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SOLUTIONS OF QUASI-HOMOGENEITY EQUATION OF AGGREGATION FUNCTIONS

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The homogeneous functions play an important role in image processing, decision making and other relevant fields. In this article, we first give solutions of quasi-homogeneity equation of aggregation functions completely, and then we introduce the concept of triple generator of a quasi-homogeneous aggregation function, which is applied to construct a quasi-homogeneous aggregation function.

Keywords: aggregation function, quasi-homogeneity, triple generator

Classification: 03B52, 03E72

1. INTRODUCTION

In image processing, decision making and other relevant fields, the homogeneous functions play an important role, see, e. g., [1, 3, 4]. Thus, it is very valuable to study them from the theoretical point of view. A function $A : [0,1]^2 \to [0,1]$ is said to be homogeneous of order $k > 0$ if it satisfies $A(\lambda x, \lambda y) = \lambda^k A(x,y)$ for all $x, y, \lambda \in [0,1]$. So far, there are many articles for investigating the homogeneity of particular aggregation functions. For example, Rückschlossová presented a complete characterization of homogeneous aggregation functions [7]. On the other hand, a more relaxed homogeneity, called a quasi-homogeneity, was introduced by Ebanks in [2]. Quasi-homogeneous t-norms are defined by $T(\lambda x, \lambda y) = \varphi^{-1}(\psi(\lambda)\varphi(T(x,y)))$ for all $x, y, \lambda \in [0,1]$ where $\psi : [0,1] \to [0,1]$ is a function and $\varphi : [0,1] \to [0,\infty)$ is a continuous injection and T is a triangular norm [2]. Just replacing the triangular norm T by a copula, quasi-homogeneous copulas were similarly defined by Mayor, Mesiar and Torrens [6]. Su, Zong and Mesiar [8] investigated the characterizations of homogeneous and quasi-homogeneous aggregation functions, respectively. Wang and Zhu [10] studied the pseudo-homogeneous overlap and grouping functions. One can easily see that all the results of [8, 10] are suitable for an aggregation function whose diagonal function is continuous. Recently, Su, Zong, Mesiar and Halaš [9] improved the results of [8]. This article will focuses on a more general discussion on solutions of quasi-homogeneity equation of aggregation functions, which not only provides a clear classification of quasi-homogeneous aggregation functions, but also gives the selection of parameter functions and their impact on the solutions of quasi-

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homogeneity equation of aggregation functions. We also provide a way to construct quasi-homogeneous binary aggregation functions by using single-argument functions.

The rest of this article are organized as follows. In Section 2, we completely provide solutions of quasi-homogeneity equation of aggregation functions, which show a clear classification of quasi-homogeneous aggregation functions. In Section 3, we introduce the concept of triple generator of a quasi-homogeneous aggregation function, which is applied to construct a quasi-homogeneous aggregation function. A conclusion is drawn in Section 4.

2. SOLUTIONS OF QUASI-HOMOGENEITY EQUATION OF AGGREGATION FUNCTIONS

This section is devoted to study solutions of quasi-homogeneity equations of aggregation functions.

We first review two important results that are necessary in the following discussion.

Lemma 2.1. (see Kuczma [5], Theorem 13.1.6) Let D be one of the sets $(0,1), [0,1)$, $(-1,1)$, $(-1,0)\cup(0,1), (1,\infty), (0,\infty), [0,\infty), (\infty,0)\cup(0,\infty), R$. A function $f : D \to R$ is a continuous solution of the multiplicative Cauchy equation $f(xy) = f(x)f(y)$ if and only if either $f = 0$, or $f = 1$, or for any $x \in D$, f has one of the following forms: $f(x) = |x|^c$, $f(x) = |x|^c sgn(x)$ with a certain $c \in R$. And, if $0 \in D$, then $c > 0$.

Lemma 2.2. (see Kuczma [5], Theorem 13.1.9) Let D be one of the sets $(0,1), [0,1)$, $(-1,1)$, $(-1,0)\cup(0,1), (1,\infty), (0,\infty), [0,\infty), (\infty,0)\cup(0,\infty), R$, and let $f : D \to R$ is a solution of the multiplicative Cauchy equation $f(xy) = f(x)f(y)$. If f is measurable, then it is continuous in $D\backslash\{0\}$.

Moreover, recall that a function $A : [0,1]^2 \to [0,1]$ is called an *aggregation function* if it is increasing and satisfies the boundary conditions $A(0,0) = 0$ and $A(1,1) = 1$. A function $\delta_A : [0,1] \to [0,1]$ with $\delta_A(x) = A(x,x)$ is called a *diagonal function* of A, in symbols δ_A.

Definition 2.3. (Su et al. [8]) An aggregation function $A : [0,1]^2 \to [0,1]$ is said to be (φ,ψ)-quasi-homogeneous if, for all $x, y, \lambda \in [0,1]$,

$$A(\lambda x, \lambda y) = \varphi^{-1}(\psi(\lambda)\varphi(A(x,y)))$$

where $\psi : [0,1] \to [0,1]$ is an arbitrary function and $\varphi : [0,1] \to [0,\infty)$ is a continuous injection.

It is well known that a continuous injection maps a compact set to a compact set. Therefore, to ensure the existence of the inverse function φ^{-1} of function φ, we should reconsider the mapping φ since $[0,\infty)$ is not compact. This leads us to introduce the definition of a quasi-homogeneous aggregation function as follows.

