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NEW METHODS TO CONSTRUCT UNINORMS BY EXTENDING UNINORMS WITH CLOSURE OPERATORS AND T-SUPERCONORMS

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In this paper, we provide new methods to construct uninorms by extending given uninorms on a subinterval of a bounded lattice with closure operators (resp. interior operators) and t-superconorms (resp. t-subnorms). Meanwhile, these methods for uninorms generalize some known methods for uninorms in the literature. An example is also provided to show our method.

Keywords: bounded lattices, closure operators, t-superconorms, uninorms

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1. INTRODUCTION

In 1996, uninorms were introduced by Yager and Rybalov [28] on the unit interval, as a generalization of triangular norms (t-norms, for short) and triangular conorms (t-conorms, for short) [20], allowing a neutral element e to lie anywhere in $[0,1]$ rather than at 1 or 0. Since then, uninorms have been widely used in several fields, such as fuzzy set theory, fuzzy system modeling, expert systems, neural networks, fuzzy logic and so on (see, e.g., [10, 11, 12, 19, 21, 25]).

In 2015, the concept of uninorms was generalized from the unit interval $[0,1]$ to bounded lattices by Karaçal and Mesiar [17]. And some construction methods for uninorms on bounded lattices were introduced by Karaçal and Mesiar. Since then, a number of construction methods have been introduced in the literature. The constructions of uninorms are usually based on the following tools, such as t-norms (resp. t-conorms) (see, e.g., [1, 3, 4, 17, 18]), t-subnorms (resp. t-superconorms) (see, e.g., [15, 16, 30]), closure operators (resp. interior operators) (see, e.g., [6, 7, 8, 13, 22, 29]), additive generators [14] and uninorms (see, e.g., [5, 26]).

Especially, in 2023, the new methods to construct uninorms by extending given uninorms on bounded lattices were introduced by Çaylı [5], and Xiu and Zheng [26], respectively. Then, in [27], Xiu and Zheng also provided new methods to construct uninorms on bounded lattices by using a given uninorm only or a given uninorm with a t-norm (t-conorm). In this paper, we will continue to study the construction methods for uninorms on bounded lattices by extending the given uninorms based on closure operators

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(resp. interior operators) and t-superconorms (resp. t-subnorms). That is, we will try to construct uninorms by the given uninorms, closure operators (resp. interior operators) and t-superconorms (resp. t-subnorms). Meanwhile, the resulting methods for uninorms generalize some known methods for uninorms in the literature.

The structure of this paper is as follows. In Section 2, we recall some basic concepts and properties related to lattices and aggregation operators on bounded lattices, which will be used in this paper. In Section 3, we propose construction methods for uninorms on bounded lattices by extending the given uninorms based on closure operators (resp. interior operators) and t-superconorms (resp. t-subnorms). In Section 4, some conclusions are added.

2. PRELIMINARIES

Definition 2.1. (Birkhoff [2]) A lattice $(L, \leq)$ is bounded if it has top and bottom elements, which are written as 1 and 0, respectively, that is, there exist two elements $1, 0 \in L$ such that $0 \leq x \leq 1$ for all $x \in L$.

Throughout this article, unless stated otherwise, L will denote a bounded lattice with the top and bottom elements 1 and 0, respectively.

Definition 2.2. (Birkhoff [2]) Let L be a bounded lattice, $a, b \in L$ with $a \leq b$. A subinterval $[a, b]$ of L is defined as $[a, b] = \{x \in L : a \leq x \leq b\}$. Similarly, we can define $[a, b) = \{x \in L : a \leq x < b\}, (a, b] = \{x \in L : a < x \leq b\}$ and $(a, b) = \{x \in L : a < x < b\}$. If a and b are incomparable, then we use the notation $a \parallel b$.

In the following, I_a denotes the set of all incomparable elements with a, that is, $I_a = \{x \in L \mid x \parallel a\}$. I^a denotes the set of all comparable elements with a, that is, $I^a = \{x \in L \mid x \nparallel a\}$. I_a^b denotes the set of elements that are incomparable with a but comparable with b, that is, $I_a^b = \{x \in L \mid x \parallel a \text{ and } x \nparallel b\}$. $I_{a,b}$ denotes the set of elements that are incomparable with both a and b, that is, $I_{a,b} = \{x \in L \mid x \parallel a \text{ and } x \parallel b\}$. Obviously, $I_a^a = \emptyset$ and $I_{a,a} = I_a$.

Definition 2.3. (Saminger [24]) An operation $T : L^2 \to L$ is called a t-norm on L if it is commutative, associative, and increasing with respect to both variables, and it has the neutral element $1 \in L$, that is, $T(1, x) = x$ for all $x \in L$.

Definition 2.4. (Çaylı et all. [4]) An operation $S : L^2 \to L$ is called a t-conorm on L if it is commutative, associative, and increasing with respect to both variables, and it has the neutral element $0 \in L$, that is, $S(0, x) = x$ for all $x \in L$.

Definition 2.5. (Palmeira and Bedregal [23]) A binary operation $F : L^2 \to L$ is called a t-subnorm on L if it is commutative, increasing, associative, and $F(x, y) \leq x \wedge y$ for any $(x, y) \in L^2$.

Definition 2.6. (Palmeira and Bedregal [23]) A binary operation $R : L^2 \to L$ is called a t-superconorm on L if it is commutative, increasing, associative, and $x \vee y \leq R(x, y)$ for any $(x, y) \in L^2$.

Definition 2.7. (Everett [9]) A mapping $cl : L \to L$ is said to be a closure operator on L if, for all $x, y \in L$, it satisfies the following three conditions:

(1) $x \le cl(x)$;

(2) $cl(x \vee y) = cl(x) \vee cl(y)$;

(3) $cl(cl(x)) = cl(x)$.

Definition 2.8. (Ouyang and Zhang [22]) A mapping $int : L \to L$ is said to be an interior operator on L if, for all $x, y \in L$, it satisfies the following three conditions:

(1) $int(x) \le x$;

(2) $int(x \wedge y) = int(x) \wedge int(y)$;

(3) $int(int(x)) = int(x)$.

