, X(t+\epsilon), Y(t+\epsilon)\right)\right]\right) + o(\epsilon), \\ &= J(t,x,y,\pi^{\epsilon}) + o(\epsilon).\end{aligned}$$

We have already established in the first part that $W(t,x,y) = J(t,x,y,\hat{\pi})$. So,

$$J(t,x,y,\hat{\pi}) - J(t,x,y,\pi^{\epsilon}) \geq o(\epsilon),$$

hence

$$\liminf_{\epsilon\to 0}\left\{\frac{J(t,x,y,\hat{\pi}) - J(t,x,y,\pi^{\epsilon})}{\epsilon}\right\} \geq 0. \tag{4.15}$$

As a result, $\hat{\pi}$ is an equilibrium strategy. □

Remark 4.2. Since the infinitesimal generators $\mathcal{A}^{\pi}$ and $\mathcal{A}^{\hat{\pi}}$ are expressed in terms of the coefficients of the stochastic delay differential equation (3.1), which themselves depend on the delay variable z, it follows that the coefficients of the extended Hamilton–Jacobi–Bellman system (3.6) – (3.9) also inherit this z-dependence. Hence, in the general case, one cannot presume that the solutions of the extended HJB system are independent of z. The next theorem, however, establishes the necessary and sufficient conditions on the functions b, $\widetilde{b}$, σ, L and Ψ that ensure the first condition in Theorem 4.1 is satisfied.

Theorem 4.3. The extended HJB system (3.6) – (3.9) possesses a solution triple W, l and ϑ that is independent of the delay variable z, if and only if the following conditions are satisfied.

$$b\left(t,x,y,z\right)=\alpha\left(t,x,y\right)+z\beta\left(t,x,y\right), \tag{4.16}$$

and

$$\frac{\partial b}{\partial y}\left(t,x,y,z\right)=e^{\delta\lambda}\beta\left(t,x,y\right)\frac{\partial b}{\partial x}\left(t,x,y,z\right), \tag{4.17}$$

$$\frac{\partial\widetilde{b}}{\partial y}\left(t,x,y,\hat{\pi}\left(t\right)\right)=e^{\delta\lambda}\beta\left(t,x,y\right)\frac{\partial\widetilde{b}}{\partial x}\left(t,x,y,\hat{\pi}\left(t\right)\right), \tag{4.18}$$

$$\frac{\partial\sigma}{\partial y}\left(t,x,y,\hat{\pi}\left(t\right)\right)=e^{\delta\lambda}\beta\left(t,x,y\right)\frac{\partial\sigma}{\partial x}\left(t,x,y,\hat{\pi}\left(t\right)\right), \tag{4.19}$$

$$\begin{aligned}&\frac{\partial L}{\partial y}\left(T,x,y,x+\varpi y\right)+\frac{\partial\Psi}{\partial y}\left(T,x,y,x+\varpi y\right)\\&=e^{\delta\lambda}\beta\left(t,x,y\right)\left(\frac{\partial L}{\partial x}\left(T,x,y,x+\varpi y\right)+\frac{\partial\Psi}{\partial x}\left(T,x,y,x+\varpi y\right)\right),\end{aligned} \tag{4.20}$$

$$\varpi=\beta\left(t,x,y\right)e^{\delta\lambda}. \tag{4.21}$$

P r o o f. Arguing as in the classical verification approach [8], the proof first establishes the necessity of the stated conditions. Assume that W, l^{s,x_1,y_1}, l and ϑ are z-independent. The infinitesimal generator $\mathcal{A}^{\pi}$ for any smooth function ϕ is

$$\mathcal{A}^{\pi}\phi=\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial x}(b+\tilde{b})+\frac{\partial\phi}{\partial y}(x-e^{-\delta\lambda}z-\lambda y)+\frac{1}{2}\frac{\partial^2\phi}{\partial x^2}\sigma^2.$$

For $\mathcal{A}^{\pi}\phi$ to be z-independent, the coefficient of z must vanish

$$\frac{\partial\phi}{\partial x}\frac{\partial b}{\partial z}-e^{-\delta\lambda}\frac{\partial\phi}{\partial y}=0,$$

Since this must hold for arbitrary ϕ, we deduce that

$$\frac{\partial\phi}{\partial y}=e^{\delta\lambda}\frac{\partial b}{\partial z}\frac{\partial\phi}{\partial x}.$$

This implies that $\dfrac{\partial b}{\partial z}$ must be independent of z, leading to the affine decomposition

$$b(t,x,y,z)=\alpha(t,x,y)+z\beta(t,x,y).$$

The z-independence condition requires

$$\frac{\partial \phi}{\partial y} = e^{\delta \lambda} \beta \frac{\partial \phi}{\partial x},$$

for any function $\phi = W$, l^{s,x_1,y_1}, l and ϑ.

The generator $\mathcal{A}^{\pi}$ contains the term $\frac{\partial \phi}{\partial x} \tilde{b}$. For $\mathcal{A}^{\pi} \phi$ to be z-independent, $\tilde{b}$ must not introduce any new z-dependence. Applying the z-independence condition to $\phi = \tilde{b}$

$$\frac{\partial \tilde{b}}{\partial y} = e^{\delta \lambda} \beta \frac{\partial \tilde{b}}{\partial x}.$$

This ensures that the coefficient $\tilde{b}$ does not disrupt the z-independence of the generator. The diffusion term $\frac{1}{2} \frac{\partial^2 \phi}{\partial x^2} \sigma^2$ must also be z-independent. Applying the z-independence condition to $\phi = \sigma$

$$\frac{\partial \sigma}{\partial y} = e^{\delta \lambda} \beta \frac{\partial \sigma}{\partial x}.$$

This guarantees that the second-order term in the generator remains z-independent. The condition (4.20) guarantees compatibility at $t = T$ by ensuring, through the chain rule applied to the terminal condition of the extended HJB system, that the combined function $L + \Psi$ satisfies the condition (4.20).