Definition 2.4. (Zhu and Wang [11]) An aggregation function $A : [0,1]^2 \to [0,1]$ is said to be (φ,ψ)-quasi-homogeneous (quasi-homogeneous for short) if, for all $x, y, \lambda \in [0,1]$,

$$A(\lambda x, \lambda y) = \varphi^{-1}(\psi(\lambda)\varphi(A(x,y))) \tag{1}$$

where $\psi : [0,1] \to [0,1]$ is an arbitrary function and $\varphi : [0,1] \to [0,b]$ is a continuous bijection with $[0,b] \subseteq [0,\infty]$.

Lemma 2.5. If an aggregation function $A : [0,1]^2 \to [0,1]$ is (φ,ψ)-quasi-homogeneous, then

(i) $\varphi(1) \neq 0$;

(ii) $\varphi(0) = 0$;

(iii) φ is an increasing bijection.

Proof. (*i*) Suppose $\varphi(1) = 0$. Then by Eq. (1),

$$A(\lambda,\lambda) = \varphi^{-1}(\psi(\lambda)\varphi(A(1,1))) = \varphi^{-1}(0) = 1$$

for all $\lambda \in [0,1]$, which is impossible. Therefore, $\varphi(1) \neq 0$.

(*ii*) By Eq. (1),

$$\delta_A(\lambda x) = A(\lambda x,\lambda x) = \varphi^{-1}(\psi(\lambda)\varphi(A(x,x))) = \varphi^{-1}(\psi(\lambda)\varphi(\delta_A(x)))$$

for all $x,\lambda \in [0,1]$, i.e., $\delta_A(\lambda x) = \varphi^{-1}(\psi(\lambda)\varphi(\delta_A(x)))$, or equivalently,

$$\varphi(\delta_A(\lambda x)) = \psi(\lambda)\varphi(\delta_A(x)) \tag{2}$$

for all $x,\lambda \in [0,1]$. Considering $x = 0$ in Eq. (2), we have

$$\varphi(0) = \psi(\lambda)\varphi(0) \text{ for all } \lambda \in [0,1],$$

or equivalently,

$$\varphi(0)(1-\psi(\lambda)) = 0 \text{ for all } \lambda \in [0,1],$$

then

$$\varphi(0) = 0.$$

Otherwise, $\psi(\lambda) = 1$ for all $\lambda \in [0,1]$. In this case, considering $\lambda = 0$ and Eq. (1), we have

$$A(0,0) = A(x,y) \text{ for all } x,y \in [0,1],$$

a contradiction. Thus

$$\varphi(0) = 0.$$

(*iii*) Obviously, from (*i*) and (*ii*) we have that φ is an increasing bijection. □

By Lemma 2.5, if $A : [0,1]^2 \to [0,1]$ is a (φ,ψ)-quasi-homogeneous aggregation function then $\varphi(1) = b$. In what follows, we shall distinguish b by two cases $0 < b < \infty$ and $b = \infty$, respectively.

2.1. The case $0 < b < \infty$

Lemma 2.6. If an aggregation function $A : [0,1]^2 \to [0,1]$ is (φ, ψ)-quasi-homogeneous, then

(i) ψ is increasing;

(ii) $\psi(\lambda x) = \psi(\lambda)\psi(x)$ for all $\lambda, x \in [0,1]$;

(iii) $\psi(0) = 0$, $\psi(1) = 1$;

(iv) ψ is continuous on $(0,1)$.

Proof. (*i*) Considering $x = 1$ in Eq. (2), then

$$\psi(\lambda) = \frac{1}{\varphi(1)}\varphi(\delta_A(\lambda)) \text{ for all } \lambda \in [0,1]. \tag{3}$$

Lemma 2.5 and Eq. (3) imply that ψ is increasing.

(*ii*) By Eqs. (2) and (3), we have

$$\psi(\lambda x) = \psi(\lambda)\psi(x) \text{ for all } \lambda, x \in [0,1]. \tag{4}$$

(*iii*) By Eq. (4), we have

$$\psi(0) = \psi(0)\psi(0) \tag{5}$$

and

$$\psi(1) = \psi(1)\psi(1). \tag{6}$$

Eq. (5) implies $\psi(0) = 0$ or $\psi(0) = 1$, and Eq. (6) means $\psi(1) = 0$ or $\psi(1) = 1$. Then, $0 \le \psi(\lambda) \le 1$ for all $\lambda \in [0,1]$ since ψ is increasing. If $\psi(0) = 1$, then $\psi(\lambda) \equiv 1$ for all $\lambda \in [0,1]$ since ψ is increasing. In this case, $A(0,0) = \varphi^{-1}(\psi(0)\varphi(A(1,1))) = 1$, a contradiction. Therefore, $\psi(0) = 0$. Similarly, we can get $\psi(1) = 1$.

(*iv*) It follows immediately from (*i*), (*ii*) and Lemma 2.2 □

Lemma 2.7. If an aggregation function $A : [0,1]^2 \to [0,1]$ is (φ, ψ)-quasi-homogeneous, then one of the following statements hold:

(i) $\psi(x) = x^c$ for a certain $c > 0$.

(ii)
$$\psi(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1. \end{cases}$$

(iii)
$$\psi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in (0,1]. \end{cases}$$

P r o o f. By Lemmas 2.1 and 2.6, we have that for any $x \in (0,1)$, $\psi(x) = 0$ or $\psi(x) = 1$ or, $\psi(x)$ has one of the following forms: $\psi(x) = |x|^c$, $\psi(x) = |x|^c sgn(x)$, with a certain $c \in R$. Next, we investigate the function ψ by distinguishing three cases.