Definition 2.9. (Karaçal and Mesiar [17] An operation $U : L^2 \to L$ is called a uninorm on L if it is commutative, associative, and increasing with respect to both variables, and it has the neutral element $e \in L$, that is, $U(e, x) = x$ for all $x \in L$.

Theorem 2.10. (Zhao and Wu [29]) Let $e \in L \setminus \{0, 1\}$, $cl : L \to L$ be a closure operator, T be a t-norm on $[0, e]$ and S be a t-conorm on $[e, 1]$ with $S(x, y) < 1$ for all $(x, y) \in (e, 1)^2$. If $x \parallel y$ for all $x \in I_e$ and $y \in (e, 1)$, then the function $U : L^2 \to L$ defined by

$$U(x,y) = \begin{cases} T(x,y) & \text{if } (x,y) \in [0,e]^2, \\ x & \text{if } (x,y) \in I_e \times (L \setminus (\{1\} \cup I_e)) \cup [0,e) \times (e,1), \\ y & \text{if } (x,y) \in (L \setminus (\{1\} \cup I_e)) \times I_e \cup (e,1) \times [0,e), \\ cl(x) \vee cl(y) & \text{if } (x,y) \in I_e \times I_e \cup \{1\} \times L \cup L \times \{1\}, \\ S(x,y) & \text{otherwise}, \end{cases}$$

is a uninorm on L with the neutral element e.

Theorem 2.11. (Zhao and Wu [29]) Let $e \in L \setminus \{0, 1\}$, $int : L \to L$ be an interior operator, S be a t-conorm on $[e, 1]$ and T be a t-norm on $[0, e]$ with $T(x, y) > 0$ for all $(x, y) \in (0, e)^2$. If $x \parallel y$ for all $x \in I_e$ and $y \in (0, e)$, then the function $U : L^2 \to L$ defined by

$$U(x,y) = \begin{cases} S(x,y) & \text{if } (x,y) \in [e,1]^2, \\ x & \text{if } (x,y) \in I_e \times (L \setminus (\{0\} \cup I_e)) \cup (e,1] \times (0,e), \\ y & \text{if } (x,y) \in (L \setminus (\{0\} \cup I_e)) \times I_e \cup (0,e) \times (e,1], \\ int(x) \wedge int(y) & \text{if } (x,y) \in I_e \times I_e \cup \{0\} \times L \cup L \times \{0\}, \\ T(x,y) & \text{otherwise}, \end{cases}$$

is a uninorm on L with the neutral element e.

Proposition 2.12. (Ji [16]) Let $\mathcal{A}$ be a nonempty set and $A_1, A_2, \ldots, A_n$ be subsets of $\mathcal{A}$. Let H be a commutative binary operation on $\mathcal{A}$. Then H is associative on $A_1 \cup A_2 \cup \ldots \cup A_n$ if and only if all of the following statements hold:

(i) for every combination $\{i,j,k\}$ of size 3 chosen from $\{1,2,\ldots,n\}$, $H(x,H(y,z)) = H(H(x,y),z) = H(y,H(x,z))$ for all $x\in A_i, y\in A_j, z\in A_k$;

(ii) for every combination $\{i,j\}$ of size 2 chosen from $\{1,2,\ldots,n\}$, $H(x,H(y,z)) = H(H(x,y),z)$ for all $x\in A_i, y\in A_i, z\in A_j$;

(iii) for every combination $\{i,j\}$ of size 2 chosen from $\{1,2,\ldots,n\}$, $H(x,H(y,z)) = H(H(x,y),z)$ for all $x\in A_i, y\in A_j, z\in A_j$;

(iv) for every $i\in\{1,2,\ldots,n\}$, $H(x,H(y,z)) = H(H(x,y),z)$ for all $x,y,z\in A_i$.

3. NEW METHODS TO CONSTRUCT UNINORMS BY EXTENDING GIVEN UNINORMS

In this section, we mainly construct new uninorms by extending given uninorms based on closure operators (resp. interior operators) and t-superconorms (resp. t-subnorms).

First, we propose a new construction method for uninorms on bounded lattices by extending a given uninorm by a closure operator and a t-superconorm under some conditions. Moreover, we obtain that these conditions are necessary and sufficient under some additional restraints on the bounded lattices. Meanwhile, these restraints always exist in a bounded lattice.

Theorem 3.1. Let $a\in L\setminus\{0,1\}$, $cl: L\to L$ be a closure operator, U^* be a uninorm on $[0,a]$ with a neutral element e and R be a t-superconorm on $[a,1]$ with $R(a,a)=a$. Suppose that the following conditions hold:

(1) $U^*(x,y)\notin[0,e]$ for all $(x,y)\in I_e^a\times I_e^a$,

(2) $U^*(x,y)=\begin{cases} x & \text{if } (x,y)\in[0,e)\times(I_e^a\cup(e,a]),\\ y & \text{if } (x,y)\in(I_e^a\cup(e,a])\times[0,e),\end{cases}$

(3) $x\parallel y$ for all $x\in I_{e,a}$ and $y\in I_e^a\cup I_a^e\cup(a,1)$.

Then the function $U: L^2\to L$ defined by

$$U(x,y)=\begin{cases} U^*(x,y) & \text{if } (x,y)\in[0,a]^2,\\ x & \text{if } (x,y)\in I_{e,a}\times(L\setminus(\{1\}\cup I_{e,a}))\cup[0,e)\times(I_a^e\cup(a,1))\\ & \qquad\cup(I_a^e\cup(a,1))\times\{e\},\\ y & \text{if } (x,y)\in(L\setminus(\{1\}\cup I_{e,a}))\times I_{e,a}\cup(I_a^e\cup(a,1))\times[0,e)\\ & \qquad\cup\{e\}\times(I_a^e\cup(a,1)),\\ cl(x)\vee cl(y) & \text{if } (x,y)\in I_{e,a}\times I_{e,a}\cup\{1\}\times L\cup L\times\{1\},\\ R(x\vee a,y\vee a) & \text{otherwise,}\end{cases}$$

is a uninorm on L with the neutral element e if and only if $R(x,y)<1$ for all $(x,y)\in(a,1)^2$ and $x\vee a<1$ for all $x\in I_a^e$.

