MIGRATIVITY OF CONTINUOUS T-CONORMS WITH RESPECT TO ORDINAL SUM IMPLICATIONS

Xinxin Yan and Hongjun Zhou

The topic of migrativity among aggregation functions is of significant interest from both theoretical and practical perspectives within the field of fuzzy set theory. Nonetheless, there is a scarcity of characterizations in the existing literature concerning the migrativity of ordinal sum implications, especially when the ordinal summands are positioned along the major diagonal line of $[0,1]^2$, and this area has not been thoroughly investigated. The present paper aims to fill this gap by conducting a detailed study on the migrativity of t-conorms with respect to ordinal sum implications. We provide the structural solutions to the migrative functional equation for t-conorms with respect to ordinal sum implications, which depend on the position of parameter α within the range of natural negation N. The characterizations under which t-conorms are α-migrative with respect to ordinal sum implications are obtained by presenting ordinal sum representations of the underlying functions.

Keywords: Migrativity, T-conorm, Ordinal sum implication

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1. INTRODUCTION

Fuzzy implications, as a fundamental category of logical connectives within fuzzy set theory, have found widespread applications across various domains of fuzzy mathematics. These domains encompass mathematical fuzzy logic [12, 26, 56], approximate reasoning [19, 20, 28], fuzzy inference and control systems [14, 40], image processing [29] and fuzzy relation equations [8, 17], among others. Over the past few decades, fuzzy implications have received considerable attention from both theoretical and practical viewpoints. Given their importance, it is necessary to select appropriate fuzzy implications. As a result, the construction techniques of new categories of fuzzy implications have long been a popular and intriguing research topic, one of these renowned techniques is the ordinal sum construction. It's worth noting that the ordinal sum construction technique was elucidated by Clifford [13] within the context of abstract semi-groups, specifically aimed at constructing novel semi-groups. However, the ordinal sum construction for fuzzy implications was not fully recognized by scholars until Su et al. [45] proposed the ordinal sum implications. To achieve this, numerous categories of ordinal sums of fuzzy implications have been proposed [5, 6, 18, 36, 43, 57], which are constructed as a form

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of ordinal sum based on a given set of implications. Zhou [55] introduced two general construction techniques for generating the general ordinal sum of fuzzy implications, designated as "Implication Complementing" and "Implication Reconstructing". These techniques aim to unify the ordinal sum implications presented in the existing literature. As we have seen, it is necessary to explore their properties, including migrativity etc.

The migrative functional equation was proposed by Mesiar and Novák [37], to find t-norm solutions of the equation $T(\alpha x, y) = T(x, \alpha y)$ that are strict and distinct from $T(x, y) = \alpha xy$ on $(0, 1)^2$, where $\alpha \in (0, 1)$. This equation acts as a bridge between the generalized associativity equation [1, 2, 3] and the recursiveness equation [15]. Furthermore, the migrativity is closely related to algebraic properties such as the Lipschitz property, bisymmetry, associativity, as well as additive generators [9, 10]. Its applications in constructing convex combinations and analytically characterizing novel fuzzy logic connectives [24, 27, 31, 41, 42, 54] are particularly noteworthy. Additionally, it holds potential for use in fields such as image processing and decision making [10, 16, 29, 38, 39]. To date, the findings related to the migrativity of aggregation functions and their generalizations are extensively documented in the existing literature, including aggregation functions [33, 34], t-norms [25], t-subnorms [49], semi-copulas [21, 38], uni-norms [23, 35, 46, 48, 52, 62], null-norms [50, 58, 63], 2-uninorms [22, 32], as well as overlap functions and grouping functions [44, 51, 54, 59, 60, 61], among others.

Moreover, the migrativity of fuzzy implications [7, 11, 43, 47, 53, 57] has been extensively studied, leading to a detailed characterization of the analytic structures of the underlying functions. Notably, based on the study of generalized law of importation for fuzzy implications, Baczyński et al. [7] proposed the concept of α-migrativity of fuzzy implications and explored the relationship between the two. Furthermore, the α-migrativity for t-conorms S with respect to Reichenbach implication I_{RC} was presented:

$$S(1 - \alpha + \alpha x, y) = S(x, 1 - \alpha + \alpha y), \quad x, y \in [0, 1].$$

Pan et al. [43] put forward the α-migrativity for continuous t-conorms S with respect to general fuzzy implications I by the following formula:

$$S(I(\alpha, x), y) = S(x, I(\alpha, y)), \quad x, y \in [0, 1].$$

Particularly, they characterized the solutions of above migrative functional equation for continuous t-conorms with respect to specific fuzzy implications, including Reichenbach implication, Zhou implication, Łukasiewicz implication and Kleene-Dienes implication, and explored the migrativity of continuous t-conorms with respect to general fuzzy implications. Subsequently, Zhou et al. [57] conducted an in-depth study on the migrativity of t-conorms with respect to N-ordinal sum implication and portrayed the structure of the solutions under migrative functional equation.

However, since the ordinal summands of N-ordinal sum implications are positioned along the minor diagonal line of $[0, 1]^2$, it is crucial to also investigate the migrativity of t-conorms with respect to ordinal sum implications where the summands are aligned along the major diagonal of $[0, 1]^2$. Building on the ordinal sum implications proposed by Zhou [55], we will further explore the migrativity of t-conorms with respect to these implications. To sum up, the following objectives form the main motivation of this paper:

(i) As previously mentioned, the t-conorms that are α-migrative with respect to four specific fuzzy implications, as well as general fuzzy implications, are either continuous Archimedean or can be represented as an ordinal sum of continuous t-conorms. Furthermore, it is noteworthy that the analytic structures of fuzzy implications fulfilling the migrative functional equation typically take the form of ordinal sum, rather than necessarily being (S, N)-implications. This observation underscores the importance of investigating the migrativity of t-conorms with respect to ordinal sum implications.

(ii) Pan et al. [43] characterized the solutions to the migrative functional equation for t-conorms S with respect to the general fuzzy implications I under the constraint $N_I(\alpha) = \alpha$ (see [43, Theorems 4.6 and 4.8]), which indicated the relationship of $I(\alpha, \cdot)$ and $S(\alpha, \cdot)$. In this paper, we will overcome the limitations imposed by this constraint and provide characterizations of the analytic structures of the underlying functions without such restrictions. By Theorems 3.3, 3.4, 3.6, 3.7, 3.9, 3.10 and 3.14, and Remarks 3.5 (iii) and 3.18, these results are more general than those in [43]. This approach highlights the compatibility between the ordinal sum structures of t-conorms and those of ordinal sum implications, as revealed by the migrative functional equation. Moreover, from the viewpoint of application, ordinal sum implications can be utilized to interpret IF-THEN rules by providing different summands for distinct regions, as necessitated when designing a controller to automatically regulate the speed of a vehicle [55]. When designing such a controller, different control rules and parameters can be assigned to each speed range or driving scenario based on their specific requirements. These rules and parameters operate similarly to distinct summands, each functioning within its designated region to collectively achieve precise control over the vehicle's speed. Hence, it is necessary to investigate the migrativity of ordinal sum implications.

(iii) From the theoretical point of view, the research value of ordinal sum implications lies not only in their strong relationships with given fuzzy implications but also in their capability to provide the structural description of solutions to the migrative functional equation. For an N-ordinal sum implications $I_\star = (\langle a_k, b_k, I_k\rangle, N)_{k\in\Lambda}$, continuous Archimedean t-conorms are not migrative with respect to $I_\star$ for any $N(\alpha) \in (0, 1)$ (see [57, Proposition 3.4]). However, for an ordinal sum implication $I = (\langle a_k, b_k, I_k\rangle, I_*)_{k\in\Lambda}$, by Remark 3.17, continuous Archimedean t-conorm is α-migrative with respect to ordinal sum implication I for $N(\alpha) \in [0, 1]$. Especially, if $\alpha \in \{a_k, b_k\}$, then the structure of solutions to the migrative functional equation for continuous t-conorms, with respect to N-ordinal sum implications, is simply the form of $\max(N(\alpha), \cdot)$. In this paper, by Theorems 3.3, 3.4 and 3.14, the ordinal sum decompositions for the underlying t-conorms are more profound. Moreover, Theorems 3.7 and 3.10 indicate that the structure of $I_*(\alpha, \cdot)$ is no longer of the form $\max(N(\alpha), \cdot)$. These results are more general than those found in [43, 57]. From the applied point of view, the investigation of the migrativity of t-conorms is of paramount significance in domains such as image processing, decision making and classification. Based on the above findings, the present paper will endeavour to undertake a comprehensive analysis of the migrativity of t-conorms with respect to ordinal sum implications.

The remaining sections are organized as follows. In Section 2, some fundamental definitions and properties that are directly pertinent to this paper are presented for review. In Section 3, characterizations of the migrativity of t-conorms with respect to ordinal sum implications are obtained, depending on the position of parameter α within the range of natural negation N. In Section 4, a sketch of further considerations is shown.

2. PRELIMINARIES

This section is dedicated to introducing fundamental definitions and relevant properties that are essential for comprehending the content of this paper. Throughout the paper, the abbreviation "iff" will be used as a shorthand for "if and only if", and the symbol "$\mathbb{N}$" will consistently denote the set of natural numbers.

Definition 2.1. (Klement, Mesiar and Pap [30]) A binary function $S : [0,1]^2 \to [0,1]$ is known as a t-conorm, if it is commutative, associative and non-decreasing in both variables, and has the neutral element 0.

We represent by $\mathcal{S_C}$ the set of all continuous t-conorms.

Proposition 2.2. (Klement, Mesiar and Pap [30]) The following results concern the properties and ordinal sum representation of t-conorms.

(i) A t-conorm S is known as Archimedean, if for each $(x, y) \in (0,1)^2$ there is an $n \in \mathbb{N}$ with $x_S^{(n)} > y$, where $x_S^{(n)} = S\left(x, x_S^{(n-1)}\right)$ and $x_S^{(0)} = 0$.

(ii) Let $\{S_\gamma\}_{\gamma\in\Lambda}$ be a family of t-conorms, $\{(c_\gamma, d_\gamma)\}_{\gamma\in\Lambda}$ a family of non-empty, pairwise disjoint open subintervals of $[0,1]$, where Λ is a finite or countable infinite index set. Then the function $S : [0,1]^2 \to [0,1]$, expressed by

$$S(x,y) = \begin{cases} c_\gamma + (d_\gamma - c_\gamma)S_\gamma(\eta,\theta), & x,y \in [c_\gamma, d_\gamma], \\ \max(x,y), & \text{otherwise}, \end{cases} \tag{1}$$

and represented as $S = (\langle c_\gamma, d_\gamma, S_\gamma\rangle)_{\gamma\in\Lambda}$, where $\eta = \frac{x-c_\gamma}{d_\gamma-c_\gamma}$, $\theta = \frac{y-c_\gamma}{d_\gamma-c_\gamma}$, is a t-conorm. Conversely, every continuous t-conorm can be uniquely represented as an ordinal sum of continuous Archimedean t-conorms.

(iii) For $S \in \mathcal{S_C}$ and $a \in (0,1)$. Then a is an idempotent element of S iff $S(a,y) = \max(a,y)$ for all $y \in [0,1]$ iff $S = (\langle 0, a, S_{\gamma_1}\rangle, \langle a, 1, S_{\gamma_2}\rangle)$ with $S_{\gamma_1}, S_{\gamma_2} \in \mathcal{S_C}$.