(i) If for any $x \in (0,1)$, $\psi(x)$ has one of the following forms: $\psi(x) = |x|^c$, $\psi(x) = |x|^c sgn(x)$, with a certain $c \in R$, then by Lemma 2.6, $\psi(x) = x^c$ with a certain $c > 0$.

(ii) If $\psi(x) = 0$ for any $x \in (0,1)$, then, by (iii) and (iv) of Lemma 2.6,

$$\psi(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1. \end{cases}$$

(iii) If $\psi(x) = 1$ for any $x \in (0,1)$, then, by (iii) and (iv) of Lemma 2.6, we have

$$\psi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in (0,1]. \end{cases}$$

□

Lemma 2.8. If an aggregation function $A : [0,1]^2 \to [0,1]$ is (φ,ψ)-quasi-homogeneous with $\psi(x) = x^c$ for a certain $c > 0$, then the following statements hold:

(i) δ_A is an increasing bijection.

(ii) $\varphi(x) = \varphi(1)(\delta_A^{-1}(x))^c$.

P r o o f. (i) From Eq. (1) we have for any $x \in [0,1]$,

$$A(x,x) = \varphi^{-1}(x^c\varphi(A(1,1))) = \varphi^{-1}(x^c\varphi(1)) \text{ for a certain } c > 0,$$

then for any $x \in [0,1]$,

$$\delta_A(x) = \varphi^{-1}(x^c\varphi(1)) \text{ for a certain } c > 0. \tag{7}$$

Obviously, δ_A is an increasing bijection since φ is an increasing bijection.

(ii) Using Eq. (7), we obtain that $\varphi(x) = \varphi(1)(\delta_A^{-1}(x))^c$. □

Theorem 2.9. An aggregation function $A : [0,1]^2 \to [0,1]$ is (φ,ψ)-quasi-homogeneous with $\psi(x) = x^c$ and $\varphi(x) = \varphi(1)(\delta_A^{-1}(x))^c$ for a certain $c > 0$ if and only if it is $(\widetilde{\varphi},\widetilde{\psi})$-quasi-homogeneous with $\widetilde{\psi}(x) = x^\alpha$ and $\widetilde{\varphi}(x) = (\delta_A^{-1}(x))^\alpha$ for any $\alpha > 0$.

P r o o f. Indeed, for any $x, y \in [0,1]$,

$$\varphi^{-1}(\psi(\lambda)\varphi(A(x,y))) = \delta_A\Big(\Big(\frac{\lambda^c\varphi(1)(\delta_A^{-1}(A(x,y)))^c}{\varphi(1)}\Big)^{\frac{1}{c}}\Big) = \delta_A(\lambda\delta_A^{-1}(A(x,y)))$$

and

$$\widetilde{\varphi}^{-1}(\widetilde{\psi}(\lambda)\widetilde{\varphi}(A(x,y))) = \delta_A((\lambda^\alpha(\delta_A^{-1}(A(x,y)))^\alpha)^{\frac{1}{\alpha}}) = \delta_A(\lambda\delta_A^{-1}(A(x,y))),$$

i. e., $\varphi^{-1}(\psi(\lambda)\varphi(A(x,y))) = \widetilde{\varphi}^{-1}(\widetilde{\psi}(\lambda)\widetilde{\varphi}(A(x,y)))$. Therefore, $A : [0,1]^2 \to [0,1]$ is (φ,ψ)-quasi-homogeneous with $\psi(x) = x^c$ and $\varphi(x) = \varphi(1)(\delta_A^{-1}(x))^c$ for a certain $c > 0$ if and only if it is $(\widetilde{\varphi},\widetilde{\psi})$-quasi-homogeneous with $\widetilde{\psi}(x) = x^\alpha$ and $\widetilde{\varphi}(x) = (\delta_A^{-1}(x))^\alpha$ for any $\alpha > 0$. □

Notice that by Theorem 2.9 and Lemma 2.8, a $(\widetilde{\varphi},\widetilde{\psi})$-quasi-homogeneous aggregation function is equivalent to a (φ,ψ)-quasi-homogeneous one, thus we can use the simplification version $(x, \delta_A^{-1}(x))$-quasi-homogeneous instead of $(x^c, \varphi(1)(\delta_A^{-1}(x))^c)$-quasi-homogeneous since α is any real number that is greater than 0. Therefore, in what follows, when we discuss the (φ,ψ)-quasi-homogeneity of aggregation function A with $\psi(x) = x^c$ and $\varphi(x) = \varphi(1)(\delta_A^{-1}(x))^c$ for a certain $c > 0$, we can always presuppose $\psi(x) = x$ and $\varphi(x) = \delta_A^{-1}(x)$ for any $x \in [0,1]$.

Theorem 2.10. An aggregation function $A : [0,1]^2 \to [0,1]$ is (φ,ψ)-quasi-homogeneous if and only if one of the following statements holds:

(i) The diagonal δ_A is an increasing bijection and there exist two increasing functions $h, g : [0,1] \to [0,1]$ fulfilling that $h(1) = g(1) = 1$, $\frac{\delta_A^{-1}(h(x))}{x}$ and $\frac{\delta_A^{-1}(g(x))}{x}$ are decreasing on $(0,1]$ such that for all $x, y \in [0,1]$

$$A(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ \delta_A(y\delta_A^{-1}(h(\frac{x}{y}))) & \text{if } x \le y \text{ and } y \ne 0, \\ \delta_A(x\delta_A^{-1}(g(\frac{y}{x}))) & \text{if } y \le x \text{ and } x \ne 0, \end{cases} \tag{8}$$

$\psi(x) = x$ and $\varphi(x) = \delta_A^{-1}(x)$. Moreover, in this case, the quasi-homogeneous aggregation function A is continuous.

