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AN IMPROVED DESCENT DIRECTION FOR PATH-FOLLOWING ALGORITHM IN MONOTONE LINEAR COMPLEMENTARITY PROBLEMS

Linda Menniche, Billel Zaoui and Djamel Benterki

We present a new full-Newton step feasible interior-point method for solving monotone linear complementarity problems. We derive an efficient search direction by applying an algebraic transformation to the central path system. Furthermore, we prove that the proposed method solves the problem within polynomial time. Notably, the algorithm achieves the best-known iteration bound, namely $O(\sqrt{n}\log\frac{n}{\epsilon})$-iterations. Finally, comparative numerical simulations illustrate the effectiveness of the proposed algorithm.

Keywords: Interior-point method, Monotone linear complementarity problem, Descent direction

Classification: 90C51, 90C33, 65K05

1. INTRODUCTION

The monotone linear complementarity problem (LCP) is an important problem in optimization and mathematics, where the goal is to find a solution that satisfies specific linear constraints and complementarity conditions. This problem has applications in various fields, including game theory, economics and engineering [2].

Primal-dual interior-point methods (IPMs) become one of the most active methods for solving wide classes of optimization problems thanks to their polynomial complexity and their numerical efficiency [16, 21].

The determination of the search direction plays a key role in the case of IPMs. Therefore, Darvay in [4] proposed a new technique for finding search directions for linear optimization (LO). This technique is based on an algebraic equivalent transformation (AET) with a square root function applied to the centering equation of the system which characterizes the central path. The new search directions are obtained by applying Newton's method to the resulting system. Later, Achache [1], Wang and Bai [17, 18, 19] and Wang et al. [20] extended Darvay's approach to convex quadratic programming (CQP), second-order cone optimization (SOCO), semidefinite optimization (SDO), symmetric cone optimization (SCO) and the $P_*(\kappa)$-LCP, respectively. Moreover, Kheirfam and Haghighi [9] proposed the function $\psi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$ in the AET technique to solve the

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$P_*(\kappa)$-LCPs. In addition, Darvay et al. in [3] took the function $\psi(t) = t - \sqrt{t}$ in this technique to present new primal-dual interior point algorithm (IPA) for LO. For more related papers about the AET technique, we refer the reader to [5, 6, 8, 11, 22, 23, 24], etc.

In [10], Kheirfam and Nasrollahi used the AET technique with the power function $\psi_q(t) = t^{q/2}(q \geq 1)$, to develop a full-Newton short-step IPA for LO. They introduced a set of new search directions based on the parameter q. Additionally, with the choices $\tau = \frac{2}{(q-1)^2+2}$ and $\theta = \frac{1}{((2q-1)^3-5q)\sqrt{n}}$ for $q \geq 3$, they derived an iteration bound given by $\mathcal{O}\left(\sqrt{n}\log\frac{\mu_0\left(n+\frac{4q(q-2)}{((q-1)^2+2)^2}\right)}{\epsilon}\right)$. It is worth noting that the work of Kheirfam and Nasrollahi [10] relies on some earlier contributions (see e.g., [4, 12, 15, 16]). Additionally, from the iteration bound in [10], we see that when q becomes very large, θ becomes very small. This makes the rate $1-\theta$ (which controls the decrease of the barrier parameter) approach one, leading to slower convergence and potentially even divergence of the algorithm. Therefore, using a large value of q in $\psi_q(t) = t^{q/2}$ can result in poor numerical performance. To address this, Grimes and Achache in [7] reconsidered the analysis of their IPA for LO in the context of monotone LCP. They proposed a non-parametric univariate function, $\psi(t) = t^{5/2}$, to improve the numerical results of these algorithms.

Motivated by the works mentioned above, we propose a new full-Newton step feasible IPM for monotone LCPs, based on the AET technique. In our approach, we use the function $\psi(t) = t^{5/3}$ to derive new efficient search directions via a full-Newton step in the transformed system. We show that the new algorithm has polynomial complexity with an iteration bound of $\mathcal{O}(\sqrt{n}\log\frac{n}{\epsilon})$. Furthermore, to validate the effectiveness of our algorithm, we conducted numerical tests to compare its performance with that of Grimes and Achache [7], as well as tests using problems from the quadprog test collection [14].

The paper is organized as follows: Section 2 presents the concept of central path and derives the classical search direction. In Section 3, we describe the new search direction and the algorithm framework. Section 4 analyzes the complexity of the algorithm. Section 5 provides numerical experiments and comparisons. Finally, Section 6 concludes with a summary and future works.

The notations used in this paper are as follows: $\mathbb{R}^n$ denotes the set of n-dimensional real vectors. $\mathbb{R}^{m\times n}$ is the set of $m\times n$ matrices. The componentwise product (Hadamard product) of vectors x and y is denoted as $xy = (x_1y_1, x_2y_2, \ldots, x_ny_n)^T$, and their elementwise division is given by $\frac{x}{y} = \left(\frac{x_1}{y_1}, \frac{x_2}{y_2}, \ldots, \frac{x_n}{y_n}\right)^T$ with $y_i \neq 0$ for all $i = 1, \ldots, n$. For a vector x and a scalar p, the elementwise power is denoted by $x^p = (x_1^p, x_2^p, \ldots, x_n^p)^T$. The Euclidean norm and the infinity norm are denoted by $\|\cdot\|$ and $\|\cdot\|_\infty$, respectively.

2. PROBLEM FORMULATION

The LCP requires the computation of a vector pair $(x, y) \in \mathbb{R}^{2n}$ satisfying

$$x \geq 0, \quad y \geq 0, \quad y = Mx + q, \quad x^Ty = 0, \tag{1}$$

where M is a given $n \times n$ matrix and q is a given vector in $\mathbb{R}^n$.
The last equation in system (1), known as the *complementarity condition*, can also be written as $xy = 0$, where xy denotes the Hadamard product of the vectors x and y. Thus, system (1) can be rewritten as

$$x \geq 0, \quad y \geq 0, \quad y = Mx + q, \quad xy = 0, \tag{2}$$

Throughout the paper, we will use the following notation for various subsets associated with the LCP

- $F(\text{LCP}) = \{(x, y) \in \mathbb{R}^{2n} : x \geq 0, y \geq 0, y = Mx + q\}$, representing the affine feasibility set of the LCP.
- $F^0(\text{LCP}) = \{(x, y) \in \mathbb{R}^{2n} : x > 0, y > 0, y = Mx + q\}$, denoting the strictly feasible affine set of the LCP.
- $\mathcal{S}(\text{LCP}) = \{(x, y) \in F(\text{LCP}) : xy = 0\}$, indicating the solution set of the LCP.

