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NEW CONSTRUCTIONS OF NULLNORMS ON BOUNDED TRELLISES

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In this paper, we focus on the construction of nullnorms on bounded trellises. The features of the element that acts as the annihilator of a nullnorm are discussed and the relevant results show that the element acting as the annihilator must not be included in any cycle. Drawing upon this revelation, we propose some new methods for constructing nullnorms on bounded trellises, which are different from those given by Xiu et al. Additionally, some illustrative examples are provided to facilitate a more comprehensive understanding.

Keywords: nullnorm, trellis, order-preserving mapping, t-(co)norm

Classification: 03B52, 03E72, 03G10

1. INTRODUCTION

Nullnorms, as a generalization of t-norms and t-conorms, emerged from Calvo et al.'s study of the Frank equation for uninorms [7]. They possess a unique absorbing element that can exist anywhere in the unit interval, called the annihilator. The nullnorms have important theoretical research value due to the fact that they are composed of a t-norm and a t-conorm, and they also have a wide range of applications, such as fuzzy logic, decision making, expert systems, neural networks and so on [13, 14, 20, 21]. For the above reasons, there has been a lot of research about nullnorms on the unit interval [11, 12, 27, 28, 29, 30, 39].

Since incomparability is very common in practical applications, in recent years, scholars prefer to study aggregation operators on more general structures, especially on bounded lattices [1]. Initially, Karaçal et al. [25] extended nullnorms to an arbitrary bounded lattice and proved the existence of nullnorms on arbitrary bounded lattices. Subsequently, a series of studies about nullnorms on bounded lattices have sprung up, mainly focusing on the construction and representation of nullnorms [2, 3, 4, 5, 6, 15, 22, 23, 32, 33, 35, 36, 37].

As we know, associativity plays a key role in the development of lattice theory. In 1971, Skala [31] introduced a generalization of lattices, namely trellises (weakly associative lattices). This generalization allows for most theorems of lattice theory to hold without assuming associativity. It is undeniable that the existence of transitivity does bring great convenience to mathematical reasoning. However, it is actually shown that

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the absence of transitivity is also very important in some phenomena. For example, in ecosystems, non-transitive communities allow competing species to coexist, which helps maintain biodiversity [24]. The classic non-transitive system involves a tripartite community of competing species that exhibit a relationship analogous to the game rock-paper-scissors, where rock dominates scissors, scissors outcompete paper, and paper prevails over rock. Therefore, the study of non-transitive relations is very necessary. Furthermore, in the study of random variables [8, 9], the absence of transitivity gives rise to a cycle-transitivity framework. The lack of transitivity has two main effects: the presence of cycles and incomparability. Specifically, if X is better than Y, Y is better than Z, and Z is better than X, this leads to a cycle, or X is better than Y, Y is better than Z, but X and Z are not comparable, thus creating incomparability.

Due to the aforementioned factors, scholars have initiated investigations into structures featuring non-transitive relations, such as pseudo-ordered sets and trellises. The trellises are also known as weakly associative lattices, tournament lattices or non-associative lattices in some literature [10, 16, 17, 18]. Although the absence of transitivity of the relation does not affect the existence of the meets and the joins, it results in the meets and the joins no longer satisfying associativity. Recently, Zedam et al. [38] extended the concept of a t-norm to bounded pseudo-ordered sets and in particular on bounded trellises and gave some examples. As mentioned above, due to the lack of transitivity, the meet operation on a proper bounded trellis is no longer a t-norm, and it has been shown that there may be multiple maximal t-norms or no maximal t-norms on a bounded trellis. They also provided a method for constructing t-norms on a bounded $\wedge$-semi-trellis based on interior operators. Inspired by this, Kong et al. [26] introduced the notion of uninorms on bounded trellises. They found that only those elements that are middle-transitive on a bounded trellis can act as the neutral element of a uninorm, and provided several methods for constructing uninorms on bounded trellises by means of interior operators and closure operators, respectively, based on the assumption that the neutral element is left-transitive, right-transitive, or not in any cycles. Xiu et al. [34] introduced the notion of nullnorms to bounded trellises, they found that the element acting as an annihilator must be middle-transitive, and gave the corresponding construction methods. In this paper, we focus on the construction of nullnorms on bounded trellises. In particular, we assert that the annihilator of a nullnorm on a bounded trellis cannot exist in any cycle, and present several new methods for constructing a nullnorm on a bounded trellis by means of the order-preserving mappings and t-(co)norms. In addition, some illustrative examples for new construction methods are provided.

The remaining sections of this paper are structured as follows. In Section 2, we recall some essential definitions and results about trellises which will be utilized in this paper. In Section 3, we review the basic properties of nullnorms on bounded trellises and further explore the necessary conditions for the element that acts as the annihilator of a nullnorm, then a series of new methods to construct nullnorms on bounded trellises and some illustrative examples are presented. Finally, we give the conclusion in Section 4.

2. PRELIMINARIES

In this section, we recall some essential definitions and results about trellises which will be used in this paper, more details can be found in [17, 19, 31].

Definition 2.1. (Skala [31]) Let X be a nonempty set. A pseudo-order on X is a binary relation $\trianglelefteq$ on X such that, for all $x, y \in X$,

(i) $x \trianglelefteq x$ (reflexivity);

(ii) $x \trianglelefteq y$ and $y \trianglelefteq x$ implies $x = y$ (antisymmetry).

A nonempty set X is called a pseudo-ordered set (psoset, for short) if it is equipped with a pseudo-order $\trianglelefteq$, we denote it by $P = (X, \trianglelefteq)$.

Let $P = (X, \trianglelefteq)$ be a psoset and $x, y \in X$. If $x \trianglelefteq y$ and $x \neq y$, then we write $x \lhd y$; if $x \trianglelefteq y$ does not hold, then we write $x \ntrianglelefteq y$. If $x \ntrianglelefteq y$ and $y \ntrianglelefteq x$, we say that x and y are incomparable, in which case we denote $x \parallel y$. The set of all elements in X that are incomparable to x is denoted by I_x. We write $x \lesssim y$ if there exists a finite sequence $(x_1, ..., x_n)$ of elements from X such that $x \trianglelefteq x_1 \trianglelefteq ... \trianglelefteq x_n \trianglelefteq y$. Take a subset Y of X and $a, b \in Y$, we write $a \lesssim_Y b$ if there exists a finite sequence $(y_1, ..., y_n)$ of elements from Y such that $a \trianglelefteq y_1 \trianglelefteq ... \trianglelefteq y_n \trianglelefteq b$. If for all $a, b \in Y$, both $a \lesssim_Y b$ and $b \lesssim_Y a$ hold, then Y is called a cycle. Obviously, any singleton $Y = \{x\}$ is a trivial cycle. Owing to the antisymmetry of $\trianglelefteq$, any non-trivial cycle contains at least three elements.

Similarly to partially ordered sets (posets, for short), the representation of a finite pseudo-ordered set can be achieved through a Hasse diagram with the following distinction: if x and y are not related, while in a poset this would be implied by transitivity, then x and y are joined by a dashed edge. If $x \lesssim y$ and $y \trianglelefteq x$, then x and y are joined by a directed edge going from y to x.

The concepts of the maximal/minimal element, the greatest/smallest element, the upper/lower bound, the smallest upper bound (supremum), the greatest lower bound (infimum) for psosets are defined in the same way as the corresponding concepts for posets. Take a subset A of a psoset $P = (X, \trianglelefteq)$, the antisymmetry of the pseudo-order implies that if A has an supremum (resp. infimum), then it is unique, and is denoted by $\bigvee A$ (resp. $\bigwedge A$). If $A = \{a, b\}$, then we write $a \vee b$ (called join) instead of $\bigvee\{a, b\}$ and $a \wedge b$ (called meet) instead of $\bigwedge\{a, b\}$.

Definition 2.2. (Kong and Zhao [26]) A bounded psoset is a psoset that has a smallest element denoted by 0 and a greatest element denoted by 1, i.e. $0 \trianglelefteq x \trianglelefteq 1$, for all $x \in X$. We denote it by $P = (X, \trianglelefteq, 0, 1)$.

Definition 2.3. (Gladstien [19]) A $\wedge$-semi-trellis (resp. $\vee$-semi-trellis) is a psoset $P = (X, \trianglelefteq)$ such that $x \wedge y$ (resp. $x \vee y$) exists for all $x, y \in X$. A trellis is a psoset that is both a $\wedge$-semi-trellis and a $\vee$-semi-trellis and we denote it by $P = (X, \trianglelefteq, \wedge, \vee)$. If a trellis is not a lattice, then we call it a proper trellis. Clearly, $x \trianglelefteq y$ is defined as $x \wedge y = x$ (or $x \vee y = y$) for a trellis. A bounded $\wedge$-semi-trellis is denoted by $P = (X, \trianglelefteq, \wedge, 0, 1)$. A bounded $\vee$-semi-trellis is denoted by $P = (X, \trianglelefteq, \vee, 0, 1)$. A bounded trellis is denoted by $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$.

In this paper, we only consider the bounded trellis containing at least three elements.

Definition 2.4. (Kong and Zhao [26]) Let $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis, $a, b \in X, a \trianglelefteq b$. A subinterval $[a, b]$ of X is defined as $[a, b] = \{x \in X | a \trianglelefteq x \trianglelefteq b\}$. Similarly, $(a, b] = \{x \in X | a \lhd x \trianglelefteq b\}$, $[a, b) = \{x \in X | a \trianglelefteq x \lhd b\}$ and $(a, b) = \{x \in X | a \lhd x \lhd b\}$.

Definition 2.5. (Skala [31]) Let $P = (X, \trianglelefteq)$ be a psoset. An element $b \in X$ is said to be:

(i) right-transitive, if $b \trianglelefteq x \trianglelefteq y$ implies $b \trianglelefteq y$, for all $x, y \in X$;

(ii) left-transitive, if $x \trianglelefteq y \trianglelefteq b$ implies $x \trianglelefteq b$, for all $x, y \in X$;

(iii) middle-transitive, if $x \trianglelefteq b \trianglelefteq y$ implies $x \trianglelefteq y$, for all $x, y \in X$;

(iv) transitive, if it is right-transitive, left-transitive and middle-transitive.