Conversely, suppose that the structural conditions (4.16)–(4.21) are in force. Recall that the auxiliary function ϑ is governed by the Kolmogorov backward equation

$$\mathcal{A}^{\hat{\pi}} \vartheta(t, x, y) = 0, \tag{4.22}$$

with terminal condition $\vartheta(T, x, y) = x + \varpi y$, where the generator is given by

$$\mathcal{A}^{\hat{\pi}} \vartheta = \frac{\partial \vartheta}{\partial t} + \frac{\partial \vartheta}{\partial x}(b + \tilde{b}) + \frac{\partial \vartheta}{\partial y}(x - e^{-\delta \lambda} z - \lambda y) + \frac{1}{2} \frac{\partial^2 \vartheta}{\partial x^2} \sigma^2.$$

Differentiating (4.22) with respect to z we obtain from the decomposition (4.16) and the fact that $\varpi = e^{\delta \lambda} \beta$

$$\frac{\partial \vartheta}{\partial y} e^{-\delta \lambda} - \frac{\partial \vartheta}{\partial x} \beta = 0$$

Inserting this into (4.22), this equation now takes the form

$$\frac{\partial \vartheta}{\partial t} + \frac{\partial \vartheta}{\partial x}(\alpha + \tilde{b}) + \frac{\partial \vartheta}{\partial y}(x - \lambda y) + \frac{1}{2} \frac{\partial^2 \vartheta}{\partial x^2} \sigma^2 = 0.$$

Noting also that from the terminal condition, we compute $\frac{\partial \vartheta}{\partial x} = 1$, $\frac{\partial \vartheta}{\partial y} = \varpi$. The z-independence condition $\frac{\partial \vartheta}{\partial y} = e^{\delta \lambda} \beta \frac{\partial \vartheta}{\partial x}$ reduces to: $\varpi = e^{\delta \lambda} \beta$.

The coefficients in $\mathcal{A}^{\hat{\pi}}\vartheta$ depend only on (t,x,y) (since α, $\tilde{b}$, σ are z-independent by assumption). The terminal condition $\vartheta(T,x,y) = x + \varpi y$ is manifestly independent of z. Then the solution ϑ to the PDE $\mathcal{A}^{\hat{\pi}}\vartheta = 0$ inherits its z-independence from the coefficients and terminal condition.

The remaining terms in the extended HJB system are $\mathcal{A}^{\hat{\pi}}l$, $\mathcal{A}^{\hat{\pi}}l^{t,x,y}$, $\mathcal{A}^{\hat{\pi}}(\Psi \circ \vartheta)$, and $\mathcal{H}^{\hat{\pi}}\vartheta$. Since ϑ is z-independent and both Ψ and the generators share the same structure as $\mathcal{A}^{\hat{\pi}}\vartheta$, all these terms inherit z-independence. The extended HJB equation for W is therefore a supremum over an expression from which z has been eliminated, ensuring its solution W is independent of z. □

5. APPLICATION IN MEAN-VARIANCE PORTFOLIO WITH STATE DEPENDENT RISK AVERSION WITH DELAY

We assume that an investor can invest in a financial market, in which two securities are trated continuously, one of them is a bond with price $B_0(s)$ at time $s \in [0,T]$ governed by

$$\frac{dB_0(s)}{B_0(s)} = r_0(s)ds,\ B_0(0) = b_0 > 0, \tag{5.1}$$

where $r_0 : [0,T] \to (0,+\infty)$ represents a deterministic function denoting the risk-free rate. The additional asset, termed as risky stocks, is characterized by its price process B_1 which follows the following stochastic differential equation

$$\frac{dB_1(s)}{B_1(s)} = r_1(s)ds + \sigma(s)d\mathcal{W}(s),\ B_1(0) = b_1 > 0, \tag{5.2}$$

where $r_1 : [0,T] \to (0,+\infty)$ and $\sigma : [0,T] \longrightarrow \mathbb{R}$ represent the appreciation rate and the volatility of the risky stock, respectively. $\mathcal{W}$ is a one-dimensional standard Brownian motion.

A trading strategy is a one-dimensional stochastic process denoted by π, where $\pi(s)$ represents the amount invested in the risky stock at time $s \in [0,T]$. The dollar amount invested in the bond at time s is given by $X(s) - \pi(s)$, where X is the wealth process associated with the strategy π and the initial capital x_0. The evolution of X can then be described from (5.1) and (5.2) as

$$\begin{cases} dX(s) = & \{dX(s) - \pi(s)\}\dfrac{dB_0(s)}{B_0(s)} + \pi(s)\dfrac{dB_1(s)}{B_1(s)}, s \in [0,T], \\ X(0) = & x_0. \end{cases} \tag{5.3}$$

Accordingly, the wealth process solves the following SDE

$$\begin{cases} dX(s) = & \{r_0(s)X(s) + \rho(s)\pi(s)\}\, ds + \pi(s)\sigma(s)d\mathcal{W}(s),\ s \in [0,T], \\ X(0) = & x_0, \end{cases} \tag{5.4}$$

where $\rho = r_1 - r_0$. Traditionally, the investor's wealth process is modeled by a stochastic differential equation without delay, as formulated in (5.3) or, equivalently, (5.4). For optimal time-consistent solutions under this framework, we refer the reader to [4].

The equation (5.4) thus describe the classical formulation of wealth dynamics in the mean–variance setting. This standard model assumes that the investor's wealth depends only on current allocations between the bond and the stock, with no influence from past values. Such a formulation has been extensively analyzed in the literature (see, e. g., [4]) and provides a natural reference point for comparison. Nevertheless, in many practical situations the evolution of wealth exhibits path dependence. Indeed, investors frequently adjust their strategies according to past performance: if recent returns have been favorable, the investor may choose to distribute dividends to stakeholders; by contrast, if performance has been poor, additional capital injections may be required to offset losses. These considerations highlight the need for a richer modeling framework that can account for memory effects. To this end, we extend the standard dynamics (5.4) by introducing delay terms that incorporate information from the recent history of the wealth process. Specifically, we define two auxiliary processes that capture different aspects of the past. Following [23, 24], we introduce two auxiliary processes

$$Y(s) = \int_{-\delta}^{0} e^{\lambda\tau} X(s+\tau)\,\mathrm{d}\tau, \qquad Z(s) = X(s-\delta),$$

where $\lambda > 0$ is an averaging parameter and $\delta > 0$ is the delay period. Here, $Y(s)$ represents the exponentially weighted average of the portfolio's performance over the past interval $[s-\delta, s]$, thus summarizing its recent trend, while $Z(s)$ reflects the lagged wealth at time $s-\delta$, providing a pointwise delayed measure. Using these processes, we model capital inflows and outflows through a function $h\big(s, X(s)-Y(s), X(s)-Z(s)\big)$, where $X(s)-Y(s)$ captures the deviation of current wealth from its recent average and $X(s)-Z(s)$ measures its deviation from the delayed state. This formulation reflects realistic investor behavior: large deviations from past values may trigger dividend distributions in the case of gains or capital injections in the case of losses. Following the approach of [23, 24], the resulting delayed wealth dynamics take the form of the stochastic delay differential equation

$$\begin{cases} dX(s) = & \{r_0(s)X(s) + \pi(s)\,\rho(s) - h(s, X(s)-Y(s), X(s)-Z(s)))\}\,ds \\ & +\sigma(s)\pi(s)d\mathcal{W}(s), \text{ for } s \in [0,T], \\ X(s) = & \xi(s),\ s \in [-\delta, 0],\ \xi_0(s) \in \mathcal{C}([-\delta,0];\mathbb{R}). \end{cases} \tag{5.5}$$