Moreover, we make the following assumptions for the LCP

- **Assumption 1:** The strictly feasible affine set $F^0(\text{LCP}) \neq \emptyset$.
- **Assumption 2:** The matrix M is positive semidefinite. Under this condition, the LCP is called *monotone*.

Given these assumptions, the set $\mathcal{S}(\text{LCP})$ is nonempty and convex.

2.1. The classical central path approach for LCP

The main idea behind path-following IPMs is to introduce a parameter μ to create a sequence of feasible points that converge to a solution. This is achieved by considering a parametrized version of the system

$$\begin{cases} y = Mx + q, \\ xy = \mu e, \quad x \geq 0, \quad y \geq 0, \end{cases} \tag{3}$$

where e is the all-ones vector of length n. For any parameter $\mu > 0$, system (3) defines the so-called *central path* of the monotone LCP.

Under the previously mentioned assumptions, there exists a unique solution $(x(\mu), y(\mu))$ to system (3) for each $\mu > 0$, as shown in [13]. We refer to the solutions $(x(\mu), y(\mu))$ as the μ-centers of the monotone LCP. Notably, as μ approaches zero, $(x(\mu), y(\mu))$ converges to a solution of (2).

By applying Newton's method to system (3), we can develop a classical path-following algorithm to approximate this central path.

In the next section, we introduce a new variant to improve the approximation of the central path.

3. NEW SEARCH DIRECTION

Following [4], the AET technique for computing a new search direction for interior point algorithms (IPAs) is based on the transformation of the centrality equation $xy = \mu e$ in (3) to the new equation $\psi\left(\frac{xy}{\mu}\right) = \psi(e)$, where $\psi : (0, +\infty) \to \mathbb{R}$ is a continuously differentiable and invertible function, i.e., ψ^{-1} exists. Then, the system (3) is transformed to the following system

$$\begin{cases} y = Mx + q, \\ \psi\left(\frac{xy}{\mu}\right) = \psi(e), \quad x, y \geq 0. \end{cases} \tag{4}$$

Using Newton's method to solve the nonlinear system (4), we obtain the following linear system

$$\begin{cases} \Delta y - M\Delta x = 0, \\ \frac{1}{\mu} y\psi'\left(\frac{xy}{\mu}\right)\Delta x + \frac{1}{\mu} x\psi'\left(\frac{xy}{\mu}\right)\Delta y = \psi(e) - \psi\left(\frac{xy}{\mu}\right). \end{cases} \tag{5}$$

Here, Δx and Δy denote the search directions, ψ' denotes the derivative of ψ.
We introduce the following notation

$$v = \sqrt{\frac{xy}{\mu}},\ d = \sqrt{\frac{x}{y}},\ d_x = \frac{v\Delta x}{x} \text{ and } d_y = \frac{v\Delta y}{y}. \tag{6}$$

Here, we obtain

$$\mu v(d_x + d_y) = y\Delta x + x\Delta y,\ d_x d_y = \frac{\Delta x \Delta y}{\mu}.$$

We can easily verify that the system (5) is written in the following form

$$\begin{cases} d_y - \bar{M} d_x = 0, \\ d_x + d_y = P_v. \end{cases} \tag{7}$$

Where $P_v = \frac{\psi(e) - \psi(v^2)}{v\psi'(v^2)}$ and $\bar{M} = DMD$ with $D = diag(d)$.
In this paper, we shall consider $\psi : (0, +\infty) \longrightarrow \mathbb{R}$, such that $\psi(t) = t^{\frac{5}{3}}$. Then

$$P_v = \frac{3}{5}(v^{-\frac{7}{3}} - v). \tag{8}$$

To analyze the algorithm, we define a proximity measure to the central path as follows

$$\delta(v) = \delta(x, y, \mu) = \frac{5}{3} \parallel P_v \parallel = \parallel v^{-\frac{7}{3}} - v \parallel . \tag{9}$$

where $\parallel . \parallel$ denotes the Euclidean norm.
It is clear that

$$\delta(v) = 0 \Leftrightarrow v = e \Leftrightarrow xy = \mu e.$$

Now, we describe the corresponding algorithm as follows.

Algorithm 1 Interior point algorithm for monotone LCP

Require: Accuracy parameter $\varepsilon > 0$; Barrier update parameter $0 < \theta < 1$; Threshold parameter $0 < \tau < 1$; Initial point $(x^0, y^0) \in F^0(\text{LCP})$ and $\mu^0 > 0$ such that $\delta(x^0, y^0, \mu^0) \leq \tau$.

1: **Initialization:** Set $x = x^0$, $y = y^0$, $\mu = \mu^0$.
2: **while** $x^T y > \varepsilon$ **do**
3: Update $\mu = (1 - \theta)\mu$.
4: Solve the system (7) and use (6) to obtain $(\Delta x, \Delta y)$.
5: Update $x = x + \Delta x$ and $y = y + \Delta y$.
6: **end while**

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4. ANALYSIS OF THE ALGORITHM

In this section, we present a detailed analysis of the proposed algorithm. We aim to show its main theoretical features, such as maintaining strict feasibility and effectively reducing the duality gap. Using key lemmas, we prove the algorithm's convergence and provide bounds on the number of iterations needed to achieve a given accuracy.

We start with the following technical lemma which will be useful throughout our analysis.

Lemma 4.1. Let (d_x, d_y) be a solution of system (7) with $\delta = \delta(x, y, \mu)$, $\mu > 0$. Then

$$0 \leq d_x^T d_y \leq \frac{9}{50}\delta^2 \tag{10}$$

and

$$\|d_x d_y\|_\infty \leq \frac{9}{100}\delta^2, \quad \|d_x d_y\| \leq \frac{9}{50\sqrt{2}}\delta^2, \tag{11}$$

where $\|\cdot\|_\infty$ denotes the infinity norm.

