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A NOTE ON MEASURABLE MODIFICATIONS

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We present, with purely didactic aims, a simple and essentially self-contained proof of two necessary and sufficient conditions for existence of a measurable modification of a stochastic process with values in a separable complete metric space. Existence of a measurable modification of a stochastic process continuous in probability is an immediate consequence.

Keywords: stochastic processes, measurable modification

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1. INTRODUCTION

A necessary and sufficient condition for existence of a measurable modification of a stochastic process was essentially found in the early papers [3, 14] and also the basic structure of the proof – to use approximations with simple functions and suitable measurable selectors to find a measurable version of their limit – is well established. However, it seems hard, as out teaching experience shows, to find a reference to a version of the result and its proof that is reasonably self-contained, elementary and covers processes with values in Polish spaces (which are needed, e. g., in the theory of stochastic partial differential equations) and we aim at providing such a reference. The starting point for us was the seminal paper [2] by D. Cohn, however, treating non-compact state spaces we had to supply a different proof. Proofs in the particular case of processes continuous in probability which meet the above criteria are available, but the argument in the general case is not very different or more difficult and the result is worth being known.

J. Hoffmann–Jørgensen later found a rather different necessary and sufficient condition in terms of two-dimensional marginals. For completeness, we decided to include a version of his result into our paper, although in this case it was not necessary to make substantial changes in the standard proof.

We consider stochastic processes indexed by a measurable space S since, first, the proof is exactly the same as for the particular choice $S = \mathbb{R}_{\geq 0}$ and, secondly, our theorem then applies to various classes of random fields. Processes indexed by metric spaces were considered already in [2] and the books [9, 10], but the metric structure was required mainly in a construction of a separable modification. (We do not consider separable

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modifications in this paper referring the reader to e. g. [17, § IV.2] or [2] for a discussion of separable processes.)

2. MAIN RESULTS

Let $(U, \varkappa)$ be a separable complete metric space, $(\Omega, \mathscr{F}, \mathsf{P})$ a probability space (not necessarily complete), and let the space of all equivalence classes of U-valued $\mathscr{F}$-measurable functions on Ω be denoted by $L^0(\mathsf{P}; U)$ or $L^0(\mathscr{F}; U)$. (Equivalence is, of course, defined as equality P-almost surely.) If $f\colon \Omega \longrightarrow U$ is an $\mathscr{F}$-measurable function, its equivalence class will be denoted (following [7]) by $f^\bullet \in L^0(\mathsf{P}; U)$. Set $\varkappa_1 = \varkappa \wedge 1$, in the space $L^0(\mathsf{P}; U)$ we define a metric

$$d(f^\bullet, g^\bullet) = \int_\Omega \varkappa_1(f(\omega), g(\omega))\, \mathrm{d}\mathsf{P}(\omega), \quad f^\bullet, g^\bullet \in L^0(\mathsf{P}; U);$$

d plainly does not depend on the choice of representatives in equivalence classes. It is well known that $(L^0(\mathsf{P}; U), d)$ is a complete metric space and convergence in the metric d is just the convergence in probability (cf. e. g. [15, § 6.1]).

Both finite and countable infinite sets are called countable in the sequel. If S is a metric space, it is always equipped with its Borel σ-algebra. We say that a σ-algebra $\mathscr{G} \subseteq \mathscr{F}$ is P-countably generated if there exists a countable system $\mathscr{C} \subseteq \mathscr{G}$ such that for any $A \in \mathscr{G}$ a set $A' \in \sigma(\mathscr{C})$ may be found satisfying $\mathsf{P}(A \triangle A') = 0$. In this case we say that $\mathscr{C}$ P-generates $\mathscr{G}$. (Plainly, if $\mathscr{G}$ is generated by a countable algebra then it is P-countably generated.)

Theorem 2.1. Let $X = (X_s,\ s \in S)$ be a U-valued stochastic process defined on the probability space $(\Omega, \mathscr{F}, \mathsf{P})$ and indexed by a measurable space $(S, \mathscr{S})$. Set

$$\mathscr{X} : S \longrightarrow L^0(\mathsf{P}; U), \ s \longmapsto X_s^\bullet.$$

Then the following conditions are equivalent:

(a) X has a $\mathscr{S} \otimes \mathscr{F}$-measurable modification.

(b) The mapping $\mathscr{X}$ is $\mathscr{S}$-measurable with a separable range $\operatorname{Rng} \mathscr{X}$.