To make the problem affordable, we assume that h has a linear structure as follows

$$h(s, X(s)-Y(s), X(s)-Z(s)) = \alpha(s)(X(s)-Y(s)) + \beta\,(X(s)-Z(s))\,, \tag{5.6}$$

where $\alpha : [0,T] \longrightarrow \mathbb{R}_+$ is a deterministic uniformly bounded function, $\beta \geq 0$ is a constant such that $r_0(s) - \alpha(s) - \beta > 0$. Invoking (5.6) in equation (5.5) , we obtain the wealth process should satisfies the following SDDE

$$\begin{cases} dX(s) = & (\mu(s)X(s) + \rho(s)\pi(s) + \alpha Y(s) + \beta Z(s))\,ds \\ & +\sigma(s)\pi(s)d\mathcal{W}(s),\ s \in [0,T] \\ X(s) = & \xi(s-t),\ s \in [t-\delta, 0], \end{cases} \tag{5.7}$$

where $\xi \in \mathcal{C}([-\delta,0];\mathbb{R})$ and $\mu = r_0 - \alpha - \beta$. According to Lemma 2.1 in [23], for any admissible strategy π, the state equation (5.7) has a unique solution X.

For any fixed initial state $(t,\xi) \in [0,T] \times \mathcal{C}([-\delta,0];\mathbb{R})$, the purpuse is to choose an investment strategy π by maximization of the conditional expectation of terminal wealth and average wealth over the period $[t-\delta,T]$, while trying at the same time minimize financial risk. Interpreting risk as the conditional variance. So the optimization problem is therefore to maximize the following utility

$$\begin{aligned}\bar{J}(t,\xi,\pi) &= J(t,x,y,\pi), \\ &= \mathbb{E}_{t,x,y}[X(T)+\varpi Y(T)] - \frac{\gamma(x,y)}{2} Var_{t,x,y}[X(T)+\varpi Y(T)],\end{aligned} \tag{5.8}$$

where $\mathbb{E}_{t,x,y}[\cdot] = \mathbb{E}[\cdot \mid X(t)=x,\, Y(t)=y]$ and $Var_{t,x,y}[\cdot] = Var[\cdot \mid X(t)=x,\, Y(t)=y]$, subject to $\mathcal{U}[0,T]$, where X satisfies (5.7), $\varpi \in \mathbb{R}$ is the weight between $X(T)$ and $Y(T)$. As in Björk et al. [4], we define the deterministic function

$$\gamma(x,y) = \frac{\gamma}{x+\varpi y}, \tag{5.9}$$

as a state dependent risk aversion where x is the current wealth and $\gamma > 0$. Consequently, in this section, we consider an objective function whose dependence is restricted to x and y instead of the whole initial path. More precisely, we suppose that The mean-variance optimization problem becomes

$$W(t,x,y) = \sup_{\pi} \left\{ \mathbb{E}_{t,x,y}[X(T)+\varpi Y(T)] - \frac{\gamma(x,y)}{2} Var_{t,x,y}[X(T)+\varpi Y(T)] \right\}. \tag{5.10}$$

Before formulating the extended HJB equations and the associated verification theorem for equilibriums, we give firstly the infenitesimal generator corresponding to the above model. For any feedback strategy π the operator is defined for any function $f \in \mathcal{C}^{1,2,1}(\,[0,T]\times\mathbb{R}^2)$ the generator $\mathcal{A}^{\pi}$ is defined as follows

$$\begin{aligned}\mathcal{A}^{\pi} f(t,x,y) = \; & \frac{\partial f}{\partial t}(t,x,y) + \frac{\partial f}{\partial y}(t,x,y)\left\{x - e^{-\delta\lambda} z - \lambda y\right\} \\ & + \frac{\partial f}{\partial x}(t,x,y)\left\{\mu(t)x + \pi(t)\rho(t) + \alpha(t)y + \beta z\right\} \\ & + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left(\pi(t)\sigma(t)\right)^2 .\end{aligned}$$

We aim to explicitly establish the investment equilibrium for the optimization problem (5.10). We accomplish this by formulating the extended Hamilton–Jacobi–Bellman equations. The derived system, along with its corresponding verification theorem, succinctly defines and applies the investment equilibrium. Then, we have to form the risk aversion function (5.9), first we have $\frac{\partial \gamma}{\partial x}(x,y) = -\frac{\gamma}{(x+\varpi y)^2}$, $\frac{\partial \gamma}{\partial y}(x,y) = -\frac{\varpi\gamma}{(x+\varpi y)^2}$, and $\frac{\partial^2 \gamma}{\partial x^2}(x,y) = \frac{2\gamma}{(x+\varpi y)^3}$. Note that the solution of the extended HJB equations in this case is given by

$$\begin{cases} W(t,x,y) = l(t,x,y,x,y) + \dfrac{\gamma}{2(x+\varpi y)}\vartheta^2(t,x,y), \\ \Psi(t,x,y,\vartheta(t,x,y)) = \dfrac{\gamma}{2(x+\varpi y)}\vartheta^2(t,x,y), \\ l(t,x,y,x,y) = \mathbb{E}_{t,x,y}[X(T)+\varpi Y(T)] - \dfrac{\gamma}{2(x+\varpi y)}\mathbb{E}_{t,x,y}[(X(T)+\varpi Y(T))^2], \\ \vartheta(t,x,y) = \mathbb{E}_{t,x,y}[X(T)+\varpi Y(T)]. \end{cases}$$