In this paper, the set of right-transitive elements of X, the set of left-transitive elements of X and the set of middle-transitive elements of X are denoted by $X^{rtr}, X^{ltr}, X^{mtr}$, respectively.

Definition 2.6. (Zedam and Baets [38]) Let $P = (X, \trianglelefteq, 0, 1)$ be a bounded psoset. A binary operation H on P is called:

(i) commutative, if $H(x, y) = H(y, x)$, for all $x, y \in X$;

(ii) associative, if $H(H(x, y), z) = H(x, H(y, z))$, for all $x, y, z \in X$;

(iii) right-increasing, if $y \trianglelefteq z$ implies $H(x, y) \trianglelefteq H(x, z)$, for all $x, y, z \in X$;

(iv) left-increasing, if $y \trianglelefteq z$ implies $H(y, x) \trianglelefteq H(z, x)$, for all $x, y, z \in X$;

(v) increasing, if $x \trianglelefteq y$ and $z \trianglelefteq t$ implies $H(x, z) \trianglelefteq H(y, t)$, for all $x, y, z, t \in X$.

Definition 2.7. (Zedam and Baets [38]) Let $P = (X, \trianglelefteq, 0, 1)$ be a bounded psoset. A binary operation $T : X^2 \to X$ is called a triangular norm (t-norm, for short) on P if it is commutative, associative, increasing and has 1 as the neutral element, i.e. $T(x, 1) = x$, for all $x \in X$.

Definition 2.8. (Zedam and Baets [38]) Let $P = (X, \trianglelefteq, 0, 1)$ be a bounded psoset. A binary operation $S : X^2 \to X$ is called a triangular conorm (t-conorm, for short) on P if it is commutative, associative, increasing and has 0 as the neutral element, i.e. $S(x, 0) = x$, for all $x \in X$.

We introduce the concept of order-preserving mapping on a psoset similar to a poset.

Definition 2.9. Let $P_1 = (X, \trianglelefteq)$ and $P_2 = (Y, \trianglelefteq)$ be psosets. A mapping $f : X \to Y$ is said to be order-preserving if $x \trianglelefteq y$ implies $f(x) \trianglelefteq f(y)$ for any $x, y \in X$.

3. NULLNORMS ON BOUNDED TRELLISES

In this section, we recall the concept and basic properties of a nullnorm on a bounded trellis, explore the necessary conditions for the element to act as the annihilator, and present several new methods for constructing nullnorms on bounded trellises.

For convenience, we denote $I_a^r = \{x \in I_a | x \lesssim a\}$, $I_a^l = \{x \in I_a | a \lesssim x\}$, $I_a^* = \{x \in I_a | x \notin (I_a^r \cup I_a^l)\}$, $I_a^1 = \{x \in I_a | x \lesssim a$ and x is joined to a by a dashed edge $\}$, $I_a^2 = \{x \in I_a | a \lesssim x$ and a is joined to x by a dashed edge $\}$, $I_a^3 = \{x \in I_a |$ x and a are not joined by a dashed edge $\}$, and $K = \{x \in X^{mtr} |$ x does not belong to any cycle $\}$. Obviously, $I_a^1 \subseteq I_a^r$, $I_a^2 \subseteq I_a^l$, $I_a^* \subseteq I_a^3$, $I_a = I_a^r \cup I_a^l \cup I_a^* = I_a^1 \cup I_a^2 \cup I_a^3$.

Example 3.1. Let $P_1 = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis given by Hasse diagram in Figure 1, where $X = \{0, a, b, c, d, e, f, g, h, 1\}$. Then $b \lesssim d$ and $d \trianglelefteq b$, $b \ntrianglelefteq d$, $a \ntrianglelefteq g$, $\{b, c, d\}$ is a cycle, $[0, a] = \{0, b, c, a\}$, $[a, 1] = \{a, f, h, 1\}$, $I_a^r = \{d\}$, $I_a^1 = \{\emptyset\}$, $I_a^l = I_a^2 = \{g\}$, $I_a^* = \{e\}$, $I_a^3 = \{d, e\}$.

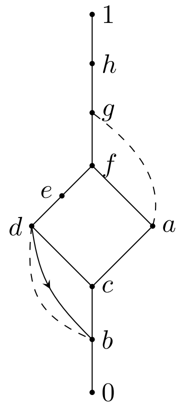

Fig. 1: Hasse diagram of the bounded trellis P_1.

Definition 3.2. (Xiu and Zheng [34]) Let $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis. A binary operation $V : X^2 \to X$ is called a nullnorm on P if it is commutative, associative, increasing and there exists an element $a \in X$, called the annihilator of V, such that $V(0, x) = x$ for any $x \in [0, a]$ and $V(1, x) = x$ for any $x \in [a, 1]$.

It is clear that the nullnorm degenerates to the t-norm when $a = 0$ and to the t-conorm when $a = 1$. In this paper, we only consider the case $a \in X \backslash \{0, 1\}$.

3.1. Basic properties of nullnorms on bounded trellises

Proposition 3.3. (Xiu and Zheng [34]) Let $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis, $a \in X \backslash \{0, 1\}$ and V be a nullnorm on P with annihilator a. Then the following statements hold:

(i) $V|_{[0,a]^2}$ is a t-conorm on $[0, a]$.

(ii) $V|_{[a,1]^2}$ is a t-norm on $[a, 1]$.

(iii) $V(x, y) = a$ for all $(x, y) \in [0, a] \times [a, 1]$.

(iv) $a \trianglelefteq V(x, y)$ for all $(x, y) \in [a, 1]^2 \cup ([a, 1] \times I_a) \cup (I_a \times [a, 1])$.

(v) $V(x, y) \trianglelefteq a$ for all $(x, y) \in [0, a]^2 \cup ([0, a] \times I_a) \cup (I_a \times [0, a])$.

(vi) $V(x, y) \trianglelefteq y$ for all $(x, y) \in X \times [a, 1]$.

(vii) $V(x, y) \trianglelefteq x$ for all $(x, y) \in [a, 1] \times X$.

(viii) $x \trianglelefteq V(x,y)$ for all $(x,y) \in [0,a] \times X$.

(ix) $y \trianglelefteq V(x,y)$ for all $(x,y) \in X \times [0,a]$.

(x) $x \vee y \trianglelefteq V(x,y)$ for all $(x,y) \in [0,a]^2$.

(xi) $V(x,y) \trianglelefteq x \wedge y$ for all $(x,y) \in [a,1]^2$.

(xii) $(x \wedge a) \vee (y \wedge a) \trianglelefteq V(x,y)$ for all $(x,y) \in ([0,a] \times I_a) \cup (I_a \times [0,a]) \cup I_a^2$.

(xiii) $V(x,y) \trianglelefteq (x \vee a) \wedge (y \vee a)$ for all $(x,y) \in ([a,1] \times I_a) \cup (I_a \times [a,1]) \cup I_a^2$.

Proposition 3.4. (Xiu and Zheng [34]) Let $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis, $a \in X \backslash \{0,1\}$ and V be a nullnorm on P with annihilator a.

(i) If $x \in I_a^1$, then $V(x,y) = a$ for all $y \in [a,1]$.

(ii) If $x \in I_a^2$, then $V(x,y) = a$ for all $y \in [0,a]$.

Remark 3.5. In fact, Proposition 3.4 (i) is valid for the case where $x \in I_a^r$ and Proposition 3.4 (ii) is valid for the case where $x \in I_a^l$. Thus, we have the following proposition.

Proposition 3.6. Let $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis, $a \in X \backslash \{0,1\}$ and V be a nullnorm on P with annihilator a.

(i) If $x \in I_a^r$, then $V(x,y) = a$ for all $y \in [a,1]$.

(ii) If $x \in I_a^l$, then $V(x,y) = a$ for all $y \in [0,a]$.

Proof. The proof is similar to the proof of Proposition 3.2 in [34]. □

Proposition 3.7. (Xiu and Zheng [34]) Let $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis, $a \in X \backslash \{0,1\}$ and V be a nullnorm on P with annihilator a. Then $a \in X^{mtr}$.

Remark 3.8. (Kong and Zhao [26]) Let $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis. If $a \in X^{mtr}$, then $x \triangleleft y$ for all $x \in (0,a), y \in (a,1)$.

Proposition 3.9. Let $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis, $a \in X^{mtr} \backslash \{0,1\}$ and V be a nullnorm on P with annihilator a. Then a is not in any cycle in P, i.e. $a \in K \backslash \{0,1\}$.

Proof. Suppose that $a \triangleleft x_1 \triangleleft x_2 \triangleleft x_3 \triangleleft \cdots \triangleleft x_{n-2} \triangleleft x_{n-1} \triangleleft x_n \triangleleft a$, where $x_i \in X, i = 1, 2, \ldots, n$. Clearly, $x_i \notin \{0,1\}, x_1 \in (a,1), x_n \in (0,a)$. Since $a \in X^{mtr}$, according to Remark 3.8, we have $x_2 \notin (0,a), x_{n-1} \notin (a,1)$ and further $n \geq 3$.

Now, we prove $n \neq 3$. Suppose that $a \triangleleft x_1 \triangleleft x_2 \triangleleft x_3 \triangleleft a$, then $x_2 \notin (0,a) \cup (a,1)$, we only need to talk about the case where $x_2 \in I_a$. Since $a \in X^{mtr}$, then $x_3 \triangleleft x_1$. Since $x_2 \in I_a^l$, then we have $V(0, x_2) = a$ and $V(x_3, x_2) = a$ by Proposition 3.6 (ii). According to the increasing property of V, we can get

$$a = V(0, x_2) \trianglelefteq V(1, x_2) \trianglelefteq V(1, x_3) = a,$$

therefore, $V(1,x_2)=a$. Also we have

$$a=V(x_2,x_3)\trianglelefteq V(1,x_1)\trianglelefteq V(1,x_2)=a,$$

then $V(1,x_1)=a$. However, by the definition of the nullnorm, we have $V(1,x_1)=x_1$, it contradicts with $x_2\in I_a$. Thus $n>3$.

