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CONSTRUCTING MIXED UNINORMS ON BOUNDED LATTICES

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In this paper, we present the definition of mixed uninorms and propose several methods for constructing two special classes of mixed uninorms on bounded lattices through t-subnorms and t-superconorms. These methods generalize $\mathbb{U}_{\min}$, $\mathbb{U}_{\max}$, $\mathbb{U}^1_{\min}$ and $\mathbb{U}^0_{\max}$ on bounded lattices that have been previously discussed in the literature. Some examples are given to construct mixed uninorms on bounded lattices.

Keywords: bounded lattices, uninorms, T-superconorm, T-subnorm

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1. INTRODUCTION

Schweizer and Skla [26] first proposed triangular conorms (t-conorms) and triangular norms (t-norms) on the real unit interval [0,1]. These operators serve as generalizations of disjunctive and conjunctive operations, respectively, within the context of classical two-valued logical connectives. They are extensively utilized in fuzzy logic, fuzzy set theory, multi-criteria decision support, and various branches of information science [12, 17, 28]. Rybalov and Yager [29] introduced uninorms on the real unit interval [0,1] to generalize both triangular conorms and triangular norms, allowing their neutral or unit elements to reside anywhere within the interval. Specifically, if the neutral element or unit is 1, the uninorm corresponds to a t-norm, while if the neutral element or unit is 0, it corresponds to a t-conorm. Fodor and Rybalov [15] demonstrated that uninorms are integration of t-conorms and t-norms. This structural characteristic enables uninorms to be applicable across various fields, including the aggregation of fuzzy information, expert systems [25], fuzzy set theory [16], neural networks [31], and other areas such as pseudo-analysis, measure theory, and fuzzy mathematical morphology.

Due to the fact that bounded lattices exhibit a greater degree of generality than real unit intervals $[0, 1]$, some scholars have expressed significant interest in uninorms defined on bounded lattices. Mesiar and Karacal [22] were the first to construct uninorms on bounded lattices in 2015. Since then, numerous researchers have employed various ways of constructing uninorms, including derived from t-conorms and t-norms [1, 3, 6, 7, 8, 9, 10, 14, 13, 22], t-superconorms and t-subnorms [18, 21, 30], interior operators and

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closure operators [11, 19, 24, 32], additive generators [20], and pre-existing uninorms [27] and so on. Notably, Zhang et al. [30] characterized two classes of uninorms (denoted by $\mathbb{U}_{\max}$ and $\mathbb{U}_{\min}$, respectively) and unified all related construction methods. We find that among all the methods of uninorms on bounded lattices in the literature, almost all the values of uninorms on $A(e) = (e,1] \times [0,e) \cup [0,e) \times (e,1]$ are max or min except for uninorms of $\mathbb{U}^0_{\max}$ and $\mathbb{U}^1_{\min}$, respectively. Then we naturally propose a question: What is the composition of uninorms on bounded lattices if the values of uninorms on $A(e)$ is mixed, i. e., $\mathbb{U}(x_1, y_1) = \min(x_1, y_1)$ for some $(x_1, y_1) \in A(e)$ and $\mathbb{U}(x_2, y_2) = \max(x_2, y_2)$ for $(x_2, y_2) \in A(e)$. This is the motivation of the work.

In the current work, we introduce the definitions of two special types of mixed uninorms, i. e., $\mathbb{U}^1_{\text{mix}}$ and $\mathbb{U}^2_{\text{mix}}$, on a bounded lattices and give their construction methods under some conditions. Our methods can generalize $\mathbb{U}_{\max}$, $\mathbb{U}_{\min}$, $\mathbb{U}^0_{\max}$ and $\mathbb{U}^1_{\min}$ proposed by Zhang et al. [30].

Below is a detailed outline of the work's structure. The fundamental concepts and pivotal conclusions pertaining to various aggregation functions on bounded lattices are revisited and elucidated in Section 2. This section aims to provide a comprehensive understanding of the essential characteristics and properties of uninorms. The definition of two specific types of mixed uninorms on bounded lattices is introduced, along with two innovative constructions for them, in Section 3. Finally, in the concluding section, our analysis will culminate in several conclusions, which will not only summarize the key findings of our study but also propose promising avenues for future scholarly pursuits in this domain.