Proof. Necessity: Let U be a uninorm on L with the neutral element e. We prove that $R(x,y) < 1$ for all $(x,y) \in (a,1)^2$ and $x \vee a < 1$ for all $x \in I_a^e$.

(1) $R(x,y) < 1$ for all $(x,y) \in (a,1)^2$.

Assume that there exists $(x,y) \in (a,1)^2$ such that $R(x,y) = 1$. Then $U(x,U(y,0)) = U(x,0) = 0$ and $U(U(x,y),0) = U(R(x,y),0) = U(1,0) = 1$. Since $0 \neq 1$, this leads to a contradiction with the associativity property of $U(x,y)$. Thus, $R(x,y) < 1$ for all $(x,y) \in (a,1)^2$.

(2) $x \vee a < 1$ for all $x \in I_a^e$.

Assume that there exists $x \in I_a^e$ such that $x \vee a = 1$. Then $U(x,U(x,0)) = U(x,0) = 0$ and $U(U(x,x),0) = U(R(x \vee a, x \vee a),0) = U(1,0) = 1$. Since $0 \neq 1$, this leads to a contradiction with the associativity property of $U(x,y)$. Thus, $x \vee a < 1$ for all $x \in I_a^e$.

Sufficiency: By the definition of U, we can obtain that the commutativity of U and the fact that e is the neutral element of U hold. Hence, we prove only the monotonicity and the associativity of U.

I. Monotonicity: We need to prove that if $x \leq y$, then $U(x,z) \leq U(y,z)$ for all $z \in L$. If $x = 1$, then $U(x,z) = 1 = U(y,z)$ for all $z \in L$. If $y = 1$, then $U(x,z) \leq 1 = U(y,z)$ for all $x,z \in L$. If $z = 1$, then $U(x,z) = 1 = U(y,z)$ for all $x,y \in L$. It is obvious that $U(x,z) \leq U(y,z)$ if both x and y belong to one of the intervals $[0,e), \{e\}, I_e^a, (e,a], I_a^e, I_{e,a}$ or $(a,1)$ for all $z \in L$ or $z \in \{e\}$ for all $x,y \in L \setminus \{1\}$. Next we only need to prove the remaining cases.

1. $x \in [0,e)$
   - 1.1. $y \in \{e\}$
     - 1.1.1. $z \in [0,e) \cup I_e^a \cup (e,a]$
       $U(x,z) = U^*(x,z) \leq U^*(y,z) = U(y,z)$
     - 1.1.2. $z \in I_a^e \cup (a,1)$
       $U(x,z) = x \leq z = U(y,z)$
     - 1.1.3. $z \in I_{e,a}$
       $U(x,z) = z = U(y,z)$
   - 1.2. $y \in I_e^a \cup (e,a]$
     - 1.2.1. $z \in [0,e) \cup I_e^a \cup (e,a]$
       $U(x,z) = U^*(x,z) \leq U^*(y,z) = U(y,z)$
     - 1.2.2. $z \in I_a^e \cup (a,1)$
       $U(x,z) = x \leq R(y \vee a, z \vee a) = U(y,z)$
     - 1.2.3. $z \in I_{e,a}$
       $U(x,z) = z = U(y,z)$
   - 1.3. $y \in I_a^e$
     - 1.3.1. $z \in [0,e)$
       $U(x,z) = U^*(x,z) \leq z = U(y,z)$
     - 1.3.2. $z \in I_e^a \cup (e,a]$
       $U(x,z) = U^*(x,z) = x < a < R(y \vee a, z \vee a) = U(y,z)$
     - 1.3.3. $z \in I_a^e \cup (a,1)$
       $U(x,z) = x < a < R(y \vee a, z \vee a) = U(y,z)$
     - 1.3.4. $z \in I_{e,a}$
       $U(x,z) = z = U(y,z)$

1.4. $y \in I_{e,a}$
1.4.1. $z \in [0,e)$
$U(x,z) = U^*(x,z) \leq x < y = U(y,z)$
1.4.2. $z \in I_e^a \cup (e,a] \cup I_a^e \cup (a,1)$
$U(x,z) = x < y = U(y,z)$
1.4.3. $z \in I_{e,a}$
$U(x,z) = z \leq cl(y) \vee cl(z) = U(y,z)$
1.5. $y \in (a,1)$
1.5.1. $z \in [0,e)$
$U(x,z) = U^*(x,z) \leq z = U(y,z)$
1.5.2. $z \in I_e^a \cup (e,a]$
$U(x,z) = U^*(x,z) \leq x < a < R(y \vee a, z \vee a) = U(y,z)$
1.5.3. $z \in I_a^e \cup (a,1)$
$U(x,z) = x < a < R(y \vee a, z \vee a) = U(y,z)$
1.5.4. $z \in I_{e,a}$
$U(x,z) = z = U(y,z)$

2. $x \in I_e^a$
2.1. $y \in (e,a]$
2.1.1. $z \in [0,e) \cup I_e^a \cup (e,a]$
$U(x,z) = U^*(x,z) \leq U^*(y,z) = U(y,z)$
2.1.2. $z \in I_a^e \cup (a,1)$
$U(x,z) = R(x \vee a, z \vee a) \leq R(y \vee a, z \vee a) = U(y,z)$
2.1.3. $z \in I_{e,a}$
$U(x,z) = z = U(y,z)$
2.2. $y \in I_a^e$
2.2.1. $z \in [0,e)$
$U(x,z) = U^*(x,z) = z = U(y,z)$
2.2.2. $z \in I_{e,a}$
$U(x,z) = z = U(y,z)$
2.2.3. $z \in I_e^a \cup (e,a]$
$U(x,z) = U^*(x,z) \leq a < R(y \vee a, z \vee a) = U(y,z)$
2.2.4. $z \in I_a^e \cup (a,1)$
$U(x,z) = R(x \vee a, z \vee a) \leq R(y \vee a, z \vee a) = U(y,z)$
2.3. $y \in (a,1)$
2.3.1. $z \in [0,e) \cup I_{e,a}$
$U(x,z) = z = U(y,z)$
2.3.2. $z \in I_e^a \cup (e,a]$
$U(x,z) = U^*(x,z) \leq a < R(y \vee a, z \vee a) = U(y,z)$
2.3.3. $z \in I_a^e \cup (a,1)$
$U(x,z) = R(x \vee a, z \vee a) \leq R(y \vee a, z \vee a) = U(y,z)$