Now, we talk about the case where $n\geq 4$. From what has been discussed above, we have $x_2\in(a,1)\cup I_a, x_{n-1}\notin(a,1)$. If $x_2\in I_a$, then $x_2\in I_a^r$, and by Proposition 3.6 (i), we have $V(x_2,1)=a$. By the increasing property of V, we can get

$$x_1=V(x_1,1)\trianglelefteq V(x_2,1)=a,$$

it contradicts with $x_1\in(a,1)$. Thus $x_2\in(a,1)$. Because of $x_2\lhd x_3$, we have $x_3\in(a,1)\cup I_a$. Similarly, it can be proved that $x_i\in(a,1), i=3,4,\dots,n-1$, it contradicts with $x_{n-1}\notin(a,1)$. To sum up, a is not in any cycle in P. □

Remark 3.10. According to the Proposition 3.9, we can obtain that if there is a nullnorm with annihilator a on a bounded trellis $P=(X,\trianglelefteq,\wedge,\vee,0,1)$, then $I_a^r\cap I_a^l=\emptyset$, $A=\{x\in I_a^l|\ \exists y\in(0,a)\cup I_a^r\cup I_a^*\text{ s.t. } x\lhd y\}=\emptyset$, $B=\{x\in I_a^*|\ \exists y\in(0,a)\cup I_a^r\text{ s.t. } x\lhd y\}=\emptyset$, and $C=\{x\in(a,1)|\ \exists y\in(0,a)\cup I_a^r\cup I_a^*\text{ s.t. } x\lhd y\}=\emptyset$.

3.2. Some methods for constructing nullnorms on bounded trellises

Now, we present some new methods for constructing nullnorms on bounded trellises.

Theorem 3.11. Let $P=(X,\trianglelefteq,\wedge,\vee,0,1)$ be a bounded trellis, $a\in K\backslash\{0,1\}$, T be a t-norm on $[a,1]$, S be a t-conorm on $[0,a]$, $f:X\to[a,1]$ be an order-preserving mapping satisfying $f(x)=x$ for all $x\in[a,1]$, $g:X\to[0,a]$ be an order-preserving mapping satisfying $g(x)=x$ for all $x\in[0,a]$. Then the binary operation V defined by (1) is a nullnorm with annihilator a on P.

$$V(x,y)=\begin{cases}S(g(x),g(y)), & (x,y)\in([0,a]\cup I_a^r)^2,\\ T(f(x),f(y)), & (x,y)\in([a,1]\cup I_a^l\cup I_a^*)^2,\\ a, & \text{otherwise.}\end{cases} \tag{1}$$

Proof. It is obvious that V is commutative and has an annihilator a. Thus we only need to prove the increasing property and associativity of V.

Let $x,y,z,t\in X$ such that $x\trianglelefteq y, z\trianglelefteq t$, we verify $V(x,z)\trianglelefteq V(y,t)$.

1. $x\in[0,a]\cup I_a^r, z\in[0,a]\cup I_a^r$, then $V(x,z)=S(g(x),g(z))\trianglelefteq a$.
   1.1. If $y\in[0,a]\cup I_a^r, t\in[0,a]\cup I_a^r$, then $V(x,z)=S(g(x),g(z))\trianglelefteq S(g(y),g(t))=V(y,t)$.
   1.2. If $y\in[0,a]\cup I_a^r, t\in(a,1]\cup I_a^l\cup I_a^*$ or $y\in(a,1]\cup I_a^l\cup I_a^*, t\in[0,a]\cup I_a^r$, then $V(x,z)=S(g(x),g(z))\trianglelefteq a=V(y,t)$.
   1.3. If $y,t\in(a,1]\cup I_a^l\cup I_a^*$, then $V(x,z)=S(g(x),g(z))\trianglelefteq a\trianglelefteq T(f(y),f(t))=V(y,t)$, further $V(x,z)\trianglelefteq V(y,t)$ due to $a\in X^{mtr}$.

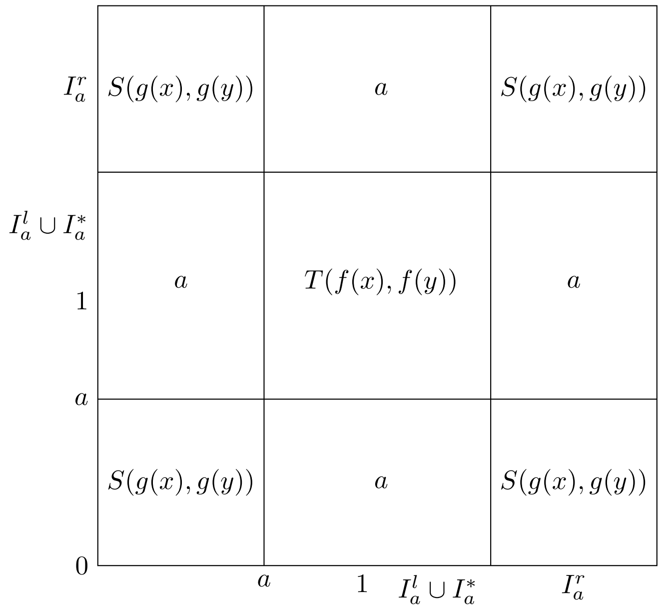

Fig. 2: The nullnorm V on the bounded trellis P in Theorem 3.11.

2. $x \in [0, a] \cup I_a^r, z \in (a, 1] \cup I_a^l$, then $t \in (a, 1] \cup I_a^l$ and $V(x, z) = a$.
   - 2.1. If $y \in [0, a] \cup I_a^r$, then $V(x, z) = a = V(y, t)$.
   - 2.2. If $y \in (a, 1] \cup I_a^l \cup I_a^*$, then $V(x, z) = a \trianglelefteq T(f(y), f(t)) = V(y, t)$.
3. $x \in [0, a] \cup I_a^r, z \in I_a^*$, then $t \in (a, 1] \cup I_a^l \cup I_a^*$ and $V(x, z) = a$.
   - 3.1. If $y \in [0, a] \cup I_a^r$, then $V(x, z) = a = V(y, t)$.
   - 3.2. If $y \in (a, 1] \cup I_a^l \cup I_a^*$, then $V(x, z) = a \trianglelefteq T(f(y), f(t)) = V(y, t)$.
4. $x \in (a, 1] \cup I_a^l, z \in [0, a] \cup I_a^r$, the proof is similar to the case 2.
5. $x, z \in (a, 1] \cup I_a^l$, then $y, t \in (a, 1] \cup I_a^l$ and $V(x, z) = T(f(x), f(z)) \trianglelefteq T(f(y), f(t)) = V(y, t)$.
6. $x \in (a, 1] \cup I_a^l, z \in I_a^*$, then $y \in (a, 1] \cup I_a^l, t \in (a, 1] \cup I_a^l \cup I_a^*$ and $V(x, z) = T(f(x), f(z)) \trianglelefteq T(f(y), f(t)) = V(y, t)$.
7. $x \in I_a^*, z \in [0, a] \cup I_a^r$, the proof is similar to the case 3.
8. $x \in I_a^*, z \in (a, 1] \cup I_a^l$, the proof is similar to the case 6.
9. $x, z \in I_a^*$, then $y, t \in (a, 1] \cup I_a^l \cup I_a^*$ and $V(x, z) = T(f(x), f(z)) \trianglelefteq T(f(y), f(t)) = V(y, t)$.

In summary, V is increasing. Now, we verify the associativity of V. Let $x, y, z \in X$.

1. $x \in [0,a] \cup I_a^r, y \in [0,a] \cup I_a^r$, then $V(x,y) = S(g(x), g(y)) \trianglelefteq a$.

   1.1. If $z \in [0,a] \cup I_a^r$, then $V(y,z) = S(g(y), g(z)) \trianglelefteq a$ and we have that
   $V(V(x,y),z) = S(g(S(g(x),g(y))),g(z)) = S(S(g(x),g(y)),g(z))$
   $= S(g(x), S(g(y),g(z))) = S(g(x), g(S(g(y),g(z)))) = V(x, V(y,z))$.

   1.2. If $z \in (a,1] \cup I_a^l \cup I_a^*$, then $V(y,z) = a$ and $V(V(x,y),z) = a = V(x,V(y,z))$.

2. $x \in [0,a] \cup I_a^r, y \in (a,1] \cup I_a^l \cup I_a^*$, then $V(x,y) = a$.

   2.1. If $z \in [0,a] \cup I_a^r$, then $V(y,z) = a$ and $V(V(x,y),z) = a = V(x,V(y,z))$.

   2.2. If $z \in (a,1] \cup I_a^l \cup I_a^*$, then $a \trianglelefteq T(f(y), f(z)) = V(y,z)$ and $V(V(x,y),z) = a = V(x,V(y,z))$.

3. $x \in (a,1] \cup I_a^l \cup I_a^*, y \in [0,a] \cup I_a^r$, then $V(x,y) = a$.

   3.1. If $z \in [0,a] \cup I_a^r$, then $V(y,z) = S(g(y),g(z)) \trianglelefteq a$ and $V(V(x,y),z) = a = V(x,V(y,z))$.

   3.2. If $z \in (a,1] \cup I_a^l \cup I_a^*$, then $V(y,z) = a$ and $V(V(x,y),z) = a = V(x,V(y,z))$.

4. $x, y \in (a,1] \cup I_a^l \cup I_a^*$, then $a \trianglelefteq T(f(x), f(y)) = V(x,y)$.

   4.1. If $z \in [0,a] \cup I_a^r$, then $V(y,z) = a$ and $V(V(x,y),z) = a = V(x,V(y,z))$.

   4.2. If $z \in (a,1] \cup I_a^l \cup I_a^*$, then $a \trianglelefteq T(f(y), f(z)) = V(y,z)$ and we have that
   $V(V(x,y),z) = T(f(T(f(x),f(y))),f(z)) = T(T(f(x),f(y)),f(z))$
   $= T(f(x), T(f(y),f(z))) = T(f(x), f(T(f(y),f(z)))) = V(x,V(y,z))$.