We represent by $\mathcal{S_{CA}}$ the set of all continuous Archimedean t-conorms, and by $\mathcal{S_{CNA}}$ the set of all continuous non-Archimedean t-conorms.

Example 2.3. (Klement, Mesiar and Pap [30]) The following are typical examples of continuous t-conorms, where $x, y \in [0,1]$.

(1) Maximum: $S_M(x, y) = \max(x, y)$.

(2) Probabilistic sum: $S_P(x, y) = x + y - xy$.

(3) Łukasiewicz: $S_{LK}(x, y) = \min(x + y, 1)$.

From now on, let us provide the basic definitions related to fuzzy implications and fuzzy negations.

Definition 2.4. (Baczyński and Jayaram [4]) A binary function $I : [0,1]^2 \to [0,1]$ is known as a *fuzzy implication*, if for all $x, y, z \in [0,1]$, it fulfills $I(0,0) = I(1,1) = 1$, $I(1,0) = 0$ and the following conditions:

(I1) $x \le y$ implies $I(y, z) \le I(x, z)$,

(I2) $y \le z$ implies $I(x, y) \le I(x, z)$.

Definition 2.5. (Baczyński and Jayaram [4]) A non-increasing unary function $N : [0,1] \longrightarrow [0,1]$ is known as a *fuzzy negation*, if it fulfills $N(0) = 1$ and $N(1) = 0$.

Example 2.6. (Baczyński and Jayaram [4]) The following are some examples of fuzzy implications, where $x, y \in [0,1]$.

(1) Reichenbach implication: $I_{RC}(x, y) = 1 - x + xy$.

(2) Łukasiewicz implication: $I_{LK}(x, y) = \min(1 - x + y, 1)$.

(3) Kleene-Dienes implication: $I_{KD}(x, y) = \max(1 - x, y)$.

(4) Weber implication: $I_{WB}(x, y) = \begin{cases} 1, & x < 1, \\ y, & \text{otherwise.} \end{cases}$

(5) $I_D(x, y) = \begin{cases} 1, & x = 0, \\ y, & \text{otherwise.} \end{cases}$

Example 2.7. (Baczyński and Jayaram [4]) The fuzzy negations N_0 and N^1 are given by

$$N_0(x) = \begin{cases} 0, & x > 0, \\ 1, & \text{otherwise,} \end{cases} \qquad N^1(x) = \begin{cases} 0, & x = 1, \\ 1, & \text{otherwise.} \end{cases}$$

In particular, fuzzy negations N_0 and N^1 are least and greatest, respectively.

Zhou proposed a general ordinal sum of fuzzy implications, known as "Implication Complementing," which has garnered significant attention. A concise summary of this ordinal sum construction technique is provided below.

Theorem 2.8. (Zhou [55]) Let $\{I_k\}_{k\in\Lambda}$ be a family of fuzzy implications, $\{[a_k, b_k]\}_{k\in\Lambda}$ a family of pairwise disjoint closed subintervals of $[0,1]$, where Λ is a finite or countable infinite index set, and let I_* be a fuzzy implication. If, for all $k \in \Lambda$, the following conditions are met:

(I_*^1) $I_*(x,y) \leq a_k + (b_k - a_k)I_k\left(1, \frac{y-a_k}{b_k-a_k}\right)$ for all $(x,y) \in (b_k, 1] \times [a_k, b_k]$.

(I_*^2) $I_*(x,y) \leq a_k + (b_k - a_k)I_k\left(\frac{x-a_k}{b_k-a_k}, 0\right)$ for all $(x,y) \in [a_k, b_k] \times [0, a_k)$.

(I_*^3) $I_*(x,y) \geq b_k$ for all $(x,y) \in ([0, a_k) \times [a_k, b_k]) \cup ([a_k, b_k] \times (b_k, 1])$.

Then the binary function $I : [0,1]^2 \to [0,1]$, defined by

$$I(x,y) = \begin{cases} a_k + (b_k - a_k)\, I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), & x, y \in [a_k, b_k] \text{ and } x \neq 0, \\ I_*(x,y), & \text{otherwise}, \end{cases} \tag{2}$$

and written as $I = (\langle a_k, b_k, I_k\rangle, I_*)_{k\in\Lambda}$, is a fuzzy implication, referred to as ordinal sum of $\{I_k\}_{k\in\Lambda}$ and I_* on $\{(a_k, b_k)\}_{k\in\Lambda}$ and shortly, ordinal sum implication. Conversely, the function I defined by Eq. (2) fulfills the aforementioned conditions $(I_*^1) - (I_*^3)$ if it is a fuzzy implication.

We represent by $\mathcal{I}_{\mathcal{OS}}$ the set of ordinal sum implications given by (2) in Theorem 2.8.

Remark 2.9. Let $I = (\langle a_k, b_k, I_k\rangle, I_*)_{k\in\Lambda} \in \mathcal{I}_{\mathcal{OS}}$ be an ordinal sum implication. If $0 \in \bigcup_{k\in\Lambda}[a_k, b_k]$, then there exists $k_0 \in \Lambda$ such that $a_{k_0} = 0$. If $1 \in \bigcup_{k\in\Lambda}[a_k, b_k]$, then there exists $k_0 \in \Lambda$ such that $b_{k_0} = 1$. In this paper, we primarily focus on the case $[a_k, b_k] \subsetneqq [0,1]$, i. e., $[a_k, b_k]$ is a proper subset of $[0,1]$ for all $k \in \Lambda$.

Proposition 2.10. Let $I = (\langle a_k, b_k, I_k\rangle, I_*)_{k\in\Lambda} \in \mathcal{I}_{\mathcal{OS}}$ be an ordinal sum implication. Then the following assertions hold:

(i) If $0 \in \bigcup_{k\in\Lambda}[a_k, b_k]$, then there exists $k_0 \in \Lambda$ such that $a_{k_0} = 0$ and

$$N_I(x) = \begin{cases} 1, & x = 0, \\ b_{k_0} N_{I_{k_0}}\left(\frac{x}{b_{k_0}}\right), & x \in (0, b_{k_0}], \\ 0, & \text{otherwise}. \end{cases}$$

(ii) If $0 \notin \bigcup_{k\in\Lambda}[a_k, b_k]$, then $N_I = N_{I_*}$.

Proof. The proof can be divided into the following two cases:

(i) If $0 \in \bigcup_{k\in\Lambda}[a_k, b_k]$, which implies that $a_{k_0} = 0$ for some $k_0 \in \Lambda$. When $x = 0$, then $N_I(0) = I(0,0) = I_*(0,0) = 1$. When $x \in (0, b_{k_0}]$, then $N_I(x) = I(x,0) = b_{k_0} I_{k_0}\left(\frac{x}{b_{k_0}}, 0\right) = b_{k_0} N_{I_{k_0}}\left(\frac{x}{b_{k_0}}\right)$. When $x \in (b_{k_0}, 1]$, then $N_I(x) = I(x,0) = I_*(x,0) = N_{I_*}(x) = 0$, since $I_*(x,0) \leq b_{k_0} I_{k_0}(1,0) = 0$.

(ii) If $0 \notin \bigcup_{k\in\Lambda}[a_k, b_k]$, then $N_I(x) = I(x,0) = I_*(x,0) = N_{I_*}(x)$ for all $x \in [0,1]$.

□

3. MIGRATIVITY OF CONTINUOUS T-CONORMS WITH RESPECT TO ORDINAL SUM IMPLICATIONS

We review some basic results on the (α, I)-migrativity of t-conorms when I is a fuzzy implication.

Definition 3.1. (Pan, Zhou and Yan [43]) Assume that $\alpha \in [0,1]$, S is a t-conorm and I is a fuzzy implication. Then S is called *α-migrative* with respect to I, or (α, I)-migrative, if they fulfill

$$S(I(\alpha, x), y) = S(x, I(\alpha, y)), \tag{3}$$

for all $x, y \in [0,1]$. Assume that for every $\alpha \in [0,1]$, S is α-migrative with respect to I, then S is called *migrative* with respect to I.

Proposition 3.2. (Pan, Zhou and Yan [43]) Assume that S is a t-conorm and I is a fuzzy implication, then the following assertions hold.

(i) If $\alpha = 0$, then Eq. (3) holds for arbitrary t-conorms S and fuzzy implications I.

(ii) If $\alpha = 1$, then Eq. (3) holds iff $I(1, y) = y$ for all $y \in [0,1]$.

(iii) If $\alpha \in [0,1]$, then Eq. (3) holds iff $I(\alpha, y) = S(N_I(\alpha), y)$ for all $y \in [0,1]$.

By Proposition 3.2, we focus primarily on the case $\alpha \in (0,1)$. For an ordinal sum implication $I = (\langle a_k, b_k, I_k\rangle, I_*)_{k\in\Lambda}$, if there exists $k_0 \in \Lambda$ such that $a_{k_0} = 0$ and $b_{k_0} = 1$, or if $\Lambda = \varnothing$, then I degenerate to the given fuzzy implication I_{k_0} or I_*. In this case, for more details about the migrativity of continuous t-conorms S with respect to I, please refer to [43].

In the following discussion, the underlying fuzzy implication I in Eq. (3) is specifically limited to being an ordinal sum implication, which is defined as $I = (\langle a_k, b_k, I_k\rangle, I_*)_{k\in\Lambda}$ in Eq. (2). The results of this paper will illustrate the interrelationship between the ordinal sum structures of t-conorms and those of I, with the aid of Eq. (3). We define $\gamma_a = \min\{a_k | k \in \Lambda\}$, $\gamma_b = \min\{b_k | k \in \Lambda\}$, $\eta_a = \max\{a_k | k \in \Lambda\}$ and $\eta_b = \max\{b_k | k \in \Lambda\}$. We first explore the α-migrativity of t-conorms S with respect to I with $\alpha \in [a_k, b_k]$. Specifically, for some $k_0 \in \Lambda$, we consider the following three cases:

(1) $\alpha = a_{k_0} > 0$;

(2) $\alpha \in (a_{k_0}, b_{k_0})$;

(3) $\alpha = b_{k_0} < 1$,

In what follows, we shall explore the case $\alpha = a_{k_0} > 0$ for some $k_0 \in \Lambda$.

Theorem 3.3. Let $I = (\langle a_k, b_k, I_k\rangle, I_*)_{k\in\Lambda} \in \mathcal{I}_{\mathcal{OS}}$ be an ordinal sum implication and $S \in \mathcal{S}_{\mathcal{C}}$ a continuous t-conorm. If $\alpha = a_{k_0}$ for some $k_0 \in \Lambda$ with $a_{k_0} = \gamma_a > 0$ and $N_I(\alpha) \in (0, a_{k_0})$, then S is a_{k_0}-migrative with respect to I iff $S = (\langle 0, b_{k_0}, S_1\rangle, \langle b_{k_0}, 1, S_2\rangle)$

with $S_1, S_2 \in \mathcal{S}_{\mathcal{C}}$, and $I_*(a_{k_0}, \cdot)$ can be expressed on $[0, a_{k_0}) \cup (b_{k_0}, 1]$ in the following form:

$$I_*(a_{k_0}, y) = \begin{cases} b_{k_0} S_1 \left(\frac{N_{I_*}(a_{k_0})}{b_{k_0}}, \frac{y}{b_{k_0}} \right), & y \in [0, a_{k_0}), \\ y, & y \in (b_{k_0}, 1], \end{cases} \tag{4}$$

where $S_1(x, y) = 1$ for all $(x, y) \in E$, thereinto $E = \left\{ (x, y) \middle| \frac{N_{I_*}(a_{k_0})}{b_{k_0}} \leq \min(x, y) \leq \frac{a_{k_0}}{b_{k_0}} \leq \max(x, y) \leq 1 \right\} \bigcup \left\{ (x, y) \middle| \frac{a_{k_0}}{b_{k_0}} \leq x, y \leq 1 \right\}$.