P r o o f. To prove the first inequality in (10), we use the relationships given in (5) and (6), we get

$$d_x^T d_y = \frac{1}{\mu}(\Delta x)^T \Delta y = \frac{1}{\mu}(\Delta x)^T M \Delta x \geq 0,$$

this last inequality holds because M is a positive semidefinite matrix.

For the second inequality in (10), from (8), we have

$$\|P_v\|^2 = \|d_x + d_y\|^2 = \|d_x\|^2 + \|d_y\|^2 + 2d_x^T d_y \geq 2d_x^T d_y.$$

Since $\|P_v\|^2 = \frac{9}{25}\delta^2$, it follows that

$$2d_x^T d_y \leq \frac{9}{25}\delta^2.$$

Then

$$d_x^T d_y \leq \frac{9}{50}\delta^2.$$

Thus, the first part of the lemma is proved.
Now, we have

$$d_x d_y = \frac{1}{4}\left((d_x + d_y)^2 - (d_x - d_y)^2\right).$$

Furthermore, it is evident that

$$\|d_x + d_y\|^2 = \|d_x - d_y\|^2 + 4d_x^T d_y.$$

Since $d_x^T d_y \geq 0$, we conclude that

$$\|d_x - d_y\| \leq \|d_x + d_y\|.$$

Thus, we get

$$\begin{aligned}
\|d_x d_y\|_\infty &= \frac{1}{4}\left\|(d_x + d_y)^2 - (d_x - d_y)^2\right\|_\infty \\
&\leq \frac{1}{4}\max\left(\|d_x + d_y\|_\infty^2; \|d_x - d_y\|_\infty^2\right) \\
&\leq \frac{1}{4}\max\left(\|d_x + d_y\|^2; \|d_x - d_y\|^2\right) \\
&\leq \frac{1}{4}\|d_x + d_y\|^2 = \frac{1}{4}\|P_v\|^2 = \frac{1}{4}\left(\frac{9}{25}\delta^2\right) = \frac{9}{100}\delta^2.
\end{aligned}$$

Next, for the final statement in the second part of the lemma, we have

$$\begin{aligned}
\|d_x d_y\|^2 = e^T(d_x d_y)^2 &= \frac{1}{16}e^T\left((d_x + d_y)^2 - (d_x - d_y)^2\right)^2 \\
&= \frac{1}{16}\left\|(d_x + d_y)^2 - (d_x - d_y)^2\right\|^2 \\
&\leq \frac{1}{16}\left(\|d_x + d_y\|^4 + \|d_x - d_y\|^4\right) \\
&\leq \frac{1}{8}\|d_x + d_y\|^4 = \frac{1}{8}\|P_v\|^4 = \frac{1}{8}\left(\frac{3}{5}\delta\right)^4
\end{aligned}$$

Hence, $\|d_x d_y\| \leq \sqrt{\frac{1}{8}\left(\frac{3}{5}\right)^4 \delta^4} = \frac{1}{2\sqrt{2}}\left(\frac{3}{5}\right)^2 \delta^2 = \frac{9}{50\sqrt{2}}\delta^2.$
This completes the proof. □

The following lemma ensures that the iterates generated by the algorithm after a full-Newton step remain in the strictly feasible region of the LCP under a certain proximity measure.

Lemma 4.2. let $\delta = \delta(x, y, \mu) < \frac{10}{3}$, then $(x_+, y_+) = (x + \Delta x, y + \Delta y) \in F^0(LCP)$.

P r o o f. Let $0 \leq \alpha \leq 1$ and $(x, y) \in F^0(LCP)$. We define

$$x_+(\alpha) = x + \alpha\Delta x, \quad y_+(\alpha) = y + \alpha\Delta y.$$

Thus,

$$x_+(\alpha)y_+(\alpha) = xy + \alpha(x\Delta y + y\Delta x) + \alpha^2 \Delta x \Delta y.$$

From equations (6) and (7), we get

$$\begin{aligned} x_+(\alpha)y_+(\alpha) &= \mu v^2 + \alpha\mu v P_v + \alpha^2 \mu d_x d_y \\ &= \mu\left((1-\alpha)v^2 + \alpha\left(v^2 + vP_v + \alpha d_x d_y\right)\right). \end{aligned} \tag{12}$$

Now, the inequality $x_+(\alpha)y_+(\alpha) > 0$ holds if

$$v^2 + vP_v + \alpha d_x d_y > 0.$$

From equations (11) and (8) with $\delta < \frac{10}{3}$, we have

$$\begin{aligned} v^2 + vP_v + \alpha d_x d_y &\geq v^2 + vP_v - \alpha \|d_x d_y\|_\infty e \\ &\geq v^2 + vP_v - \alpha \frac{9\delta^2}{100} e \\ &= v^2 + v\frac{3}{5}\left(v^{-\frac{7}{3}} - v\right) - \alpha\frac{9\delta^2}{100} e \\ &> \frac{2}{5}v^2 + \frac{3}{5}v^{-\frac{4}{3}} - e. \end{aligned}$$

Clearly, $x_+(\alpha)y_+(\alpha) > 0$ holds if

$$\frac{2}{5}v^2 + \frac{3}{5}v^{-\frac{4}{3}} - e \geq 0.$$

Let $F(t) = \frac{2}{5}t^2 + \frac{3}{5}t^{-\frac{4}{3}} - 1$ for all $t > 0$. Then

$$F'(t) = \frac{4}{5}t - \frac{4}{5}t^{-\frac{7}{3}}, \quad F''(t) = \frac{4}{5} + \frac{28}{15}t^{-\frac{10}{3}}, \quad \forall t > 0.$$

Since F is strictly convex, it has a unique minimum. Moreover, $F'(1) = 0$, which implies that $F(t)$ reaches its minimum at $t = 1$. Since a strictly convex function cannot go below its minimum, we conclude that

$$F(t) \geq F(1) = 0 \quad \forall t > 0.$$

Thus, we have

$$\frac{2}{5}v^2 + \frac{3}{5}v^{-\frac{4}{3}} - e > 0.$$

Therefore, $x_+(\alpha)y_+(\alpha) > 0$ for all $0 \leq \alpha \leq 1$, meaning that $x_+(\alpha)$ and $y_+(\alpha)$ do not change sign for any $0 \leq \alpha \leq 1$. Additionally, we have $x_+(0) = x > 0$ and $y_+(0) = y > 0$. Then, $x_+(1) = x_+ > 0$ and $y_+(1) = y_+ > 0$, ensuring that x_+ and y_+ are strictly feasible. □

Lemma 4.3. If $\delta = \delta(x, y, \mu) < \dfrac{10}{3}$, then $\min(v_+) \geq \dfrac{3}{10}\sqrt{\dfrac{100}{9} - \delta^2}$, where $v_+ = \sqrt{\dfrac{x_+y_+}{\mu}}$.