(ii) There exist two constants $\alpha, \beta \in [0,1]$ such that for all $x, y \in [0,1]$

$$A(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ 1 & \text{if } x,y \in (0,1], \\ \alpha & \text{if } x = 0, y \in (0,1], \\ \beta & \text{if } y = 0, x \in (0,1], \end{cases} \tag{9}$$

$\psi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in (0,1] \end{cases}$ and $\varphi : [0,1] \to [0,b]$ is an increasing bijection.

(iii) There exist two increasing functions $h, g : [0,1] \to [0,1]$ fulfilling that $h(1) = g(1) = 1$ such that for all $x, y \in [0,1]$

$$A(x,y) = \begin{cases} 1 & \text{if } (x,y) = (1,1), \\ 0 & \text{if } x,y \in [0,1), \\ g(y) & \text{if } x = 1, y \in [0,1), \\ h(x) & \text{if } y = 1, x \in [0,1), \end{cases} \tag{10}$$

$\psi(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1 \end{cases}$ and $\varphi : [0,1] \to [0,b]$ is an increasing bijection.

P r o o f. Suppose that the aggregation function A is (φ, ψ)-quasi-homogeneous. Then, by Lemmas 2.6, 2.7 and Theorem 2.9, we have $\psi(x) = x, \varphi(x) = \delta_A^{-1}(x)$ or

$$\psi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in (0,1], \end{cases} \quad \text{or } \psi(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1. \end{cases}$$

Next, we investigate the representation of quasi-homogeneous aggregation function A by distinguishing three cases.

Case 1. If $\psi(x) = x$ and $\varphi(x) = \delta_A^{-1}(x)$ for any $x \in [0,1]$, then from Eq.(1),

$$\begin{aligned} A(x,y) &= \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ A(y \cdot \frac{x}{y}, y \cdot 1) & \text{if } x \le y \text{ and } y \neq 0, \\ A(x \cdot 1, x \cdot \frac{y}{x}) & \text{if } y \le x \text{ and } x \neq 0 \end{cases} \\ &= \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ \delta_A(y\delta_A^{-1}(A(\frac{x}{y}, 1))) & \text{if } x \le y \text{ and } y \neq 0, \\ \delta_A(x\delta_A^{-1}(A(1, \frac{y}{x}))) & \text{if } y \le x \text{ and } x \neq 0. \end{cases} \end{aligned}$$

Consider the functions $g, h : [0,1] \to [0,1]$ defined by $g(x) = A(1,x)$ and $h(x) = A(x,1)$, respectively. Then $g(1) = h(1) = 1$, g and h are increasing. Moreover,

$$A(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ \delta_A(y\delta_A^{-1}(h(\frac{x}{y}))) & \text{if } x \le y \text{ and } y \neq 0, \\ \delta_A(x\delta_A^{-1}(g(\frac{y}{x}))) & \text{if } y \le x \text{ and } x \neq 0. \end{cases}$$

Now, we show that $\frac{\delta_A^{-1}(h(x))}{x}$ is decreasing for all $x \in (0,1]$. Suppose $\frac{\delta_A^{-1}(h(x))}{x}$ is not decreasing for all $x \in (0,1]$. Then there exist $0 < y_1 < z_1 \le 1$ such that $\frac{\delta_A^{-1}(h(y_1))}{y_1} < \frac{\delta_A^{-1}(h(z_1))}{z_1}$. Let $0 < x \le y < z \le 1$ with $\frac{x}{y} = z_1$ and $\frac{x}{z} = y_1$. Then one has that $\frac{\delta_A^{-1}(h(\frac{x}{z}))}{\frac{x}{z}} < \frac{\delta_A^{-1}(h(\frac{x}{y}))}{\frac{x}{y}}$, which leads to $x\frac{\delta_A^{-1}(h(\frac{x}{z}))}{\frac{x}{z}} < x\frac{\delta_A^{-1}(h(\frac{x}{y}))}{\frac{x}{y}}$. Thus $z\delta_A^{-1}(h(\frac{x}{z})) < y\delta_A^{-1}(h(\frac{x}{y}))$. So, for $0 < x \le y < z \le 1$, we have that

$$z\delta_A^{-1}(A(\frac{x}{z}, 1)) < y\delta_A^{-1}(A(\frac{x}{y}, 1)).$$

This follows that $\delta_A(z\delta_A^{-1}(A(\frac{x}{z}, 1))) < \delta_A(y\delta_A^{-1}(A(\frac{x}{y}, 1)))$, i.e., $A(x,z) < A(x,y)$, a contradiction. Therefore, $\frac{\delta_A^{-1}(h(x))}{x}$ is decreasing for all $x \in (0,1]$. Similarly, we can prove that $\frac{\delta_A^{-1}(g(x))}{x}$ is decreasing for all $x \in (0,1]$.

From the preceding proof, in order to prove the continuity of the quasi-homogeneous aggregation function A it is enough to verify that both functions g and h are continuous, respectively. Next, we just prove that the function g is continuous, the continuity of the function h being analogous.

We first prove that the function g is left-continuous. In fact, let $x_0 \in (0,1]$. Then for any $0 < x \le x_0$ there always exists a $\lambda \in [0,1]$ such that $0 \le \lambda x_0 \le x \le x_0$. Thus

$$A(\lambda, \lambda x_0) \le A(1, x) \le A(1, x_0).$$

On the other hand,

$$\lim_{\lambda \nearrow 1} A(\lambda, \lambda x_0) = \lim_{\lambda \nearrow 1} \delta_A(\lambda \delta_A^{-1}(A(1, x_0))) = A(1, x_0).$$

Thus

$$g(x_0^-) = \lim_{x \nearrow x_0} g(x) = \lim_{x \nearrow x_0} A(1, x) = \lim_{\lambda \nearrow 1} A(\lambda, \lambda x_0) = A(1, x_0) = g(x_0),$$

i. e., g is left-continuous.