3. $x \in \{e\}$
3.1. $y \in (e,a]$
3.1.1. $z \in [0,e) \cup I_e^a \cup (e,a]$
$U(x,z) = U^*(x,z) \leq U^*(y,z) = U(y,z)$
3.1.2. $z \in I_a^e \cup (a,1)$

$U(x, z) = z \leq R(y \vee a, z \vee a) = U(y, z)$

3.1.3. $z \in I_{e,a}$

$U(x, z) = z = U(y, z)$

3.2. $y \in I_a^e \cup (a, 1)$

3.2.1. $z \in [0, e) \cup I_{e,a}$

$U(x, z) = z = U(y, z)$

3.2.2. $z \in I_e^a \cup (e, a] \cup I_a^e \cup (a, 1)$

$U(x, z) = z < R(y \vee a, z \vee a) = U(y, z)$

4. $x \in (e, a], y \in I_a^e \cup (a, 1)$

4.1. $z \in [0, e) \cup I_{e,a}$

$U(x, z) = z = U(y, z)$

4.2. $z \in I_e^a \cup (e, a]$

$U(x, z) = U^*(x, z) \leq a < R(y \vee a, z \vee a) = U(y, z)$

4.3. $z \in I_a^e \cup (a, 1)$

$U(x, z) = R(x \vee a, z \vee a) \leq R(y \vee a, z \vee a) = U(y, z)$

5. $x \in I_a^e, y \in (a, 1)$

5.1. $z \in [0, e) \cup I_{e,a}$

$U(x, z) = z = U(y, z)$

5.2. $z \in I_e^a \cup (e, a] \cup I_a^e \cup (a, 1)$

$U(x, z) = R(x \vee a, z \vee a) \leq R(y \vee a, z \vee a) = U(y, z)$

Therefore, the monotonicity property of U holds.

II. Associativity: We need to prove that $U(x, U(y, z)) = U(U(x, y), z)$ for all $x, y, z \in L$. If at least one of x, y, z equals to 1 or e, then $U(x, U(y, z)) = U(U(x, y), z)$ holds. Then we need to verify the following cases by Proposition 2.12.

1. If $x, y, z \in [0, e) \cup I_e^a \cup (e, a]$, then since U^* is associative, we have $U(x, U(y, z)) = U(U(x, y), z) = U(y, U(x, z))$.

2. If $x, y, z \in I_a^e \cup (a, 1)$, then since R is associative, we have $U(x, U(y, z)) = U(U(x, y), z)$.

3. If $x, y, z \in I_{e,a}$, then $U(x, U(y, z)) = U(x, cl(y) \vee cl(z)) = cl(x) \vee cl(y) \vee cl(z) = U(cl(x) \vee cl(y), z) = U(U(x, y), z)$.

4. If $x, y \in [0, e)$ and $z \in I_a^e \cup (a, 1)$, then $U(x, U(y, z)) = U(x, y) = U^*(x, y) = U(U^*(x, y), z) = U(U(x, y), z)$.

5. If $x, y \in I_e^a \cup (e, a]$ and $z \in I_a^e \cup (a, 1)$, then $U(x, U(y, z)) = U(x, R(y \vee a, z \vee a)) = R(x \vee a, R(y \vee a, z \vee a)) = R(a, R(a, z \vee a)) = R(R(a, a), z \vee a) = R(a, z \vee a) = R(U^*(x, y) \vee a, z \vee a) = U(U^*(x, y), z) = U(U(x, y), z)$ and $U(y, U(x, z)) = U(y, R(x \vee a, z \vee a)) = R(y \vee a, R(x \vee a, z \vee a)) = R(a, R(a, z \vee a)) = R(R(a, a), z \vee a) = R(a, z \vee a)$. Thus $U(x, U(y, z)) = U(U(x, y), z) = U(y, U(x, z))$.

6. If $x, y \in [0, e) \cup I_e^a \cup (e, a]$ and $z \in I_{e,a}$, then $U(x, U(y, z)) = U(x, z) = z = U(U^*(x, y), z) = U(U(x, y), z)$ and $U(y, U(x, z)) = U(y, z) = z$. Thus $U(x, U(y, z)) = U(U(x, y), z) = U(y, U(x, z))$.

7. If $x, y \in I_a^e$ and $z \in I_{e,a}$, then $U(x, U(y, z)) = U(x, z) = z = U(R(x \vee a, y \vee a), z) = U(U(x, y), z)$.

8. If $x, y \in I_{e,a}, z \in (a, 1)$ and $cl(x) \vee cl(y) = 1$, then $U(x, U(y, z)) = U(x, y) = cl(x) \vee cl(y) = 1 = U(1, z) = U(cl(x) \vee cl(y), z) = U(U(x, y), z)$;

If $x, y \in I_{e,a}$ $z \in (a, 1)$ and $cl(x) \vee cl(y) \neq 1$, then $U(x, U(y, z)) = U(x, y) = cl(x) \vee cl(y) = U(cl(x) \vee cl(y), z) = U(U(x, y), z)$.

9. If $x \in [0, e)$ and $y, z \in I_a^e \cup (a, 1)$, then $U(x, U(y, z)) = U(x, R(y \vee a, z \vee a)) = x = U(x, z) = U(U(x, y), z)$ and $U(y, U(x, z)) = U(y, x) = x$. Thus $U(x, U(y, z)) = U(U(x, y), z) = U(y, U(x, z))$.

10. If $x \in I_e^a \cup (e, a]$ and $y, z \in I_a^e \cup (a, 1)$, then since R is associative, we have $U(x, U(y, z)) = U(U(x, y), z) = U(y, U(x, z))$.

11. If $x \in [0, e) \cup I_e^a \cup (e, a] \cup I_a^e, y, z \in I_{e,a}$ and $cl(y) \vee cl(z) = 1$, then $U(x, U(y, z)) = U(x, cl(y) \vee cl(z)) = U(x, 1) = 1 = cl(y) \vee cl(z) = U(U(x, y), z)$;

If $x \in [0, e) \cup I_e^a \cup (e, a] \cup I_a^e, y, z \in I_{e,a}$ and $cl(y) \vee cl(z) \neq 1$, then $U(x, U(y, z)) = U(x, cl(y) \vee cl(z)) = cl(y) \vee cl(z) = U(y, z) = U(U(x, y), z)$.