The derivatives of W evaluted at (t,x,y) are the following

$$\begin{aligned} \frac{\partial W}{\partial t} &= \frac{\partial l}{\partial t} + \frac{\gamma}{x+\varpi y}\vartheta\frac{\partial \vartheta}{\partial t}, \\ \frac{\partial W}{\partial x} &= \frac{\partial l}{\partial x} + \frac{\partial l}{\partial x_1} - \frac{\gamma}{2(x+\varpi y)^2}\vartheta^2 + \frac{\gamma}{x+\varpi y}\vartheta\frac{\partial \vartheta}{\partial x}, \\ \frac{\partial^2 W}{\partial x^2} &= \frac{\partial^2 l}{\partial x^2} + \frac{\partial^2 l}{\partial x_1^2} + 2\frac{\partial^2 l}{\partial x \partial x_1} + \frac{\gamma}{(x+\varpi y)^3}\vartheta^2 - 2\frac{\gamma}{(x+\varpi y)^2}\vartheta\frac{\partial \vartheta}{\partial x} \\ &\quad + \frac{\gamma}{x+\varpi y}\frac{\partial^2 \vartheta}{\partial x^2} + \frac{\gamma}{x+\varpi y}\vartheta\frac{\partial^2 \vartheta}{\partial x^2}, \\ \frac{\partial W}{\partial y} &= \frac{\partial l}{\partial y} + \frac{\partial l}{\partial y_1} + \frac{\gamma}{x+\varpi y}\vartheta\frac{\partial \vartheta}{\partial y} - \frac{\gamma\varpi}{2(x+\varpi y)^2}\vartheta^2, \end{aligned} \tag{5.12}$$

where l and it's derivatives are evaluated at (t,x,y,x,y) while ϑ and it's derivatives are evaluted at (t,x,y). Inserting the above expressions into (3.6) − (3.9), we get

$$\begin{cases} \dfrac{\partial l}{\partial t} + \dfrac{\gamma}{x+\varpi y}\vartheta\dfrac{\partial \vartheta}{\partial t} + \sup\limits_{\pi}\Bigg\{ \left(\dfrac{\partial l}{\partial y_1} + \dfrac{\gamma}{x+\varpi y}\vartheta\dfrac{\partial \vartheta}{\partial y}\right)\left\{x - e^{-\delta\lambda}z - \lambda y\right\} \\ + \left(\dfrac{\partial l}{\partial x} + \dfrac{\gamma}{x+\varpi y}\vartheta\dfrac{\partial \vartheta}{\partial x}\right)\{\mu(t)x + \pi(t)\rho(t) + \alpha(t)y + \beta z\} \\ + \dfrac{1}{2}\left(\dfrac{\partial^2 l}{\partial x^2} + \dfrac{\gamma}{x+\varpi y}\vartheta\dfrac{\partial^2 \vartheta}{\partial x^2}\right)(\pi(t)\sigma(t))^2 \Bigg\} = 0, \\ \dfrac{\partial l}{\partial t} + \dfrac{\partial l}{\partial y}\left\{x - e^{-\delta\lambda}z - \lambda y\right\} + \dfrac{\partial l}{\partial x}\{\mu(t)x + \hat{\pi}(t)\rho(t) + \alpha(t)y + \beta z\} \\ + \dfrac{1}{2}\dfrac{\partial^2 l}{\partial x^2}(\hat{\pi}(t)\sigma(t))^2 = 0, \\ \dfrac{\partial \vartheta}{\partial t} + \dfrac{\partial \vartheta}{\partial y}\left\{x - e^{-\delta\lambda}z - \lambda y\right\} + \dfrac{\partial \vartheta}{\partial x}\{\mu(t)x + \hat{\pi}(t)\rho(t) + \alpha(t)y + \beta z\} \\ + \dfrac{1}{2}\dfrac{\partial^2 \vartheta}{\partial x^2}(\hat{\pi}(t)\sigma(t))^2 = 0. \end{cases} \tag{5.13}$$

Theorem 5.1. For the mean-variance problem (5.10) the equilibrium investment strategy is given for $t \in [0,T]$ by

$$\hat{\pi}(t) = c(t)(x+\varpi y) + k(t) \vee 0, \tag{5.14}$$

where the functions c and k are given by the following integral equations

$$\begin{cases} c(t) = & \dfrac{\rho(t)}{\sigma^2(t)\gamma}\left(e^{\int_t^T -\eta(t)\,\mathrm{d}u} + \gamma e^{\int_t^T (A(u)-\eta(u))\,\mathrm{d}u} - \gamma\right), \\ k(t) = & \dfrac{\rho(t)}{\sigma^2(t)}\left(e^{-\int_t^T \eta(t)\,\mathrm{d}u}\int_t^T e^{\int_s^T A(u)du}\left(B\left(s\right) + C\left(s\right)\chi(s)\right)\,\mathrm{d}s \right. \\ & \left. \qquad - \int_t^T e^{-\int_t^s \eta(u)\,\mathrm{d}u}\,\mathrm{d}s\right). \end{cases} \tag{5.15}$$

Here $A(t) = r_0(t) - \alpha(t) - \beta + \varpi + \rho(t)c(t)$, $B(t) = \rho(s)k(t)$, $C(t) = \sigma(t)c(t)$, $\chi(t) = \sigma(t)k(t)$ and $\eta(t) = A(t) + C^2\left(t\right).$

P r o o f. We consider an affine feedback control policy of the form

$$\hat{\pi}(s) = c(s)\left(X(s) + \varpi Y(s)\right) + k(s),$$

where c and k are deterministic functions. Substituting this policy into the wealth dynamics governed by the stochastic delay differential equation (5.7) yields the following expansion

$$\begin{aligned} dX(s) = &\left(\mu(s)X(s) + \rho(s)c(s)\left(X(s) + \varpi Y(s)\right) + \rho(s)k(s)\right. \\ &\left. + \alpha(s)Y(s) + \beta Z(s)\right)ds \\ &+ \left(\sigma(s)c(s)\left(X(s) + \varpi Y(s)\right) + \sigma(s)k(s)\right)d\mathcal{W}(s), \end{aligned}$$

To reduce the inherent infinite-dimensionality of the problem, we introduce an aggregated state variable $X(s) + \varpi Y(s)$ and utilizing the dynamics of $Y(s)$, given by $dY(s) = \left(X(s) - \lambda Y(s) - e^{-\delta\lambda}Z(s)\right)ds$, then we have

$$\begin{aligned} d\left(X(s) + \varpi Y(s)\right) = &\left(\left(\mu(s) + \varpi\right)X(s) + \left(\alpha(s) - \varpi\lambda\right)Y(s) + \left(\beta - \varpi e^{-\delta\lambda}\right)Z(s)\right. \\ &\left. + \rho(s)c(s)\left(X(s) + \varpi Y(s)\right) + \rho(s)k(s)\right)ds \\ &+ \left(\sigma(s)c(s)\left(X(s) + \varpi Y(s)\right) + \sigma(s)k(s)\right)d\mathcal{W}(s). \end{aligned} \tag{5.16}$$

A necessary condition for the problem to become finite-dimensional is that the dynamics of $X(s) + \varpi Y(s)$ must not depend explicitly on the delayed state Z. From equation (5.16), this necessitates the elimination of its coefficient, then $\varpi = \beta e^{\delta\lambda}$.