(c) The σ-algebra $\sigma(X_s,\ s \in S)$ is P-countably generated and there exists $\mathscr{G} \subseteq \mathscr{B}(U)$ closed under formation of finite intersections, generating $\mathscr{B}(U)$, $U \in \mathscr{G}$, and such that the function

$$S \longrightarrow \mathbb{R}, \ r \longmapsto \mathsf{P}\big\{(X_s, X_r) \in A \times B\big\} \tag{1}$$

is $\mathscr{S}$-measurable for every $s \in S$ and $A, B \in \mathscr{G}$.

Corollary 2.2. Assume that S is a separable metric space and $X = (X_s,\ s \in S)$ a U-valued stochastic process defined on $(\Omega, \mathscr{F}, \mathsf{P})$ continuous in probability. Then X admits a measurable modification.

Let us recall that a mapping η from a measurable space $(T, \mathscr{T})$ into a metric space V is called $\mathscr{T}$-measurable provided $\eta^{-1}(B) \in \mathscr{T}$ for every Borel set $B \subseteq V$, the stochastic process X is called $\mathscr{S} \otimes \mathscr{F}$-measurable if the mapping $S \times \Omega \longrightarrow U$, $(s, \omega) \longmapsto X(s, \omega)$ is $\mathscr{S} \otimes \mathscr{F}$-measurable and that a process $\tilde{X} = (\tilde{X}_s,\ s \in S)$ is a modification of X provided $\mathsf{P}\{\omega \in \Omega;\ X(s, \omega) = \tilde{X}(s, \omega)\} = 1$ for every $s \in S$.

Remark 2.3. (An important one.) If S is a Polish space (i. e. a separable completely metrisable space), in particular any interval in $\mathbb{R}_{\geq 0}$, the hypothesis that $\mathscr{X}$ has a separable range is superfluous, as the following theorem shows: if $f \colon Y \longrightarrow Z$ is a Borel mapping from a Polish space Y into an arbitrary metric space Z then f has a separable range. The proof is short and easy provided one knows (the rather nontrivial fact) that uncountable analytic sets in a Polish space have cardinality $2^{\aleph_0}$. Up to our best knowledge, this theorem appeared for the first time in [2, Remark] as a result of Calvin Moore communicated to the author by R. M. Dudley. Apparently independently the same result with an essentially identical proof is given in the book [16, Lemma 9.1] (in a chapter written by S. G. Simpson) and as Simpson's result it is reproduced in the textbook [22, Theorem 4.3.8].

3. PROOFS

Before proving Theorem 2.1 we first establish two auxiliary lemmas. The first of them is a version of a functional form of the monotone class theorem for metric space valued functions. We state it separately since it might be useful also elsewhere. (For a simple proof of the classical functional monotone class theorem see e. g. [11, Theorem 1.4] or [8, Theorem B.1.7].)

Lemma 3.1. Let $\mathscr{A}$ be an algebra of subsets of a set S, U a separable metric space and $\boldsymbol{H}$ a space of functions $S \longrightarrow U$ satisfying the following conditions:

(i) $\boldsymbol{H}$ is closed under pointwise convergence on S;

(ii) $\boldsymbol{H} \supseteq \boldsymbol{R}$, where $\boldsymbol{R}$ is the set of all U-valued $\mathscr{A}$-measurable functions on S taking only finitely many values.

Then $\boldsymbol{H}$ contains all $\sigma(\mathscr{A})$-measurable U-valued functions.

($\mathscr{A}$ is not a σ-algebra, but $\mathscr{A}$-measurability of a function taking only finitely many values has an obvious meaning.)

Proof. We set $\mathscr{S} = \sigma(\mathscr{A})$ and whenever $\{V_1, \dots, V_r\}$ is a partition of S into disjoint sets and $v_1, \dots, v_r \in U$ we shall denote – in the proofs of this and the next lemmas only – by $\bigoplus_{j=1}^{r} v_j \mathbf{1}_{V_j}$ the function taking value v_j on the set V_j, $1 \leq j \leq r$.