12. If $x \in I_{e,a}$ and $y, z \in (a, 1)$, then $U(x, U(y, z)) = U(x, R(y \vee a, z \vee a)) = x = U(x, z) = U(U(x, y), z)$.

13. If $x \in [0, e), y \in I_e^a \cup (e, a]$ and $z \in I_a^e \cup (a, 1)$, then $U(x, U(y, z)) = U(x, R(y \vee a, z \vee a)) = x = U(x, z) = U(U(x, y), z)$ and $U(y, U(x, z)) = U(y, x) = x$. Thus $U(x, U(y, z)) = U(U(x, y), z) = U(y, U(x, z))$.

14. If $x \in [0, e), y \in I_a^e$ and $z \in I_{e,a}$, then $U(x, U(y, z)) = U(x, z) = z = U(x, z) = U(U(x, y), z)$ and $U(y, U(x, z)) = U(y, z) = z$. Thus $U(x, U(y, z)) = U(U(x, y), z) = U(y, U(x, z))$.

15. If $x \in [0, e), y \in I_{e,a}$ and $z \in (a, 1)$, then $U(x, U(y, z)) = U(x, y) = y = U(y, z) = U(U(x, y), z)$ and $U(y, U(x, z)) = U(y, x) = y$. Thus $U(x, U(y, z)) = U(U(x, y), z) = U(y, U(x, z))$.

16. If $x \in I_e^a, y \in I_a^e$ and $z \in I_{e,a}$, then $U(x, U(y, z)) = U(x, z) = z = U(R(x \vee a, y \vee a), z) = U(U(x, y), z)$ and $U(y, U(x, z)) = U(y, z) = z$. Thus $U(x, U(y, z)) = U(U(x, y), z) = U(y, U(x, z))$.

17. If $x \in (e, a], y \in I_a^e$ and $z \in I_{e,a}$, then $U(x, U(y, z)) = U(x, z) = z = U(R(x \vee a, y \vee a), z) = U(U(x, y), z)$ and $U(y, U(x, z)) = U(y, z) = z$. Thus $U(x, U(y, z)) = U(U(x, y), z) = U(y, U(x, z))$.

18. If $x \in I_e^a \cup (e, a] \cup I_a^e, y \in I_{e,a}$ and $z \in (a, 1)$, then $U(x, U(y, z)) = U(x, y) = y = U(y, z) = U(U(x, y), z)$ and $U(y, U(x, z)) = U(y, R(x \vee a, z \vee a)) = y$. Thus $U(x, U(y, z)) = U(U(x, y), z) = U(y, U(x, z))$.

Combining the above cases, we obtain that $U(x, U(y, z)) = U(U(x, y), z)$ for all $x, y, z \in L$ by Proposition 2.12. Therefore, U is a uninorm on L with the neutral element e. □

In the following, we investigate the additional conditions under which the sufficient conditions (1),(2) and (3) in Theorem 3.1 are necessary. The key point is that these requirements on L are relatively weak and can always be fulfilled.

Lemma 3.2. If $(a, 1) \neq \emptyset$ and $I_a^e \neq \emptyset$, then the condition $U^*(x, y) \notin [0, e]$ for all $(x, y) \in I_e^a \times I_e^a$ is necessary.

P r o o f. First, we give the proof of $U^*(x,y) \notin [0,e)$ for all $(x,y) \in I_e^a \times I_e^a$. Assume that there exists $(x,y) \in I_e^a \times I_e^a$ such that $U^*(x,y) \in [0,e)$. Since $(a,1) \neq \emptyset$, there exists $z \in (a,1)$. Then $U(x,U(y,z)) = U(x,R(y \vee a, z \vee a)) = R(x \vee a, R(y \vee a, z \vee a)) = R(a, R(a, z \vee a)) = R(a,z)$ and $U(U(x,y),z) = U(U^*(x,y),z) = U^*(x,y)$. Since $U^*(x,y) \neq R(a,z)$, this is a contradiction with the associativity property of U.

Next we prove that $U^*(x,y) \neq e$ for all $(x,y) \in I_e^a \times I_e^a$. Assume that there exists $(x,y) \in I_e^a \times I_e^a$ such that $U^*(x,y) = e$. Since $I_a^e \neq \emptyset$, there exists $z \in I_a^e$. Then $U(x,U(y,z)) = U(x,R(y \vee a, z \vee a)) = R(x \vee a, R(y \vee a, z \vee a)) = R(a, R(a, z \vee a)) = R(a, z \vee a)$ and $U(U(x,y),z) = U(e,z) = z$. Since $R(a, z \vee a) \neq z$, this is a contradiction with the associativity property of U.

Therefore, we can obtain that $U^*(x,y) \notin [0,e]$ for all $(x,y) \in I_e^a \times I_e^a$. □

Lemma 3.3. If $(a,1) \neq \emptyset$, then the conditions that $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a \cup I_a^e \cup (a,1)$ and $U^*(x,y) = \begin{cases} x & \text{if } (x,y) \in [0,e) \times (I_e^a \cup (e,a]), \\ y & \text{if } (x,y) \in (I_e^a \cup (e,a]) \times [0,e), \end{cases}$ are necessary.

P r o o f. (1) First, we give the proof of $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_a^e \cup (a,1)$. Assume that there exist $x \in I_{e,a}$ and $y \in I_a^e \cup (a,1)$ such that $x \nparallel y$, i.e., $x < y$. Then $U(0,x) = x$ and $U(0,y) = 0$. Since $x \nleq 0$, this is a contradiction with the monotonicity property of U. Thus $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_a^e \cup (a,1)$.

Next, we give the proof of $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a$. Assume that there exist $x \in I_{e,a}$ and $y \in I_e^a$ such that $x \nparallel y$, i.e., $y < x$. Since $(a,1) \neq \emptyset$, there exists $z \in (a,1)$. Then $U(z,x) = x$ and $U(z,y) = R(z \vee a, y \vee a) = R(z,a)$. Since $x \parallel R(z,a)$ from the above proof, this contradicts the monotonicity property of U. Thus, $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a$.