Proof. ($\Rightarrow$) Suppose that S is a_{k_0}-migrative with respect to I, letting $x = b_{k_0}$ in Eq. (3), then $S(b_{k_0}, y) = S(b_{k_0}, b_{k_0})$ for all $y \in [a_{k_0}, b_{k_0}]$. And, $S(N_{I_*}(a_{k_0}), y)$ can be expressed in the following form:

$$S(N_{I_*}(a_{k_0}), y) = I(a_{k_0}, y) = \begin{cases} b_{k_0}, & y \in [a_{k_0}, b_{k_0}], \\ I_*(a_{k_0}, y), & \text{otherwise}. \end{cases} \tag{5}$$

Since $S(N_{I_*}(a_{k_0}), \cdot)$ is continuous, then, by Proposition 3.2 (iii), $I(a_{k_0}, \cdot)$ also is continuous. Combined with $I(a_{k_0}, 0) \in (0, a_{k_0})$ and $I(a_{k_0}, a_{k_0}) = b_{k_0}$, then there exists $c \in (0, a_{k_0})$ such that $I(a_{k_0}, c) = a_{k_0}$, leading to

$$\begin{aligned} S(a_{k_0}, a_{k_0}) &= S(a_{k_0}, I(a_{k_0}, c)) = S(I(a_{k_0}, a_{k_0}), c) = S(b_{k_0}, c) = S(I(a_{k_0}, b_{k_0}), c) \\ &= S(b_{k_0}, I(a_{k_0}, c)) = S(b_{k_0}, a_{k_0}) = S(b_{k_0}, b_{k_0}). \end{aligned}$$

Next, we complete the proof by showing that b_{k_0} is an idempotent element of S.

Note that $(N_{I_*}(a_{k_0}))_S^{(2)} > N_{I_*}(a_{k_0})$. Suppose to the contrary that $(N_{I_*}(a_{k_0}))_S^{(2)} = N_{I_*}(a_{k_0})$, then by Proposition 2.2, we obtain that $S(N_{I_*}(a_{k_0}), y) = \max(N_{I_*}(a_{k_0}), y)$ for all $y \in [0, 1]$, which means that $S(N_{I_*}(a_{k_0}), a_{k_0}) = a_{k_0} \neq b_{k_0}$, so, $(N_{I_*}(a_{k_0}))_S^{(2)} > N_{I_*}(a_{k_0})$, i. e. $(N_{I_*}(a_{k_0}))_S^{(2)} \in (N_{I_*}(a_{k_0}), b_{k_0}]$. Moreover, we have that

$$S\left(b_{k_0}, (N_{I_*}(a_{k_0}))_S^{(2)}\right) = S\left(S\left(b_{k_0}, N_{I_*}(a_{k_0})\right), N_{I_*}(a_{k_0})\right) = S\left(b_{k_0}, N_{I_*}(a_{k_0})\right) = b_{k_0}.$$

If $(N_{I_*}(a_{k_0}))_S^{(2)} \in [a_{k_0}, b_{k_0}]$, then b_{k_0} is idempotent for S.

If $(N_{I_*}(a_{k_0}))_S^{(2)} \in (N_{I_*}(a_{k_0}), a_{k_0})$, suppose that there exists $m \in \mathbb{N}^+$ such that $(N_{I_*}(a_{k_0}))_S^{(2^m)} < a_{k_0}$ and $(N_{I_*}(a_{k_0}))_S^{(2^{m+1})} \geq a_{k_0}$, then

$$\begin{aligned} S(b_{k_0}, a_{k_0}) &\leq S\left(b_{k_0}, (N_{I_*}(a_{k_0}))_S^{(2^{m+1})}\right) \\ &= S\left(b_{k_0}, S\left((N_{I_*}(a_{k_0}))_S^{(2^m)}, (N_{I_*}(a_{k_0}))_S^{(2^m)}\right)\right) \\ &= b_{k_0}, \end{aligned}$$

and $S(b_{k_0}, a_{k_0}) \geq b_{k_0}$. Hence, b_{k_0} is idempotent for S, which implies that $S(x, y) = b_{k_0}$ for all $(x, y) \in ([N_{I_*}(a_{k_0}), b_{k_0}] \times [a_{k_0}, b_{k_0}]) \cup ([a_{k_0}, b_{k_0}] \times [N_{I_*}(a_{k_0}), a_{k_0}))$.

In addition, since b_{k_0} is an idempotent element of S, then from Proposition 2.2, it follows that $S = (\langle 0, b_{k_0}, S_1\rangle, \langle b_{k_0}, 1, S_2\rangle)$ with $S_1, S_2 \in \mathcal{S_C}$, where $S_1(x,y) = 1$ for all $(x,y) \in \left\{(x,y) \middle| \frac{N_{I_*}(a_{k_0})}{b_{k_0}} \le \min(x,y) \le \frac{a_{k_0}}{b_{k_0}} \le \max(x,y) \le 1\right\} \bigcup \left\{(x,y) \middle| \frac{a_{k_0}}{b_{k_0}} \le x, y \le 1\right\}$.

Finally, we need to verify the expression of $I_*(a_{k_0}, \cdot)$ on $[0, a_{k_0}) \cup (b_{k_0}, 1]$. By Eq. (5), then

$$I_*(a_{k_0}, y) = S(N_{I_*}(a_{k_0}), y) = \begin{cases} b_{k_0} S_1\left(\frac{N_{I_*}(a_{k_0})}{b_{k_0}}, \frac{y}{b_{k_0}}\right), & y \in [0, a_{k_0}), \\ y, & y \in (b_{k_0}, 1]. \end{cases}$$

($\Leftarrow$) Under the assumptions, then for all $x, y \in [0,1]$, we have

$$\begin{aligned} S(I(a_{k_0}, x), y) &= \begin{cases} S(b_{k_0}, y), & x \in [a_{k_0}, b_{k_0}], \\ S(I_*(a_{k_0}, x), y), & \text{otherwise}, \end{cases} \\ &= \begin{cases} S\left(b_{k_0} S_1\left(\frac{N_{I_*}(a_{k_0})}{b_{k_0}}, \frac{x}{b_{k_0}}\right), y\right), & x \in [0, a_{k_0}), \\ S(b_{k_0}, y), & x \in [a_{k_0}, b_{k_0}], \\ S(x,y), & x \in (b_{k_0}, 1], \end{cases} \\ &= S\left(S(N_{I_*}(a_{k_0}), x), y\right) \\ &= S\left(x, S(N_{I_*}(a_{k_0}), y)\right) \\ &= S(x, I(a_{k_0}, y)). \end{aligned}$$

Hence, S is a_{k_0}-migrative with respect to I. □

Theorem 3.4. Let $I = (\langle a_k, b_k, I_k\rangle, I_*)_{k\in\Lambda} \in \mathcal{I_{OS}}$ be an ordinal sum implication and $S \in \mathcal{S_C}$ a continuous t-conorm. If $\alpha = a_{k_0}$ for some $k_0 \in \Lambda$ with $a_{k_0} = \gamma_a > 0$ and $N_I(\alpha) \in [a_{k_0}, b_{k_0}]$, then S is a_{k_0}-migrative with respect to I iff $S = (\langle 0, b_{k_0}, S_1\rangle, \langle b_{k_0}, 1, S_2\rangle)$ with $S_1, S_2 \in \mathcal{S_C}$, and $I_*(a_{k_0}, \cdot)$ can be expressed on $[0, a_{k_0}) \cup (b_{k_0}, 1]$ in the following form:

$$I_*(a_{k_0}, y) = \begin{cases} b_{k_0} S_1\left(\frac{N_{I_*}(a_{k_0})}{b_{k_0}}, \frac{y}{b_{k_0}}\right), & y \in [0, a_{k_0}), \\ y, & y \in (b_{k_0}, 1], \end{cases} \tag{6}$$

where $S_1(x,y) = 1$ for all $(x,y) \in F$, thereinto $F = \left\{(x,y) \middle| \frac{a_{k_0}}{b_{k_0}} \le \min(x,y) \le \frac{N_{I_*}(a_{k_0})}{b_{k_0}} \le \max(x,y) \le 1\right\} \bigcup \left\{(x,y) \middle| \frac{N_{I_*}(a_{k_0})}{b_{k_0}} \le x, y \le 1\right\}$.

Proof. ($\Rightarrow$) Suppose that S is a_{k_0}-migrative with respect to I, then from Theorem 3.3, it follows that $S(b_{k_0}, a_{k_0}) = S(b_{k_0}, b_{k_0})$ and $S(N_{I_*}(a_{k_0}), a_{k_0}) = S(N_{I_*}(a_{k_0}), b_{k_0}) = b_{k_0}$. Since $b_{k_0} \le S(b_{k_0}, a_{k_0}) \le S(b_{k_0}, N_{I_*}(a_{k_0})) = b_{k_0}$, then $S(b_{k_0}, a_{k_0}) = b_{k_0}$, which implies that $S(x,y) = b_{k_0}$ for all $(x,y) \in ([a_{k_0}, b_{k_0}] \times [N_{I_*}(a_{k_0}), b_{k_0}]) \cup ([N_{I_*}(a_{k_0}), b_{k_0}] \times [a_{k_0}, N_{I_*}(a_{k_0})])$. Moreover, because b_{k_0} is idempotent for S, then Proposition 2.2 implies that $S = (\langle 0, b_{k_0}, S_1\rangle, \langle b_{k_0}, 1, S_2\rangle)$ with $S_1, S_2 \in \mathcal{S_C}$, where $S_1(x,y) = 1$ for all $(x,y) \in F$, thereinto $F = \Big\{(x,y)\Big|\frac{a_{k_0}}{b_{k_0}} \le \min(x,y) \le \frac{N_{I_*}(a_{k_0})}{b_{k_0}} \le \max(x,y) \le 1\Big\} \bigcup \Big\{(x,y)\Big|\frac{N_{I_*}(a_{k_0})}{b_{k_0}} \le x,y \le 1\Big\}$.

Next, we need to verify the expression of $I_*(a_{k_0},\cdot)$ on $[0,a_{k_0}) \cup (b_{k_0},1]$, it follows from Eq. (5) that

$$I_*(a_{k_0},y) = S(N_{I_*}(a_{k_0}),y) = \begin{cases} b_{k_0} S_1\left(\frac{N_{I_*}(a_{k_0})}{b_{k_0}}, \frac{y}{b_{k_0}}\right), & y \in [0,a_{k_0}), \\ y, & y \in (b_{k_0},1]. \end{cases}$$

(⇐) The sufficiency can be proven directly by referring to Theorem 3.3. □

Remark 3.5. Let $I = (\langle a_k, b_k, I_k\rangle, I_*)_{k\in\Lambda} \in \mathcal{I}_{\mathcal{OS}}$ be an ordinal sum implication, $S \in \mathcal{S}_{\mathcal{C}}$ a continuous t-conorm and $a_{k_0} = \gamma_a > 0$ for some $k_0 \in \Lambda$.

(i) If $N_I(a_{k_0}) = b_{k_0}$, then S is a_{k_0}-migrative with respect to I iff $I(a_{k_0},y) = S(b_{k_0},y) = \max(b_{k_0},y)$ for all $y \in [0,1]$. This conclusion coincides with [57, Theorem 3.5 (1)]. It is evident that Theorems 3.3 and 3.4 are more general than [57, Theorem 3.5 (1)].