□

Theorem 3.12. Let $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis, $a \in K \backslash \{0,1\}$, T be a t-norm on $[a,1]$, S be a t-conorm on $[0,a]$, $f : X \to [a,1]$ be an order-preserving mapping satisfying $f(x) = x$ for all $x \in [a,1]$, $g : X \to [0,a]$ be an order-preserving mapping satisfying $g(x) = x$ for all $x \in [0,a]$. Then the binary operation V defined by (2) is a nullnorm with annihilator a on P.

$$V(x,y) = \begin{cases} S(g(x), g(y)), & (x,y) \in ([0,a] \cup I_a^r \cup I_a^*)^2, \\ T(f(x), f(y)), & (x,y) \in ([a,1] \cup I_a^l)^2, \\ a, & \text{otherwise}. \end{cases} \tag{2}$$

Proof. It is obvious that V is commutative and has an annihilator a. Thus we only need to prove the increasing property and associativity of V.

Let $x, y, z, t \in X$ such that $x \trianglelefteq y, z \trianglelefteq t$, we verify $V(x,z) \trianglelefteq V(y,t)$.

1. $x \in [0,a] \cup I_a^r, z \in [0,a] \cup I_a^r$, then $V(x,z) = S(g(x), g(z)) \trianglelefteq a$.

   1.1. If $y \in [0,a] \cup I_a^r \cup I_a^*, t \in [0,a] \cup I_a^r \cup I_a^*$, then $V(x,z) = S(g(x), g(z)) \trianglelefteq S(g(y), g(t)) = V(y,t)$.

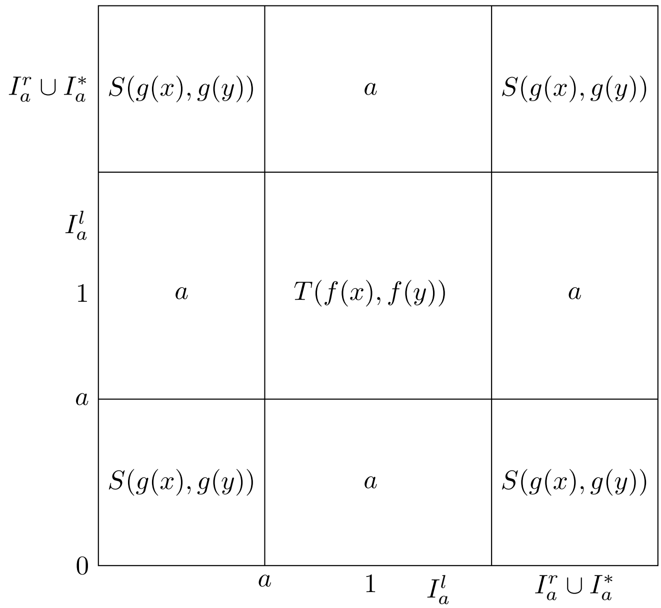

Fig. 3: The nullnorm V on the bounded trellis P in Theorem 3.12.

1.2. If $y \in [0,a] \cup I_a^r \cup I_a^*, t \in (a,1] \cup I_a^l$ or $y \in (a,1] \cup I_a^l, t \in [0,a] \cup I_a^r \cup I_a^*$, then $V(x,z) = S(g(x), g(z)) \trianglelefteq a = V(y,t)$.

1.3. If $y, t \in (a,1] \cup I_a^l$, then $V(x,z) = S(g(x), g(z)) \trianglelefteq a \trianglelefteq T(f(y), f(t)) = V(y,t)$, further $V(x,z) \trianglelefteq V(y,t)$ due to $a \in X^{mtr}$.

2. $x \in [0,a] \cup I_a^r, z \in (a,1] \cup I_a^l$, then $t \in (a,1] \cup I_a^l$ and $V(x,z) = a$.

2.1. If $y \in [0,a] \cup I_a^r \cup I_a^*$, then $V(x,z) = a = V(y,t)$.

2.2. If $y \in (a,1] \cup I_a^l$, then $V(x,z) = a \trianglelefteq T(f(y), f(t)) = V(y,t)$.

3. $x \in [0,a] \cup I_a^r, z \in I_a^*$, then $t \in (a,1] \cup I_a^l \cup I_a^*$ and $V(x,z) = S(g(x), g(z)) \trianglelefteq a$.

3.1. If $y \in [0,a] \cup I_a^r \cup I_a^*, t \in (a,1] \cup I_a^l$, then $V(x,z) \trianglelefteq a = V(y,t)$.

3.2. If $y \in [0,a] \cup I_a^r \cup I_a^*, t \in I_a^*$, then we have $V(x,z) = S(g(x), g(z)) \trianglelefteq S(g(y), g(t)) = V(y,t)$.

3.3. If $y \in (a,1] \cup I_a^l, t \in (a,1] \cup I_a^l$, then $V(x,z) \trianglelefteq a \trianglelefteq T(f(y), f(t)) = V(y,t)$, further we have $V(x,z) \trianglelefteq V(y,t)$ due to $a \in X^{mtr}$.

3.4. If $y \in (a,1] \cup I_a^l, t \in I_a^*$, then $V(x,z) \trianglelefteq a = V(y,t)$.

4. $x \in (a,1] \cup I_a^l, z \in [0,a] \cup I_a^r$, the proof is similar to the case 2.

5. $x, z \in (a,1] \cup I_a^l$, then $y, t \in (a,1] \cup I_a^l$ and $V(x,z) = T(f(x), f(z)) \trianglelefteq T(f(y), f(t)) = V(y,t)$.

6. $x \in (a,1] \cup I_a^l, z \in I_a^*$, then $y \in (a,1] \cup I_a^l, t \in (a,1] \cup I_a^l \cup I_a^*$ and $V(x,z) = a$.
   - 6.1. If $t \in (a,1] \cup I_a^l$, then $V(x,z) = a \trianglelefteq T(f(y), f(t)) = V(y,t)$.
   - 6.2. If $t \in I_a^*$, then $V(x,z) = a = V(y,t)$.
7. $x \in I_a^*, z \in [0,a] \cup I_a^r$, the proof is similar to the case 3.
8. $x \in I_a^*, z \in (a,1] \cup I_a^l$, the proof is similar to the case 6.
9. $x, z \in I_a^*$, then $y, t \in (a,1] \cup I_a^l \cup I_a^*$ and $V(x,z) = S(g(x), g(z)) \trianglelefteq a$.
   - 9.1. If $y, t \in (a,1] \cup I_a^l$, then $V(x,z) \trianglelefteq a \trianglelefteq T(f(y), f(t)) = V(y,t)$, further we have $V(x,z) \trianglelefteq V(y,t)$ due to $a \in X^{mtr}$.
   - 9.2. If $y \in (a,1] \cup I_a^l, t \in I_a^*$ or $y \in I_a^*, t \in (a,1] \cup I_a^l$, then $V(x,z) \trianglelefteq a = V(y,t)$.
   - 9.3. If $y, t \in I_a^*$, then $V(x,z) = S(g(x), g(z)) \trianglelefteq S(g(y), g(t)) = V(y,t)$.

In summary, V is increasing. Now, we verify the associativity of V. Let $x, y, z \in X$.

1. $x, y \in [0,a] \cup I_a^r \cup I_a^*$, then $V(x,y) = S(g(x), g(y)) \trianglelefteq a$.
   - 1.1. If $z \in [0,a] \cup I_a^r \cup I_a^*$, then $V(y,z) = S(g(y), g(z)) \trianglelefteq a$ and we have that
     $V(V(x,y),z) = S(g(S(g(x),g(y))),g(z)) = S(S(g(x),g(y)),g(z))$
     $= S(g(x), S(g(y),g(z))) = S(g(x), g(S(g(y),g(z)))) = V(x, V(y,z))$.
   - 1.2. If $z \in (a,1] \cup I_a^l$, then $V(y,z) = a$ and $V(V(x,y),z) = a = V(x,V(y,z))$.
2. $x \in [0,a] \cup I_a^r \cup I_a^*, y \in (a,1] \cup I_a^l$, then $V(x,y) = a$.
   - 2.1. If $z \in [0,a] \cup I_a^r \cup I_a^*$, then $V(y,z) = a$ and $V(V(x,y),z) = a = V(x,V(y,z))$.
   - 2.2. If $z \in (a,1] \cup I_a^l$, then $a \trianglelefteq T(f(y), f(z)) = V(y,z)$ and $V(V(x,y),z) = a = V(x,V(y,z))$.
3. $x \in (a,1] \cup I_a^l, y \in [0,a] \cup I_a^r \cup I_a^*$, then $V(x,y) = a$.
   - 3.1. If $z \in [0,a] \cup I_a^r \cup I_a^*$, then $V(y,z) = S(g(y), g(z)) \trianglelefteq a$ and $V(V(x,y),z) = a = V(x,V(y,z))$.
   - 3.2. If $z \in (a,1] \cup I_a^l$, then $V(y,z) = a$ and $V(V(x,y),z) = a = V(x,V(y,z))$.
4. $x, y \in (a,1] \cup I_a^l$, then $a \trianglelefteq T(f(x), f(y)) = V(x,y)$.
   - 4.1. If $z \in [0,a] \cup I_a^r \cup I_a^*$, then $V(y,z) = a$ and $V(V(x,y),z) = a = V(x,V(y,z))$.
   - 4.2. If $z \in (a,1] \cup I_a^l$, then $a \trianglelefteq T(f(y), f(z)) = V(y,z)$ and we have that
     $V(V(x,y),z) = T(f(T(f(x),f(y))),f(z)) = T(T(f(x),f(y)),f(z))$
     $= T(f(x), T(f(y),f(z))) = T(f(x), f(T(f(y),f(z)))) = V(x,V(y,z))$.