Combining the above proofs, we can obtain that $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a \cup I_a^e \cup (a,1)$.

(2) First, we prove $U^*(x,y) = \begin{cases} x & \text{if } (x,y) \in [0,e) \times (e,a], \\ y & \text{if } (x,y) \in (e,a] \times [0,e). \end{cases}$ Then we just give the proof of $U^*(x,y) = x$ for all $(x,y) \in [0,e) \times (e,a]$, and the other case can be obtained immediately by the commutativity property of U^*. From the definition of U, we know that $U(x,y) = U^*(x,y) = x$ for all $(x,y) \in [0,e) \times \{e\}$ and $U(x,y) = x$ for all $(x,y) \in [0,e) \times (a,1)$. Since the monotonicity of U, $U(x,y) = x$ for all $(x,y) \in [0,e) \times (e,a]$. By the definition of U, $U^*(x,y) = U(x,y) = x$ for all $(x,y) \in [0,e) \times (e,a]$. Thus $U^*(x,y) = \begin{cases} x & \text{if } (x,y) \in [0,e) \times (e,a], \\ y & \text{if } (x,y) \in (e,a] \times [0,e). \end{cases}$

Next, we prove $U^*(x,y) = \begin{cases} x & \text{if } (x,y) \in [0,e) \times I_e^a, \\ y & \text{if } (x,y) \in I_e^a \times [0,e). \end{cases}$ Now we give the proof of $U^*(x,y) = x$ for all $(x,y) \in [0,e) \times I_e^a$, and the other case can be proved immediately by the commutativity property of U^*. From the above proof, we can obtain that $U^*(x,y) \in [0,e)$ for all $(x,y) \in [0,e) \times \{a\} \cup \{a\} \times [0,e)$. Since the monotonicity of U^*, $U^*(x,y) \in [0,e)$ for all $(x,y) \in [0,e) \times I_e^a$. Assume that there exists

$(x,y) \in [0,e) \times I_e^a$ such that $U^*(x,y) \neq x$. Since $(a,1) \neq \emptyset$, there exists $z \in (a,1)$. Then $U(x,U(y,z)) = U(x,R(y \vee a, z \vee a)) = x$ and $U(U(x,y),z) = U(U^*(x,y),z) = U^*(x,y)$. Since $U^*(x,y) \neq x$, this is a contradiction with the associativity property of U. Thus
$U^*(x,y) = \begin{cases} x & \text{if } (x,y) \in [0,e) \times I_e^a, \\ y & \text{if } (x,y) \in I_e^a \times [0,e). \end{cases}$ □

According to the results in Theorem 3.1 and Lemmas 3.2 and 3.3, we can obtain the following result eventually. That is, if $(a,1) \neq \emptyset$ and $I_a^e \neq \emptyset$, then all the conditions are necessary and sufficient.

Theorem 3.4. Let $a \in L \setminus \{0,1\}$, $cl : L \to L$ be a closure operator, U^* be a uninorm on $[0,a]$ with a neutral element e and R be a t-superconorm on $[a,1]$ with $R(a,a) = a$. Assume $(a,1) \neq \emptyset$ and $I_a^e \neq \emptyset$. Then the function $U : L^2 \to L$ defined by

$$U(x,y) = \begin{cases} U^*(x,y) & \text{if } (x,y) \in [0,a]^2, \\ x & \text{if } (x,y) \in I_{e,a} \times (L \setminus (\{1\} \cup I_{e,a})) \cup [0,e) \times (I_a^e \cup (a,1)) \\ & \qquad \cup (I_a^e \cup (a,1)) \times \{e\}, \\ y & \text{if } (x,y) \in (L \setminus (\{1\} \cup I_{e,a})) \times I_{e,a} \cup (I_a^e \cup (a,1)) \times [0,e) \\ & \qquad \cup \{e\} \times (I_a^e \cup (a,1)), \\ cl(x) \vee cl(y) & \text{if } (x,y) \in I_{e,a} \times I_{e,a} \cup \{1\} \times L \cup L \times \{1\}, \\ R(x \vee a, y \vee a) & \text{otherwise}, \end{cases}$$

is a uninorm on L with the neutral element e if and only if the following conditions hold:

(1) $U^*(x,y) \notin [0,e]$ for all $(x,y) \in I_e^a \times I_e^a$,

(2) $U^*(x,y) = \begin{cases} x & \text{if } (x,y) \in [0,e) \times (I_e^a \cup (e,a]), \\ y & \text{if } (x,y) \in (I_e^a \cup (e,a]) \times [0,e), \end{cases}$

(3) $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a \cup I_a^e \cup (a,1)$,

(4) $R(x,y) < 1$ for all $(x,y) \in (a,1)^2$,

(5) $x \vee a < 1$ for all $x \in I_a^e$.

In Theorem 3.1, if taking $e = a$, then $[0,a] = [0,e]$, $I_{e,a} = I_e$, $I_e^a \cup I_a^e \cup (e,a] = \emptyset$ and U^* is a t-norm on $[0,e]$. Thus, the conditions (1), (2), (5) and the condition $x \parallel y$ for all $x \in I_{e,a}$ and $y \in I_e^a \cup I_a^e$ in (3) naturally hold. In this case, we can obtain the following result.

Corollary 3.5. Let $e \in L \setminus \{0,1\}$, $cl : L \to L$ be a closure operator, T be a t-norm on $[0,e]$ and R be a t-superconorm on $[e,1]$ with $R(e,e) = e$ and U be the binary operation on L defined by

$$U(x,y)=\begin{cases} T(x,y) & \text{if } (x,y)\in[0,e]^2,\\ x & \text{if } (x,y)\in I_e\times(L\setminus(\{1\}\cup I_e))\cup[0,e)\times(e,1)\\ & \qquad\qquad \cup\,(e,1)\times\{e\},\\ y & \text{if } (x,y)\in(L\setminus(\{1\}\cup I_e))\times I_e\cup(e,1)\times[0,e)\\ & \qquad\qquad \cup\,\{e\}\times(e,1),\\ cl(x)\vee cl(y) & \text{if } (x,y)\in I_e\times I_e\cup\{1\}\times L\cup L\times\{1\},\\ R(x\vee e,y\vee e) & \text{otherwise.}\end{cases}$$

Suppose that $x \parallel y$ for all $x\in I_e$ and $y\in(e,1)$. Then U is a uninorm on L with the neutral element e if and only if $R(x,y)<1$ for all $(x,y)\in(e,1)^2$.