Now, we prove that the function g is right-continuous. Indeed, from the monotonicity of the aggregation function A we have that

$$\lim_{x \searrow x_0} g(x) \geq g(x_0)$$

for any $x_0 \in (0,1]$ and $x_0 < x < 1$. If $\lim_{x \searrow x_0} g(x) > g(x_0)$, i. e., $g(x_0^+) > g(x_0)$,then for any $\kappa \in [0,1]$,

$$\begin{aligned}
\lim_{x \searrow x_0} A(\kappa, \kappa x) &= \lim_{x \searrow x_0} \delta_A(\kappa \delta_A^{-1}(A(1, x))) \\
&= \lim_{x \searrow x_0} \delta_A(\kappa \delta_A^{-1}(g(x))) \\
&= \delta_A(\kappa \delta_A^{-1}(g(x_0^+))) \\
&> \delta_A(\kappa \delta_A^{-1}(g(x_0))) \\
&= \delta_A(\kappa \delta_A^{-1}(A(1, x_0))) \\
&= A(\kappa, \kappa x_0).
\end{aligned} \tag{11}$$

In particular, let $\lambda_0 \in (0,1)$. Then from Eq.(11) we have $\lim_{x \searrow x_0} A(\lambda_0, \lambda_0 x) > A(\lambda_0, \lambda_0 x_0)$, i. e.

$$\delta_A(\lambda_0 \delta_A^{-1}(g(x_0^+))) > \delta_A(\lambda_0 \delta_A^{-1}(g(x_0))). \tag{12}$$

Thus for any $0 < \lambda < \lambda_0 < 1$,

$$\begin{aligned}
A(\lambda, \lambda_0 x_0) &\geq \lim_{x \searrow x_0} A(\lambda, \lambda x) \\
&= \lim_{x \searrow x_0} \delta_A(\lambda \delta_A^{-1}(A(1, x))) \\
&= \lim_{x \searrow x_0} \delta_A(\lambda \delta_A^{-1}(g(x))) \\
&= \lim_{x \searrow x_0} \delta_A(\lambda \delta_A^{-1}(g(x_0^+))) \\
&> \delta_A(\lambda \delta_A^{-1}(g(x_0))) \quad \text{Eq. (12)} \\
&= \delta_A(\lambda \delta_A^{-1}(A(1, x_0))) \\
&= A(\lambda, \lambda x_0).
\end{aligned} \tag{13}$$

Write $\lim_{x \searrow x_0} A(\lambda, \lambda x) = A(\lambda, \lambda x_0^+)$. Then

$$\begin{aligned}\lim_{\lambda \nearrow \lambda_0} A(\lambda, \lambda x_0^+) &= \lim_{\lambda \nearrow \lambda_0} \delta_A(\lambda \delta_A^{-1}(A(1, x_0^+))) \\ &= \delta_A(\lambda_0 \delta_A^{-1}(A(1, x_0^+))) \\ &= \delta_A(\lambda_0 \delta_A^{-1}(g(x_0^+)))\end{aligned} \tag{14}$$

and

$$\begin{aligned}\lim_{\lambda \nearrow \lambda_0} A(\lambda, \lambda x_0) &= \lim_{\lambda \nearrow \lambda_0} \delta_A(\lambda \delta_A^{-1}(A(1, x_0))) \\ &= \delta_A(\lambda_0 \delta_A^{-1}(A(1, x_0))) \\ &= \delta_A(\lambda_0 \delta_A^{-1}(g(x_0))).\end{aligned} \tag{15}$$

Thus from Eq. (12), (13), (14) and (15) we have

$$\lim_{\lambda \nearrow \lambda_0} A(\lambda, \lambda_0 x_0) > A(\lambda_0, \lambda_0 x_0), \tag{16}$$

i. e., $A(\cdot, \lambda_0 x_0)$ is not left-continuous at λ_0. Meanwhile, obviously, for any $x_1, y_1 \in (0, 1]$ and $0 < x < x_1$ there always exists a $\lambda \in [0, 1]$ such that $0 \leq \lambda x_1 \leq x \leq x_1$. Then it follows from the monotonicity of the aggregation function A that

$$A(\lambda x_1, \lambda y_1) \leq A(x, y_1) \leq A(x_1, y_1).$$

Thus

$$\lim_{x \nearrow x_1} A(x, y_1) = \lim_{\lambda \nearrow 1} A(\lambda x_1, \lambda y_1) = A(x_1, y_1),$$

i. e.,

$$\lim_{x \nearrow x_1} A(x, y_1) = A(x_1, y_1),$$

contrary to Eq.(16). In short, g is right-continuous.