(ii) Based on the characterizations described Theorems 3.3 and 3.4 for the migrativity of S with respect to I, we have that $I(a_{k_0},y) = I_*(a_{k_0},y) = y$ for all $y \in (b_{k_0},1]$. However, for some $k_1 \in \Lambda$, $I(a_{k_0},y) = I_*(a_{k_0},y) = y < b_{k_1} = I(a_{k_1},y)$ for all $y \in (a_{k_1},b_{k_1})$, which is a contradiction. Therefore, according to the above fact, if S is a_{k_0}-migrative with respect to I, then $a_{k_0} = \gamma_a = \eta_a$, i. e., $\Lambda = \{k_0\}$.

(iii) For some $k_1 \in \Lambda$, by the structure of I, it implies that $N_I(a_{k_1}) \in [0,a_{k_0}]$. Then we split into the following three cases to explore the a_{k_1}-migrativity of S with respect to I.

- $N_I(a_{k_1}) = 0$. It is obvious that S is not a_{k_1}-migrative with respect to I.
- $N_I(a_{k_1}) = a_{k_0}$. If S is a_{k_1}-migrative with respect to I, then $I(a_{k_1},y) = \max(a_{k_0},y)$ for all $y \in [0,1]$, leading to $b_{k_1} = I(a_{k_1},a_{k_1}) = \max(a_{k_0},a_{k_1}) = a_{k_1}$, which is a contradiction.
- $N_I(a_{k_1}) \in (0,a_{k_0})$. Then by Theorem 3.3, S is a_{k_1}-migrative with respect to I iff $S = (\langle 0,b_{k_1},S_1\rangle, \langle b_{k_1},1,S_2\rangle)$ with $S_1,S_2 \in \mathcal{S}_C$, and $I_*(a_{k_1},\cdot)$ can be expressed on $[0,a_{k_1}) \cup (b_{k_1},1]$ in the following form:

$$I_*(a_{k_1},y) = \begin{cases} b_{k_1} S_1\left(\frac{N_{I_*}(a_{k_1})}{b_{k_1}}, \frac{y}{b_{k_1}}\right), & y \in [0,a_{k_1}), \\ y, & y \in (b_{k_1},1], \end{cases}$$

where $S_1(x,y) = 1$ for all $(x,y) \in E$, thereinto $E = \Big\{(x,y) \mid \frac{N_{I_*}(a_{k_1})}{b_{k_1}} \le \min(x,y) \le \frac{a_{k_1}}{b_{k_1}} \le \max(x,y) \le 1\Big\} \bigcup \Big\{(x,y) \mid \frac{a_{k_1}}{b_{k_1}} \le x,y \le 1\Big\}$. Note that: $I(b_{k_0},y) \ge b_{k_1} S_1\left(\frac{N_{I_*}(a_{k_1})}{b_{k_1}}, \frac{y}{b_{k_1}}\right)$ for all $y \in [a_{k_0},b_{k_0}]$ and $a_{k_1} = \eta_a$.

In the following, we will further explore the case $\alpha \in (a_{k_0}, b_{k_0})$ for some $k_0 \in \Lambda$ with $N_I(\alpha) \in (a_{k_0}, b_{k_0})$.

Theorem 3.6. Let $I = (\langle a_k, b_k, I_k\rangle, I_*)_{k\in\Lambda} \in \mathcal{I}_{\mathcal{OS}}$ be an ordinal sum implication and $S \in \mathcal{S}_{\mathcal{C}}$ a continuous t-conorm. Suppose that $\alpha \in (a_{k_0}, b_{k_0})$ for some $k_0 \in \Lambda$ with $N_I(\alpha) \in (a_{k_0}, b_{k_0})$ and $a_{k_0} = \gamma_a \geq 0$. If $I(\alpha, N_I(\alpha)) = N_I(\alpha)$, then S is α-migrative with respect to I iff the following assertions hold:

(i) $S(N_I(\alpha), y) = \max(N_I(\alpha), y)$ for all $y \in [0, 1]$.

(ii) $I_*(\alpha, y) = \max(N_I(\alpha), y)$ for all $y \in [0, a_{k_0}) \cup (b_{k_0}, 1]$.

(iii) $I_{k_0}\left(\frac{\alpha - a_{k_0}}{b_{k_0} - a_{k_0}}, y\right) = \max\left(\frac{N_I(\alpha) - a_{k_0}}{b_{k_0} - a_{k_0}}, y\right)$ for all $y \in [0, 1]$.

Proof. ($\Rightarrow$) Suppose that S is α-migrative with respect to I, then $S(N_I(\alpha), N_I(\alpha)) = I(\alpha, N_I(\alpha)) = N_I(\alpha)$, which implies that $N_I(\alpha)$ is idempotent for S. Then by Proposition 2.2, $S(N_I(\alpha), y) = \max(N_I(\alpha), y)$ for all $y \in [0, 1]$, which indicates that for all $y \in [0, a_{k_0}) \cup (b_{k_0}, 1]$, $I_*(\alpha, y) = I(\alpha, y) = S(N_I(\alpha), y) = \max(N_I(\alpha), y)$, and for all $y \in [a_{k_0}, b_{k_0}]$,

$$a_{k_0} + (b_{k_0} - a_{k_0}) I_{k_0}\left(\frac{\alpha - a_{k_0}}{b_{k_0} - a_{k_0}}, \frac{y - a_{k_0}}{b_{k_0} - a_{k_0}}\right) = S(N_I(\alpha), y) = \max(N_I(\alpha), y),$$

i. e., $I_{k_0}\left(\frac{\alpha - a_{k_0}}{b_{k_0} - a_{k_0}}, y\right) = \max\left(\frac{N_I(\alpha) - a_{k_0}}{b_{k_0} - a_{k_0}}, y\right)$ for all $y \in [0, 1]$.

($\Leftarrow$) Suppose that items (i)-(iii) are valid, then for all $x, y \in [0, 1]$,

$$\begin{aligned} S(I(\alpha, x), y) &= \begin{cases} S\left(a_{k_0} + (b_{k_0} - a_{k_0}) I_{k_0}\left(\frac{\alpha - a_{k_0}}{b_{k_0} - a_{k_0}}, \frac{x - a_{k_0}}{b_{k_0} - a_{k_0}}\right), y\right), & x \in [a_{k_0}, b_{k_0}], \\ S(I_*(\alpha, x), y), & \text{otherwise}, \end{cases} \\ &= \begin{cases} S\left(a_{k_0} + (b_{k_0} - a_{k_0}) \max\left(\frac{N_I(\alpha) - a_{k_0}}{b_{k_0} - a_{k_0}}, \frac{x - a_{k_0}}{b_{k_0} - a_{k_0}}\right), y\right), & x \in [a_{k_0}, b_{k_0}], \\ S(\max(N_I(\alpha), x), y), & \text{otherwise}, \end{cases} \\ &= S\left(\max(N_I(\alpha), x), y\right) \\ &= S(x, I(\alpha, y)). \end{aligned}$$

Therefore, S is α-migrative with respect to I. □

Theorem 3.7. Let $I = (\langle a_k, b_k, I_k\rangle, I_*)_{k\in\Lambda} \in \mathcal{I}_{\mathcal{OS}}$ be an ordinal sum implication and $S \in \mathcal{S}_{\mathcal{C}}$ a continuous t-conorm. Suppose that $\alpha \in (a_{k_0}, b_{k_0})$ for some $k_0 \in \Lambda$ with $N_I(\alpha) \in (a_{k_0}, b_{k_0})$ and $a_{k_0} = \gamma_a \geq 0$. If $I(\alpha, N_I(\alpha)) > N_I(\alpha)$, then S is α-migrative with respect to I iff $S = (\cdots, \langle u, v, S_2\rangle, \cdots)$ with $N_I(\alpha) \in (u, v)$ and $S_2 \in \mathcal{S}_{\mathcal{CA}}$, where $u = \sup\{x \in [0, N_I(\alpha)) | S(x, x) = x\}$, $v = \inf\{x \in (N_I(\alpha), 1] | S(x, x) = x\} \leq b_{k_0}$, and one of the following assertions holds:

(i) If $u < a_{k_0}$, then

$$I_*(\alpha, y) = \begin{cases} N_I(\alpha), & y \in [0, u], \\ u + (v-u) S_2\left(\frac{N_I(\alpha)-u}{v-u}, \frac{y-u}{v-u}\right), & y \in (u, a_{k_0}), \\ y, & y \in (b_{k_0}, 1], \end{cases} \tag{7}$$

and

$$I_{k_0}\left(\frac{\alpha - a_{k_0}}{b_{k_0} - a_{k_0}}, y\right) = \begin{cases} \frac{u-a_{k_0}}{b_{k_0}-a_{k_0}} + \frac{v-u}{b_{k_0}-a_{k_0}} S_2(p, q), & y \in \left[0, \frac{v-a_{k_0}}{b_{k_0}-a_{k_0}}\right], \\ y, & \text{otherwise}, \end{cases} \tag{8}$$

where $p = \frac{N_I(\alpha)-u}{v-u}$ and $q = \frac{a_{k_0}+(b_{k_0}-a_{k_0})y-u}{v-u}$.

(ii) If $u \geq a_{k_0}$, then $I_*(\alpha, y) = \max(N_I(\alpha), y)$ for all $y \in [0, a_{k_0}) \cup (b_{k_0}, 1]$, and

$$I_{k_0}\left(\frac{\alpha - a_{k_0}}{b_{k_0} - a_{k_0}}, y\right) = \begin{cases} \frac{N_I(\alpha)-a_{k_0}}{b_{k_0}-a_{k_0}}, & y \in \left[0, \frac{u-a_{k_0}}{b_{k_0}-a_{k_0}}\right), \\ \frac{u-a_{k_0}}{b_{k_0}-a_{k_0}} + \frac{v-u}{b_{k_0}-a_{k_0}} S_2(p, q), & y \in \left[\frac{u-a_{k_0}}{b_{k_0}-a_{k_0}}, \frac{v-a_{k_0}}{b_{k_0}-a_{k_0}}\right], \\ y, & \text{otherwise}, \end{cases} \tag{9}$$

where $p = \frac{N_I(\alpha)-u}{v-u}$ and $q = \frac{a_{k_0}+(b_{k_0}-a_{k_0})y-u}{v-u}$.

In particular, if $u = a_{k_0}$ and $v = b_{k_0}$, then S is α-migrative with respect to I iff S can be expressed in the form of $S = (\cdots, \langle a_{k_0}, b_{k_0}, S_2\rangle, \cdots)$ with $S_2 \in \mathcal{S}_{\mathcal{CA}}$, and S_2 is $\frac{\alpha-a_{k_0}}{b_{k_0}-a_{k_0}}$-migrative with respect to I_{k_0} and $I_*(\alpha, y) = \max(N_I(\alpha), y)$ for all $y \in [0, a_{k_0}) \cup (b_{k_0}, 1]$, where $N_I(\alpha) = a_{k_0} + (b_{k_0} - a_{k_0}) N_{I_{k_0}}\left(\frac{\alpha-a_{k_0}}{b_{k_0}-a_{k_0}}\right)$.