Proof. We know from Lemma 4.2 that $x_+ > 0$ and $y_+ > 0$, thus $v_+ = \sqrt{\frac{x_+ y_+}{\mu}}$ is well-defined.
Setting $\alpha = 1$ in (12) and using (8), we obtain

$$v_+^2 = v^2 + vP_v + d_x d_y = \frac{2}{5}v^2 + \frac{3}{5}v^{-\frac{4}{3}} + d_x d_y. \tag{13}$$

According to the previous lemma, we know that if $\delta < \frac{10}{3}$, then

$$\frac{2}{5}v^2 + \frac{3}{5}v^{-\frac{4}{3}} - e > 0,$$

which implies that

$$v_+^2 \geq e + d_x d_y.$$

Using (11), we get

$$\begin{aligned} v_+^2 &\geq e + d_x d_y \\ &\geq (e - \|d_x d_y\|_\infty e) \\ &\geq \left(1 - \frac{9}{100}\delta^2\right) e \\ &\geq \frac{9}{100}\left(\frac{100}{9} - \delta^2\right) e. \end{aligned}$$

Thus, we conclude that

$$\min(v_+) \geq \frac{3}{10}\sqrt{\frac{100}{9} - \delta^2}.$$

This completes the proof of the lemma. □

We state the following lemma which will be used in the next part of the analysis.

Lemma 4.4. ([3], Lemma 5.2) Let $f : [d, \infty) \longrightarrow (0, \infty)$ be a decreasing function with $d > 0$, furthermore, let us consider the positive vector v of length n such that $\min(v) > d$. Then

$$\|f(v)(e - v^2)\| \leq f\big(\min(v)\big)\|e - v^2\| \leq f(d)\|e - v^2\|.$$

In the following lemma, we prove the local quadratic convergence of the full-Newton step.

Lemma 4.5. Let $(x, y) \in F^0(LCP)$ and $\delta < \frac{10}{3}$. Then

$$\delta^+ = \delta(x_+, y_+, \mu) \leq \frac{\left(\frac{3}{10}\right)^{-\frac{7}{3}}\left(\frac{100}{9} - \delta^2\right)^{-\frac{7}{6}} - \frac{3}{10}\left(\frac{100}{9} - \delta^2\right)^{\frac{1}{2}}}{\frac{9}{100}\delta^2}\left(\frac{2}{5} + \frac{9}{50\sqrt{2}}\right)\delta^2,$$

Moreover, if $\delta \leq \frac{1}{4}$, so $\delta^+ \leq \delta^2$.

P r o o f . From the definition of the proximity measure, we have

$$\delta^+ = \|v_+^{-\frac{7}{3}} - v_+\| = \left\| \frac{v_+^{-\frac{7}{3}} - v_+}{e - v_+^2}(e - v_+^2) \right\|.$$

We define the function

$$f(t) = \frac{t^{-\frac{7}{3}} - t}{1 - t^2}.$$

Next, we compute the derivative of f

$$f'(t) = \frac{-\frac{7}{3}t^{-\frac{10}{3}} + \frac{13}{3}t^{-\frac{4}{3}} - t^2 - 1}{(1 - t^2)^2}.$$

Let us define the function

$$f_1(t) = -\frac{7}{3}t^{-\frac{10}{3}} + \frac{13}{3}t^{-\frac{4}{3}} - t^2 - 1,$$

we calculate the first and the second derivatives of this function, we get

$$f_1'(t) = \frac{70}{9}t^{-\frac{13}{3}} - \frac{52}{9}t^{-\frac{7}{3}} - 2t,$$

and

$$f_1''(t) = -\frac{910}{27}t^{-\frac{16}{3}} + \frac{364}{27}t^{-\frac{10}{3}} - 2.$$

We observe that $f_1'(t) = 0$ for $t = 1$, and $f_1''(1) = \frac{-600}{27} < 0$. Hence, $t = 1$ is a maximum of $f_1(t)$, which implies that

$$f_1(t) \leq f_1(1) = 0, \quad \forall t > 0.$$

Thus, we conclude that

$$f'(t) = \frac{f_1(t)}{(1 - t^2)^2} < 0, \quad \forall t > 0, t \neq 1.$$

Then, f is a decreasing function on $(0, +\infty)$. By applying Lemma 4.4, we can write

$$\delta^+ = \|f(v_+)(e - v_+^2)\| \leq f\left(\min(v_+)\right) \|e - v_+^2\|,$$

where

$$f(\min(v_+)) \leq f\left(\frac{3}{10}\sqrt{\frac{100}{9} - \delta^2}\right) = \frac{\left(\frac{3}{10}\sqrt{\frac{100}{9} - \delta^2}\right)^{-\frac{7}{3}} - \frac{3}{10}\sqrt{\frac{100}{9} - \delta^2}}{1 - \left(\frac{3}{10}\sqrt{\frac{100}{9} - \delta^2}\right)^2}$$

$$= \frac{\left(\frac{3}{10}\right)^{-\frac{7}{3}}\left(\frac{100}{9} - \delta^2\right)^{-\frac{7}{6}} - \frac{3}{10}\left(\frac{100}{9} - \delta^2\right)^{\frac{1}{2}}}{\frac{9}{100}\delta^2}.$$

Moreover, we have

$$\begin{aligned}\|e - v_+^2\| &= \|e - (v^2 + vP_v + d_x d_y)\| \\ &= \|e - v^2 - vP_v - d_x d_y\| \\ &\leq \|e - \frac{2}{5}v^2 - \frac{3}{5}v^{-\frac{4}{3}}\| + \|d_x d_y\|.\end{aligned}$$