With $\varpi = \beta e^{\delta\lambda}$, the drift simplifies to

$$\left(\mu(s) + \varpi\right)X(s) + \left(\alpha(s) - \varpi\lambda\right)Y(s) + \rho(s)c(s)\left(X(s) + \varpi Y(s)\right) + \rho(s)k(s).$$

For the dynamics to remain closed on the aggregated state $X(s) + \varpi Y(s)$, this linear form must be representable as $\pounds(s)\left(X(s) + \varpi Y(s)\right)$ for some deterministic function $\pounds$. This requirement leads to the condition $\alpha(s) - \varpi\lambda = \varpi\left(\mu(s) + \varpi\right).$ Hence the coditions $\varpi = \beta e^{\delta\lambda}$ and $\alpha(s) = \varpi(\mu(s) + \varpi + \lambda)$ guarantee that the controlled wealth dynamics evolve exclusively in terms of the finite-dimensional state variable $X(s)+\varpi Y(s)$, without explicit delayed dependence.

So, we obtain

$$\begin{aligned} d\left(X(s) + \varpi Y(s)\right) &= \left\{A(s)\left(X(s) + \varpi Y(s)\right) + B(t)\right\}ds \\ &+ \left\{C(s)\left(X(s) + \varpi Y(s)\right) + \chi(t)\right\}d\mathcal{W}(s), \end{aligned} \tag{5.17}$$

Next, we calculate $\mathbb{E}\left[X(T)+\varpi Y(T)\right]$ and $\mathbb{E}\left[\left(X(T)+\varpi Y(T)\right)^2\right]$, where $X(s)=x$ and $Y(s)=y$. We first construct the following exponential martingale

$$d\Upsilon(t)=\Upsilon(t)\left(\left\{-A(t)+C^2(t)\right\}dt-C(t)d\mathcal{W}(t)\right),$$

this implies that

$$\Upsilon(t)=\Upsilon(0)\exp\left\{\int_0^t\left(\left\{-A(s)+\frac{1}{2}C^2(s)\right\}\mathrm{d}s+C(s)\,\mathrm{d}\mathcal{W}(s)\right)\right\},$$

then

$$\frac{\Upsilon(t)}{\Upsilon(T)}=\exp\left\{\int_t^T\left(\left\{A(s)-\frac{1}{2}C^2(s)\right\}\mathrm{d}s+C(s)\,\mathrm{d}\mathcal{W}(s)\right)\right\}. \tag{5.18}$$

From Itô's formula to $(X(s)+\varpi Y(s))\,\Upsilon(t)$ we get

$$d\left((X(t)+\varpi Y(t))\,\Upsilon(t)\right)=\Upsilon(t)\left\{(B(t)+C(t)\chi(t))\,dt+\chi(t)d\mathcal{W}(t)\right\}.$$

Next, by taking expectations and integrating the above equation from t to T, we obtain

$$\begin{aligned}&X(T)+\varpi Y(T)\\&=(x+\varpi y)\frac{\Upsilon(t)}{\Upsilon(T)}+\int_t^T\left\{\left(\frac{\Upsilon(s)}{\Upsilon(T)}\right)\left((B\,(s)+C\,(s)\,\chi(s))\;\mathrm{d}s+\chi(s)\,\mathrm{d}\mathcal{W}(s)\right)\right\}.\end{aligned} \tag{5.19}$$

Now we make the Ansatz

$$\begin{aligned}&\mathbb{E}\left[X(T)+\varpi Y(T)\right]=P_1(t)(x+\varpi y)+Q_1(t),\\&\mathbb{E}\left[\left(X(T)+\varpi Y(T)\right)^2\right]=S(t)\left(x+\varpi y\right)^2+P_2(t)(x+\varpi y)+Q_2\left(t\right),\end{aligned}$$

note that $\mathbb{E}\left[\dfrac{\Upsilon(t)}{\Upsilon(T)}\right]=e^{\int_t^T A(u)\,\mathrm{d}u}$. So, it yields

$$P_1(t)=e^{\int_t^T A(u)\,\mathrm{d}u}, \tag{5.20}$$

and

$$Q_1(t)=\int_t^T e^{\int_s^T A(u)du}\left(B\,(s)+C\,(s)\,\chi(s)\right)\mathrm{d}s. \tag{5.21}$$

By (5.19), we can derive

$$\begin{aligned}&\left(X(T)+\varpi Y(T)\right)^2\\&=(x+\varpi y)^2\left(\frac{\Upsilon(t)}{\Upsilon(T)}\right)^2+\left(\int_t^T\frac{\Upsilon(s)}{\Upsilon(T)}\left\{\left(B\,(s)+C\,(s)\,\chi(s)\right)\,\mathrm{d}s+\int_t^T\chi(s)\,\mathrm{d}\mathcal{W}(s)\right\}\right)^2\\&+2(x+\varpi y)\frac{\Upsilon(t)}{\Upsilon(T)}\left(\int_t^T\left\{\frac{\Upsilon(s)}{\Upsilon(T)}\left((B\,(s)+C\,(s)\,\chi(s))\;\mathrm{d}s+\chi(s)d\mathcal{W}(s)\right)\right\}\right).\end{aligned}$$

Hence, we get

$$S(t) = e^{\int_t^T \left(A(u)+\left(A(u)+C^2(u)\right)\right) \mathrm{d}u} = e^{\int_t^T (A(u)+\eta(u)) \mathrm{d}u},$$

where $\eta(t) = A(t) + C^2(t)$, and

$$P_2(t) = 2\mathbb{E}\left[\frac{\Upsilon(t)}{\Upsilon(T)}\left(\int_t^T \frac{\Upsilon(s)}{\Upsilon(T)}\left\{(B(s) + C(s)\chi(s))\,\mathrm{d}s + \int_t^T \chi(s)\,\mathrm{d}\mathcal{W}(s)\right\}\right)\right]$$

from (5.18) we get

$$P_2(t) = 2\int_t^T e^{\int_t^s A(u)\,\mathrm{d}u} e^{\int_s^T (A(u)+\eta(u))\,\mathrm{d}u} (B(s) + C(s)\chi(s))\,\mathrm{d}s, \tag{5.23}$$

and

$$Q_2(t) = \mathbb{E}\left[\left(\int_t^T \frac{\Upsilon(s)}{\Upsilon(T)}(B(s) + C(s)\chi(s))\,\mathrm{d}s + \chi(s)\,\mathrm{d}\mathcal{W}(s)\right)^2\right]. \tag{5.24}$$