Case 2. If

$$\psi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in (0, 1], \end{cases}$$

then from Eq.(1), for an arbitrary increasing bijection $\varphi : [0, 1] \to [0, b]$,

$$A(x, x) = \varphi^{-1}(\psi(x)\varphi(A(1, 1))) = A(1, 1) = 1, x \in (0, 1].$$

So

$$A(x, x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in (0, 1] \end{cases}$$

and

$$A(x, y) = 1 \text{ for all } x, y \in (0, 1].$$

Moreover, $A(0, y) = A(y \cdot 0, y \cdot 1) = \varphi^{-1}(\psi(y)\varphi(A(0, 1))) = A(0, 1)$ for any $y \in (0, 1]$ and $A(x, 0) = A(x \cdot 1, x \cdot 0) = \varphi^{-1}(\psi(x)\varphi(A(1, 0))) = A(1, 0)$ for any $x \in (0, 1]$. Let $\alpha = A(0, 1)$ $\beta = A(1, 0)$. Therefore,

$$A(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ 1 & \text{if } x, y \in (0, 1], \\ \alpha & \text{if } x = 0, y \in (0, 1], \\ \beta & \text{if } y = 0, x \in (0, 1]. \end{cases}$$

Case 3. If

$$\psi(x) = \begin{cases} 0 & \text{if } x \in [0, 1), \\ 1 & \text{if } x = 1, \end{cases}$$

Then from Eq.(1), for an arbitrary increasing bijection $\varphi : [0, 1] \to [0, b]$,

$$A(x, x) = \varphi^{-1}(\psi(x)\varphi(A(1, 1))) = \varphi^{-1}(0) = 0, x \in [0, 1).$$

So

$$A(x, x) = \begin{cases} 0 & \text{if } x \in [0, 1), \\ 1 & \text{if } x = 1 \end{cases}$$

and

$$A(x, y) = 0 \text{ for all } x, y \in [0, 1).$$

Consider the functions $g, h : [0, 1] \to [0, 1]$ defined by $g(x) = A(1, x)$ and $h(x) = A(x, 1)$, respectively. Therefore,

$$A(x, y) = \begin{cases} 1 & \text{if } (x, y) = (1, 1), \\ 0 & \text{if } x, y \in [0, 1), \\ g(y) & \text{if } x = 1, y \in [0, 1), \\ h(x) & \text{if } y = 1, x \in [0, 1). \end{cases}$$

Conversely, one can directly verify that the aggregation function A is (φ, ψ)-quasi-homogeneous if one of $(i), (ii), (iii)$ holds. □

Notice that Theorem 2.10 describes the (φ, ψ)-quasi-homogeneity of aggregation function A in the case $0 < b < \infty$ completely. One can easily see that Theorem 3 of [9] concerns the (φ, ψ)-quasi-homogeneity of aggregation function A when $b < \infty$.

2.2. The case $b = \infty$

In the following we assume $0 \cdot \infty = 0$ by convention.

Theorem 2.11. An aggregation function $A : [0, 1]^2 \to [0, 1]$ is (φ, ψ)-quasi-homogeneous if and only if one of the following statements holds:

(i) There exist two constants $\alpha, \beta \in [0, 1]$ such that for all $x, y \in [0, 1]$,

$$A(x, y) = \begin{cases} 0 & \text{if } (x, y) = (0, 0), \\ 1 & \text{if } x, y \in (0, 1], \\ \alpha & \text{if } x = 0, y \in (0, 1], \\ \beta & \text{if } y = 0, x \in (0, 1], \end{cases}$$

$$\psi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in (0,1] \end{cases}$$

and $\varphi : [0,1] \to [0,b]$ is an increasing bijection.

(ii) There exist two increasing functions $h, g : [0,1] \to [0,1]$ fulfilling that $h(1) = g(1) = 1$ such that for all $x, y \in [0,1]$,

$$A(x,y) = \begin{cases} 1 & \text{if } (x,y) = (1,1), \\ 0 & \text{if } x, y \in [0,1), \\ g(y) & \text{if } x = 1, y \in [0,1), \\ h(x) & \text{if } y = 1, x \in [0,1), \end{cases}$$

$$\psi(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1 \end{cases}$$

and $\varphi : [0,1] \to [0,b]$ is an increasing bijection.

P r o o f. Considering $x = 1$ in Eq. (2), we have

$$\varphi(\delta_A(\lambda)) = \psi(\lambda)\varphi(1) \tag{17}$$

for all $\lambda \in [0,1]$. By Eqs. (2) and (17), we have

$$\varphi(1)\psi(\lambda x) = \varphi(1)\psi(\lambda)\psi(x) \text{ for all } \lambda, x \in [0,1].$$

Then

$$\psi(\lambda x) = \psi(\lambda)\psi(x) \text{ for all } \lambda, x \in [0,1]. \tag{18}$$

Considering $\lambda = 1$ in Eq. (17), we have $\varphi(\delta_A(1)) = \psi(1)\varphi(1)$, i.e., $\varphi(1) = \psi(1)\varphi(1)$. Then

$$\psi(1) = 1. \tag{19}$$

Considering $\lambda = 0$ in Eq. (17), we have $\varphi(\delta_A(0)) = \psi(0)\varphi(1)$, i.e., $\varphi(0) = \psi(0)\varphi(1)$. Then, by *(ii)* of Lemma 2.5 we have

$$\psi(0) = 0. \tag{20}$$

In the following, we prove that $\delta_A(\lambda) = 1$ or $\delta_A(\lambda) = 0$ for any $\lambda \in (0,1)$. In fact, if $\delta_A(\lambda) < 1$ for a $\lambda \in (0,1)$, then $\varphi(\delta_A(\lambda)) < \infty$. By Eq. (17), we have

$$\psi(\lambda) = 0.$$

Thus

$$\psi(\lambda)\varphi(1) = 0. \tag{21}$$

So

$$\varphi(\delta_A(\lambda)) = \psi(\lambda)\varphi(1) = 0.$$

Then from Lemma 2.5, $\delta_A(\lambda) = 0$ since φ is a bijection. In short, $\delta_A(\lambda) = 1$ or $\delta_A(\lambda) = 0$ for any $\lambda \in (0,1)$. Consequently, we distinguish three cases as below.