Remark 3.6. In Corollary 3.5, if we take the t-superconorm R as a t-conorm, then we can obtain the uninorm in Theorem 2.10. Moreover, compared with Theorem 2.10, the condition $R(x,y)<1$ for all $(x,y)\in(e,1)^2$ in Corollary 3.5 is both sufficient and necessary. This shows that our method generalizes the known method in the literature.

The next example illustrates the construction method of uninorms on bounded lattices in Theorem 3.1.

Example 3.7. Given a bounded lattice $L_1=\{0,b,m,e,c,q,a,n,d,s,f,k,l,1\}$ depicted in Figure 1, a uninorm $U^*:[0,a]^2\to[0,a]$ shown in Table 1, a t-superconorm R on $[a,1]$ defined by $R(x,y)=x\vee y$ for all $x,y\in[a,1]$, and a closure operator $cl:L_1\to L_1$ defined by $cl(x)=x\vee k$ for all $x\in L_1$. It is easy to see that L_1, U^*, R and cl satisfy the conditions in Theorem 3.1. By Theorem 3.1, we can obtain a uninorm U on L_1 with the neutral element e, as shown in Table 2.

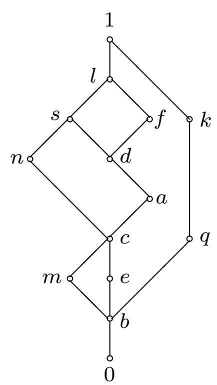

Fig. 1. The lattice L_1.

U^*	0	b	e	m	c	a
0	0	0	0	0	0	0
b	0	b	b	b	b	b
e	0	b	e	m	c	a
m	0	b	m	m	c	a
c	0	b	c	c	c	a
a	0	b	a	a	a	a

Tab. 1. The uninorm U^* on $[0, a]$.

U	0	b	e	m	c	a	d	n	s	f	l	q	k	1
0	0	0	0	0	0	0	0	0	0	0	0	q	k	1
b	0	b	b	b	b	b	b	b	b	b	b	q	k	1
e	0	b	e	m	c	a	d	n	s	f	l	q	k	1
m	0	b	m	m	c	a	d	s	s	f	l	q	k	1
c	0	b	c	c	c	a	d	s	s	f	l	q	k	1
a	0	b	a	a	a	a	d	s	s	f	l	q	k	1
d	0	b	d	d	d	d	d	s	s	f	l	q	k	1
n	0	b	n	s	s	s	s	s	s	l	l	q	k	1
s	0	b	s	s	s	s	s	s	s	l	l	q	k	1
f	0	b	f	f	f	f	f	l	l	f	l	q	k	1
l	0	b	l	l	l	l	l	l	l	l	l	q	k	1
q	q	q	q	q	q	q	q	q	q	q	q	k	k	1
k	k	k	k	k	k	k	k	k	k	k	k	k	k	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Tab. 2. The uninorm U^* on $[0, a]$.

Next, we propose a new construction method for uninorms on bounded lattices by extending a given uninorm based on an interior operator and a t-subnorm. Meanwhile, the dual result of Theorem 3.1 is given.

Theorem 3.8. Let $b \in L \setminus \{0, 1\}$, $int : L \to L$ be an interior operator, U^* be a uninorm on $[b, 1]$ with a neutral element e and F be a t-subnorm on $[0, b]$ with $F(b, b) = b$. Suppose that the following conditions hold:

(1) $U^*(x, y) \notin [e, 1]$ for all $(x, y) \in I_e^b \times I_e^b$,

(1) $U^*(x, y) = \begin{cases} x & \text{if } (x, y) \in (e, 1] \times (I_e^b \cup [b, e)), \\ y & \text{if } (x, y) \in (I_e^b \cup [b, e)) \times (e, 1], \end{cases}$

(3) $x \parallel y$ for all $x \in I_{e,b}$ and $y \in I_e^b \cup I_b^e \cup (0, b)$.

Then the function $U : L^2 \to L$ defined by

$$U(x,y) = \begin{cases} U^*(x,y) & \text{if } (x,y) \in [b,1]^2, \\ x & \text{if } (x,y) \in I_{e,b} \times (L \setminus (\{0\} \cup I_{e,b})) \cup (e,1] \times (I_b^e \cup (0,b)) \\ & \qquad \cup (I_b^e \cup (0,b)) \times \{e\}, \\ y & \text{if } (x,y) \in (L \setminus (\{0\} \cup I_{e,b})) \times I_{e,b} \cup (I_b^e \cup (0,b)) \times (e,1] \\ & \qquad \cup \{e\} \times (I_b^e \cup (0,b)), \\ int(x) \wedge int(y) & \text{if } (x,y) \in I_{e,b} \times I_{e,b} \cup \{0\} \times L \cup L \times \{0\}, \\ F(x \wedge b, y \wedge b) & \text{otherwise,} \end{cases}$$

is a uninorm on L with the neutral element e if and only if $F(x,y) > 0$ for all $(x,y) \in (0,b)^2$ and $x \wedge b > 0$ for all $x \in I_b^e$.

P r o o f. It can be proved immediately by a proof similar to Theorem 3.1. □

Dually, we give the additional conditions under which some sufficient conditions in Theorem 3.8 are necessary. The key point is that these two requirements on L are relatively weak and can always be fulfilled.

Remark 3.9. Let $b \in L \setminus \{0,1\}$ and U in Theorem 3.8 be a uninorm on L with the neutral element e.

(1) If $(0,b) \neq \emptyset$ and $I_b^e \neq \emptyset$, then the condition $U^*(x,y) \notin [e,1]$ for all $(x,y) \in I_e^b \times I_e^b$ is necessary.