2. PRELIMINARIES

In this particular section of our discussion, we will first revisit the fundamental concept of bounded lattices. Subsequently, building upon the foundation of bounded lattices, we will delve into the notions and properties of several aggregation functions, with a particular focus on the characterization and represention theorems of some classes of uninorms within the context of bounded lattices.

Definition 2.1. (Birkhoff [2]) The binary relation $\leqslant$, defined on the non-empty set P, is designated as a partial order relation when it fulfilling the three fundamental properties of antisymmetry, reflexivity, and transitivity. If for any two elements in P, there exists a greatest lower bound (also known as the infimum) and a least upper bound (also known as the supremum), then $(P, \leqslant, \wedge, \vee)$ is said to be a lattice. For the sake of convenience, such a lattice is referred to as a partially ordered lattice. If the lattice $(P, \leqslant, \wedge, \vee)$ has a least element 0 and a greatest element 1, then P is called a bounded lattice, and is denoted as $(P, \leqslant, \wedge, \vee, 0, 1)$. Unless otherwise specified, L will always refer to a bounded lattice.

Definition 2.2. (Birkhoff [2]) Let $m, n \in L$ be arbitrarily fixed with $m \leq n$. The subinterval of L is termed as $[m, n] = \{x \in L : m \leq x \leq n\}$. In the same way, we can define $[m, n)$, $(m, n]$, (m, n). The notation I_m denotes the subset of L comprising all elements that are incomparable with m, formally denoted as $I_m = \{x \in L \mid x \parallel m\}$. Correspondingly, the set $I_{m,n}$ denotes the subset of L containing all elements that are simultaneously incomparable with both m and n, mathematically expressed as $I_{m,n} = \{x \in L \mid x \parallel m \text{ and } x \parallel n\}$. Lastly, the concept of I_m^n, if applicable, represents the subset of L where elements are incomparable with m yet maintain a comparability relationship with n, that is, $I_m^n = \{x \in L \mid x \parallel m \ and \ x \nparallel n\}$. Obviously, $I_m^m = \emptyset$ and $I_{m,m} = I_m$.

Definition 2.3. (De Baets and Mesiar [4], De Coomann and Kerre [5]) Let $[m, n]$ be a subinterval of L. A binary function $\mathbb{T} : [m, n]^2 \to [m, n]$ is termed as a triangular norm (t-norm). if it fulfills associativity, commutativity, and monotonically increasing with regard to each argument and $\mathbb{T}(n, y_1) = \mathbb{T}(y_1, n) = y_1$ for all $y_1 \in [m, n]$. The t-norm $\mathbb{T}$ is called positive if $\mathbb{T}(x_1, y_1) > m$ for any $x_1, y_1 \in (m, n]$.

Dually, a binary function $\mathbb{S} : [m, n]^2 \to [m, n]$ is termed as a t-conorm on $[m, n]$ if it satisfies associativity, monotonicity, commutativity and $\mathbb{S}(x_1, m) = \mathbb{S}(m, x_1) = x_1$ for any $x_1 \in [m, n]$. The t-conorm $\mathbb{S}$ is called positive if $\mathbb{S}(x_1, y_1) < n$ for any $x_1, y_1 \in [m, n)$.

Definition 2.4. (Zhang et al. [30]) Let S be a subset of L that has a greatest (resp. smallest) element. A binary function $\mathbb{G} : S^2 \to S$ is known as a t-superconorm (resp. t-subnorm) on S if it fulfills commutativity, monotonicity, associativity, and $x_1 \vee y_1 \leq \mathbb{G}(x_1, y_1)$ (resp. $\mathbb{G}(x_1, y_1) \leq x_1 \wedge y_1$) for any $(x_1, y_1) \in S^2$

Definition 2.5. (Karaçal and Mesiar [22]) A binary operation $\mathbb{U} : L^2 \to L$ is known as a uninorm on L with neutral or unit element $e \in L \setminus \{0, 1\}$ if it fulfills associativity, commutativity, and monotonically increasing with regard to each argument and $\mathbb{U}(e, x_1) = x_1$ for any $x_1 \in L$.