Therefore, it follows that

$$\begin{aligned} &l(t, x, y, x_1, y_1) \\ &= P_1(t)(x + \varpi y) + Q_1(t) - \frac{\gamma}{2(x_1 + \varpi y_1)}\left[S(t)(x + \varpi y)^2 + P_2(t)(x + \varpi y) + Q_2(t)\right], \end{aligned} \tag{5.25}$$

with

$$\vartheta(t, x, y) = \mathbb{E}_{t,x,y}[X(T) + \varpi Y(T)] = P_1(t)(x + \varpi y) + Q_1(t). \tag{5.26}$$

Since $\hat{\pi}$ is the feedback control that realizes the supremum in the first equation of (5.13), the first-order condition implies

$$\left(\frac{\partial l}{\partial x} + \frac{\gamma}{x + \varpi y}\vartheta\frac{\partial \vartheta}{\partial x}\right)\rho(t) + \left(\frac{\partial^2 l}{\partial x^2} + \frac{\gamma}{x + \varpi y}\vartheta\frac{\partial^2 \vartheta}{\partial x^2}\right)\hat{\pi}(t)\sigma^2(t) = 0. \tag{5.27}$$

Next, we proceed to calculate the following derivatives

$$\begin{aligned} \frac{\partial l}{\partial x} &= -\frac{\gamma}{(x_1 + \varpi y_1)}S(t)(x + \varpi y) + P_1(t) - \frac{\gamma}{2(x_1 + \varpi y_1)}P_2(t), \\ \frac{\partial^2 l}{\partial x^2} &= -\frac{\gamma}{(x_1 + \varpi y_1)}S(t), \\ \frac{\partial \vartheta}{\partial x} &= P_1(t), \\ \frac{\partial^2 \vartheta}{\partial x^2} &= 0, \end{aligned} \tag{5.28}$$

substituting into (5.27), we obtain

$$0 = \left(-\gamma S(t) + P_1(t) + \gamma P_1^2(t) - \frac{\gamma}{2(x + \varpi y)}P_2(t) + \frac{\gamma}{x + \varpi y}Q_1(t)P_1(t)\right)\rho(t)$$

$$- \frac{\gamma}{(x+\varpi y)} S(t)\sigma^2(t)\hat{\pi}(t). \tag{5.29}$$

Consequently, the equilibrium investment strategy is

$$\hat{\pi}(t) = -\frac{\rho(t)}{\sigma^2(t)} \left(\frac{\left(-\gamma S(t) + P_1(t) + \gamma P_1^2(t)\right)(x+\varpi y)}{-\gamma S(t)} + \frac{Q_1(t)P_1(t) - \frac{1}{2}P_2(t)}{-S(t)} \right). \tag{5.30}$$

Hence, by comparing with the assumption to find finally the values of the functions c and k as follows

$$\begin{cases} c(t) = & -\dfrac{\rho(t)}{\sigma^2(t)} \left(\dfrac{P_1(t) - \gamma S(t) + \gamma P_1^2(t)}{-\gamma S(t)} \right), \\ k(t) = & -\dfrac{\rho(t)}{\sigma^2(t)} \left(\dfrac{Q_1(t)P_1(t) - \dfrac{1}{2}P_2(t)}{-S(t)} \right). \end{cases} \tag{5.31}$$

Thus, the proof is completed. □

Remark 5.2. 1. The equilibrium feedback strategy admits the affine form $\hat{\pi}(t) = c(t)(x+\varpi y)+k(t)$. In practice, however, such an expression may produce negative values, corresponding to short-selling of the risky asset. Since short-selling is typically ruled out in insurance and many investment contexts, it is important to ensure that the strategy remains non-negative. This is achieved by projecting the unconstrained strategy onto the non-negative real line, leading to

$$\hat{\pi}(t) = (c(t)(x+\varpi y) + k(t)) \vee 0.$$

This ensures that the resulting strategy is consistent with realistic market restrictions. On the other hand, in financial markets where short-selling is permitted, the projection is unnecessary and the unconstrained affine form is directly admissible.

2. Equations (5.15) should be interpreted as a system of coupled integral equations for the unknown functions c and k. These equations determine the time-dependent coefficients of the equilibrium control but cannot be solved in closed form because the functions A, B, C, η, χ all depend on c and k themselves. Consequently, they must be solved numerically, for example by the iterative scheme presented in the next section.

3. While the existence and uniqueness of solutions for the integral equations (5.15) are theoretically guaranteed, their complexity requires iterative numerical schemes to approximate $Q_2(\cdot)$.

4. Constructing the exponential martingale associated with the terminal wealth $X(t)+\varpi Y(t)$ involves addressing delays in the wealth evolution equation and leveraging key condition $\varpi = \beta e^{\delta\lambda}$. Simulation simplifies this otherwise challenging process.

6. NUMERICAL SIMULATIONS

6.1. Numerical scheme to approximate $Q_2(t)$

Firstly, we derive the iterative updates for $c_{n+1}(t)$ and $k_{n+1}(t)$, approximating the integrals using the trapezoidal rule. Let $\{t_0, t_1, \dots, t_N\}$ represent a discrete time grid with $t_0 = t$ and $t_N = T$, we consider $\Delta t = \frac{T}{N}$ is the grid spacing, and define the functions $A_n(t) = r_0 - \alpha - \beta + \varpi + \rho c_n(t)$, $\eta_n(t) = A_n(t) + C_n^2(t)$, $B_n(t) = \rho k_n(t)$, $C_n(t) = \sigma c_n(t)$, $\chi_n(t) = \sigma k_n(t)$. This sets the short-hand notation used in the iteration: A_n, η_n, B_n, C_n, χ_n encode the current iterate values. The final forms $c_{n+1}(t)$ and $k_{n+1}(t)$ are as follows

$$\begin{aligned} c_{n+1}(t) &= \frac{\rho}{\sigma^2\gamma}\Bigg(\exp\Bigg(-\Delta t \sum_{j=i}^{N-1} \frac{\eta_n(t_j)+\eta_n(t_{j+1})}{2}\Bigg) \\ &\qquad + \gamma \exp\Bigg(\Delta t \sum_{j=i}^{N-1} \frac{(A_n(t_j)-\eta_n(t_j)) + (A_n(t_{j+1})-\eta_n(t_{j+1}))}{2}\Bigg) - \gamma\Bigg), \\ k_{n+1}(t) &= \frac{\rho}{\sigma^2}\big(S_1(t) - S_2(t)\big), \end{aligned}$$

where $S_1(t)$ and $S_2(t)$ are given by

$$\begin{aligned} S_1(t) &= \exp\Bigg(-\Delta t \sum_{j=i}^{N-1} \frac{\eta_{n+1}(t_j)+\eta_{n+1}(t_{j+1})}{2}\Bigg) \\ &\quad \times \Delta t \sum_{j=i}^{N-1} \exp\Bigg(\Delta t \sum_{k=j}^{N-1} \frac{A_{n+1}(t_k)+A_{n+1}(t_{k+1})}{2}\Bigg) \\ &\quad \times \frac{(B_n(t_j)+C_n(t_j)\chi_n(t_j)) + (B_n(t_{j+1})+C_n(t_{j+1})\chi_n(t_{j+1}))}{2}, \\ S_2(t) &= \Delta t \sum_{j=i}^{N-1} \exp\Bigg(-\Delta t \sum_{k=i}^{j} \frac{\eta_{n+1}(t_k)+\eta_{n+1}(t_{k+1})}{2}\Bigg). \end{aligned}$$