Proof. ($\Rightarrow$) Suppose that S is α-migrative with respect to I, then $S(N_I(\alpha), N_I(\alpha)) = I(\alpha, N_I(\alpha)) > N_I(\alpha)$, which means that $N_I(\alpha)$ is not an idempotent element of S. Hence, put $u = \sup\{x \in [0, N_I(\alpha)) | S(x, x) = x\}$ and $v = \inf\{x \in (N_I(\alpha), 1] | S(x, x) = x\}$, by [43, Theorem 4.8], u and v are idempotent elements of S, then from Proposition 2.2, there exists a continuous Archimedean t-conorm S_2 such that $S = (\cdots, \langle u, v, S_2\rangle, \cdots)$.

Note that $v \leq b_{k_0}$. Suppose that $v > b_{k_0}$, since $S_2 \in \mathcal{S}_{\mathcal{CA}}$ is a continuous Archimedean t-conorm, then $b_{k_0} = a_{k_0} + (b_{k_0} - a_{k_0}) I_{k_0}\left(\frac{\alpha-a_{k_0}}{b_{k_0}-a_{k_0}}, \frac{b_{k_0}-a_{k_0}}{b_{k_0}-a_{k_0}}\right) = S(N_I(\alpha), b_{k_0}) = u + (v - u) S_2\left(\frac{N_I(\alpha)-u}{v-u}, \frac{b_{k_0}-u}{v-u}\right) > b_{k_0}$, which is a contradiction. Hence, $v \leq b_{k_0}$.

Next, we will delve into investigate the following two cases: $u < a_{k_0}$ or $u \geq a_{k_0}$.

- $u < a_{k_0}$.

  The first step is to identify expression of $I_*(\alpha, \cdot)$ on $[0, 1] \setminus [a_{k_0}, b_{k_0}]$.

  1) If $y \in (u, a_{k_0})$, then $I_*(\alpha, y) = S(N_I(\alpha), y) = u + (v - u) S_2\left(\frac{N_I(\alpha)-u}{v-u}, \frac{y-u}{v-u}\right)$.

  2) If $y \in [0, u] \cup (b_{k_0}, 1]$, then $I_*(\alpha, y) = S(N_I(\alpha), y) = \max(N_I(\alpha), y)$.

The second step is to identify the expression of $I_{k_0}\left(\frac{\alpha-a_{k_0}}{b_{k_0}-a_{k_0}},\cdot\right)$.

1) If $y\in[a_{k_0},v]$, then

$$\begin{aligned}&a_{k_0}+(b_{k_0}-a_{k_0})I_{k_0}\left(\frac{\alpha-a_{k_0}}{b_{k_0}-a_{k_0}},\frac{y-a_{k_0}}{b_{k_0}-a_{k_0}}\right)\\&=S(N_I(\alpha),y)\\&=u+(v-u)S_2\left(\frac{N_I(\alpha)-u}{v-u},\frac{y-u}{v-u}\right).\end{aligned}$$

Therefore, for all $y\in\left[0,\frac{v-a_{k_0}}{b_{k_0}-a_{k_0}}\right]$,

$$I_{k_0}\left(\frac{\alpha-a_{k_0}}{b_{k_0}-a_{k_0}},y\right)=\frac{u-a_{k_0}}{b_{k_0}-a_{k_0}}+\frac{v-u}{b_{k_0}-a_{k_0}}S_2\left(p,q\right),$$

where $p=\frac{N_I(\alpha)-u}{v-u}$ and $q=\frac{a_{k_0}+(b_{k_0}-a_{k_0})y-u}{v-u}$.

2) If $y\in(v,b_{k_0}]$, then

$$a_{k_0}+(b_{k_0}-a_{k_0})I_{k_0}\left(\frac{\alpha-a_{k_0}}{b_{k_0}-a_{k_0}},\frac{y-a_{k_0}}{b_{k_0}-a_{k_0}}\right)=S(N_I(\alpha),y)=y.$$

Therefore, for all $y\in\left(\frac{v-a_{k_0}}{b_{k_0}-a_{k_0}},1\right]$, we have that $I_{k_0}\left(\frac{\alpha-a_{k_0}}{b_{k_0}-a_{k_0}},y\right)=y$.

In conclusion, the expressions for $I_*(\alpha,\cdot)$ on $[0,1]\setminus[a_k,b_k]$ and $I_{k_0}\left(\frac{\alpha-a_{k_0}}{b_{k_0}-a_{k_0}},\cdot\right)$ are as shown in (7) and (8), respectively.

- $u\geq a_{k_0}$.

  The proof is the same as that for $u<a_{k_0}$. For all $y\in[0,1]\setminus[a_{k_0},b_{k_0}]$, $I_*(\alpha,y)=S(N_I(\alpha),y)=\max(N_I(\alpha),y)$, and $I_{k_0}\left(\frac{\alpha-a_{k_0}}{b_{k_0}-a_{k_0}},\cdot\right)$ can be expressed in the following form:

$$I_{k_0}\left(\frac{\alpha-a_{k_0}}{b_{k_0}-a_{k_0}},y\right)=\begin{cases}\frac{N_I(\alpha)-a_{k_0}}{b_{k_0}-a_{k_0}}, & y\in\left[0,\frac{u-a_{k_0}}{b_{k_0}-a_{k_0}}\right),\\ \frac{u-a_{k_0}}{b_{k_0}-a_{k_0}}+\frac{v-u}{b_{k_0}-a_{k_0}}S_2\left(p,q\right), & y\in\left[\frac{u-a_{k_0}}{b_{k_0}-a_{k_0}},\frac{v-a_{k_0}}{b_{k_0}-a_{k_0}}\right],\\ y, & \text{otherwise},\end{cases}$$

  where $p=\frac{N_I(\alpha)-u}{v-u}$ and $q=\frac{a_{k_0}+(b_{k_0}-a_{k_0})y-u}{v-u}$.

($\Leftarrow$) Suppose that $S=(\cdots,\langle u,v,S_2\rangle,\cdots)$ with $S_2\in\mathcal{S}_{\mathcal{CA}}$, I_* fulfills Eq. (7) and I_{k_0}

fulfills Eq. (8), then for all $x, y \in [0,1]$,

$$\begin{aligned}
S(I(\alpha,x),y) &= \begin{cases} S\left(a_{k_0} + (b_{k_0} - a_{k_0}) I_{k_0}\left(\frac{\alpha - a_{k_0}}{b_{k_0} - a_{k_0}}, \frac{x - a_{k_0}}{b_{k_0} - a_{k_0}}\right), y\right), & x \in [a_{k_0}, b_{k_0}], \\ S(I_*(\alpha,x),y), & \text{otherwise}, \end{cases} \\
&= \begin{cases} S(N_I(\alpha),y), & x \in [0,u], \\ S\left(u + (v-u) S_2\left(\frac{N_I(\alpha)-u}{v-u}, \frac{x-u}{v-u}\right), y\right), & x \in (u,v], \\ S(x,y), & x \in (v,1], \end{cases} \\
&= S\left(S(N_I(\alpha),x),y\right) \\
&= S\left(x, S(N_I(\alpha),y)\right) \\
&= S(x, I(\alpha,y)).
\end{aligned}$$

If $I_*(\alpha,y) = \max(N_I(\alpha),y)$ for all $y \in [0,a_{k_0}) \cup (b_{k_0},1]$ and I_{k_0} fulfills Eq. (9), then we can check the α-migrativity of S with respect to I in an analogous way.

If $u = a_{k_0}$ and $v = b_{k_0}$, then S is α-migrative with respect to I iff S can be expressed in the form of $S = (\cdots, \langle a_{k_0}, b_{k_0}, S_2\rangle, \cdots)$ with $S_2 \in \mathcal{S}_{\mathcal{CA}}$, and S_2 is $\frac{\alpha - a_{k_0}}{b_{k_0} - a_{k_0}}$-migrative with respect to I_{k_0} and $I_*(\alpha,y) = \max(N_I(\alpha),y)$ for all $y \in [0,a_{k_0}) \cup (b_{k_0},1]$, where $N_I(\alpha) = a_{k_0} + (b_{k_0} - a_{k_0})\, N_{I_{k_0}}\left(\frac{\alpha - a_{k_0}}{b_{k_0} - a_{k_0}}\right)$. □

Example 3.8. The following serve as illustrative examples pertinent to Theorems 3.6 and 3.7.

(i) Let

$$I_*(x,y) = \begin{cases} \frac{5}{4} - x, & (x,y) \in \left[\frac{5}{12}, \frac{5}{6}\right] \times \left[0, \frac{5}{12}\right), \\ \frac{5}{6}, & (x,y) \in \left[\frac{5}{12}, \frac{5}{6}\right]^2, \\ y, & (x,y) \in \left[\frac{5}{12}, \frac{5}{6}\right] \times \left(\frac{5}{6}, 1\right], \\ 1 - x, & (x,y) \in \left(\frac{5}{6}, 1\right] \times [0,1), \\ 1, & \text{otherwise}, \end{cases}$$

then $I = \left(\left\langle \frac{5}{12}, \frac{5}{6}, I_{KD}\right\rangle, I_*\right)$ is an ordinal sum implication, i. e.,

$$I(x,y) = \begin{cases} \max\left(\frac{5}{4} - x, y\right), & (x,y) \in \left[\frac{5}{12}, \frac{5}{6}\right]^2, \\ \frac{5}{4} - x, & (x,y) \in \left[\frac{5}{12}, \frac{5}{6}\right] \times \left[0, \frac{5}{12}\right), \\ y, & (x,y) \in \left[\frac{5}{12}, \frac{5}{6}\right] \times \left(\frac{5}{6}, 1\right], \\ 1 - x, & (x,y) \in \left(\frac{5}{6}, 1\right] \times [0,1), \\ 1, & \text{otherwise}. \end{cases}$$

Let $\alpha \in \left(\frac{5}{12}, \frac{5}{6}\right)$, then $N_I(\alpha) = \frac{5}{4} - \alpha$ and $I(\alpha, N_I(\alpha)) = N_I(\alpha)$. Obviously, we have that $I(\alpha,y) = \max\left(\frac{5}{4} - \alpha, y\right) = \max(N_I(\alpha),y) = S_M(N_I(\alpha),y)$ for all $y \in [0,1]$. Then, by Theorem 3.6, S_M is α-migrative with respect to I.