(i) If $\delta_A(\lambda) = 1$ for all $\lambda \in (0,1)$, then by Eq. (17) we have $\psi(\lambda) = 1$ for all $\lambda \in (0,1)$. From Eqs. (19) and (20), we have that for all $x \in [0,1]$,

$$\psi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in (0,1]. \end{cases}$$

Therefore, from Eq.(1), for an arbitrary increasing bijection $\varphi : [0,1] \to [0,b]$,

$$A(x,x) = \varphi^{-1}(\psi(x)\varphi(A(1,1))) = A(1,1) = 1, x \in (0,1].$$

So

$$A(x,x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x \in (0,1] \end{cases}$$

and

$$A(x,y) = 1 \text{ for all } x,y \in (0,1].$$

Moreover, $A(0,y) = A(y\cdot 0, y\cdot 1) = \varphi^{-1}(\psi(y)\varphi(A(0,1))) = A(0,1)$ for any $y \in (0,1]$ and $A(x,0) = A(x \cdot 1, x \cdot 0) = \varphi^{-1}(\psi(x)\varphi(A(1,0))) = A(1,0)$ for any $x \in (0,1]$. Let $\alpha = A(0,1)$ $\beta = A(1,0)$. Therefore, we have that for all $x,y \in [0,1]$,

$$A(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ 1 & \text{if } x,y \in (0,1], \\ \alpha & \text{if } x = 0, y \in (0,1], \\ \beta & \text{if } y = 0, x \in (0,1]. \end{cases}$$

(ii) If $\delta_A(\lambda) = 0$ for all $\lambda \in (0,1)$, then by Eq. (17), $\psi(\lambda) = 0$ for all $\lambda \in (0,1)$. Then by Eqs. (19) and (20),

$$\psi(x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1. \end{cases}$$

Then from Eq.(1), for an arbitrary increasing bijection $\varphi : [0,1] \to [0,b]$,

$$A(x,x) = \varphi^{-1}(\psi(x)\varphi(A(1,1))) = \varphi^{-1}(0) = 0, x \in [0,1).$$

So

$$A(x,x) = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1 \end{cases}$$

and

$$A(x,y) = 0 \text{ for all } x,y \in [0,1).$$

Consider the functions $g,h : [0,1] \to [0,1]$ defined by $g(x) = A(1,x)$ and $h(x) = A(x,1)$, respectively. Therefore,

$$A(x,y) = \begin{cases} 1 & \text{if } (x,y) = (1,1), \\ 0 & \text{if } x,y \in [0,1), \\ g(y) & \text{if } x = 1, y \in [0,1), \\ h(x) & \text{if } y = 1, x \in [0,1). \end{cases}$$

(iii) If there exists an $e \in (0,1)$ such that

$$\delta_A(\lambda) = 0 \text{ for all } \lambda \in (0,e)$$

and

$$\delta_A(\lambda) = 1 \text{ for all } \lambda \in (e,1),$$

then by Eq. (17), we have

$$\psi(\lambda) = 0 \text{ for all } \lambda \in (0,e) \tag{22}$$

and

$$\psi(\lambda) = 1 \text{ for all } \lambda \in (e,1).$$

Therefore, by Eq. (18), there exists a $\lambda \in (e,1)$ such that

$$\psi(\lambda^n) = (\psi(\lambda))^n = 1 \text{ for all } n \in N. \tag{23}$$

On the other hand, there exists an $m \in N$ such that $0 < \lambda^m < e$ since $\lim_{n\to\infty} \lambda^n = 0$. So it follows from Eq.(22) that $\psi(\lambda^m) = 0$, contrary to Eq.(23).

□

Notice that applying Theorems 2.10 and 2.11, from the point of view of the functional representation, the quasi-homogeneous aggregation functions are exactly classified into three classes as Eqs.(8), (9) and (10), respectively. One can check that the diagonal function of the first classified aggregation function is continuous and the others are non-continuous. Moreover, the constructions of the last two classes are very clear. Therefore, our next attention is how to construct the first classified quasi-homogeneous aggregation functions.

3. CONSTRUCTIONS OF QUASI-HOMOGENEOUS AGGREGATION FUNCTIONS

In this section, we first introduce a triple generator of a quasi-homogeneous aggregation function, then investigate how to construct quasi-homogeneous aggregation functions by using their triple generators.

Definition 3.1. Let $f : [0,1] \to [0,1]$ be an increasing bijection and $g, h : [0,1] \to [0,1]$ be two increasing functions. If a function $A : [0,1]^2 \to [0,1]$ given by

$$A(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0), \\ f(yf^{-1}(h(\frac{x}{y}))) & \text{if } x \le y \text{ and } y \ne 0, \\ f(xf^{-1}(g(\frac{y}{x}))) & \text{if } y \le x \text{ and } x \ne 0 \end{cases} \tag{24}$$

is a quasi-homogeneous aggregation function, then (f, g, h) is called a triple generator of a quasi-homogeneous aggregation function A, and A is said to be a quasi-homogeneous aggregation function generated by the triple (f, g, h).

Proposition 3.2. Let $f : [0,1] \to [0,1]$ be an increasing bijection on $[0,1]$ and $g, h : [0,1] \to [0,1]$ be two increasing functions fulfilling the following conditions:

(1) $h(1) = 1$, $g(1) = 1$;

(2) $\frac{f^{-1}(h(x))}{x}$ and $\frac{f^{-1}(g(x))}{x}$ are two decreasing functions on $(0,1]$.