For a uninorm $\mathbb{U}$ with neutral or unit element $e \in L \setminus \{0, 1\}$ on L, we have that $\mathbb{U}|_{[0,e]^2}$ is termed as a t-norm on $[0, e]$ and $\mathbb{U}|_{[e,1]^2}$ is termed as a t-conorm on $[e, 1]$. If $\mathbb{U}(0, 1) = 0$ is satisfied, then $\mathbb{U}$ is a conjunctive uninorm; it is disjunctive, if $\mathbb{U}(0, 1) = 1$.

Definition 2.6. (Zhang et al. [30]) Let a binary function $\mathbb{U} : L^2 \to L$ be a uninorm on L with neutral or unit element $e \in L \setminus \{0, 1\}$

(i) If $\mathbb{U}(x_0, y_0) = y_0$ for all $(x_0, y_0) \in (e, 1] \times (L \setminus [e, 1])$ holds, then $\mathbb{U}$ is termed as $\mathbb{U}_{\min}$.

(ii) If $\mathbb{U}(x_0, y_0) = y_0$ for all $(x_0, y_0) \in [0, e) \times (L \setminus [e, 1])$ holds, then $\mathbb{U}$ is termed as $\mathbb{U}_{\max}$.

(iii) If $\mathbb{U}(x_0, y_0) = y_0$ for all $(x_0, y_0) \in (e, 1) \times (L \setminus [0, e])$, and $U(1, y_0) = 1$ for all $y_0 \in (L \setminus [0, e])$ holds, then $\mathbb{U}$ is termed as $\mathbb{U}_{\min}^1$.

(iv) If $\mathbb{U}(x_0, y_0) = y_0$ for all $(x_0, y_0) \in (0, e) \times (L \setminus [0, e])$, and $\mathbb{U}(0, y_0) = 0$ for all $y_0 \in (L \setminus [0, e])$ holds, then $\mathbb{U}$ is termed as $\mathbb{U}_{\max}^0$.

Theorem 2.7. (Zhang et al. [30]) It is easy to see that $\mathbb{U} \in \mathbb{U}_{\min}$ if and only if $\mathbb{U}$ can be characterized by the following formula, where $\mathbb{S}$ be a t-conorm on $[e, 1]$, $\mathbb{F}$ be a t-subnorm on $(L \setminus [e,1])$.

$$\mathbb{U}(x_0,y_0)=\begin{cases}\mathbb{S}(x_0,y_0) & (x_0,y_0)\in[e,1]^2,\\ y_0 & (x_0,y_0)\in[e,1]\times(L\setminus[e,1]),\\ x_0 & (x_0,y_0)\in(L\setminus[e,1])\times[e,1],\\ \mathbb{F}(x_0,y_0) & (x_0,y_0)\in((L\setminus[e,1]))^2.\end{cases} \tag{1}$$

Theorem 2.8. (Zhang et al. [30]) It is easy to see that $\mathbb{U}\in\mathbb{U}_{\max}$ if and only if $\mathbb{U}$ can be characterized by the following formula, where $\mathbb{T}$ be a t-norm on $[0,e]$ and $\mathbb{G}$ be a t-superconorm on $(L\setminus[0,e])$.

$$\mathbb{U}(x_0,y_0)=\begin{cases}\mathbb{T}(x_0,y_0) & (x_0,y_0)\in[0,e]^2,\\ y_0 & (x_0,y_0)\in[0,e]\times(L\setminus[0,e]),\\ x_0 & (x_0,y_0)\in(L\setminus[0,e])\times[0,e],\\ \mathbb{G}(x_0,y_0) & (x_0,y_0)\in(L\setminus[0,e])^2.\end{cases} \tag{2}$$