(ii) Let $[a_0, b_0] = \left[\frac{1}{3}, \frac{2}{3}\right]$,

$$I_*(x,y) = \begin{cases} \frac{7}{12}, & (x,y) \in \left\{\frac{1}{2}\right\} \times \left[0, \frac{1}{4}\right], \\ \frac{1}{5}y + \frac{8}{15}, & (x,y) \in \left\{\frac{1}{2}\right\} \times \left(\frac{1}{4}, \frac{1}{3}\right), \\ y, & (x,y) \in \left\{\frac{1}{2}\right\} \times \left(\frac{2}{3}, 1\right), \\ 1, & (x,y) \in \left[0, \frac{1}{3}\right) \times [0,1) \\ & \text{or } (x,y) \in \left[\frac{1}{3}, \frac{1}{2}\right) \times \left(\frac{2}{3}, 1\right) \\ & \text{or } (x,y) \in [0,1] \times \{1\}, \\ \frac{1}{3}, & (x,y) \in \left(\frac{1}{2}, \frac{2}{3}\right] \times \left[0, \frac{1}{3}\right), \\ 0, & (x,y) \in \left(\frac{2}{3}, 1\right] \times [0,1), \\ \frac{2}{3}, & \text{otherwise,} \end{cases}$$

and

$$I_1(x,y) = \begin{cases} \frac{1}{5}y + \frac{4}{5}, & (x,y) \in \left\{\frac{1}{2}\right\} \times [0,1], \\ 0, & (x,y) \in \left(\frac{1}{2}, 1\right] \times [0,1), \\ 1, & \text{otherwise,} \end{cases}$$

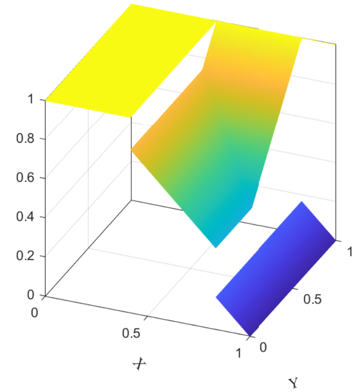
then $I = \left(\left\langle \frac{1}{3}, \frac{2}{3}, I_1 \right\rangle, I_*\right)$ is an ordinal sum implication, i. e.,

$$I(x,y) = \begin{cases} \frac{7}{12}, & (x,y) \in \left\{\frac{1}{2}\right\} \times \left[0, \frac{1}{4}\right], \\ \frac{1}{5}y + \frac{8}{15}, & (x,y) \in \left\{\frac{1}{2}\right\} \times \left(\frac{1}{4}, \frac{2}{3}\right], \\ y, & (x,y) \in \left\{\frac{1}{2}\right\} \times \left(\frac{2}{3}, 1\right), \\ \frac{2}{3}, & (x,y) \in \left[\frac{1}{3}, \frac{1}{2}\right) \times \left[0, \frac{2}{3}\right] \\ & \text{or } (x,y) \in \left(\frac{1}{2}, \frac{2}{3}\right] \times \left[\frac{2}{3}, 1\right), \\ \frac{1}{3}, & (x,y) \in \left(\frac{1}{2}, \frac{2}{3}\right] \times \left[0, \frac{2}{3}\right), \\ 0, & (x,y) \in \left(\frac{2}{3}, 1\right] \times [0,1), \\ 1, & \text{otherwise.} \end{cases}$$

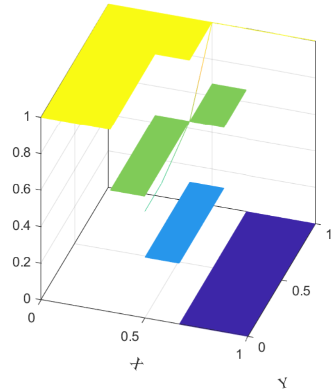
Let $\alpha = \frac{1}{2}$, then $N_I(\alpha) = \frac{7}{12}$ and $I(\alpha, N_I(\alpha)) > N_I(\alpha)$. And let $[u,v] = \left[\frac{1}{4}, \frac{2}{3}\right]$, then $S = \left(\left\langle \frac{1}{4}, \frac{2}{3}, S_P \right\rangle\right)$ is an ordinal sun t-conorm, obviously, we have that

$$I\left(\frac{1}{2}, y\right) = S\left(\frac{7}{12}, y\right) = \begin{cases} \frac{7}{12}, & y \in \left[0, \frac{1}{4}\right), \\ \frac{1}{5}y + \frac{8}{15}, & y \in \left[\frac{1}{4}, \frac{2}{3}\right], \\ y, & \text{otherwise.} \end{cases}$$

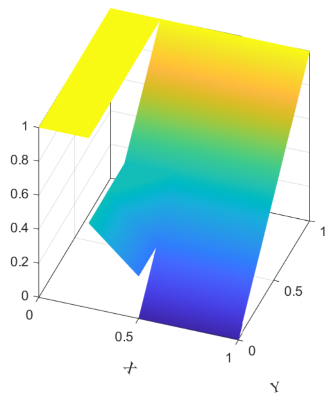
Then, by Theorem 3.7, S is α-migrative with respect to I.



Example 3.8 (i).



Example 3.8 (ii).



Example 3.8 (iii).

Fig. 1: Plots of ordinal sum implications I in Example 3.8.

(iii) Let $[a_0, b_0] = \left[\frac{1}{4}, \frac{1}{2}\right]$ and

$$I_*(x,y) = \begin{cases} 1, & (x,y) \in \left[0, \frac{1}{4}\right) \times [0,1], \\ \frac{3}{4} - x, & (x,y) \in \left[\frac{1}{4}, \frac{1}{2}\right] \times \left[0, \frac{1}{4}\right), \\ \frac{1}{2}, & (x,y) \in \left[\frac{1}{4}, \frac{1}{2}\right]^2, \\ y, & \text{otherwise}, \end{cases}$$

then $I = \left(\left\langle \frac{1}{4}, \frac{1}{2}, I_{LK} \right\rangle, I_*\right)$ is an ordinal sum implication, i. e.,

$$I(x,y) = \begin{cases} 1, & (x,y) \in \left[0, \frac{1}{4}\right) \times [0,1], \\ \frac{3}{4} - x, & (x,y) \in \left[\frac{1}{4}, \frac{1}{2}\right] \times \left[0, \frac{1}{4}\right), \\ \min\left(\frac{1}{2} - x + y, \frac{1}{2}\right), & (x,y) \in \left[\frac{1}{4}, \frac{1}{2}\right]^2, \\ y, & \text{otherwise}. \end{cases}$$

Let $\alpha \in \left(\frac{1}{4}, \frac{1}{2}\right)$, then $N_I(\alpha) = \frac{3}{4} - \alpha$ and $I(\alpha, N_I(\alpha)) > N_I(\alpha)$. And let $[u, v] = \left[\frac{1}{4}, \frac{1}{2}\right]$, then $S = \left(\left\langle \frac{1}{4}, \frac{1}{2}, S_{LK} \right\rangle\right)$ is an ordinal sun t-conorm, obviously, we have that

$$I(\alpha, y) = \begin{cases} \frac{3}{4} - \alpha, & y \in \left[0, \frac{1}{4}\right), \\ \min\left(\frac{1}{2} - \alpha + y, \frac{1}{2}\right), & y \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ y, & \text{otherwise}, \end{cases}$$
$$= S(N_I(\alpha), y).$$

Then, by Theorem 3.7, S is α-migrative with respect to I.

The plots of ordinal sum implications I in Example 3.8 are given in Fig. 1.

Furthermore, when $\alpha \in (a_{k_0}, b_{k_0})$ for some $k_0 \in \Lambda$, we continue to explore the case $N_I(\alpha) \in (0, \gamma_a]$ with $\gamma_a > 0$.

Theorem 3.9. Let $I = (\langle a_k, b_k, I_k \rangle, I_*)_{k \in \Lambda} \in \mathcal{I}_{\mathcal{OS}}$ be an ordinal sum implication and $S \in \mathcal{S}_{\mathcal{C}}$ a continuous t-conorm. If $\alpha \in (a_{k_0}, b_{k_0})$ for some $k_0 \in \Lambda$ with $N_I(\alpha) \in (0, \gamma_a)$ and $I(\alpha, N_I(\alpha)) = N_I(\alpha)$, then S is α-migrative with respect to I iff the following assertions hold:

(i) $S(N_I(\alpha), y) = \max(N_I(\alpha), y)$ for all $y \in [0, 1]$.

(ii) $I_*(\alpha, y) = \max(N_I(\alpha), y)$ for all $y \in [0, a_{k_0}) \cup (b_{k_0}, 1]$.

(iii) $I_{k_0}\left(\frac{\alpha - a_{k_0}}{b_{k_0} - a_{k_0}}, y\right) = y$ for all $y \in [0, 1]$.

Proof. The proof follows the same steps as for Theorem 3.6. □

Theorem 3.10. Let $I = (\langle a_k, b_k, I_k \rangle, I_*)_{k \in \Lambda} \in \mathcal{I}_{\mathcal{OS}}$ be an ordinal sum implication and $S \in \mathcal{S}_{\mathcal{C}}$ a continuous t-conorm. If $\alpha \in (a_{k_0}, b_{k_0})$ for some $k_0 \in \Lambda$ with $N_I(\alpha) \in (0, \gamma_a)$ and $I(\alpha, N_I(\alpha)) > N_I(\alpha)$, then S is α-migrative with respect to I iff $S = (\cdots, \langle u, v, S_2 \rangle, \cdots)$ with $N_I(\alpha) \in (u, v)$ and $S_2 \in \mathcal{S}_{\mathcal{CA}}$, where $u = \sup\{x \in [0, N_I(\alpha)) \mid S(x, x) = x\}$, $v = \inf\{x \in (N_I(\alpha), 1] \mid S(x, x) = x\} \leq b_{k_0}$, and one of the following assertions holds:

(i) If $v \leq a_{k_0}$, then $I_{k_0}\left(\frac{\alpha - a_{k_0}}{b_{k_0} - a_{k_0}}, y\right) = y$ for all $y \in [0, 1]$ and

$$I_*(\alpha, y) = \begin{cases} N_I(\alpha), & y \in [0, u), \\ u + (v - u) S_2\left(\frac{N_I(\alpha) - u}{v - u}, \frac{y - u}{v - u}\right), & y \in [u, v], \\ y, & y \in (v, a_{k_0}) \cup (b_{k_0}, 1], \end{cases} \tag{10}$$

(ii) If $a_{k_0} < v \leq b_{k_0}$, then

$$I_*(\alpha, y) = \begin{cases} N_I(\alpha), & y \in [0, u), \\ u + (v - u) S_2\left(\frac{N_I(\alpha) - u}{v - u}, \frac{y - u}{v - u}\right), & y \in [u, a_{k_0}), \\ y, & y \in (b_{k_0}, 1], \end{cases} \tag{11}$$

and

$$I_{k_0}\left(\frac{\alpha-a_{k_0}}{b_{k_0}-a_{k_0}},y\right)=\begin{cases}\frac{u-a_{k_0}}{b_{k_0}-a_{k_0}}+\frac{v-u}{b_{k_0}-a_{k_0}}S_2\left(p,q\right), & y\in\left[0,\frac{v-a_{k_0}}{b_{k_0}-a_{k_0}}\right],\\ y, & \text{otherwise},\end{cases} \tag{12}$$

where $p=\frac{N_I(\alpha)-u}{v-u}$ and $q=\frac{a_{k_0}+(b_{k_0}-a_{k_0})y-u}{v-u}$.

Proof. The proof follows the same steps as for Theorem 3.7. □

Remark 3.11. For an ordinal sum implication $I=(\langle a_k,b_k,I_k\rangle,I_*)_{k\in\Lambda}\in\mathcal{I}_{\mathcal{OS}}$, a continuous t-conorm $S\in\mathcal{S}_{\mathcal{C}}$ and $a_{k_0}=\gamma_a>0$ for some $k_0\in\Lambda$. For some $k_1\in\Lambda$, we then distinguish the following two cases to explore (α,I)-migrativity of S for $\alpha\in(a_{k_0},b_{k_0})\cup(a_{k_1},b_{k_1})$ with $N_I(\alpha)=a_{k_0}$.

- $\alpha\in(a_{k_0},b_{k_0})$. If $N_I(\alpha)=a_{k_0}$, then the equivalent characterizations for the (α,I)-migrativity of S can be obtained by Theorems 3.9 and 3.10.
- $\alpha\in(a_{k_1},b_{k_1})$. If $N_I(\alpha)=a_{k_0}$, then $I(\alpha,a_{k_0})=a_{k_0}$. By Theorem 3.9, it can be shown that S is α-migrative with respect to I iff $I(\alpha,y)=\max(a_{k_0},y)$ for all $y\in[0,1]$.