Then the function $A : [0,1]^2 \to [0,1]$ defined by Eq. (24) is a quasi-homogeneous aggregation function, i. e., (f, g, h) is the triple generator of the quasi-homogeneous aggregation function A. Moreover, the diagonal δ_A is an increasing bijection.

Proof. Obviously, $A(0,0) = 0, A(1,1) = 1$. Next, we prove the monotonicity of function A. First, we prove that $A(x,y) \leq A(x,z)$ whenever $y \leq z$. Obviously, $A(0,y) \leq A(0,z)$ whenever $y \leq z$. Considering $x \in (0,1]$ and $y \leq z$.

◇ If $y \leq z \leq x$, then

$$\begin{aligned} A(x,y) &= f(xf^{-1}(g(\frac{y}{x}))) \\ &\leq f(xf^{-1}(g(\frac{z}{x}))) \text{ by the monotonicity of } f \text{ and } g \\ &= A(x,z). \end{aligned}$$

◇ If $y \leq x \leq z$, then $\frac{f^{-1}(h(x))}{x} \geq f^{-1}(h(1))/1$, i. e., $f^{-1}(h(x)) \geq x$ since $\frac{f^{-1}(h(x))}{x}$ is decreasing on $(0,1]$. Thus

$$\begin{aligned} A(x,y) &= f(xf^{-1}(g(\frac{y}{x}))) \\ &\leq f(x) \\ &= f(z \cdot \frac{x}{z}) \\ &\leq f(zf^{-1}(h(\frac{x}{z}))) \\ &= A(x,z). \end{aligned}$$

◇ If $x \leq y \leq z$, then

$$\begin{aligned} A(x,y) &= f(yf^{-1}(h(\frac{x}{y}))) \\ &\leq f(zf^{-1}(h(\frac{x}{z}))) \text{ by the monotonicity of } \frac{f^{-1}(h(x))}{x} \\ &= A(x,z). \end{aligned}$$

Bring all observations together, we can see that $A(x,y) \leq A(x,z)$ whenever $y \leq z$. Similarly, we can prove that $A(y,x) \leq A(z,x)$ whenever $y \leq z$.

Secondly, we prove the quasi-homogeneity of A. Considering the functions $\psi(x) = x, \varphi(x) = f^{-1}(x)$ for all $x \in [0,1]$. Then by Eq. (24) for all $x, y, \lambda \in [0,1]$,

$$
\begin{aligned}
A(\lambda x, \lambda y) &= \begin{cases} 0 & \text{if } x = y = 0 \text{ or } \lambda = 0, \\ f(\lambda y f^{-1}(h(\frac{\lambda x}{\lambda y}))) & \text{if } \lambda x \leq \lambda y \text{ and } \lambda y \neq 0, \\ f(\lambda x f^{-1}(g(\frac{\lambda y}{\lambda x}))) & \text{if } \lambda y \leq \lambda x \text{ and } \lambda x \neq 0. \end{cases} \\
&= \begin{cases} 0 & \text{if } x = y = 0 \text{ or } \lambda = 0, \\ f(\lambda y f^{-1}(h(\frac{x}{y}))) & \text{if } x \leq y \text{ and } y \neq 0, \\ f(\lambda x f^{-1}(g(\frac{y}{x}))) & \text{if } y \leq x \text{ and } x \neq 0. \end{cases} \\
&= \begin{cases} 0 & \text{if } x = y = 0 \text{ or } \lambda = 0, \\ f(\lambda f^{-1}(f(y f^{-1}(h(\frac{x}{y}))))) & \text{if } x \leq y \text{ and } y \neq 0, \\ f(\lambda f^{-1}(f(x f^{-1}(g(\frac{y}{x}))))) & \text{if } y \leq x \text{ and } x \neq 0. \end{cases} \\
&= f(\lambda f^{-1}(A(x,y))) \\
&= \varphi^{-1}(\psi(\lambda)\varphi(A(x,y))).
\end{aligned}
$$

Therefore, A is (φ, ψ)-quasi-homogeneous with $\psi(x) = x, \varphi(x) = f^{-1}(x)$ for all $x \in [0,1]$. It is easy to see that $\delta_A(x,x) = f(x)$ for all $x \in [0,1]$. Thus δ_A is an increasing bijection. □

Example 3.3. Consider the functions $g, h : [0,1] \to [0,1]$ and $f : [0,1] \to [0,1]$ defined by $g(x) = x, h(x) = \frac{2x}{1+x}$ and $f(x) = x$, respectively. It is easy to verify that g, h and f satisfy the conditions of Proposition 3.2. Then by Proposition 3.2, the function $A : [0,1]^2 \to [0,1]$ with

$$
A(x,y) = \begin{cases} \frac{2xy}{x+y} & \text{if } x \leq y \text{ and } y \neq 0, \\ y & \text{if } y \leq x \end{cases}
$$

for all $x, y \in [0,1]$ is a quasi-homogeneous aggregation function whose triple generator is (f, g, h).

From Theorem 2.10, one can immediately have the following conclusion.

Proposition 3.4. Let $A : [0,1]^2 \to [0,1]$ be a quasi-homogeneous aggregation function. Then A has a triple generator (f, g, h) and the following statements hold:

(1) $g(x) = A(1,x), h(x) = A(x,1)$ for all $x \in [0,1]$;

(2) $f(x) = \delta_A(x)$ for all $x \in [0,1]$.

4. CONCLUSIONS

The contributions of this article include two aspects, one is that we completely provide solutions of quasi-homogeneity equation of aggregation functions, which classify the quasi-homogeneous aggregation functions into three classes. The other is that we construct quasi-homogeneous aggregation functions by using their triple generators.

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