Separability of U implies that any $\mathscr{S}$-measurable U-valued function can be uniformly approximated by a function with a countable range, hence it is a pointwise limit of $\mathscr{S}$-measurable functions taking only finitely many values. (If it is not clear one can consult e. g. [19, Lemma 0.4].) Assume that φ is such a function, so there exist $K \geq 1$, $u_1, \dots, u_K \in U$ and a partition $\{M_1, \dots, M_K\}$ of S into pairwise disjoint sets $M_1, \dots, M_K \in \mathscr{S}$ such that

$$\varphi = \bigoplus_{i=1}^{K} u_i \mathbf{1}_{M_i}; \tag{2}$$

we aim at proving that $\varphi \in \boldsymbol{H}$. Fix an arbitrary partition $\{A_2, \dots, A_m\}$ of S into pairwise disjoint sets $A_2, \dots, A_m \in \mathscr{A}$, $m \geq K$, and points $v_2, \dots, v_m \in U$; let us consider a function

$$\psi_N = u_1\mathbf{1}_N \oplus \bigoplus_{j=2}^m v_j\mathbf{1}_{A_j\setminus N}$$

for a set $N\in\mathscr{S}$. If $N\in\mathscr{A}$ then $\psi_N\in\boldsymbol{R}$, hence the system $\mathscr{N}=\{N\in\mathscr{S};\ \psi_N\in\boldsymbol{H}\}$ contains $\mathscr{A}$ by assumption (ii) of Lemma 3.1 and it is a monotone class. Indeed, if $N_k\in\mathscr{N}$, $N_k\nearrow N$ as $k\to\infty$, then $\psi_{N_k}\longrightarrow\psi_N$ as $k\to\infty$ on S, and analogously for monotone intersections. Consequently, $\mathscr{N}=\mathscr{S}$, in particular $\psi_{M_1}\in\boldsymbol{H}$ for an arbitrary choice of a partition $\{A_2,\dots,A_m\}$ and points $v_2,\dots,v_m$. As the next step, consider the function $\zeta_N\colon S\longrightarrow U$,

$$\zeta_N = u_1\mathbf{1}_{M_1}\oplus u_2\mathbf{1}_{N\setminus M_1}\oplus\bigoplus_{j=2}^m v_j\mathbf{1}_{(A_j\setminus N)\setminus M_1}$$

for $N\in\mathscr{S}$. If $N\in\mathscr{A}$, then $\zeta_N=\psi_N$ for the choice of the $\mathscr{A}$-measurable partition $\{N,A_2\setminus N,\dots,A_m\setminus N\}$ and points $u_2,v_2,\dots,v_m$, therefore $\zeta_N\in\boldsymbol{H}$. Using the monotone class theorem as above we find that

$$u_1\mathbf{1}_{M_1}\oplus u_2\mathbf{1}_{M_2}\oplus\bigoplus_{j=2}^m v_j\mathbf{1}_{A_j\setminus(M_1\cup M_2)}\in\boldsymbol{H}.$$

(Note that $M_2\setminus M_1=M_2$ as M_1, M_2 are disjoint.) Repeating this argument K times and taking into account that $A_j\setminus(M_1\cup\dots\cup M_K)=\emptyset$, $j=1,\dots,m$, we prove that function (2) is in $\boldsymbol{H}$. □

The second lemma concerns separability of L^0-spaces. The result is surely well known but we cannot find any reference and so we provide a proof.

Lemma 3.2. (i) If the σ-algebra $\mathscr{F}$ is P-countably generated then the space $L^0(\mathscr{F};U)$ is separable.

(ii) Let $D=\{F_\gamma;\ \gamma\in\Gamma\}$ be a separable subset of $L^0(\mathscr{F};U)$. Fix a system of representatives $f_\gamma\in F_\gamma$, $\gamma\in\Gamma$ in an arbitrary way, then the σ-algebra $\sigma(f_\gamma;\gamma\in\Gamma)$ is P-countably generated.

Note that in Part (ii) of Lemma 3.2 it is not assumed that $\mathscr{F}$ is P-countably generated.

Proof. Whenever $\{V_1,\dots,V_r\}$ is a partition of Ω into disjoint sets and $v_1,\dots,v_r\in U$ we shall again denote by $\bigoplus_{j=1}^r v_j\mathbf{1}_{V_j}$ the function taking value v_j on the set V_j, $1\le j\le r$.