Since

$$\|e - \frac{2}{5}v^2 - \frac{3}{5}v^{-\frac{4}{3}}\| = \|\varphi(v).\frac{25}{9}P_v^2\|,$$

where

$$\varphi(v) = \frac{e - \frac{2}{5}v^2 - \frac{3}{5}v^{-\frac{4}{3}}}{(v^{-\frac{7}{3}} - v)^2} = \frac{5v^{\frac{14}{3}} - 2v^{\frac{20}{3}} - 3v^{\frac{10}{3}}}{5(e - v^{\frac{10}{3}})^2}.$$

Let's consider the function $\varphi(t) = \dfrac{5t^{\frac{14}{3}} - 2t^{\frac{20}{3}} - 3t^{\frac{10}{3}}}{5(1 - t^{\frac{10}{3}})^2}$.
After some calculation, we obtain

$$\varphi'(t) = \frac{1}{3(1 - t^{\frac{10}{3}})^4}\left[14t^{\frac{11}{3}} - 8t^{\frac{17}{3}} - 6t^{\frac{7}{3}} - 6t^{\frac{31}{3}} - 18t^{\frac{27}{3}} - 8t^3\right] < 0, \quad \forall t > 0.$$

So, φ is continuous and strictly decreasing, also $\varphi(0) = 0$ and $\lim_{t\to+\infty}\varphi(t) = -\frac{2}{5}$. Then

$$-\frac{2}{5} < \varphi(t) < 0, \quad \forall t > 0.$$

Thus

$$0 < |\varphi(v_i)| < \frac{2}{5}, \quad \forall i = \overline{1, n}.$$

Additionally, we know that

$$\|P_v\|^2 = \frac{9}{25}\delta^2, \quad \|\varphi(v)\|_\infty < \frac{2}{5} \quad \text{and} \quad \|P_v^2\| \leq \|P_v\|^2.$$

Thus, we obtain

$$\|e - \frac{2}{5}v^2 - \frac{3}{5}v^{-\frac{4}{3}}\| \leq \|\varphi(v)\|_\infty.\frac{25}{9}\|P_v\|^2 = \frac{2}{5}\delta^2.$$

Moreover, from (11) we have $\|d_x d_y\| \leq \frac{9}{50\sqrt{2}}\delta^2$, so

$$\|e - v_+^2\| \leq \left(\frac{2}{5} + \frac{9}{50\sqrt{2}}\right)\delta^2. \tag{14}$$

This gives

$$\delta^+ \leq \frac{\left(\frac{3}{10}\right)^{-\frac{7}{3}}\left(\frac{100}{9} - \delta^2\right)^{-\frac{7}{6}} - \frac{3}{10}\left(\frac{100}{9} - \delta^2\right)^{\frac{1}{2}}}{\frac{9}{100}\delta^2}\left(\frac{2}{5} + \frac{9}{50\sqrt{2}}\right)\delta^2.$$

Finally, let us define the function $h(\delta) = f\left(\frac{3}{10}\sqrt{\frac{100}{9} - \delta^2}\right)$ for $\delta < \frac{1}{4}$, since h is a increasing function, we conclude that $h(\delta) < h(\frac{1}{4}) = f\left(\frac{3}{10}\sqrt{\frac{100}{9} - \frac{1}{16}}\right) = f(\frac{\sqrt{1591}}{40}) \simeq 1,8042$. Then, we get

$$\delta^+ \leq f\left(\frac{\sqrt{1591}}{40}\right)\left(\frac{2}{5} + \frac{9}{50\sqrt{2}}\right)\delta^2 \leq \delta^2.$$

Thus, we have completed the proof. □

The following lemma shows the impact of a full-Newton step on the new duality gap.

Lemma 4.6. Let $\delta = \delta(xy, \mu)$. Then, the duality gap satisfies

$$(x_+)^T y_+ \leq \mu(n + 2\delta^2).$$

In addition, if $\delta \leq \frac{1}{4}$, Then

$$(x_+)^T y_+ \leq 2\mu n.$$

Proof. From (13), we know that

$$v_+^2 = v^2 + vP_v + d_x d_y = \frac{2}{5}v^2 + \frac{3}{5}v^{-\frac{4}{3}} + d_x d_y.$$

Thus,

$$\begin{aligned} x_+ y_+ &= \mu\left(v^2 + vP_v + d_x d_y\right) \\ &= \mu\left(e + \frac{25}{9}P_v^2 \cdot \frac{\frac{2}{5}v^2 + \frac{3}{5}v^{-\frac{4}{3}} - e}{\left(v^{-\frac{7}{3}} - v\right)^2} + d_x d_y\right) \\ &\leq \mu\left(e + \frac{25}{9}P_v^2 + d_x d_y\right), \end{aligned}$$

this last inequality holds because

$$0 < \frac{\frac{2}{5}v_i^2 + \frac{3}{5}v_i^{-\frac{4}{3}} - e}{\left(v_i^{-\frac{7}{3}} - v_i\right)^2} = -\varphi(v_i) < \frac{2}{5} < 1, \quad \forall i.$$

Using the previously inequality and (10), we get

$$\begin{aligned} (x_+)^T y_+ = e^T(x_+ y_+) &\leq \mu\left(n + \frac{25}{9}\|P_v\|^2 + \frac{9}{50}\delta^2\right) \\ &\leq \mu\left(n + 2\delta^2\right). \end{aligned}$$

Finally, if $\delta < \frac{1}{4}$, then $\delta^2 < 1$. Therefore,

$$(x_+)^T y_+ \leq \mu(n + 2) \leq 2\mu n,$$

because $n + 2 \leq 2n, \quad \forall n \geq 2$, which completes the proof. □

The next lemma shows the influence of full-Newton step on the proximity measure.

Lemma 4.7. Let $(x,y)\in F^0(LCP)$ such that $\delta := \delta(xy,\mu) < \frac{10}{3}$ and $\mu^+ = (1-\theta)\mu$, where $0<\theta<1$. Then

$$\delta(v_{++}) := \delta(x_+, y_+; \mu_+) \leq \frac{(1-\theta)^{\frac{5}{3}}\left(\frac{3}{10}\sqrt{\frac{100}{9}-\delta^2}\right)^{-\frac{7}{3}} - \frac{3}{10}\sqrt{\frac{100}{9}-\delta^2}}{(1-\theta)^{\frac{3}{2}} - (1-\theta)^{\frac{1}{2}} - \frac{9}{100}(1-\theta)^{\frac{1}{2}}\delta^2}\left[\left(\frac{2}{5}+\frac{9}{50\sqrt{2}}\right)\delta^2+\theta\sqrt{n}\right].$$

Moreover, if $\delta < \frac{1}{4}$ and $\theta = \frac{1}{9\sqrt{n}}$, $n\geq 1$, then $\delta(x_+y_+;\mu_+) < \frac{1}{4}$.