(2) If $(0,b) \neq \emptyset$, then the conditions that $U^*(x,y) = \begin{cases} x & \text{if } (x,y) \in (e,1] \times (I_e^b \cup [b,e)), \\ y & \text{if } (x,y) \in (I_e^b \cup [b,e)) \times (e,1] \end{cases}$ and $x \parallel y$ for all $x \in I_{e,b}$ and $y \in I_e^b \cup I_b^e \cup (0,b)$ are necessary.

Combining the results in Theorem 3.8 and Remark 3.9, we can obtain the following result eventually.

Theorem 3.10. Let $b \in L\setminus\{0,1\}$, $int : L \to L$ be an interior operator, U^* be a uninorm on $[b,1]$ with a neutral element e and F be a t-subnorm on $[0,b]$ with $F(b,b) = b$. Suppose that $(0,b) \neq \emptyset$ and $I_b^e \neq \emptyset$. Then the function $U : L^2 \to L$ defined by

$$U(x,y) = \begin{cases} U^*(x,y) & \text{if } (x,y) \in [b,1]^2, \\ x & \text{if } (x,y) \in I_{e,b} \times (L \setminus (\{0\} \cup I_{e,b})) \cup (e,1] \times (I_b^e \cup (0,b)) \\ & \qquad \cup (I_b^e \cup (0,b)) \times \{e\}, \\ y & \text{if } (x,y) \in (L \setminus (\{0\} \cup I_{e,b})) \times I_{e,b} \cup (I_b^e \cup (0,b)) \times (e,1] \\ & \qquad \cup \{e\} \times (I_b^e \cup (0,b)), \\ int(x) \wedge int(y) & \text{if } (x,y) \in I_{e,b} \times I_{e,b} \cup \{0\} \times L \cup L \times \{0\}, \\ F(x \wedge b, y \wedge b) & \text{otherwise,} \end{cases}$$

is a uninorm on L with the neutral element e if and only if the following conditions hold:

(1) $U^*(x,y) \notin [e,1]$ for all $(x,y) \in I_e^b \times I_e^b$,

(2) $U^*(x,y) = \begin{cases} x & \text{if } (x,y) \in (e,1] \times (I_e^b \cup [b,e)), \\ y & \text{if } (x,y) \in (I_e^b \cup [b,e)) \times (e,1], \end{cases}$

(3) $x \parallel y$ for all $x \in I_{e,b}$ and $y \in I_e^b \cup I_b^e \cup (0,b)$,

(4) $F(x,y) > 0$ for all $(x,y) \in (0,b)^2$,

(5) $x \wedge b > 0$ for all $x \in I_b^e$.

In Theorem 3.8, if taking $e = b$, then $[b,1] = [e,1]$, $I_{e,b} = I_e$, $I_e^b \cup I_b^e \cup [b,e) = \emptyset$ and U^* is a t-conorm on $[e,1]$. Thus, (1), (2), (5) and the condition $x \parallel y$ for all $x \in I_{e,b}$ and $y \in I_e^b \cup I_b^e$ in (3) naturally hold. In this case, we can obtain the following result.

Corollary 3.11. Let $e \in L \backslash \{0,1\}$, $int : L \to L$ be an interior operator, S be a t-conorm on $[e,1]$, F be a t-subnorm on $[0,e]$ with $F(e,e) = e$ and U be the binary operation on L defined by

$$U(x,y) = \begin{cases} S(x,y) & \text{if } (x,y) \in [e,1]^2, \\ x & \text{if } (x,y) \in I_e \times (L \setminus (\{0\} \cup I_e)) \cup (e,1] \times (0,e) \\ & \qquad \cup\, (0,e) \times \{e\}, \\ y & \text{if } (x,y) \in (L \setminus (\{0\} \cup I_e)) \times I_e \cup (0,e) \times (e,1] \\ & \qquad \cup\, \{e\} \times (0,e), \\ int(x) \wedge int(y) & \text{if } (x,y) \in I_e \times I_e \cup \{0\} \times L \cup L \times \{0\}, \\ F(x \wedge e, y \wedge e) & \text{otherwise.} \end{cases}$$

Suppose that $x \parallel y$ for all $x \in I_e$ and $y \in (0,e)$. Then U is a uninorm on L with the neutral element e if and only if $F(x,y) > 0$ for all $(x,y) \in (0,e)^2$.

Remark 3.12. In Corollary 3.11, if we take the t-subnorm F as a t-norm, then we can obtain the uninorm in Theorem 2.11. Moreover, compared with Theorem 2.11, the condition $F(x,y) > 0$ for all $(x,y) \in (0,e)^2$ in Corollary 3.11 is both sufficient and necessary. This shows that our method generalizes the known method in the literature.

4. CONCLUSIONS

In this paper, we provided new methods to construct uninorms by extending given uninorms on a subinterval of a bounded lattice based on closure operators (resp. interior operators) and t-superconorms (resp. t-subnorms). Meanwhile, the resulting methods for uninorms on bounded lattices generalize some known methods for uninorms in the literature.

About the results in this paper, we give the following remarks.

(1) As we see, there are five conditions in Theorems 3.1 and 3.8, respectively. It seems too many. However, under the additional restraints that $(a,1) \neq \emptyset$ and $I_a^e \neq \emptyset$ on L (resp. the conditions $(0,b) \neq \emptyset$ and $I_b^e \neq \emptyset$ on L), these five conditions are necessary and

sufficient. Moreover, these two requirements on L are relatively weak and can always be fulfilled.

(2) It is the first time to construct uninorms by extending uninorms on a subinterval of a bounded lattice with closure operators (resp. interior operators) and t-superconorms (resp. t-subnorms). Meanwhile, we think that this may be the reason that lead to the conditions in Theorems 3.1 and 3.8. So, in the future, we try to construct uninorms also by extending uninorms on a subinterval of a bounded lattice by closure operators (resp. interior operators) and t-superconorms (resp. t-subnorms) with less conditions.

The methods to construct uninorms via uninorms on a subinterval of a bounded lattice were first introduced by Çaylı [5], and Xiu and Zheng [26], respectively, and then were also used by Xiu and Zheng [27]. In the future, we will continue to investigate the methods and then give the constructions and characterizations of uninorms via uninorms on bounded lattices.

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