□

Theorem 3.13. Let $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis, $a \in K \backslash \{0, 1\}$, T be a t-norm on $[a, 1]$, S be a t-conorm on $[0, a]$, $f : X \to [a, 1]$ be an order-preserving mapping satisfying $f(x) = x$ for all $x \in [a, 1]$, $g : X \to [0, a]$ be an order-preserving mapping satisfying $g(x) = x$ for all $x \in [0, a]$. Then the binary operation V defined by (3) is a nullnorm with annihilator a on P.

$$V(x,y) = \begin{cases} S(g(x), g(y)), & (x,y) \in ([0,a] \cup I_a^r)^2, \\ T(f(x), f(y)), & (x,y) \in ([a,1] \cup I_a^l)^2, \\ a, & \text{otherwise.} \end{cases} \tag{3}$$

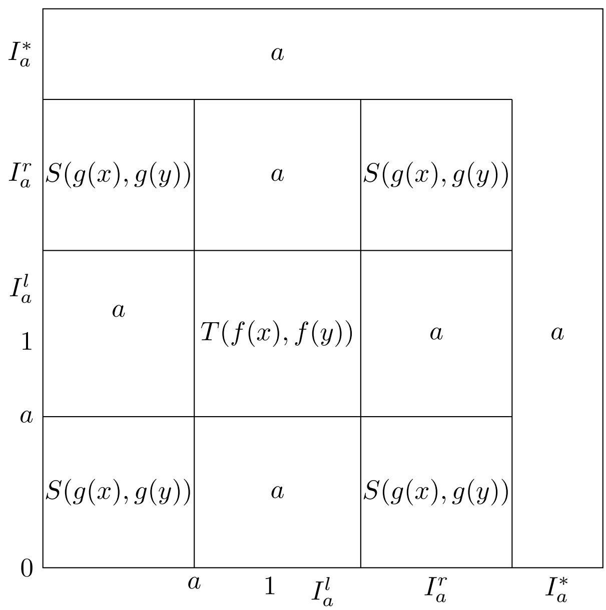

Fig. 4: The nullnorm V on the bounded trellis P in Theorem 3.13.

Proof. It is obvious that V is commutative and has an annihilator a. Thus we only need to prove the increasing property and associativity of V.

Let $x, y, z, t \in X$ such that $x \trianglelefteq y, z \trianglelefteq t$, we verify $V(x, z) \trianglelefteq V(y, t)$.

1. $x \in [0, a] \cup I_a^r, z \in [0, a] \cup I_a^r$, then $V(x, z) = S(g(x), g(z)) \trianglelefteq a$.

   1.1. If $y \in [0, a] \cup I_a^r, t \in [0, a] \cup I_a^r$, then $V(x, z) = S(g(x), g(z)) \trianglelefteq S(g(y), g(t)) = V(y, t)$.

   1.2. If $y \in [0, a] \cup I_a^r, t \in (a, 1] \cup I_a^l$ or $y \in (a, 1] \cup I_a^l, t \in [0, a] \cup I_a^r$, then $V(x, z) = S(g(x), g(z)) \trianglelefteq a = V(y, t)$.

   1.3. If $y, t \in (a, 1] \cup I_a^l$, then $V(x, z) = S(g(x), g(z)) \trianglelefteq a \trianglelefteq T(f(y), f(t)) = V(y, t)$, further $V(x, z) \trianglelefteq V(y, t)$ due to $a \in X^{mtr}$.

1.4. If $y \in I_a^*$ or $t \in I_a^*$, then $V(x,z) = S(g(x), g(z)) \trianglelefteq a = V(y,t)$.

2. $x \in [0,a] \cup I_a^r, z \in (a,1] \cup I_a^l$, then $t \in (a,1] \cup I_a^l$ and $V(x,z) = a$.

   2.1. If $y \in [0,a] \cup I_a^r \cup I_a^*$, then $V(x,z) = a = V(y,t)$.

   2.2. If $y \in (a,1] \cup I_a^l$, then $V(x,z) = a \trianglelefteq T(f(y), f(t)) = V(y,t)$.

3. $x \in [0,a] \cup I_a^r, z \in I_a^*$, then $t \in (a,1] \cup I_a^l \cup I_a^*$ and $V(x,z) = a$.

   3.1. If $y \in [0,a] \cup I_a^r \cup I_a^*, t \in (a,1] \cup I_a^l \cup I_a^*$, then $V(x,z) = a = V(y,t)$.

   3.2. If $y \in (a,1] \cup I_a^l, t \in (a,1] \cup I_a^l$, then $V(x,z) = a \trianglelefteq T(f(y), f(t)) = V(y,t)$.

   3.3. If $y \in (a,1] \cup I_a^l, t \in I_a^*$, then $V(x,z) = a = V(y,t)$.

4. $x \in (a,1] \cup I_a^l, z \in [0,a] \cup I_a^r$, the proof is similar to the case 2.

5. $x, z \in (a,1] \cup I_a^l$, then $y, t \in (a,1] \cup I_a^l$ and $V(x,z) = T(f(x), f(z)) \trianglelefteq T(f(y), f(t)) = V(y,t)$.

6. $x \in (a,1] \cup I_a^l, z \in I_a^*$, then $y \in (a,1] \cup I_a^l$ and $V(x,z) = a \trianglelefteq V(y,t)$.

7. $x \in I_a^*, z \in [0,a] \cup I_a^r$, the proof is similar to the case 3.

8. $x \in I_a^*, z \in (a,1] \cup I_a^l$, the proof is similar to the case 6.

9. $x, z \in I_a^*$, then $y, t \in (a,1] \cup I_a^l \cup I_a^*$ and $V(x,z) = a \trianglelefteq V(y,t)$.

In summary, V is increasing. Now, we verify the associativity of V.

Let $x, y, z \in X$. It is easy to verify associativity if one of the elements x, y, z belongs to I_a^*, then we only need to consider the other cases.

1. $x \in [0,a] \cup I_a^r, y \in [0,a] \cup I_a^r$, then $V(x,y) = S(g(x), g(y)) \trianglelefteq a$.

   1.1. If $z \in [0,a] \cup I_a^r$, then $V(y,z) = S(g(y), g(z)) \trianglelefteq a$, and we can obtain $V(V(x,y),z) = S(g(S(g(x),g(y))),g(z)) = S(S(g(x),g(y)),g(z))$ $= S(g(x), S(g(y),g(z))) = S(g(x), g(S(g(y),g(z)))) = V(x, V(y,z))$.

   1.2. If $z \in (a,1] \cup I_a^l$, then $V(y,z) = a$ and $V(V(x,y),z) = a = V(x,V(y,z))$.

2. $x \in [0,a] \cup I_a^r, y \in (a,1] \cup I_a^l$, then $V(x,y) = a$.

   2.1. If $z \in [0,a] \cup I_a^r$, then $V(y,z) = a$ and $V(V(x,y),z) = a = V(x,V(y,z))$.

   2.2. If $z \in (a,1] \cup I_a^l$, then $a \trianglelefteq T(f(y), f(z)) = V(y,z)$ and $V(V(x,y),z) = a = V(x,V(y,z))$.

3. $x \in (a,1] \cup I_a^l, y \in [0,a] \cup I_a^r$, then $V(x,y) = a$.

   3.1. If $z \in [0,a] \cup I_a^r$, then $V(y,z) = S(g(y), g(z)) \trianglelefteq a$ and $V(V(x,y),z) = a = V(x,V(y,z))$.

   3.2. If $z \in (a,1] \cup I_a^l$, then $V(y,z) = a$ and $V(V(x,y),z) = a = V(x,V(y,z))$.

4. $x \in (a,1] \cup I_a^l, y \in (a,1] \cup I_a^l$, then $a \trianglelefteq T(f(x), f(y)) = V(x,y)$.

4.1. If $z \in [0,a] \cup I_a^r$, then $V(y,z) = a$ and $V(V(x,y),z) = a = V(x,V(y,z))$.

4.2. If $z \in (a,1] \cup I_a^l$, then $a \trianglelefteq T(f(y),f(z)) = V(y,z)$, and we have that
$V(V(x,y),z) = T(f(T(f(x),f(y))),f(z)) = T(T(f(x),f(y)),f(z))$
$= T(f(x),T(f(y),f(z))) = T(f(x),f(T(f(y),f(z)))) = V(x,V(y,z))$.

□

Example 3.14. Let $P_2 = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis given by Hasse diagram in Figure 5, where $X = \{0,a,b,c,d,e,h,i,j,k,l,m,1\}$. Clearly, $a \in K\backslash\{0,1\}$, $[0,a] = \{0,b,d,a\}, [a,1] = \{a,i,j,l,1\}, I_a^r = \{c,e\}, I_a^l = \{k\}, I_a^* = \{m,h\}$. We can define the t-conorms S_1 and S_2 on $[0,a]$ given by Table 1 and Table 2, respectively, and define the t-norms T_1, T_2 and T_3 on $[a,1]$ given by Tables 3, 4 and 5, respectively. Given the order-preserving mappings f_1, f_2 and g_1, as shown in Tables 6, 7 and 8, respectively.

(i) With the help of the t-conorm S_1, the t-norm T_1 and the order-preserving mappings f_1 and g_1, we can construct a nullnorm V_1 with annihilator a on P_2 by Theorem 3.11, as shown in Table 9.

(ii) With the help of the t-conorm S_2, the t-norm T_2 and the order-preserving mappings f_2 and g_1, we can construct a nullnorm V_2 with annihilator a on P_2 by Theorem 3.12, as shown in Table 10.

(iii) With the help of the t-conorm S_1, the t-norm T_3 and the order-preserving mappings f_2 and g_1, we can construct a nullnorm V_3 with annihilator a on P_2 by Theorem 3.13, as shown in Table 11.