i) The proof of Part (i) resembles the standard proof that $L^1(\Omega,\mathscr{F},\mathsf{P})$ is separable if $\mathscr{F}$ is P-countably generated but some caution is necessary since $L^0(\mathscr{F};U)$ lacks a linear structure. Let us fix $\varepsilon>0$ and $f^\bullet\in L^0(\mathscr{F};U)$ arbitrarily. As U is separable, we can find an $\mathscr{F}$-measurable function $g\colon\Omega\longrightarrow U$ with a finite range such that $d(f^\bullet,g^\bullet)<\varepsilon$. Let $x_1,\dots,x_N\in U$ and a partition $\{B_1,\dots,B_N\}$ of Ω into disjoint sets from $\mathscr{F}$ be such that $g=\bigoplus_{j=1}^N x_j\mathbf{1}_{B_j}$. Let $U_0\subseteq U$ be a countable dense subset, find $y_1,\dots,y_N\in U_0$ satisfying $\varkappa(x_j,y_j)<\varepsilon$ for $j=1,\dots,N$ and set $h=\bigoplus_{j=1}^N y_j\mathbf{1}_{B_j}$. Then

$$d(g^\bullet,h^\bullet)=\sum_{j=1}^N\int_{B_j}\varkappa_1(g(\omega),h(\omega))\,\mathrm{d}\mathsf{P}(\omega)=\sum_{j=1}^N\varkappa_1(x_j,y_j)\mathsf{P}(B_j)<\varepsilon.$$

By assumption there exists a countable algebra $\mathscr{G} \subseteq \mathscr{F}$ P-generating $\mathscr{F}$. It is well known that $A_1, \dots, A_N \in \mathscr{G}$ may be found satisfying $\mathsf{P}(B_j \triangle A_j) < N^{-2}\varepsilon$, $j = 1, \dots N$, define

$$C_1 = A_1,\ C_j = A_j \setminus (A_1 \cup \dots \cup A_{j-1}),\ j = 2, \dots, N,\ C_{N+1} = \Omega \setminus \bigcup_{j=1}^{N} A_j.$$

Plainly, $C_1, \dots, C_{N+1} \in \mathscr{G}$ form a disjoint partition of Ω. Define $p\colon \Omega \longrightarrow U$ by $p = \bigoplus_{j=1}^{N+1} y_j \mathbf{1}_{C_j}$ with $y_{N+1} = y_N$, we aim at proving that $d(h^\bullet, p^\bullet) < \varepsilon$. We have

$$\begin{aligned} d(h^\bullet, p^\bullet) &= \sum_{j=1}^{N} \int_{B_j} \varkappa_1(h(\omega), p(\omega))\, \mathrm{d}\mathsf{P}(\omega) = \sum_{j=1}^{N} \int_{B_j \setminus C_j} \varkappa_1(h(\omega), p(\omega))\, \mathrm{d}\mathsf{P}(\omega) \\ &\le \sum_{j=1}^{N} \mathsf{P}(B_j \setminus C_j) \end{aligned}$$

since $h = p$ on $B_j \cap C_j$, $j = 1, \dots, N$. Clearly $\mathsf{P}(B_1 \setminus C_1) = \mathsf{P}(B_1 \setminus A_1) \le \mathsf{P}(B_1 \triangle A_1) < \varepsilon$. Further, for $j = 2, \dots, N$ we get

$$B_j \setminus C_j = B_j \setminus \bigl(A_j \setminus (A_1 \cup \dots \cup A_{j-1})\bigr) \subseteq (B_j \setminus A_j) \cup \bigcup_{i=1}^{j-1} (B_j \cap A_i) \subseteq \bigcup_{i=1}^{j} (B_i \triangle A_i),$$

we have used that if $i \ne j$, $z \in B_j \cap A_i$ then $z \notin B_i$ as $B_j \cap B_i = \emptyset$. Therefore

$$\mathsf{P}(B_j \setminus C_j) \le \sum_{i=1}^{j} \mathsf{P}(B_i \triangle A_i), \quad j = 1, \dots, N,$$

consequently

$$d(h^\bullet, p^\bullet) \le \sum_{j=1}^{N} \sum_{i=1}^{j} \mathsf{P}(B_i \triangle A_i) = \sum_{j=1}^{N} (N + 1 - j) \mathsf{P}(B_j \triangle A_j) \le N \sum_{j=1}^{N} \mathsf{P}(B_j \triangle A_j) < \varepsilon.$$

Let us denote by D the set of all functions $\psi\colon \Omega \longrightarrow U$ of the form $\psi = \bigoplus_{j=1}^{K} u_j \mathbf{1}_{E_j}$ for some $K \ge 1$, $u_1, \dots, u_K \in U_0$ and a partition $\{E_1, \dots, E_K\}$ of Ω into disjoint sets $A_1, \dots, A_K \in \mathscr{G}$. The set D is countable and we proved that for any $\varepsilon > 0$ and $f^\bullet \in L^0(\mathscr{F}; U)$ there exists $b \in D$ such that $d(f^\bullet, b^\bullet) < 3\varepsilon$. Separability of $L^0(\mathscr{F}; U)$ follows.