Remark 3.12. Let $I=(\langle a_k,b_k,I_k\rangle,I_*)_{k\in\Lambda}\in\mathcal{I}_{\mathcal{OS}}$ be an ordinal sum implication, $S\in\mathcal{S}_{\mathcal{C}}$ a continuous t-conorm and $\alpha\in(a_{k_0},b_{k_0})$ for some $k_0\in\Lambda$ with $N_I(\alpha)\in(a_{k_0},b_{k_0})$. If $I(\alpha,N_I(\alpha))=N_I(\alpha)$, then Theorem 3.6 coincides with [57, Theorem 3.7]. If $I(\alpha,N_I(\alpha))>N_I(\alpha)$ and $u\geq a_{k_0}$ in Theorem 3.7, then this conclusion reduces to [57, Theorem 3.8]. In particular, if $N_I(\alpha)=\alpha$, then Theorem 3.6 and Theorem 3.7 can be reduced to [43, Theorem 4.6] and [43, Theorem 4.8], respectively.

Corollary 3.13. Let $I=(\langle a_k,b_k,I_k\rangle,I_*)_{k\in\Lambda}\in\mathcal{I}_{\mathcal{OS}}$ be an ordinal sum implication and $S\in\mathcal{S}_{\mathcal{C}}$ a continuous t-conorm. If $\alpha\in(a_{k_0},b_{k_0})$ for some $k_0\in\Lambda$ with $N_I(\alpha)=b_{k_0}<1$ and $a_{k_0}=\gamma_a$, then S is α-migrative with respect to I iff the following assertions hold:

1) $S(b_{k_0},y)=\max(b_{k_0},y)$ for all $y\in[0,1]$.
2) $I_*(\alpha,y)=\max(b_{k_0},y)$ for all $y\in[0,a_{k_0})\cup(b_{k_0},1]$.
3) $I_{k_0}\left(\frac{\alpha-a_{k_0}}{b_{k_0}-a_{k_0}},y\right)=1$ for all $y\in[0,1]$.

In the following, we cogitate the case $\alpha=b_{k_0}<1$ for some $k_0\in\Lambda$ with $N_I(\alpha)\in(0,\gamma_a]$.

Theorem 3.14. Let $I=(\langle a_k,b_k,I_k\rangle,I_*)_{k\in\Lambda}\in\mathcal{I}_{\mathcal{OS}}$ be an ordinal sum implication, $S\in\mathcal{S}_{\mathcal{C}}$ a continuous t-conorm and $\alpha=b_{k_0}<1$ for some $k_0\in\Lambda$ with $\gamma_a>0$.

(i) If $N_I(b_{k_0}) \in (0, \gamma_a)$, the following two cases need to be considered:

(1) If $I(b_{k_0}, N_I(b_{k_0})) = N_I(b_{k_0})$, then S is b_{k_0}-migrative with respect to I iff the following assertions hold:

1) $S(N_I(b_{k_0}), y) = \max(N_I(b_{k_0}), y)$ for all $y \in [0,1]$.
2) $I_*(b_{k_0}, y) = \max(N_I(b_{k_0}), y)$ for all $y \in [0, a_{k_0}) \cup (b_{k_0}, 1]$.
3) $I_{k_0}(1, y) = y$ for all $y \in [0,1]$.

(2) If $I(b_{k_0}, N_I(b_{k_0})) > N_I(b_{k_0})$, then S is b_{k_0}-migrative with respect to I iff S can be expressed in the form of $S = (\cdots, \langle u, v, S_2\rangle, \cdots)$ with $S_2 \in \mathcal{S}_{\mathcal{CA}}$, where $u = \sup\{x \in [0, N_I(b_{k_0}))|S(x,x) = x\}$, $v = \inf\{x \in (N_I(b_{k_0}), 1]|S(x,x) = x\} \leq a_{k_0}$, $I_{k_0}(1, y) = y$ for all $y \in [0,1]$, and $I_*(b_{k_0}, \cdot)$ can be expressed on $[0, a_{k_0}) \cup (b_{k_0}, 1]$ in the following form:

$$I_*(b_{k_0}, y) = \begin{cases} u + (v-u)S_2\left(\frac{N_I(b_{k_0})-u}{v-u}, \frac{y-u}{v-u}\right), & y \in [u, v], \\ \max(N_I(b_{k_0}), y), & y \in [0,1] \setminus ([u, v] \cup [a_{k_0}, b_{k_0}]). \end{cases} \tag{13}$$

(ii) If $N_I(b_{k_0}) = \gamma_a$, then S is b_{k_0}-migrative with respect to I iff the following assertions hold:

1) $S(\gamma_a, y) = \max(\gamma_a, y)$ for all $y \in [0,1]$.
2) $I_*(b_{k_0}, y) = \max(\gamma_a, y)$ for all $y \in [0, a_{k_0}) \cup (b_{k_0}, 1]$.
3) $I_{k_0}(1, y) = y$ for all $y \in [0,1]$.

Proof. If $\alpha = b_{k_0} < 1$ for some $k_0 \in \Lambda$ and $\gamma_a = \min\{a_k | k \in \Lambda\} > 0$, then Theorem 2.8 indicates that $N_I(b_{k_0}) \leq \gamma_a$. The rest proof follows the same steps as for Theorems 3.9 and 3.10. □

Remark 3.15. Let $I = (\langle a_k, b_k, I_k\rangle, I_*)_{k\in\Lambda} \in \mathcal{I}_{\mathcal{OS}}$ be an ordinal sum implication, $S \in \mathcal{S}_{\mathcal{C}}$ a continuous t-conorm and $\alpha = b_{k_0}$ for some $k_0 \in \Lambda$. Obviously, if $N_I(b_{k_0}) = \gamma_a$, then the conclusion in Theorem 3.14 coincides with [57, Theorem 3.5 (2)].

Proposition 3.16. Let $I = (\langle a_k, b_k, I_k\rangle, I_*)_{k\in\Lambda} \in \mathcal{I}_{\mathcal{OS}}$ be an ordinal sum implication, and $S \in \mathcal{S}_{\mathcal{CA}}$ a continuous Archimedean t-conorm, then S is not α-migrative with respect to I for $\alpha \in [a_k, b_k]$ with $N_I(\alpha) \in (0, b_k]$, where $b_k < 1$.

Proof. The following proof is by contradiction. Assume that there exists $k_0 \in \Lambda$ such that $\alpha \in [a_{k_0}, b_{k_0}]$ with $b_{k_0} < 1$, and S is α-migrative with respect to I. Since S is Archimedean, then it follows from Proposition 3.2 that

$$b_{k_0} < S(N_I(\alpha), b_{k_0}) = I(\alpha, b_{k_0}) = a_{k_0} + (b_{k_0} - a_{k_0}) I_{k_0}\left(\frac{\alpha - a_{k_0}}{b_{k_0} - a_{k_0}}, \frac{b_{k_0} - a_{k_0}}{b_{k_0} - a_{k_0}}\right) = b_{k_0},$$

which is a contradiction. Therefore, S is not α-migrative with respect to I for $\alpha \in [a_k, b_k]$ with $N_I(\alpha) \in (0, b_k]$, where $b_k < 1$. □

As illustrated by Proposition 3.16, it can be demonstrated that for any $S \in \mathcal{S}_{\mathcal{CA}}$, S is not α-migrative with respect to the ordinal sum implication I under the conditions that $\alpha \in [a_k, b_k]$, where $N_I(\alpha) \in (0, b_k]$ and $b_k < 1$ for all $k \in \Lambda$.

Remark 3.17. Consider $\alpha \in \left[\frac{1}{4}, 1\right]$ and $S = S_P$. Let $I = \left(\left\langle \frac{1}{4}, 1, I_{RC} \right\rangle, I_*\right)$ be an ordinal sum implication, where

$$I_*(x,y) = \begin{cases} 1, & (x,y) \in \left[0, \frac{1}{4}\right] \times [0,1], \\ \frac{4}{3}\left(x - \frac{1}{4}\right)\left(y - \frac{1}{4}\right) - x + \frac{5}{4}, & \text{otherwise.} \end{cases}$$

Then

$$I(x,y) = \begin{cases} 1, & (x,y) \in \left[0, \frac{1}{4}\right] \times [0,1], \\ \frac{4}{3}\left(x - \frac{1}{4}\right)\left(y - \frac{1}{4}\right) - x + \frac{5}{4}, & \text{otherwise.} \end{cases}$$

It can be verified that S is α-migrative with respect to I, where t-conorm S is continuous Archimedean.

We will keep exploring the case $\alpha \notin \bigcup_{k \in \Lambda} [a_k, b_k]$.

Remark 3.18. For an ordinal sum implication $I = (\langle a_k, b_k, I_k \rangle, I_*)_{k \in \Lambda} \in \mathcal{I}_{\mathcal{OS}}$, a continuous t-conorm $S \in \mathcal{S}_{\mathcal{C}}$ and $\alpha \notin \bigcup_{k \in \Lambda} [a_k, b_k]$ with $N_I(\alpha) \in (0,1)$. For the sake of brevity, in this case, we stipulate $\gamma_a > 0$ and $\eta_b < 1$.

(i) If $\alpha > \eta_b$, then $N_I(\alpha) \in (0, \gamma_a]$. We consider the following two cases:

- $N_I(\alpha) \in (0, \gamma_a)$. Then S is α-migrative with respect to I iff $I_*(\alpha, y) = S(N_{I_*}(\alpha), y)$ for all $y \in [0,1]$. The equivalent characterizations for the (α, I)-migrativity of S can be obtained in [43].
- $N_I(\alpha) = \gamma_a$. Then S is α-migrative with respect to I iff $I_*(\alpha, y) = S(\gamma_a, y) = \max(\gamma_a, y)$ for all $y \in [0,1]$. In this case, for all $k \in \Lambda$, $I_k(b_k, y) \geq y$ for all $y \in [0,1]$. In particular, for every $\beta \in [\gamma_b, \alpha]$, S is β-migrative with respect to I iff $I(\beta, y) = S(\gamma_a, y) = \max(\gamma_a, y)$ for all $y \in [0,1]$ and $\gamma_b = \eta_b < 1$.

(ii) Suppose that there exists $k_2, k_3 \in \Lambda$ such that $a_{k_3} > b_{k_2}$ and $a_{k_i} \notin (b_{k_2}, a_{k_3})$ for all $k_i \in \Lambda$. If $\alpha \in (b_{k_2}, a_{k_3})$, then $N_I(\alpha) \in (0, \gamma_a]$. If $N_I(\alpha)$ is idempotent for S, then Remark 3.5 (iii) implies that Eq. (3) is invalid. In addition, if $N_I(\alpha)$ is not idempotent for S, then Proposition 2.2 implies that $S = (\cdots, \langle u, v, S_2 \rangle, \cdots)$ with $S_2 \in \mathcal{S}_{\mathcal{CA}}$ and $N_I(\alpha) \in (u, v)$. Suppose that S is α-migrative with respect to I, we need to consider the following three cases:

- $v \leq b_{k_2}$. Then $a_{k_3} = S(N_I(\alpha), a_{k_3}) = I(\alpha, a_{k_3}) \geq I(a_{k_3}, a_{k_3}) = b_{k_3}$, which is impossible.
- $v > b_{k_2}$. Then $b_{k_2} < S(N_I(\alpha), b_{k_2}) = I(\alpha, b_{k_2}) \leq I(b_{k_2}, b_{k_2}) = b_{k_2}$, which also is impossible.

Therefore, it can be concluded that S is not α-migrative with respect to I.