Define the iterative map Θ such that $c_{n+1}(t) = \Theta(c_n)$. Let $\Delta c_n(t) = c_n(t) - c_{n-1}(t)$. Using the mean value theorem for the exponential function, we have

$$|\exp(x) - \exp(y)| \le \max_{z\in[x,y]} |\exp(z)||x-y| = \exp(\max(x,y))|x-y|.$$

Applying this to $\Delta c_{n+1}(t)$, we find $|\Delta c_{n+1}(t)| \le K|\Delta c_n(t)|$, where K depends on the coefficients of the iterative map and satisfies $K < 1$ for sufficiently small Δt. This contraction estimate shows that successive iterates get closer, which is crucial to guarantee convergence. Hence the iterative map Θ is a contraction on the space of bounded, continuous functions with the supremum norm $\|c(t)\| = \sup_t |c(t)|$, By the Banach fixed-point theorem, Θ admits a unique fixed point $c(t)$, and the sequence $c_n(t)$ converges to it. A similar argument holds for $\Delta k_{n+1}(t) := k_{n+1}(t) - k_n(t)$.

In the second step, after iteratively approximating $c(t)$ and $k(t)$, we proceed to approximate the expectation in the expression for $Q_2(t)$ using the Euler–Maruyama method followed by a Monte Carlo approximation. From the application of the Itô isometry to the stochastic integral, the functional $Q_2(t)$ is defined by the expectation

$$Q_2(t) = \mathbb{E}\left[\left(\int_t^T \frac{\Upsilon(s)}{\Upsilon(T)}\big(B(s) + C(s)\chi(s)\big)\,\mathrm{d}s\right)^2 + \int_t^T \left(\frac{\Upsilon(s)}{\Upsilon(T)}\chi(s)\right)^2 \mathrm{d}s\right].$$

Let the interval $[t, T]$ be partitioned into N equally spaced subintervals with step size $\Delta t = (T - t)/N$ and grid points $t = t_0 < t_1 < \cdots < t_N = T$. The Brownian motion is discretized as $\mathcal{W}(t_{k+1}) = \mathcal{W}(t_k) + \sqrt{\Delta t}\,\xi_k, \qquad \xi_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$, where $\Delta\mathcal{W}_k = \mathcal{W}(t_{k+1}) - \mathcal{W}(t_k)$. The stochastic factor Υ is simulated backwards in time along each sample path. For a given path indexed by m, the backward recursion is

$$\Upsilon^{(m)}(t_k) = \Upsilon^{(m)}(t_{k+1})\,\exp\left\{A(t_k)\Delta t - \frac{1}{2}C^2(t_k)\Delta t + C(t_k)\Delta\mathcal{W}_k^{(m)}\right\},$$

starting from a prescribed terminal value $\Upsilon^{(m)}(t_N)$, where $\Delta\mathcal{W}_n^m$ is the Wiener increment for the mth path. This multiplicative structure preserves positivity and is consistent with the underlying continuous-time dynamics.

For a single path m using the trapezoidal rule yields the pathwise estimator

$$Q_{2,m}(t) = \sum_{k=0}^{N-1}\left(\frac{\Upsilon^{(m)}(t_k)}{\Upsilon^{(m)}(t_N)}\chi(t_k)\right)^2 \Delta t + \left(\sum_{k=0}^{N-1}\frac{\Upsilon^{(m)}(t_k)}{\Upsilon^{(m)}(t_N)}\big(B(t_k) + C(t_k)\chi(t_k)\big)\Delta t\right)^2.$$

Once we have approximated the above expression, we next compute the expectation using the Monte Carlo simulation method, the expectation $Q_2(t)$ is estimated by averaging over M independent sample paths

$$Q_2(t) \approx \frac{1}{M}\sum_{m=1}^{M} Q_{2,m}(t).$$

Without loss of generality , we assume for the remainder of this section that: $\delta = 0.1$, $\lambda = 0.5$, $\alpha(s) = 0.2$, $\beta = 0.1$, $\rho(s) = 0.3$, $\sigma(s) = 0.5$, $\mu(s) = 0.05$, $r_0(s) = 0.1$, $\varpi = 0.3$.

Figure 1 illustrates the simulation of $Q_2(t)$ over the time interval $[0, 1]$, a stochastic process, using the Euler–Maruyama method (blue curve) and its expected value estimated via Monte Carlo simulation (red dashed line). The blue curve shows a single realization, highlighting stochastic fluctuations, while the red dashed line represents the average behavior over multiple simulations, providing insight into the system's expected dynamics under randomness.

6.2. Effect of risk aversion coefficient on wealth and control processes

Figure 2 provides a visual illustration of the relationship between the equilibrium control strategy and the resulting wealth process. The figure consists of two panels: the top panel displays the evolution of the equilibrium wealth process $X(s)$ over time, while

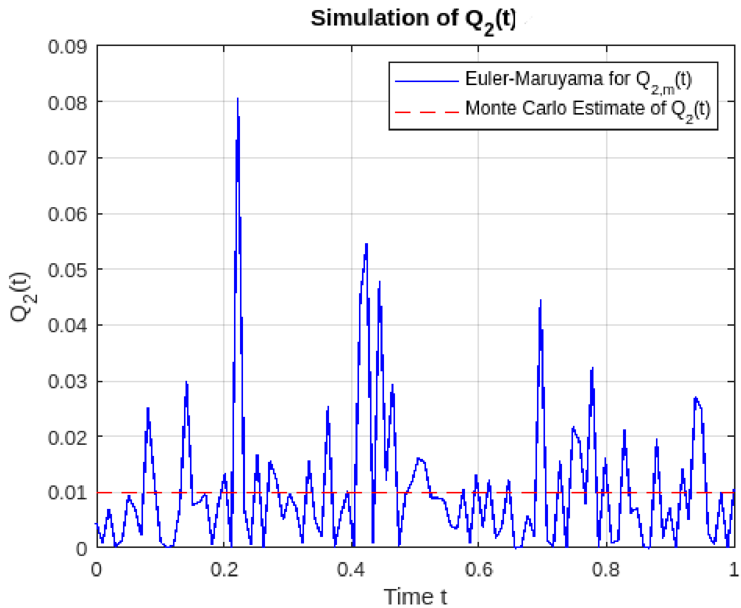
Simulation of Q2(t)
Euler-Maruyama for Q2,m(t)
Monte Carlo Estimate of Q2(t)
Q2(t)
Time t
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0
0.2
0.4
0.6
0.8
1

Fig. 1.