ii) Let $D_0 = \{F_{\gamma(n)}, n \ge 1\}$ be a countable dense subset of D. Set $\tilde{\mathscr{A}} = \sigma(f_{\gamma(n)}, n \ge 1)$, the σ-algebra $\tilde{\mathscr{A}}$ is plainly countably generated as $\mathscr{B}(U)$ is countably generated. Let $\mathscr{N} = \{N \in \mathscr{F};\ \mathsf{P}(F) = 0\}$ and define $\mathscr{A} = \sigma(\tilde{\mathscr{A}} \cup \mathscr{N})$; the σ-algebra $\mathscr{A}$ is P-countably generated. If $F_\beta \in D$ then there exist $F_{\gamma(n_k)} \in D_0$ such that $d(F_\beta, F_{\gamma(n_k)}) \longrightarrow 0$ as $k \to \infty$, hence $f_{\gamma(n_k)} \longrightarrow \varphi$ in P-probability for a function $\varphi \in F_\beta$. Obviously, $f_{\gamma(n_k)}$ are $\tilde{\mathscr{A}}$-measurable, thus φ is $\tilde{\mathscr{A}}$-measurable and $\{\varphi \ne f_\beta\} \in \mathscr{N}$, therefore f_β is $\mathscr{A}$-measurable. Consequently,

$$\sigma(f_{\gamma(n)}, n \ge 1) = \tilde{\mathscr{A}} \subseteq \sigma(f_\gamma, \gamma \in \Gamma) \subseteq \mathscr{A} = \sigma(\tilde{\mathscr{A}} \cup \mathscr{N}),$$

necessarily, $\sigma(f_\gamma, \gamma \in \Gamma)$ is P-countably generated. $\square$

P r o o f of Theorem 2.1. i) First let us assume that the condition (b) is satisfied, we aim at showing that (a) holds. By (b), there exists a sequence of $\mathscr{S}$-measurable mappings $H_n\colon S \longrightarrow L^0(\mathsf{P};U)$, $n \geq 1$, with a countable range such that

$$\sup_{s\in S} d\big(H_n(s), X_s^\bullet\big) < \frac{1}{2^{2n}} \quad \text{for any } n \geq 1.$$

As H_n's are simple functions, for any $n \geq 1$ there exist a countable partition $(B(n,k))_{k\in K_n}$ of S into sets from $\mathscr{S}$ and points $H_{n,k} \in L^0(\mathsf{P};U)$, $k \in K_n$, where $K_n \subseteq \mathbb{N}$, such that

$$H_n(s) = H_{n,k} \quad \text{for } s \in B(n,k),\ k \in K_n.$$

Let us choose $\mathscr{F}$-measurable functions $h_{n,k}\colon \Omega \longrightarrow U$ satisfying $h_{n,k}^\bullet = H_{n,k}$, $n \geq 1$, $k \in K_n$, and define

$$X_n\colon S \times \Omega \longrightarrow U,\ \ X_n(s,\omega) = h_{n,k}(\omega) \quad \text{for } s \in B(n,k),\ k \in K_n,\ n \geq 1.$$

For any $A \in \mathscr{B}(U)$ we have

$$\big\{(s,\omega) \in S \times \Omega;\ X_n(s,\omega) \in A\big\} = \bigcup_{k\in K_n} \big(B(n,k) \times h_{n,k}^{-1}(A)\big) \in \mathscr{S} \otimes \mathscr{F},$$

so X_n is a measurable U-valued process. Set

$$\Gamma = \big\{(s,\omega) \in S \times \Omega;\ \exists \lim_{n\to\infty} X_n(s,\omega) \text{ in } U\big\}.$$

Since U is complete,

$$\Gamma = \bigcap_{\alpha=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k,r=m}^{\infty} \Big\{(s,\omega) \in S \times \Omega;\ \varkappa(X_k(s,\omega), X_r(s,\omega)) < \frac{1}{\alpha}\Big\} \in \mathscr{S} \otimes \mathscr{F},$$

hence choosing an arbitrary point $u_0 \in U$ and defining $\tilde{X}$ by

$$\tilde{X}(s,\omega) = \begin{cases} \lim\limits_{n\to\infty} X_n(s,\omega), & (s,\omega) \in \Gamma, \\ u_0, & \text{otherwise}, \end{cases}$$

we get an $\mathscr{S} \otimes \mathscr{F}$-measurable process. It remains to prove that $\tilde{X}$ is a modification of X. To this end, fix $s \in S$. Since $X_n(s)^\bullet = H_n(s)$, we have