(iii) If $\alpha < \gamma_a$, then the following assertions hold.

- $N_I(\alpha)$ is idempotent for S. Then S is α-migrative with respect to I iff $N_I(\alpha) \in [\eta_b, 1)$ and $I_*(\alpha, y) = \max(N_I(\alpha), y)$ for all $y \in [0, 1]$.
- $N_I(\alpha)$ is not idempotent for S. Then, by Proposition 2.2, it implies that $S = (\cdots, \langle u, v, S_2\rangle, \cdots)$ with $S_2 \in \mathcal{S}_{\mathcal{CA}}$ and $N_I(\alpha) \in (u, v)$. Suppose that S is α-migrative with respect to I, then by Remark 3.5 (ii), it follows that $v \geq \eta_b$ and $S(N_I(\alpha), a_k) \geq b_k$ for all $k \in \Lambda$. In particular, we need to consider the following classification of $N_I(\alpha)$:
  1) If $N_I(\alpha) \in [a_{k_i}, b_{k_i})$ for some $k_i \in \Lambda$, combined with the definition of I, then we have $u < a_{k_i}$.
  2) If $N_I(\alpha) \in (0, \gamma_a) \cup (\eta_b, 1) \cup (\bigcup_{k_i \in \Lambda}\{b_{k_i}\})$, then $u \in [0, N_I(\alpha))$.
  3) Suppose that there exist $m_0, n_0 \in \mathbb{N}$ such that $b_{k_{m_0}} < a_{k_{n_0}}$ for some $k_{m_0}, k_{n_0} \in \Lambda$. If $N_I(\alpha) \in \left(b_{k_{m_0}}, a_{k_{n_0}}\right)$ and $a_{k_i} \notin \left(b_{k_{m_0}}, a_{k_{n_0}}\right)$ for all $k_i \in \Lambda$, then $u \in [0, N_I(\alpha))$.

Therefore, combined with the ordinal sum representation of S, the equivalent characterizations for the (α, I)-migrativity of S can be obtained.

Drawing upon the insights gained from our prior analysis, we gave the equivalent characterizations of migrativity on continuous t-conorms with respect to ordinal sum implications with $N_I(\alpha) \in (0, 1)$. In the following, we will turn our attention to a study of $N_I(\alpha) \in \{0, 1\}$.

Theorem 3.19. Let $I = (\langle a_k, b_k, I_k\rangle, I_*)_{k\in\Lambda} \in \mathcal{I}_{\mathcal{OS}}$ be an ordinal sum implication, S a t-conorm and $\alpha \in [0, 1]$ with $N_I(\alpha) = 0$, then the following assertions hold.

(i) If $\alpha > \eta_b$ with $\eta_b < 1$, then S is α-migrative with respect to I iff $I_*(\alpha, y) = y$ and $I_k(1, y) \geq y$ for all $y \in [0, 1]$.

(ii) If $\alpha \in (a_{k_0}, b_{k_0}]$ for some $k_0 \in \Lambda$, then S is α-migrative with respect to I iff the following assertions hold:

(1) $I_*(\alpha, y) = y$ for all $y \in [0, 1] \setminus [a_{k_0}, b_{k_0}]$.

(2) $I_{k_0}\left(\frac{\alpha - a_{k_0}}{b_{k_0} - a_{k_0}}, y\right) = y$ for all $y \in [0, 1]$.

(3) $b_{k_0} = \eta_b$.

(iii) If $\alpha = a_{k_0} > 0$ for some $k_0 \in \Lambda$, then S is not α-migrative with respect to I.

(iv) If $\alpha < \gamma_a$ with $\gamma_a > 0$ and S is α-migrative with respect to I, then $\Lambda = \emptyset$. In particular, if $\Lambda = \emptyset$ and $N_{I_*} = N_0$, then S is migrative with respect to I iff $I = I_* = I_D$.

(v) If there exists $k_0, k_1 \in \Lambda$ such that $\alpha \in (b_{k_0}, a_{k_1})$ and $(b_{k_0}, a_{k_1}) \cap \left(\bigcup_{k_i \in \Lambda}[a_{k_i}, b_{k_i}]\right) = \emptyset$ for all $k_i \in \Lambda$, then S is not α-migrative with respect to I.

Proof.

(i) Suppose that $\alpha > \eta_b$ with $\eta_b < 1$, and S is α-migrative with respect to I with $N_I(\alpha) = 0$, then $I_*(\alpha, y) = I(\alpha, y) = S(N_I(\alpha), y) = S(0, y) = y$ for all $y \in [0,1]$. Moreover, for all $k \in \Lambda$, $I(b_k, y) \geq y$ for all $y \in [0,1]$, i.e., $I_k(1, y) \geq y$ for all $y \in [0,1]$. Conversely, if $I_*(\alpha, y) = y$ for all $y \in [0,1]$, then we have $I(\alpha, y) = I_*(\alpha, y) = y = S(0, y) = S(N_I(\alpha), y)$ for all $y \in [0,1]$, which implies that S is α-migrative with respect to I.

(ii) Let $\alpha \in (a_{k_0}, b_{k_0}]$ for some $k_0 \in \Lambda$, suppose that S is α-migrative with respect to I, then by Proposition 3.2 (iii), we have $I(\alpha, y) = y$ for all $y \in [0,1]$, thus, $I_*(\alpha, y) = y$ for all $y \in [0,1] \backslash [a_{k_0}, b_{k_0}]$ and $I_{k_0}\left(\frac{\alpha - a_{k_0}}{b_{k_0} - a_{k_0}}, y\right) = y$ for all $y \in [0,1]$. And, from Remark 3.5, it follows that item (ii)-(3) is valid. Conversely, if $I_*(\alpha, y) = y$ for all $y \in [0,1] \backslash [a_{k_0}, b_{k_0}]$ and $I_{k_0}\left(\frac{\alpha - a_{k_0}}{b_{k_0} - a_{k_0}}, y\right) = y$ for all $y \in [0,1]$, then $I(\alpha, y) = y = S(N_I(\alpha), y)$ for all $y \in [0,1]$, which implies that S is α-migrative with respect to I.

(iii) Suppose that $\alpha = a_{k_0} > 0$ for some $k_0 \in \Lambda$, and S is α-migrative over I, then $b_{k_0} = I(a_{k_0}, a_{k_0}) = S(N_I(a_{k_0}), a_{k_0}) = a_{k_0}$, which is a contradiction. Hence, S is not α-migrative with respect to I.

(iv) Suppose that there exists $k_0 \in \Lambda$ such that $\Lambda \neq \emptyset$, which implies that $[a_{k_0}, b_{k_0}] \cap [0,1] \neq \emptyset$, then $I(a_{k_0}, y) = b_{k_0}$ for all $y \in [a_{k_0}, b_{k_0}]$. Since S is α-migrative with respect to I with $\alpha < \gamma_a$, then $I(\alpha, y) = y$ for all $y \in [0,1]$. Thus it can be seen that $I(\alpha, y) = y < b_{k_0} = I(a_{k_0}, y)$ for all $y \in (a_{k_0}, b_{k_0})$, which is invalid. Hence, $\Lambda = \emptyset$.

In particular, if $\Lambda = \emptyset$ and $N_{I_*} = N_0$, then $I = I_*$. Suppose that S is α-migrative with respect to I for every $\alpha \in (0,1]$, then $I(x, y) = I_*(x, y) = y$ for all $(x, y) \in (0,1] \times [0,1]$, and combined with the definition of I, we have $I = I_* = I_D$. On the contrary, it can be easily proved.

(v) The proof follows the same steps as for item (iv).

□

Remark 3.20. Let $I = (\langle a_k, b_k, I_k \rangle, I_*)_{k \in \Lambda} \in \mathcal{I}_{\mathcal{OS}}$ be an ordinal sum implication and S a t-conorm, then the following assertions hold.

(i) If there exists $k_0 \in \Lambda$ such that $b_{k_0} = 1$ and $a_{k_0} > 0$ with $N_I(a_{k_0}) = 0$, then for every $\alpha \in (a_{k_0}, 1]$, S is α-migrative over I iff $I_{k_0} = I_D$ and $I_*(x, y) = y$ for all $x \in (a_{k_0}, 1]$ and $y \in [0, a_{k_0})$.

(ii) If $N_I(b_{k_0}) = 0$ with $b_{k_0} = \eta_b$ for some $k_0 \in \Lambda$, then for every $\alpha \in (b_{k_0}, 1]$, S is α-migrative with respect to I iff $I(x, y) = I_*(x, y) = y$ for all $x \in (b_{k_0}, 1]$ and $y \in [0,1]$.

Proposition 3.21. Let $I = (\langle a_k, b_k, I_k\rangle, I_*)_{k\in\Lambda} \in \mathcal{I}_{\mathcal{OS}}$ be an ordinal sum implication, $S \in \mathcal{S}$ a t-conorm and $\alpha \in [0,1]$ with $N_I(\alpha) = 1$, then the following assertions hold.

(i) If $\alpha = \gamma_a$, then for every $\beta \in [0, \alpha]$, S is β-migrative with respect to I.

(ii) If there exists $\alpha \in [a_{k_0}, b_{k_0})$ such that $N_I(\alpha) = 1$ for some $k_0 \in \Lambda$ with $a_{k_0} > 0$, then for every $\beta \in [0, \alpha]$, S is β-migrative with respect to I. In particular, if $N_I = N^1$, then S is migrative with respect to I iff $I = I_{WB}$.

Proof.

(i) Suppose that $\alpha = \gamma_a$, which implies that $I(x, y) = I_*(x, y) = 1$ for all $x \in [0, \alpha]$ and $y \in [0, 1]$, then by Proposition 3.2 (iii), one has that $I(\beta, y) = I_*(\beta, y) = 1 = S(N_I(\beta), y)$ for all $\beta \in [0, \alpha]$ and $y \in [0, 1]$, i.e., for every $\beta \in [0, \alpha]$, S is β-migrative with respect to I.

(ii) Suppose that there exists $\alpha \in [a_{k_0}, b_{k_0})$ such that $N_I(\alpha) = 1$ for some $k_0 \in \Lambda$ with $a_{k_0} > 0$, which implies that $b_{k_0} = 1$ and $I(x, y) = 1$ for all $x \in [0, \alpha]$ and $y \in [0, 1]$, then from the proof of item (i), it follows that for every $\beta \in [0, \alpha]$, S is β-migrative with respect to I. In particular, if $N_I = N^1$, one obtains that $\Lambda = \emptyset$ or there exists $k_0 \in \Lambda$ such that $I_{k_0}(x, y) = 1$ for all $x \in [0, 1)$ and $y \in [0, 1]$ with $b_{k_0} = 1$, and combined with Proposition 3.2 (2), then S is migrative with respect to I iff $I = I_{WB}$.

□

4. CONCLUDING REMARKS

In this paper, we characterized the migrativity of t-conorms with respect to ordinal sum implications depending on the position of α, and gave the structures of the solution of the migrative functional equation for ordinal sum decompositions of the underlying functions. For further investigations, we will focus on the study on migrativity of uninorms with respect to ordinal sum implications.

DECLARATION OF COMPETING INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Xinxin Yan, School of Mathematics and Statistics, Shaanxi Normal University, Xi'an 710119. P. R. China.
e-mail: xxy06071226@163.com

Hongjun Zhou, Corresponding author. School of Mathematics and Statistics, Shaanxi Normal University, Xi'an 710119. P. R. China.
e-mail: hjzhou@snnu.edu.cn