Theorem 2.9. (Zhang et al. [30]) Let $\mathbb{U}$ be the binary operation on L defined by the following formula, where $\mathbb{S}$ be a t-conorm on $[e,1]$, $\mathbb{F}$ be a t-subnorm on $(L\setminus[e,1])$.

$$\mathbb{U}(x_0,y_0)=\begin{cases}\mathbb{S}(x_0,y_0) & (x_0,y_0)\in[e,1]^2,\\ y_0 & (x_0,y_0)\in[e,1)\times((L\setminus[e,1])),\\ x_0 & (x_0,y_0)\in(L\setminus[e,1])\times[e,1),\\ 1 & (x_0,y_0)\in\{1\}\times(L\setminus[e,1])\cup(L\setminus[e,1])\times\{1\},\\ \mathbb{F}(x_0,y_0) & (x_0,y_0)\in(L\setminus[e,1])^2.\end{cases} \tag{3}$$

Then $\mathbb{U}\in\mathbb{U}^1_{\min}$ if and only if $\mathbb{S}$ is positive.

Theorem 2.10. (Zhang et al. [30]) Let $\mathbb{U}$ be the binary operation on L defined by the following formula, where $\mathbb{T}$ be a t-norm on $[0,e]$, $\mathbb{G}$ be a t-superconorm on $(L\setminus[0,e])$.

$$\mathbb{U}(x_0,y_0)=\begin{cases}\mathbb{T}(x_0,y_0) & (x_0,y_0)\in[0,e]^2,\\ y_0 & (x_0,y_0)\in(0,e]\times(L\setminus[0,e]),\\ x_0 & (x_0,y_0)\in(L\setminus[0,e])\times(0,e],\\ 0 & (x_0,y_0)\in\{0\}\times(L\setminus[0,e])\cup(L\setminus[0,e])\times\{0\},\\ \mathbb{G}(x_0,y_0) & (x_0,y_0)\in(L\setminus[0,e])^2.\end{cases} \tag{4}$$

Then $\mathbb{U}\in\mathbb{U}^0_{\max}$ if and only if $\mathbb{T}$ is positive.

Theorem 2.11. (Klement et al. [23]) Let $(L,\leqslant)$ be a totally ordered index set and $\{(\mathbb{M}_s,*_s)\}_{s\in I}$ be a family of semigroups. Suppose that for all $s,t\in I$ with $s<t$, it either holds that (1) $\mathbb{M}_s\cap\mathbb{M}_t=\emptyset$ or (2) $\mathbb{M}_s\cap\mathbb{M}_t=\{m_{st}\}$,where m_{st} is both the neutral element of $(\mathbb{M}_s,*_s)$ and the annihilator of $(\mathbb{M}_t,*_t)$, and for all $r\in I$ with $s<r<t$, it holds that $\mathbb{M}_r=\{m_{st}\}$. Let $\mathbb{M}=\bigcup\limits_{s\in I}\mathbb{M}_s$ and define the binary operation $*$ on $\mathbb{M}$ by

$$x_1{*}y_1=\begin{cases}x_1 *_s y_1 & (x_1,y_1)\in\mathbb{M}_s^2,\\ x_1 & (x_1,y_1)\in\mathbb{M}_s\times\mathbb{M}_t \text{ and } s<t,\\ y_1 & (x_1,y_1)\in\mathbb{M}s\times\mathbb{M}_t \text{ and } t<s.\end{cases}$$

Then $(\mathbb{M},*)$ is a semigroup, termed as the ordinal sum of$\{(\mathbb{M}_s,*_s)\}_{s\in I}$.