$$d\big(X_n(s)^\bullet, X(s)^\bullet\big) = \int_\Omega \min\big(1, \varkappa(X_n(s,\omega), X(s,\omega))\big)\mathrm{d}\mathsf{P}(\omega) < \frac{1}{2^{2n}},$$

which implies that

$$\mathsf{P}\Big\{\omega \in \Omega;\ \varkappa(X_n(s,\omega), X(s,\omega)) > \frac{1}{2^n}\Big\} < \frac{1}{2^n}$$

for all $n \geq 1$. Invoking the Borel-Cantelli lemma we find that $(X_n(s,\omega))_{n=1}^\infty$ converges to $X(s,\omega)$ for P-almost all $\omega \in \Omega$ and $X_s = \tilde{X}_s$ P-almost surely follows by the definition of $\tilde{X}$.

ii) Now assume that X has a measurable modification, we shall prove that the condition (b) is satisfied. As a process and its modification define the same mapping $S \longrightarrow L^0(\mathsf{P};U)$, we may (and will) assume that X itself is measurable.

Let us denote by $\mathscr{A}$ the algebra generated by measurable rectangles in $\mathscr{S} \otimes \mathscr{F}$, its elements are finite disjoint unions of measurable rectangles. Plainly, $\mathscr{A}$ generates $\mathscr{S} \otimes \mathscr{F}$. Let $\boldsymbol{R}$ be the set of all functions $\chi\colon S \times \Omega \longrightarrow U$ such that there exist $L \geq 1$, $u_1, \dots, u_L \in U$ and a partition $\{A_1, \dots, A_L\}$ of $S \times \Omega$ into pairwise disjoint sets from $\mathscr{A}$ satisfying $\chi = u_j$ on A_j, $1 \leq j \leq L$. Consider the system $\boldsymbol{G}$ of all functions $\xi\colon S \times \Omega \longrightarrow U$ such that the corresponding mapping $S \longrightarrow L^0(\Omega;U)$, $s \longmapsto \xi(s,\cdot)^\bullet$ is $\mathscr{S}$-measurable and has a separable range. We aim at proving that all $\mathscr{S} \otimes \mathscr{F}$-measurable U-valued functions are in $\boldsymbol{G}$, however, it can be checked easily that $\boldsymbol{R} \subseteq \boldsymbol{G}$ and that $\boldsymbol{G}$ is closed under pointwise convergence, thus invoking Lemma 3.1 we immediately see that (b) is satisfied.

iii) Further we shall show that (a) implies (c). We have already proved that $\operatorname{Rng}\mathscr{X}$ is separable, thus the σ-algebra $\sigma(X_s,\, s \in S)$ is P-countably generated. Clearly, the probability $\mathsf{P}\{(X_s, X_r) \in A \times B\}$ is the same for X and its measurable modification, so we may assume that X is measurable and then the $\mathscr{S}$-measurability of (1) follows by the Fubini theorem.

iv) Finally, we claim that if (c) is satisfied then (b) holds. Set $\mathscr{T} = \sigma(X_s,\, s \in S)$ then $\operatorname{Rng}\mathscr{X} \subseteq L^0(\mathscr{T};U)$ and from Lemma 3.2 we know that $L^0(\mathscr{T};U)$ is separable, hence $\operatorname{Rng}\mathscr{X}$ is separable as well. Let $\boldsymbol{R}$ be the set of all real-valued bounded functions g on $U \times U$ such that the function $r \longmapsto \mathsf{E}g(X_s, X_r)$ is $\mathscr{S}$-measurable for any $s \in S$. Obviously, $\boldsymbol{R}$ is a vector space containing constants and closed under pointwise convergence of uniformly bounded sequences. Moreover, $\mathbf{1}_{A\times B} \in \boldsymbol{R}$ whenever $A, B \in \mathscr{G}$ by (1), thus by the functional form of the monotone class theorem $\boldsymbol{R}$ contains all bounded $\mathscr{B}(U) \otimes \mathscr{B}(U)$-measurable functions, in particular $\varkappa_1 \in \boldsymbol{R}$. Consequently, $r \longmapsto \mathsf{E}\varkappa_1(X_s, X_r) = d(X_s^\bullet, X_r^\bullet)$ is $\mathscr{S}$-measurable for all $s \in S$. Let $W \subseteq L^0(\mathscr{F};U)$ be an arbitrary open set, separability of $\operatorname{Rng}\mathscr{X}$ implies that there exist $s_m \in S$ and $\varepsilon_m > 0$, $m \in \mathbb{N}$, such that

$$W \cap \operatorname{Rng}\mathscr{X} = \bigcup_{m\in\mathbb{N}} \bigl\{F \in \operatorname{Rng}\mathscr{X};\ d(X(s_m)^\bullet, F) < \varepsilon_m\bigr\},$$

whence

$$\mathscr{X}^{-1}(W) = \bigcup_{m\in\mathbb{N}} \bigl\{r \in S;\ d(X(s_n)^\bullet, X(r)^\bullet) < \varepsilon_m\bigr\} \in \mathscr{S}.$$