P r o o f. Let

$$v_{++} = \sqrt{\frac{x_+y_+}{\mu_+}} = \sqrt{\frac{x_+y_+}{(1-\theta)\mu}} = \frac{1}{\sqrt{(1-\theta)}}v_+. \tag{15}$$

From the definition of the proximity measure, we have

$$\delta(v_{++}) = \delta(x_+, y_+, \mu_+) = \|v_{++}^{-\frac{7}{3}} - v_{++}\| = \left\|\frac{v_{++}^{-\frac{7}{3}} - v_{++}}{(e-v_{++}^2)}(e-v_{++}^2)\right\|. \tag{16}$$

Let us compute the two expression of previous norm. From (15), we get

$$\begin{aligned} v_{++}^{-\frac{7}{3}} - v_{++} &= \left(\frac{1}{(1-\theta)^{\frac{1}{2}}}v_+\right)^{-\frac{7}{3}} - \left(\frac{1}{(1-\theta)^{\frac{1}{2}}}v_+\right). \\ &= \frac{1}{(1-\theta)^{\frac{1}{2}}}\left[(1-\theta)^{\frac{5}{3}}v_+^{-\frac{7}{3}} - v_+\right], \end{aligned} \tag{17}$$

also

$$e - v_{++}^2 = \frac{(1-\theta)e - v_+^2}{1-\theta}. \tag{18}$$

Substituting (17) and (18) into (16), we get

$$\delta(v_{++}) = \left\|\frac{1}{\sqrt{(1-\theta)}}\left[\frac{(1-\theta)^{\frac{5}{3}}v_+^{-\frac{7}{3}} - v_+}{(1-\theta)e - v_+^2}\right]\left[(1-\theta)e - v_+^2\right]\right\|.$$

Define the function

$$g(t) = \frac{(1-\theta)^{\frac{5}{3}}t^{-\frac{7}{3}} - t}{(1-\theta) - t^2}, \quad \forall t>0.$$

Since $g'(t)<0$ for all $t>0$, the function g is decreasing. Using Lemmas 4.3 and 4.4, we deduce

$$\begin{aligned} \delta(v_{++}) &< \frac{1}{\sqrt{(1-\theta)}}g(\min(v_+)\|(1-\theta)e - v_+^2\| \\ &< \frac{1}{\sqrt{(1-\theta)}}\frac{(1-\theta)^{\frac{5}{3}}\left(\frac{3}{10}\sqrt{\frac{100}{9}-\delta^2}\right)^{-\frac{7}{3}} - \frac{3}{10}\sqrt{\frac{100}{9}-\delta^2}}{(1-\theta) - \left(\frac{3}{10}\sqrt{\frac{100}{9}-\delta^2}\right)^2}\|(1-\theta)e - v_+^2\|. \end{aligned} \tag{19}$$

Furthermore, by using (14), we get

$$\begin{aligned}\|(1-\theta)e - v_+^2\| &\le \|e - v_+^2\| + \|\theta e\| \\ &\le \left(\frac{2}{5} + \frac{9}{50\sqrt{2}}\right)\delta^2 + \theta\sqrt{n}. \end{aligned} \tag{20}$$

Substituting (20) into (19), we obtain

$$\delta(v_{++}) < \frac{(1-\theta)^{\frac{5}{3}}\left(\frac{3}{10}\sqrt{\frac{100}{9}-\delta^2}\right)^{-\frac{7}{3}} - \frac{3}{10}\sqrt{\frac{100}{9}-\delta^2}}{(1-\theta)^{\frac{3}{2}} - (1-\theta)^{\frac{1}{2}} - \frac{9}{100}(1-\theta)^{\frac{1}{2}}\delta^2}\left[\left(\frac{2}{5} + \frac{9}{50\sqrt{2}}\right)\delta^2 + \theta\sqrt{n}\right].$$

Now, suppose that $\delta < \frac{1}{4}$ and $\theta = \frac{1}{9\sqrt{n}}$, we have

$$\left(\frac{2}{5} + \frac{9}{50\sqrt{2}}\right)\delta^2 + \theta\sqrt{n} = \left(\frac{2}{5} + \frac{9}{50\sqrt{2}}\right)\delta^2 + \frac{1}{9} < \left(\frac{2}{5} + \frac{9}{50\sqrt{2}}\right)\frac{1}{16} + \frac{1}{9}.$$

Let us define the following function

$$f_2(\delta) = \frac{(1-\theta)^{\frac{5}{3}}\left(\frac{3}{10}\sqrt{\frac{100}{9}-\delta^2}\right)^{-\frac{7}{3}} - \frac{3}{10}\sqrt{\frac{100}{9}-\delta^2}}{(1-\theta)^{\frac{3}{2}} - (1-\theta)^{\frac{1}{2}} - \frac{9}{100}(1-\theta)^{\frac{1}{2}}\delta^2}.$$

The function f_2 is increasing for each $\delta < \frac{1}{4}$, because

$$f_2(t) = \frac{1}{\sqrt{(1-\theta)}}g(h_1(t)),$$

since $h_1(t) = \dfrac{3}{10}\sqrt{\frac{100}{9} - t^2}$ and $g(t) = \dfrac{(1-\theta)^{\frac{5}{3}}t^{-\frac{7}{3}} - t}{(1-\theta) - t^2}$ are both decreasing. So

$$f_2(\delta) < f_2\left(\frac{1}{4}\right),$$

where

$$f_2\left(\frac{1}{4}\right) = \frac{(1-\theta)^{\frac{5}{3}}\left(\frac{\sqrt{1591}}{40}\right)^{-\frac{7}{3}} - \frac{\sqrt{1591}}{40}}{(1-\theta)^{\frac{3}{2}} - \frac{1609}{1600}(1-\theta)^{\frac{1}{2}}}.$$