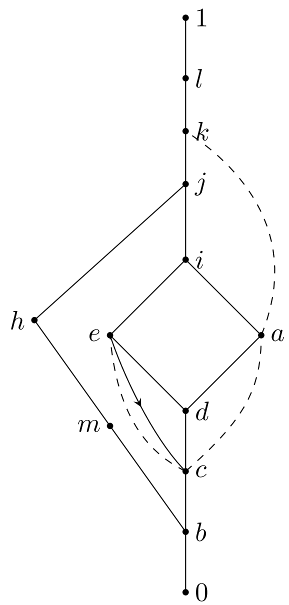

Fig. 5: Hasse diagram of the bounded trellis P_2.

S_1	0	b	d	a
0	0	b	d	a
b	b	b	d	a
d	d	d	d	a
a	a	a	a	a

Tab. 1: The t-conorm S_1.

S_2	0	b	d	a
0	0	b	d	a
b	b	d	d	a
d	d	d	d	a
a	a	a	a	a

Tab. 2: The t-conorm S_2.

T_1	a	i	j	l	1
a	a	a	a	a	a
i	a	i	i	i	i
j	a	i	j	j	j
l	a	i	j	l	l
1	a	i	j	l	1

Tab. 3: The t-norm T_1.

T_2	a	i	j	l	1
a	a	a	a	a	a
i	a	i	i	i	i
j	a	i	i	i	j
l	a	i	i	i	l
1	a	i	j	l	1

Tab. 4: The t-norm T_2.

T_3	a	i	j	l	1
a	a	a	a	a	a
i	a	i	i	i	i
j	a	i	j	j	j
l	a	i	j	j	l
1	a	i	j	l	1

Tab. 5: The t-norm T_3.

x	0	b	d	a	i	j	l	1	c	e	k	m	h
$f_1(x)$	a	a	a	a	i	j	l	1	a	a	l	j	j

Tab. 6: The order-preserving mapping f_1.

x	0	b	d	a	i	j	l	1	c	e	k	m	h
$f_2(x)$	a	a	a	a	i	j	l	1	a	a	j	a	a

Tab. 7: The order-preserving mapping f_2.

x	0	b	d	a	i	j	l	1	c	e	k	m	h
$g_1(x)$	0	b	d	a	a	a	a	a	d	d	a	b	a

Tab. 8: The order-preserving mapping g_1.

Remark 3.15. In Theorem 3.11, although we require that the domain of the order-preserving mappings f and g be X, in fact, the values of f on $[0,a) \cup I_a^r$ and g on $(a,1] \cup I_a^l \cup I_a^*$ do not affect the construction of the nullnorm, so we can restrict the domains of f and g to $[a,1] \cup I_a^l \cup I_a^*$ and $[0,a] \cup I_a^r$, respectively. Similarly, in Theorem 3.12, we can restrict the domains of f and g to $[a,1] \cup I_a^l$ and $[0,a] \cup I_a^r \cup I_a^*$, respectively. In Theorem 3.13, we can restrict the domains of f and g to $[a,1] \cup I_a^l$ and $[0,a] \cup I_a^r$, respectively.

In Theorem 3.13, if $I_a^l = I_a^r = \emptyset$, f and g are given by $f(x) = x$ for all $x \in [a,1]$ and $g(x) = x$ for all $x \in [0,a]$, respectively, then we can draw the following corollary.

Corollary 3.16. Let $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis satisfying $I_a^r \cup I_a^l = \emptyset$, $a \in K \backslash \{0,1\}$, T be a t-norm on $[a,1]$, S be a t-conorm on $[0,a]$. Then the binary operation V defined by (4) is a nullnorm with annihilator a on P:

V_1	0	b	d	a	i	j	l	1	c	e	k	m	h
0	0	b	d	a	a	a	a	a	d	d	a	a	a
b	b	b	d	a	a	a	a	a	d	d	a	a	a
d	d	d	d	a	a	a	a	a	d	d	a	a	a
a	a	a	a	a	a	a	a	a	a	a	a	a	a
i	a	a	a	a	i	i	i	i	a	a	i	i	i
j	a	a	a	a	i	j	j	j	a	a	j	j	j
l	a	a	a	a	i	j	l	l	a	a	l	j	j
1	a	a	a	a	i	j	l	1	a	a	l	j	j
c	d	d	d	a	a	a	a	a	d	d	a	a	a
e	d	d	d	a	a	a	a	a	d	d	a	a	a
k	a	a	a	a	i	j	l	l	a	a	l	j	j
m	a	a	a	a	i	j	j	j	a	a	j	j	j
h	a	a	a	a	i	j	j	j	a	a	j	j	j

Tab. 9: The nullnorm V_1 in Example 3.14 (i).

V_2	0	b	d	a	i	j	l	1	c	e	k	m	h
0	0	b	d	a	a	a	a	a	d	d	a	b	a
b	b	d	d	a	a	a	a	a	d	d	a	d	a
d	d	d	d	a	a	a	a	a	d	d	a	d	a
a	a	a	a	a	a	a	a	a	a	a	a	a	a
i	a	a	a	a	i	i	i	i	a	a	i	a	a
j	a	a	a	a	i	i	i	j	a	a	i	a	a
l	a	a	a	a	i	i	i	l	a	a	i	a	a
1	a	a	a	a	i	j	l	1	a	a	j	a	a
c	d	d	d	a	a	a	a	a	d	d	a	d	a
e	d	d	d	a	a	a	a	a	d	d	a	d	a
k	a	a	a	a	i	i	i	j	a	a	i	a	a
m	b	d	d	a	a	a	a	a	d	d	a	d	a
h	a	a	a	a	a	a	a	a	a	a	a	a	a

Tab. 10: The nullnorm V_2 in Example 3.14 (ii).

V_3	0	b	d	a	i	j	l	1	c	e	k	m	h
0	0	b	d	a	a	a	a	a	d	d	a	a	a
b	b	b	d	a	a	a	a	a	d	d	a	a	a
d	d	d	d	a	a	a	a	a	d	d	a	a	a
a	a	a	a	a	a	a	a	a	a	a	a	a	a
i	a	a	a	a	i	i	i	i	a	a	i	a	a
j	a	a	a	a	i	j	j	j	a	a	j	a	a
l	a	a	a	a	i	j	j	l	a	a	j	a	a
1	a	a	a	a	i	j	l	1	a	a	j	a	a
c	d	d	d	a	a	a	a	a	d	d	a	a	a
e	d	d	d	a	a	a	a	a	d	d	a	a	a
k	a	a	a	a	i	j	j	j	a	a	j	a	a
m	a	a	a	a	a	a	a	a	a	a	a	a	a
h	a	a	a	a	a	a	a	a	a	a	a	a	a

Tab. 11: The nullnorm V_3 in Example 3.14 (iii).

$$V(x,y)=\begin{cases} S(x,y), & (x,y)\in[0,a]^2,\\ T(x,y), & (x,y)\in[a,1]^2,\\ a, & \text{otherwise}.\end{cases} \tag{4}$$

In fact, a bounded lattice is a special bounded trellis. Let $a\in X\backslash\{0,1\}$ and $P=(X,\trianglelefteq,\wedge,\vee,0,1)$ be a bounded trellis. If P is a bounded lattice, then the mappings $s:X\to[a,1]$ and $t:X\to[0,a]$ given by $s(x)=x\vee a$ and $t(x)=x\wedge a$ are order-preserving mappings satisfying $s(x)=x$ for all $x\in[a,1]$ and $t(x)=x$ for all $x\in[0,a]$. If P is a proper bounded trellis, then s and t are usually not order-preserving mappings on P (see Examples 3.17 and 3.18), however, if we restrict the domain of s to $[a,1]\cup I_a^l\cup I_a^*$ or $[a,1]\cup I_a^l$, then s is an order-preserving mapping satisfying $s(x)=x$ for all $x\in[a,1]$ when P satisfies certain conditions, as shown in the following Proposition 3.19. Similarly, if we restrict the domain of t to $[0,a]\cup I_a^r\cup I_a^*$ or $[0,a]\cup I_a^r$, then t is an order-preserving mapping satisfying $t(x)=x$ for all $x\in[0,a]$ when P satisfies certain conditions, as shown in the following Proposition 3.20.

Example 3.17. Consider the bounded trellis $P_1=(X,\trianglelefteq,\wedge,\vee,0,1)$ given by the Hasse diagram in Figure 1, where $X=\{0,a,b,c,d,e,f,g,h,1\}$.

(i) Define the mapping $s_1:X\to[c,1]$ as $s_1(x)=x\vee c$ for all $x\in X$. Since $d\lhd b$ and $s_1(d)=d\vee c=d\ntrianglelefteq c=b\vee c=s_1(b)$, then s_1 is not an order-preserving mapping.

(ii) Define the mapping $t_1:X\to[0,b]$ as $t_1(x)=x\wedge b$ for all $x\in X$. Since $c\lhd d$ and $t_1(c)=c\wedge b=b\ntrianglelefteq d=d\wedge b=t_1(d)$, then t_1 is not an order-preserving mapping.

Example 3.18. Consider the bounded trellis $P_2=(X,\trianglelefteq,\wedge,\vee,0,1)$ given by the Hasse diagram in Figure 5, where $X=\{0,a,b,c,d,e,h,i,j,k,l,m,1\}$.

(i) Define the mapping $s_2:X\to[d,1]$ as $s_2(x)=x\vee d$ for all $x\in X$. Since $e\lhd c$ and $s_2(e)=e\vee d=e\ntrianglelefteq d=c\vee d=s_2(c)$, then s_2 is not an order-preserving mapping.

(ii) Define the mapping $t_2:X\to[0,c]$ as $t_2(x)=x\wedge c$ for all $x\in X$. Since $d\lhd e$ and $t_2(d)=d\wedge c=c\ntrianglelefteq e=e\wedge c=t_2(e)$, then t_2 is not an order-preserving mapping.