We see that $\mathscr{X}$ is $\mathscr{S}$-measurable. □

P r o o f of Corollary 2.2. Continuity in probability means that the mapping $\mathscr{X}\colon S \longrightarrow L^0(\mathsf{P};U)$, $s \longmapsto X_s^\bullet$ is continuous, so obviously Borel measurable, and a continuous image of a separable metric space is separable, as may be checked very easily, whence we see that the condition (b) from Theorem 2.1 is satisfied. □

Remark 3.3. Separability of U is plainly used in the proof of Theorem 2.1 whenever approximation with simple functions is needed. In other parts of the proof it appears in a hidden way: it yields that $\varkappa$ is $\mathscr{B}(U)\otimes\mathscr{B}(U)$-measurable and hence the metric d is well-defined. (The metric $\varkappa$ is of course $\mathscr{B}(U\times U)$-measurable, but $\mathscr{B}(U\times U) \supsetneqq \mathscr{B}(U)\otimes\mathscr{B}(U)$ may happen in the non-separable case. Corresponding examples may be found e. g. in [20, Examples 15.5–15.7].)

Remark 3.4. Our aim in this remark is to compare Theorem 2.1 and its Corollary 2.2 with some results frequently quoted in textbook; in no case we intend to survey available results on measurable modification. We still consider a U-valued stochastic process $X = (X_s,\ s \in S)$ defined on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$ and the associated mapping $\mathscr{X}$ introduced in Theorem 2.1. We shall denote by λ the Lebesgue measure on $\mathbb{R}$ and by $\mathscr{U}^\mu$ a completion of a σ-algebra $\mathscr{U}$ with respect to a measure μ defined on it.

Originally, measurable processes were defined (sometimes implicitly) as functions on the product space measurable with respect to the completed product σ-algebra. J. L. Doob in [6, Theorem II.2.6] considered the case when S is a measurable subset of $\mathbb{R}$, $U = [-\infty,\infty]$, and showed that X continuous in probability λ-almost everywhere has a $(\mathscr{B}(S)\otimes\mathscr{F})^{\lambda\otimes\mathsf{P}}$-measurable modification. This result was extended in [9, Theorem IV.3.1] and [10, Theorem III.3.1] to the case of a separable locally compact metric space U and a complete separable metric space S equipped with a complete σ-finite measure μ. A process X continuous in probability μ-almost everywhere was shown to admit a $(\mathscr{B}(S)\otimes\mathscr{F})^{\mu\otimes\mathsf{P}}$-measurable modification. In early papers, usually separable measurable modifications are considered, which may yield additional restrictions on U; in particular, in [9, 10] it is noted that completeness of U is sufficient if separability of the modification is not required, but no details are provided. A direct precursor of Theorem 2.1 is a result of Kawada [14, Theorem 2]: for $S = [0,1]$, $U = \mathbb{R}$ it is shown that X has a $(\mathscr{B}(S)\otimes\mathscr{F})^{\lambda\otimes\mathsf{P}}$-measurable modification iff $\mathscr{X}$ is λ-almost everywhere a limit of simple functions. (See also [3, Proposition 34].)

With the current definition of a measurable process, J. Neveu in [18, Chapter III.4] proved that for $U = [-\infty,\infty]$ and S an interval in $[-\infty,\infty]$ continuity in probability of X implies existence of a measurable modification. Due to a particular choice of the state space, limit superior may be used as a measurable selector in the construction of the modification. Theorem 2.1 (without the condition (c)) appeared in [3, Proposition 32] in the case when $S = \mathbb{R}$ and U is a separable metric space. (Note, however, that completeness of U is tacitly used in the proof; completeness is in fact essential, see [1, Example 8.6]. Moreover, separability of U is not assumed explicitly but built in via an additional hypothesis on X that is always satisfied in separable spaces. It is shown in [1, § 8] that, unfortunately, [3, Propositin 32] need not be valid in non-separable spaces U.) The same result for $S = \mathbb{R}_{\geq 0}$ and $U = [-\infty,\infty]$ appears in the book [5, Théorème IV.30]. The condition (b) is stated there directly as existence of measurable step functions converging uniformly in L^0 to $\mathscr{X}$ and weak convergence in L^1 is used to find a product measurable version of the limit.