If $n \ge 1$, then $0 < \theta \le \frac{1}{9}$. Consider the function

$$f_3(\theta) = \frac{(1-\theta)^{\frac{5}{3}}\left(\frac{\sqrt{1591}}{40}\right)^{-\frac{7}{3}} - \frac{\sqrt{1591}}{40}}{(1-\theta)^{\frac{3}{2}} - \frac{1609}{1600}(1-\theta)^{\frac{1}{2}}},\ \forall 0 < \theta \le \frac{1}{9}$$

Since $f_3(\theta)$ is increasing, we obtain

$$f_3(\theta) \le f_3\left(\frac{1}{9}\right) = \frac{\left(\frac{8}{9}\right)^{\frac{5}{3}}\left(\frac{\sqrt{1591}}{40}\right)^{-\frac{7}{3}} - \frac{\sqrt{1591}}{40}}{\left(\frac{8}{9}\right)^{\frac{3}{2}} - \frac{1609}{1600}\frac{2\sqrt{2}}{3}}.$$

Finally, we get

$$\delta(v_{++}) \leq \left[\frac{\left(\frac{8}{9}\right)^{\frac{5}{3}}\left(\frac{\sqrt{1591}}{40}\right)^{-\frac{7}{3}} - \frac{\sqrt{1591}}{40}}{\left(\frac{8}{9}\right)^{\frac{3}{2}} - \frac{1609}{1600}\frac{2\sqrt{2}}{3}}\right]\left[\left(\frac{2}{5} + \frac{9}{50\sqrt{2}}\right)\frac{1}{16} + \frac{1}{9}\right] \simeq 0,2225 < \frac{1}{4}.$$

This completes the proof. □

The following lemma establishes an upper bound on the number of iterations required for the algorithm.

Lemma 4.8. Suppose that the pair $(x^0, y^0) \in F^0(LCP)$ such as $\delta(x^0, y^0, \mu^0) < \frac{1}{4}$, for a fixed $\mu^0 > 0$. Moreover, let (x^k, y^k) be the point obtained after k iterations. The inequality $(x^k)^T y^k \leq \epsilon$ is satisfied when

$$k \geq \frac{1}{\theta} \log\left(\frac{2n\mu^0}{\epsilon}\right).$$

Proof. After k iterations, we have $\mu^k = (1-\theta)^k \mu^0$, Lemma 4.6 implies that

$$(x^k)^T y^k \leq 2\mu^k n \leq (1-\theta)^k 2n\mu^0.$$

Hence, the inequality $(x^k)^T y^k \leq \epsilon$ holds if

$$(1-\theta)^k 2n\mu_0 \leq \epsilon.$$

Taking logarithm, we get

$$k \log(1-\theta) \leq \log \epsilon - \log 2n\mu_0.$$

As $-\log(1-\theta) \geq \theta,\ \forall\ 0 < \theta < 1$, then the above inequality holds if

$$k\theta \geq \log 2n\mu_0 - \log \epsilon = \log \frac{2n\mu_0}{\epsilon}.$$

Hence the result. □

Theorem 4.9. Using the default $\theta = \frac{1}{9\sqrt{n}}, n \geq 1,\ \tau = \frac{1}{4}$ and $\mu_0 = \frac{1}{2}$. Then the obtained algorithm requires at most $O(\sqrt{n} \log \frac{n}{\epsilon})$ iterations for getting the ϵ-approximate solution for LCP.

Proof. Using $\theta = \frac{1}{9\sqrt{n}},\ \mu_0 = \frac{1}{2}$ in the previously lemma, the result holds. □

5. NUMERICAL EXPERIMENTS

In this section, we present comparative numerical tests between our proposed algorithm 1 with its parameter $\theta_{th} = \frac{1}{9\sqrt{n}}$ and the algorithm of Grimes and Achache [7], which uses $\theta_{th} = \frac{1}{35\sqrt{2n}}$. Both algorithms are tested under the same accuracy parameter $\epsilon = 10^{-4}$.

For the numerical experiments, we use two examples with fixed sizes and one example with a variable size of monotone LCPs. The tests are implemented using MATLAB R2009b. We refer to the algorithm of Grimes and Achache as "$M1$" and to our algorithm 1 as "$M2$". The results compare the number of iterations, "Iter," required to find an ϵ-approximate solution and the computation time, "T(s)," measured in seconds.

Example 5.1. Let us consider the following monotone LCP.

$$M = \begin{pmatrix} 6 & 6 & 4 & 3 & 2 \\ 8 & 21 & 14 & 10 & 12 \\ 4 & 14 & 13 & 5 & 9 \\ 4 & 10 & 5 & 6 & 5 \\ 3 & 12 & 8 & 4 & 10 \end{pmatrix}, q = \begin{pmatrix} -20.5 \\ -64.5 \\ -44.5 \\ -29.5 \\ -36.5 \end{pmatrix}.$$

The starting points for the algorithms are as follows

$$x^0 = (1,1,1,1,1)^T,\ y^0 = (0.5, 0.5, 0.5, 0.5, 0.5)^T.$$

An obtained solution was given as follows

$$x^* = (0.6364, 2.3222, 0.5847, 0.0001, 0.2046)^T$$

$$y^* = (0, 0, 0, 0.2149, 0.00001)^T.$$

The numerical results of this example are summarized in Table 1

Tab. 1. Comparative results for Example 5.1

$M1$		$M2$	
Iter	T(s)	Iter	T(s)
1116	0.084491	199	0.022823

Example 5.2.

$$M = \begin{pmatrix} 1 & 0 & -0.5 & 0 & 1 & 3 & 0 \\ 0 & 0.5 & 0 & 0 & 2 & 1 & -1 \\ -0.5 & 0 & 1 & 0.5 & 1 & 2 & -4 \\ 0 & 0 & 0.5 & 0.5 & 1 & -1 & 0 \\ -1 & -2 & -1 & -1 & 0 & 0 & 0 \\ -3 & -1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 & 0 \end{pmatrix}, q = \begin{pmatrix} -1 \\ 3 \\ 1 \\ -1 \\ 5 \\ 6 \\ 1.5 \end{pmatrix}.$$

We consider the following initial point

$$x^0 = (0.98, 0.14, 0.31, 1.84, 0.32, 0.12, 0.17)^T, y^0 = Mx^0 + q.$$

An obtained solution was given as follows

$$x^* = (1, 0, 0, 2, 0, 0, 0)^T,$$

$$y^* = (0, 3, 1.5, 0, 2, 5, 1.5)^T.$$

The numerical results of this example are summarized in Table 2

Tab. 2. Comparative results for Example 5.2

$M1$		$M2$	
Iter	T(s)	Iter	T(s)
1366	0.115969	244	0.024603

Example 5.3. (with variable dimension)

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 5 & 6 & \cdots & 6 \\ 2 & 6 & 9 & \cdots & 10 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 2 & 6 & 10 & \cdots & 4n-3 \end{pmatrix}, \quad q = -Me + e.$$

The starting points are $x^0 = y^0 = e$.
The obtained results of this example for different sizes of n were summarized in Table 3.