For convenience, we denote

$$L_a^1=\{x\in I_a^r|\ \exists y\in(0,a)\cup I_a^r \text{ s.t. } x\lhd y\},$$
$$L_a^2=\{x\in(0,a)\cup I_a^r|\ \exists y\in I_a^r \text{ s.t. } y\lhd x\},$$
$$H_a^1=\{x\in I_a^r\cup I_a^*|\ \exists y\in I_a^* \text{ s.t. } x\lhd y\},$$
$$H_a^2=\{x\in I_a^*|\ \exists y\in I_a^r\cup I_a^* \text{ s.t. } y\lhd x\},$$
$$O_a^1=\{x\in I_a^l|\ \exists y\in(a,1)\cup I_a^l \text{ s.t. } y\lhd x\},$$
$$O_a^2=\{x\in(a,1)\cup I_a^l|\ \exists y\in I_a^l \text{ s.t. } x\lhd y\},$$
$$R_a^1=\{x\in I_a^l\cup I_a^*|\ \exists y\in I_a^* \text{ s.t. } y\lhd x\},$$
$$R_a^2=\{x\in I_a^*|\ \exists y\in I_a^l\cup I_a^* \text{ s.t. } x\lhd y\}.$$

Proposition 3.19. Let $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis, $a \in X \backslash \{0, 1\}$, then the following items hold:

(1) The mapping $s : ([a, 1] \cup I_a^l \cup I_a^*) \rightarrow [a, 1]$ given by $s(x) = x \vee a$ is an order-preserving mapping satisfying $s(x) = x$ for all $x \in [a, 1]$ if one of the following items is satisfied:

(i) $O_a^1 \cup R_a^1 \subseteq X^{mtr}$;

(ii) $O_a^2 \cup R_a^2 \subseteq X^{rtr}$;

(iii) $(a, 1) \subseteq X^{ltr}$.

(2) The mapping $s : ([a, 1] \cup I_a^l) \rightarrow [a, 1]$ given by $s(x) = x \vee a$ is an order-preserving mapping satisfying $s(x) = x$ for all $x \in [a, 1]$ if one of the following items is satisfied:

(i) $O_a^1 \subseteq X^{mtr}$;

(ii) $O_a^2 \subseteq X^{rtr}$;

(iii) $(a, 1) \subseteq X^{ltr}$.

Proof.

(1) Let $x, y \in [a, 1] \cup I_a^l \cup I_a^*$ such that $x \trianglelefteq y$, we verify $s(x) = x \vee a \trianglelefteq y \vee a = s(y)$. We only prove the case (i), the cases (ii) and (iii) can be proved similarly.

1) If $x, y \in [a, 1]$, then $s(x) = x \trianglelefteq y = s(y)$;

2) If $x \in [a, 1], y \in I_a^l$, since $x \trianglelefteq y \trianglelefteq y \vee a$ and $O_a^1 \subseteq X^{mtr}$, then $s(x) = x \trianglelefteq y \vee a = s(y)$;

3) If $x \in I_a^l \cup I_a^*, y \in [a, 1]$, since $x \trianglelefteq y$ and $a \trianglelefteq y$, then $s(x) = x \vee a \trianglelefteq y = s(y)$;

4) If $x \in I_a^l, y \in I_a^l$, since $x \trianglelefteq y \trianglelefteq y \vee a, a \trianglelefteq y \vee a$ and $O_a^1 \subseteq X^{mtr}$, then we have $x \trianglelefteq y \vee a$ and $s(x) = x \vee a \trianglelefteq y \vee a = s(y)$;

5) If $x \in I_a^*, y \in I_a^l$, since $x \trianglelefteq y \trianglelefteq y \vee a, a \trianglelefteq y \vee a$ and $R_a^1 \subseteq X^{mtr}$, then we have $x \trianglelefteq y \vee a$ and $s(x) = x \vee a \trianglelefteq y \vee a = s(y)$;

6) If $x, y \in I_a^*$, since $x \trianglelefteq y \trianglelefteq y \vee a, a \trianglelefteq y \vee a$ and $R_a^1 \subseteq X^{mtr}$, then $x \trianglelefteq y \vee a$ and $s(x) = x \vee a \trianglelefteq y \vee a = s(y)$.

(2) The proof is similarly to (1).

□

Proposition 3.20. Let $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis, $a \in X \backslash \{0, 1\}$, then the following items hold:

(1) The mapping $t : ([0, a] \cup I_a^r \cup I_a^*) \rightarrow [0, a]$ given by $t(x) = x \wedge a$ is an order-preserving mapping satisfying $t(x) = x$ for all $x \in [0, a]$ if one of the following items is satisfied:

(i) $L_a^1 \cup H_a^1 \subseteq X^{mtr}$;
(ii) $L_a^2 \cup H_a^2 \subseteq X^{ltr}$;
(iii) $(0, a) \subseteq X^{rtr}$.

(2) The mapping $t : ([0, a] \cup I_a^r) \to [0, a]$ given by $t(x) = x \wedge a$ is an order-preserving mapping satisfying $t(x) = x$ for all $x \in [0, a]$ if one of the following items is satisfied:

(i) $L_a^1 \subseteq X^{mtr}$;
(ii) $L_a^2 \subseteq X^{ltr}$;
(iii) $(0, a) \subseteq X^{rtr}$.

Proof.

(1) Let $x, y \in [0, a] \cup I_a^r \cup I_a^*$ such that $x \trianglelefteq y$, we verify $t(x) = x \wedge a \trianglelefteq y \wedge a = t(y)$. We only prove the case (ii), the cases (i) and (iii) can be proved similarly.

1) If $x, y \in [0, a]$, then $t(x) = x \trianglelefteq y = t(y)$;
2) If $x \in [0, a], y \in I_a^r \cup I_a^*$, since $x \trianglelefteq y$ and $x \trianglelefteq a$, then $t(x) = x \trianglelefteq y \wedge a = t(y)$;
3) If $x \in I_a^r, y \in [0, a]$, since $x \wedge a \trianglelefteq x \trianglelefteq y$ and $L_a^2 \subseteq X^{ltr}$, then $t(x) = x \wedge a \trianglelefteq y = t(y)$;
4) If $x \in I_a^r, y \in I_a^r$, since $x \wedge a \trianglelefteq x \trianglelefteq y, x \wedge a \trianglelefteq a$ and $L_a^2 \subseteq X^{ltr}$, then we have $x \wedge a \trianglelefteq y$ and $t(x) = x \wedge a \trianglelefteq y \wedge a = t(y)$;
5) If $x \in I_a^r, y \in I_a^*$, since $x \wedge a \trianglelefteq x \trianglelefteq y, x \wedge a \trianglelefteq a$ and $H_a^2 \subseteq X^{ltr}$, then we have $x \wedge a \trianglelefteq y$ and $t(x) = x \wedge a \trianglelefteq y \wedge a = t(y)$;
6) If $x, y \in I_a^*$, since $x \wedge a \trianglelefteq x \trianglelefteq y, x \wedge a \trianglelefteq a$ and $H_a^2 \subseteq X^{ltr}$, then we have $x \wedge a \trianglelefteq y$ and $t(x) = x \wedge a \trianglelefteq y \wedge a = t(y)$.

(2) The proof is similarly to (1).

□

From Remark 3.15 and Propositions 3.19 and 3.20, in Theorem 3.11, let f and g be given by $f(x) = x \vee a$ for all $x \in [a, 1] \cup I_a^l \cup I_a^*$ and $g(x) = x \wedge a$ for all $x \in [0, a] \cup I_a^r$, respectively, then we can draw the following corollary.

Corollary 3.21. Let $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis, $a \in K \backslash \{0, 1\}$, T be a t-norm on $[a, 1]$, S be a t-conorm on $[0, a]$. Then the binary operation V defined by (5) is a nullnorm with annihilator a on P if the following items are satisfied:

(i) $L_a^1 \subseteq X^{mtr}$ or $L_a^2 \subseteq X^{ltr}$ or $(0, a) \subseteq X^{rtr}$;

(ii) $O_a^1 \cup R_a^1 \subseteq X^{mtr}$ or $O_a^2 \cup R_a^2 \subseteq X^{rtr}$ or $(a, 1) \subseteq X^{ltr}$.

$$V(x, y) = \begin{cases} S(x \wedge a, y \wedge a), & (x, y) \in ([0, a] \cup I_a^r)^2, \\ T(x \vee a, y \vee a), & (x, y) \in ([a, 1] \cup I_a^l \cup I_a^*)^2, \\ a, & \text{otherwise.} \end{cases} \tag{5}$$

In Theorem 3.12, let f and g be given by $f(x) = x \vee a$ for all $x \in [a, 1] \cup I_a^l$ and $g(x) = x \wedge a$ for all $x \in [0, a] \cup I_a^r \cup I_a^*$, respectively, then we can draw the following corollary.

Corollary 3.22. Let $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis, $a \in K\backslash\{0, 1\}$, T be a t-norm on $[a, 1]$, S be a t-conorm on $[0, a]$. Then the binary operation V defined by (6) is a nullnorm with annihilator a on P if the following items are satisfied:

(i) $L_a^1 \cup H_a^1 \subseteq X^{mtr}$ or $L_a^2 \cup H_a^2 \subseteq X^{ltr}$ or $(0, a) \subseteq X^{rtr}$;

(ii) $O_a^1 \subseteq X^{mtr}$ or $O_a^2 \subseteq X^{rtr}$ or $(a, 1) \subseteq X^{ltr}$.

$$V(x, y) = \begin{cases} S(x \wedge a, y \wedge a), & (x, y) \in ([0, a] \cup I_a^r \cup I_a^*)^2, \\ T(x \vee a, y \vee a), & (x, y) \in ([a, 1] \cup I_a^l)^2, \\ a, & \text{otherwise.} \end{cases} \tag{6}$$

In Theorem 3.13, let f and g be given by $f(x) = x \vee a$ for all $x \in [a, 1] \cup I_a^l$ and $g(x) = x \wedge a$ for all $x \in [0, a] \cup I_a^r$, respectively, then we can draw the following corollary.