Our statement of Theorem 2.1 (still without condition (c)) is taken from [2, Theorem 3], Corollary 2.2 is given there as Theorem 2. However, in [2] only compact spaces U are considered and our proof of sufficiency of the condition (b) in Theorem 2.1, based on the an argument from [19], is completely different from Cohn's one, which is short and very elegant, but uses a lemma on measurable selectors which seems to be specific for compact spaces. Let us note, however, that the direct proof of Corollary 2.2 (avoiding Theorem 2.1) given in [4, Proposition 3.2] in the case of a separable Banach space U is close in spirit to our proof. On the other hand, our proof of necessity of (b) follows the paper [2] rather closely, but we distilled Lemma 3.1 from Cohn's proof and we provide more details. (In [2], processes are indexed by a separable metric space S, but the metric in not used when measurable – and not measurable separable – modifications are dealt with.)

The condition (c) from Theorem 2.1 was proposed by J. Hoffmann–Jørgensen, see [12], in a slightly different form: instead of requiring that $\sigma(X_s,\ s \in S)$ is countably generated a suitable separability condition was stated in terms of weak convergence of measures $(X_s, X_r)_{\#}\mathsf{P}$, $(s,r) \in S^2$. The paper [2] is used in [12] and so it is assumed there that U is a compact metric space and S a metric separable space but the short and ingenious proof clearly remains valid under hypotheses of Theorem 2.1. (It is remarked in [12] that it is so, but without further discussion.) We follow the paper [12] rather closely, altering only the discussion of separability in L^0-spaces. The condition (c), in the present form, is thoroughly studied in [13, Chapter 5] under rather general hypotheses on the spaces S and U. (Cf. also the paper [21, Theorem 1], where the condition was rediscovered.)

Remark 3.5. Let U be an uncountable Polish space, set $I = [0,1]$. By the Borel isomorphism theorem (see e. g. [22, Theorem 3.3.13]) there exists a Borel measurable bijection $\iota\colon U \longrightarrow I$ with a Borel measurable inverse ι^{-1}. Let X be a U-valued stochastic process. If the process $\iota(X) = (\iota(X_s),\ s \in S)$ admits a measurable modification $\tilde{X}$ then $\iota^{-1}(\tilde{X})$ is a measurable modification of X, so we may content ourselves with compact metric spaces U in Theorem 2.1 once it is checked that the process $\iota(X)$ satisfies the condition (b) of Theorem 2.1 if X does so. It is true, but no straightforward proof is known to us. For example one may proceed in the following way: Any Borel measurable mapping $j\colon U \longrightarrow I$ defines a mapping $\ell(j)\colon L^0(\mathsf{P};U) \longrightarrow L^0(\mathsf{P};I)$, $f^\bullet \longmapsto (j \circ f)^\bullet$, we must prove that $\ell(j)$ is Borel measurable. It is not difficult to show that a continuous j leads to a continuous $\ell(j)$ and that a pointwise convergent sequence (j_k) of Borel measurable mappings from U to I gives rise to a pointwise convergent sequence $(\ell(j_k))$ of associated mappings from $L^0(\mathsf{P};U)$ to $L^0(\mathsf{P};I)$. Hence we may use transfinite induction to show that a Baire function j of any class $\alpha < \omega_1$ defines a Borel mapping $\ell(j)$. The set of all I-valued Baire function coincides with the set of all Borel measurable functions (see e. g. [22, Theorem 3.1.36]), in particular, $\ell(\iota)$ is a Borel mapping, in other words, $\iota(X)$ satisfies the condition (b). (Note that Remark 2.3 is needed to show separability of the range of $\ell(\iota)(\mathscr{X})$.) (In [13] one can find related considerations in a bit different context. It should be emphasized that [13] contains many results deeper and more general than those in our paper, however, their proofs often require quite advanced tools.)

The Borel isomorphism theorem is used in [21] and [13] also to replace L^0-spaces with L^2-spaces and L^1-spaces, respectively, so that well-known criteria for separability of these spaces may be applied.

In our opinion, our direct proof of Theorem 2.1 for an arbitrary Polish space U corresponds much better to our aim of providing an elementary and self-contained proof.

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