Tab. 3. Comparative results for Example 5.3

n	$M1$		$M2$	
	Iter	T(s)	Iter	T(s)
10	1797	0.1968	322	0.0374
25	3070	1.0811	554	0.3104
50	4587	6.2128	829	0.9149
100	6832	43.8086	1237	5.6703
500	17065	5515.8748	3097	1001.9191

Remark 5.4. The obtained results via our algorithm 1 ($M2$) show its efficiency compared to the algorithm of Grimes and Achache ($M1$). This efficiency is measured by a smaller number of iterations and reduced computation time registered in $M2$.
The efficiency of our approach $M2$ becomes increasingly evident as the dimension of the problem grows, as illustrated in Table 3.

5.1. Improvement of the algorithm

Since the parameter θ used in both algorithms is dependent on the dimension of the problem n (in the denominator), it decreases as n grows, leading to slower convergence. To improve this, we propose comparing the two algorithms $M1$ and $M2$ with fixed values of θ chosen from the set $\{0.1, 0.3, 0.5, 0.7, 0.9\}$.
The results of the previously examples 5.1, 5.2 and 5.3 are summarized in Table 4.

Tab. 4. Comparative results for fixed value of θ

Example	n	θ	$M1$		$M2$	
			Iter	T(s)	Iter	T(s)
Example 5.1	05	0.1	97	0.030379	97	0.011122
		0.3	29	0.006749	29	0.006653
		0.5	21	0.005521	16	0.005566
		0.7	21	0.005659	15	0.005104
		0.9	21	0.005345	15	0.005162
Example 5.2	07	0.1	100	0.136678	100	0.124180
		0.3	30	0.016931	30	0.015831
		0.5	21	0.006876	16	0.006114
		0.7	21	0.006335	12	0.005090
		0.9	21	0.006245	12	0.004569
Example 5.3	50	0.1	125	0.213962	125	0.197624
		0.3	38	0.080153	37	0.079202
		0.5	27	0.070043	20	0.051982
		0.7	26	0.046911	15	0.036607
		0.9	26	0.045135	15	0.035956
	100	0.1	132	0.931735	132	0.935253
		0.3	40	0.283016	39	0.287833
		0.5	28	0.201628	21	0.162714
		0.7	28	0.195372	16	0.119690
		0.9	28	0.188696	16	0.113299
	500	0.1	147	58.121058	147	59.896615
		0.3	44	18.294447	44	17.869089
		0.5	31	11.676487	23	9.082781
		0.7	31	11.545788	18	7.269387
		0.9	31	11.473889	17	6.654589
	1000	0.1	154	374.00098	154	371.581444
		0.3	46	111.844714	46	112.215909
		0.5	33	81.198650	24	58.907683
		0.7	32	80.103787	19	47.175033
		0.9	32	78.317369	18	44.956709

Remark 5.5. Using fixed values of θ, the results show clear improvements compared to the theoretical θ_{th} for both algorithms. When θ increases, the computation time and the number of iterations decrease. In addition, our approach gives better results than the approach of Grimes and Achache [7] and these advantages become more noticeable as θ and the problem size grow.

Now, to confirm our previous observations, we will solve some problems from the quadprog test collection (`https://CRAN.R-project.org/package=quadprog`) using different values of θ. The obtained results are summarized in Table 5.

Tab. 5. Comparative results for some quadprog problems

Problem	θ	$M1$		$M2$	
		Iter	T(s)	Iter	T(s)
Zecevic 2	θ_{th}	707	0.0548	126	0.0123
	0.1	69	0.0087	69	0.0058
	0.3	21	0.0049	21	0.0047
	0.5	15	0.0046	11	0.0042
	0.7	15	0.0042	9	0.0031
	0.9	15	0.0040	8	0.0028
Tame	θ_{th}	379	0.0376	68	0.0082
	0.1	43	0.0065	43	0.0061
	0.3	14	0.0046	13	0.0042
	0.5	10	0.0039	7	0.0030
	0.7	9	0.0036	6	0.0029
	0.9	9	0.0033	5	0.0025
Genhs 28	θ_{th}	1512	0.2511	272	0.0531
	0.1	69	0.0163	69	0.0156
	0.3	21	0.0093	21	0.0087
	0.5	16	0.0061	12	0.0054
	0.7	16	0.0057	10	0.0051
	0.9	16	0.0055	10	0.0047

6. CONCLUSION

In this paper, we have introduced an efficient feasible full-Newton step interior point algorithm to solve monotone linear complementarity problems. We have demonstrated that the resulting algorithm solves the problems in polynomial time, ensuring both efficiency and robustness. Furthermore, numerical experiments highlight the superior performance of our approach. Additionally, the implementation of our approach, with fixed values of the update parameter θ, results in a significant reduction in both number of iterations and computation time.

In the future, we will extend this approach to more generalized problems, such as semidefinite linear complementarity problems.

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Linda Menniche, LMPA, BP 98, Ouled Aissa, University of Mohammed Seddik Ben Yahia, Jijel, 18000, Algeria
e-mail: l_menniche@yahoo.fr

Billel Zaoui, LIM, Department of Mathematics, Faculty of exact sciences, Akli Mouhand Oulhadj University of Bouira, 10000, Algeria
e-mail: b.zaoui@univ-bouira.dz

Djamel Benterki, LMFN, Setif-1 University Ferhat Abbas, 19000, Algeria
e-mail: djbenterki@univ-setif.dz