Corollary 3.23. Let $P = (X, \trianglelefteq, \wedge, \vee, 0, 1)$ be a bounded trellis, $a \in K\backslash\{0, 1\}$, T be a t-norm on $[a, 1]$, S be a t-conorm on $[0, a]$. Then the binary operation V defined by (7) is a nullnorm with annihilator a on P if the following items are satisfied:

(i) $L_a^1 \subseteq X^{mtr}$ or $L_a^2 \subseteq X^{ltr}$ or $(0, a) \subseteq X^{rtr}$;

(ii) $O_a^1 \subseteq X^{mtr}$ or $O_a^2 \subseteq X^{rtr}$ or $(a, 1) \subseteq X^{ltr}$.

$$V(x, y) = \begin{cases} S(x \wedge a, y \wedge a), & (x, y) \in ([0, a] \cup I_a^r)^2, \\ T(x \vee a, y \vee a), & (x, y) \in ([a, 1] \cup I_a^l)^2, \\ a, & \text{otherwise.} \end{cases} \tag{7}$$

Remark 3.24. We now discuss the relationships between our methods and those in [34].

(i) In Corollary 3.21, if $I_a^r = I_a^1$, then the resulting nullnorm coincides with the one determined by Theorem 3.1 in [34].

(ii) In Corollary 3.22, if $I_a^l = I_a^2$, then the resulting nullnorm coincides with the one determined by Theorem 3.2 in [34].

(iii) In Corollary 3.23, if $I_a^* = \emptyset$ and $I_a^r = I_a^1$, then the resulting nullnorm coincides with the one determined by Theorem 3.1 in [34].

(iv) In Corollary 3.23, if $I_a^* = \emptyset$ and $I_a^l = I_a^2$, then the resulting nullnorm coincides with the one determined by Theorem 3.2 in [34].

4. CONCLUSION

In this paper, we further discuss the characteristics of the element acting as the annihilator of a nullnorm on a bounded trellis. Based on t-(co)norms and order-preserving mappings, we propose a series of new methods for constructing nullnorms on bounded trellises and present some examples to illustrate. In addition, we illustrate that in some special cases, the nullnorms obtained by our methods coincide with those obtained by Xiu et al. In the future, we will further consider other methods for constructing nullnorms on bounded trellises, and it would be better if the assumptions could be relaxed.

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REFERENCES

[1] G. Birkhoff: Lattice Theory. American Mathematical Society Colloquium Publications, Rhode Island 1973.

[2] G. D. Çaylı and F. Karaçal: Idempotent nullnorms on bounded lattices. Inf. Sci. *425* (2018), 154–163. DOI:10.1016/j.ins.2017.10.003

[3] G. D. Çaylı: Construction methods for idempotent nullnorms on bounded lattices. Appl. Math. Comput. *366* (2020), 124746. DOI:10.1016/j.amc.2019.124746

[4] G. D. Çaylı: Nullnorms on bounded lattices derived from t-norms and t-conorms. Inf. Sci. *512* (2020), 1134–1154. DOI:10.1016/j.ins.2019.10.059

[5] G. D. Çaylı: Some results about nullnorms on bounded lattices. Fuzzy Sets Syst. *386* (2020), 105–131. DOI:10.1016/j.fss.2019.03.010

[6] G. D. Çaylı: Generating nullnorms on some special classes of bounded lattices via closure and interior operators. Inf. Sci. *552* (2021), 118–141. DOI:10.1016/j.ins.2020.11.016

[7] T. Calvo, B. De Baets, and J. Fodor: The functional equations of Frank and Alsina for uninorms and nullnorms. Fuzzy Sets Syst. *120* (2001), 385–394. DOI:10.1016/S0165-0114(99)00125-6

[8] B. De Baets, H. De Meyer, B. De Schuymer, and S. Jenei: Cyclic evaluation of transitivity of reciprocal relations. Soc. Choice Welf. *26* (2006), 217–238. DOI:10.1007/s00355-006-0093-3

[9] B. De Baets, K. De Loof, and H. De Meyer: A frequentist view on cycle-transitivity of reciprocal relations. Fuzzy Sets Syst. *281* (2015), 198–218. DOI:10.1016/j.fss.2015.06.020

[10] J. Donald and J. Arrigo: Non-associative modular lattices. Arch. Math. *22* (1971), 18–23. DOI:10.1007/BF01222532

[11] J. Drewniak, P. Drygaś, and E. Rak: Distributivity between uninorms and nullnorms. Fuzzy Sets Syst. *159* (2008), 1646–1657. DOI:10.1016/j.fss.2007.09.015

[12] P. Drygaś: A characterization of idempotent nullnorms. Fuzzy Sets Syst. *145* (2004), 455–461. DOI:10.1016/S0165-0114(03)00259-8

[13] D. Dubois and H. Prade: A review of fuzzy set aggregation connectives. Inf. Sci. *36* (1985), 85–121. DOI:10.1016/0020-0255(85)90027-1

[14] D. Dubois and H. Prade: Fundamentals of Fuzzy Sets. Kluwer Academic Publishers, Boston, 2000.

[15] Ü. Ertuğrul: Construction of nullnorms on bounded lattices and an equivalence relation on nullnorms. Fuzzy Sets Syst. *334* (2018), 94–109. DOI:10.1016/j.fss.2017.07.020

[16] E. Fried: Tournaments and non-associative lattices. Ann. Univ. Sci. Bp. Rolando Eötvös Nomin., Sect. Math. *13* (1970), 151–164.

[17] E. Fried and G. Grätzer: Some examples of weakly associative lattices. Colloq. Math. *27* (1973), 215–221. DOI:10.4064/CM-27-2-215-221

[18] E. Fried and V. T. Sós: Weakly associative lattices and projective planes. Algebra Univers. *5* (1975), 114–119. DOI:10.1007/BF02485240

[19] K. Gladstien: A characterization of complete trellises of finite length. Algebra Univers. *3* (1973), 341–344. DOI:10.1007/BF02945138

[20] P. Hájek, T. Havránek, and R. Jiroušek: Uncertain Information Processing in Expert Systems. CRC Press, Boca Raton, 1992.

[21] W. Homenda, A. Jastrzebska, and W. Pedrycz: Multicriteria decision making inspired by human cognitive processes. Appl. Math. Comput. *290* (2016), 392–411. DOI:10.1016/j.amc.2016.05.041

[22] X. J. Hua: New constructions of nullnorms on bounded lattices. Fuzzy Sets Syst. *439* (2022), 126–141. DOI:10.1016/j.fss.2021.09.010

[23] M. A. İnce, F. Karaçal, and R. Mesiar: Medians and nullnorms on bounded lattices. Fuzzy Sets Syst. *289* (2016), 74–81. DOI:10.1016/j.fss.2015.05.015

[24] B. Kerr, M. A. Riley, M. W. Feldman, and B. J. M. Bohannan: Local dispersal promotes biodiversity in a real-life game of rock-paper-scissors. Nature *418* (2002), 171–174. DOI:10.1038/nature00823

[25] F. Karaçal, M. A. İnce, and R. Mesiar: Nullnorms on bounded lattice. Inf. Sci. *325* (2015), 227–236. DOI:10.1016/j.ins.2015.06.052

[26] Y. Kong and B. Zhao: Uninorms on bounded trellises. Fuzzy Sets Syst. *481* (2024), 108898. DOI:10.1016/j.fss.2024.108898

[27] G. Li, H. W. Liu, and Y. Su: On the conditional distributivity of nullnorms over uninorms. Inf. Sci. *317* (2015), 157–169. DOI:10.1016/j.ins.2015.04.040

[28] M. Mas, M. Monserrat, D. Ruiz, and J. Torrens: Migrative uninorms and nullnorms over t-norms and t-conorms. Fuzzy Sets Syst. *26* (2015), 20–32. DOI:10.1016/j.fss.2014.05.012

[29] F. Qin and B. Zhao: The distributive equations for idempotent uninorms and nullnorms. Fuzzy Sets Syst. *155* (2005), 446–458. DOI:10.1016/j.fss.2005.04.010

[30] J. S. Qiao and B. Q. Hu: On the migrativity of uninorms and nullnorms over overlap and grouping functions. Fuzzy Sets Syst. *346* (2018), 1–54. DOI:10.1016/j.fss.2017.11.012

[31] H. Skala: Trellis theory. Algebra Univers. *1* (1971), 218–233. DOI:10.1007/BF02944982

[32] X. R. Sun and H. W. Liu: Representation of nullnorms on bounded lattices. Inf. Sci. *539* (2020), 269–276. DOI:10.1016/j.ins.2020.06.013

[33] Z. D. Wang, Y. Ouyang, H. P. Zhang, and B. De Baets: A note on the representation theorem for nullnorms on bounded lattices. Fuzzy Sets Syst. *472* (2023), 108680. DOI:10.1016/j.fss.2023.108680

[34] Z. Y. Xiu and X. Zheng: Nullnorms on bounded trellises. arXiv preprint arXiv: 2408.09321, 2024. DOI:10.48550/arXiv.2408.09321

[35] H. P. Zhang, Z. D. Wang, Y. Ouyang, and B. De Baets: A representation of nullnorms on a bounded lattice in terms of beam operations. Fuzzy Sets Syst. *427* (2022), 149–160. DOI:10.1016/j.fss.2020.11.004

[36] H. P. Zhang, Y. Ouyang, Z. D. Wang, and B. De Baets: A characterization of idempotent nullnorms on bounded lattices. Inf. Sci. *586* (2022), 676–687. DOI:10.1016/j.ins.2021.12.004

[37] H. P. Zhang, Y. Ouyang, Z. D. Wang, and B. De Baets: A complete representation theorem for nullnorms on bounded lattices with ample illustrations. Fuzzy Sets Syst. *439* (2022), 157–169. DOI:10.1016/j.fss.2021.08.012

[38] L. Zedam and B. De Baets: Triangular norms on bounded trellises. Fuzzy Sets Syst. *462* (2023), 108468. DOI:10.1016/j.fss.2023.01.003

[39] W. W. Zong, Y. Su, and H. W. Liu: Migrative property for nullnorms. Int. J. Uncertain. Fuzziness Knowl.-Based Syst. *22* (2014), 749–759. DOI:10.1142/S021848851450038X

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