3. CONSTRUCTIONS OF MIXED UNINORMS

In the present segment of our discussion, we initiate an in-depth exploration of two special types of mixed uninorms within the context of bounded lattices, i. e., $\mathbb{U}(x_1, y_1) = \min(x_1, y_1)$ for some $(x_1, y_1) \in A(e)$ and $\mathbb{U}(x_2, y_2) = \max(x_2, y_2)$ for some $(x_2, y_2) \in A(e)$. We commence by providing precise definitions for these two classes of mixed uninorms, namely, the first class of mixed uninorms and the second class of mixed uninorms (see Definitions 3.1 and 3.9, respectively). Building upon these definitions, we then propose a series of innovative construction methods for these two special types of mixed uninorms. Our findings reveal that these construction methods possess the capability to generalize and extend $\mathbb{U}_{\max}$, $\mathbb{U}_{\min}$, $\mathbb{U}^0_{\max}$ and $\mathbb{U}^1_{\min}$ proposed by Zhang et al. [30], thereby contributing to a deeper understanding and broader appreciation of the mathematical structures involved.

Definition 3.1. Let $\mathbb{U} : L^2 \to L$ be a uninorm with neutral or unit element $e \in L\backslash\{0, 1\}$ on L. We call $\mathbb{U}$ a uninorm of the first class of mixed uninorms, if there exists $t \in [e, 1]$ such that the following conditions are satisfied:

(i) $\mathbb{U}(x_0, y_0) = x_0$ for all $x_0 \in [0, e)$ and $y_0 \in [e, t)$;

(ii) $\mathbb{U}(x_0, y_0) = y_0$ for all $x_0 \in [0, e)$ and $y_0 \in (t, 1]$;

(iii) $\mathbb{U}(x_0, t) \in \{x_0, t\}$ for all $x_0 \in [0, e)$.

Here, we use $\mathbb{U}^1_{\mathrm{mix}}$ to denote the first class of mixed uninorms on bounded lattices.

Theorem 3.2. Let $e \in L\backslash\{0, 1\}$ and $t \in [e, 1]$. Suppose that $\mathbb{F}$ is a t-subnorm on $[0, e) \cup I^t_e$, $\mathbb{S}$ is a t-conorm on $[e, t]$ and $\mathbb{R}$ is a t-superconorm on $L \setminus [0, t]$. If $I^e_t = \emptyset$, then the following operation $\mathbb{U}_1 : L^2 \to L$ is a uninorm on L with neutral or unit element e, where

$$\mathbb{U}_1(x, y) = \begin{cases} \mathbb{S}(x, y) & (x, y) \in [e, t]^2, \\ \mathbb{F}(x, y) & (x, y) \in ([0, e) \cup I^t_e)^2, \\ y & (x, y) \in [e, t] \times (L \setminus [e, t]), \\ x & (x, y) \in (L \setminus [e, t]) \times [e, t], \\ y & (x, y) \in ([0, e] \cup I^t_e) \times (L \setminus [0, t]), \\ x & (x, y) \in (L \setminus [0, t]) \times ([0, e] \cup I^t_e), \\ \mathbb{R}(x, y) & (x, y) \in (L \setminus [0, t])^2. \end{cases} \tag{5}$$

Proof. It is clearly established that $\mathbb{U}_1$ is commutative and that e serves as a neutral element.

(i) Monotonicity. Let $x_1, y_1, z_1 \in L$ be arbitrarily fixed with $x_1 < y_1$. We need to prove that $\mathbb{U}_1(x_1, z_1) \leq \mathbb{U}_1(y_1, z_1)$

1. $x_1 \in [0, e)$
   1.1. $y_1 \in [0, e) \cup I^t_e$
      1.1.1. $z_1 \in [0, e) \cup I^t_e$
         $\mathbb{U}_1(x_1, z_1) = \mathbb{F}(x_1, z_1) \leqslant \mathbb{F}(y_1, z_1) = \mathbb{U}_1(y_1, z_1)$.
      1.1.2. $z_1 \in [e, t]$
         $\mathbb{U}_1(x_1, z_1) = x_1 < y_1 = \mathbb{U}_1(y_1, z_1)$.
      1.1.3. $z_1 \in L\setminus [0, t]$
         $\mathbb{U}_1(x_1, z_1) = z_1 = \mathbb{U}_1(y_1, z_1)$.