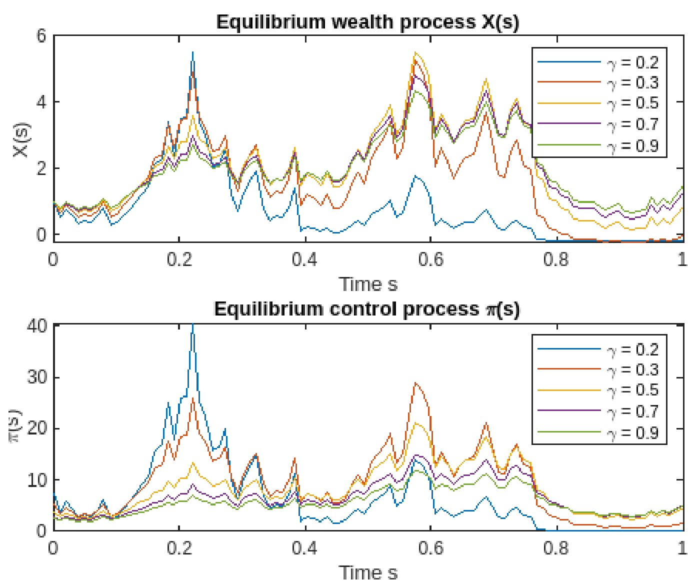
Equilibrium wealth process X(s)
X(s)
Time s
γ = 0.2
γ = 0.3
γ = 0.5
γ = 0.7
γ = 0.9
Equilibrium control process π(s)
π(s)
Time s

Fig. 2.

the bottom panel depicts the associated equilibrium control $\pi(s)$, which represents the strategy employed to manage investment and risk. Both plots are generated for five distinct values of the risk–aversion parameter $\gamma \in \{0.2, 0.3, 0.5, 0.7, 0.9\}$ and we consider the parameter values: $T = 3$, $N = 100$, $\Delta t = 0.03$, $\xi(s) = 1$ for $s \in [-\delta, 0]$. In the mean–variance framework, γ quantifies the investor's attitude toward risk: smaller values (e. g., $\gamma = 0.2$) indicate lower risk aversion and thus a greater willingness to adopt aggressive strategies in pursuit of higher returns, whereas larger values (e. g., $\gamma = 0.9$) correspond to more conservative behavior, emphasizing stability and lower volatility. The figure clearly highlights how changes in γ affect both the control and the wealth. Specifically, the bottom panel shows that as γ increases, the control $\pi(s)$ becomes smoother and its magnitude decreases, reflecting the investor's reluctance to take on substantial risk. This adjustment in the control process directly translates into the wealth dynamics shown in the top panel, since $\pi(s)$ enters the stochastic differential equation with delay that governs $X(s)$. Consequently, a more cautious control strategy results in a more stable and less volatile wealth trajectory. In this sense, the bottom panel drives the top panel: the control determines the wealth outcome, and the two–panel layout of Figure 2 explicitly conveys this cause–effect relationship.

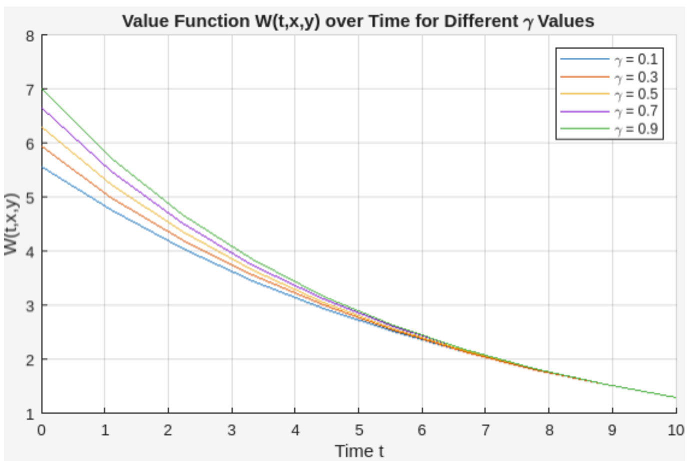

Fig. 3.

Figure 3 illustrates the time evolution of the equilibrium value function W under a stochastic framework, highlighting its dependence on varying levels of risk aversion γ. The simulation, based on numerical methods, evaluates the dynamic interactions between deterministic factors and equilibrium control strategies over a fixed horizon $T = 10$. The results show that W decreases monotonically in time, reflecting the gradual reduction in expected utility as the horizon approaches. Higher values of γ correspond to more conservative trajectories, leading to larger initial values of W and emphasizing the trade-off between wealth accumulation and stability. As time approaches the terminal horizon, the differences between the curves diminish, since all strategies are ultimately constrained by the same terminal condition.

CONCLUSION

This paper introduces a comprehensive game-theoretic framework for addressing time-inconsistent stochastic control problems that involve delays. By extending the standard Bellman equation into a system of nonlinear equations, we establish subgame-perfect Nash equilibrium strategies for delayed processes with functional objectives. The theoretical contributions are demonstrated through the application of a mean-variance portfolio problem, incorporating state-dependent risk aversion and delays, which showcases the practical applicability of our approach.

Our findings significantly advance the literature on time-inconsistent control problems by offering a structured method to handle delays, which are frequently encountered in real-world applications, particularly in financial and economic models. Through the case study of mean-variance portfolio optimization with state-dependent risk aversion, we show how our framework can be employed to derive optimal time-consistent strategies.

This work also paves the way for future research in several directions. Extending the results to other utility functions and exploring multi-agent systems with time-inconsistent behaviors would offer deeper insights into decision-making dynamics in uncertain environments. Moreover, the practical implementation of these strategies in finance, economics, and engineering will require further development of numerical methods and computational techniques to effectively solve the extended HJB equations.

Our results open up opportunities to explore better numerical methods for solving the extended HJB equations. For example, Fouque and Zhang [13] used deep learning to handle mean field control problems, showing how machine learning can effectively manage high-dimensional challenges in stochastic control. Their work demonstrates that deep learning can simplify complex calculations. Inspired by their approach, our framework could be further developed to create efficient strategies for a wider range of delayed stochastic systems.

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