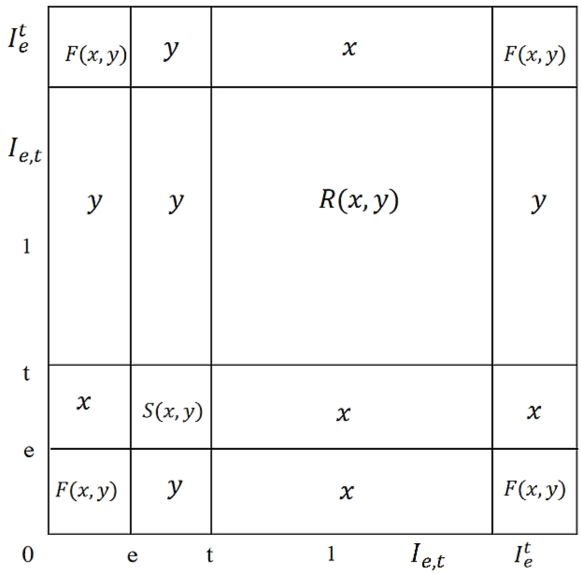

Fig. 1. The structure of uninorm U_1 in Theorem 3.2.

1.2. $y_1 \in [e, t]$

1.2.1. $z_1 \in [0, e) \cup I_e^t$

$\mathbb{U}_1(x_1, z_1) = \mathbb{F}(x_1, z_1) \leqslant z_1 = \mathbb{U}_1(y_1, z_1)$.

1.2.2. $z_1 \in [e, t]$

$\mathbb{U}_1(x_1, z_1) = x_1 < y_1 \leqslant \mathbb{S}(y_1, z_1) = \mathbb{U}_1(y_1, z_1)$.

1.2.3. $z_1 \in L \setminus [0, t]$

$\mathbb{U}_1(x_1, z_1) = z_1 = \mathbb{U}_1(y_1, z_1)$.

1.3. $y_1 \in L \setminus [0, t]$

1.3.1. $z_1 \in [0, e) \cup I_e^t$

$\mathbb{U}_1(x_1, z_1) = \mathbb{F}(x_1, z_1) \leqslant x_1 < y_1 = \mathbb{U}_1(y_1, z_1)$.

1.3.2. $z_1 \in [e, t]$

$\mathbb{U}_1(x_1, z_1) = x_1 < y_1 = \mathbb{U}_1(y_1, z_1)$.

1.3.3. $z_1 \in L \setminus [0, t]$

$\mathbb{U}_1(x_1, z_1) = z_1 \leqslant \mathbb{R}(y_1, z_1) = \mathbb{U}_1(y_1, z_1)$.

2. $x_1 \in [e, t]$. Then $y_1 \in [e, t] \cup (t, 1]$.

2.1. $y_1 \in [e, t]$

2.1.1. $z_1 \in [e, t]$

$\mathbb{U}_1(x_1, z_1) = \mathbb{S}(x_1, z_1) \leq \mathbb{S}(y_1, z_1) = \mathbb{U}_1(y_1, z_1)$.

2.1.2. $z_1 \in L \setminus [e, t]$

$\mathbb{U}_1(x_1, z_1) = z_1 = \mathbb{U}_1(y_1, z_1)$.

2.2. $y_1 \in (t, 1]$

2.2.1. $z_1 \in [0, e) \cup I_e^t$

$\mathbb{U}_1(x_1, z_1) = z_1 < y_1 = \mathbb{U}_1(y_1, z_1)$.

2.2.2. $z_1 \in [e,t]$
$\mathbb{U}_1(x_1,z_1) = \mathbb{S}\,(x_1,z_1) \leq t < y_1 = \mathbb{U}_1(y_1,z_1)$.
2.2.3. $z_1 \in L\setminus\,[0,t]$
$\mathbb{U}_1(x_1,z_1) = z_1 \leqslant \mathbb{R}\,(y_1,z_1) = \mathbb{U}_1(y_1,z_1)$.

3. $x_1 \in I_e^t$. Then $y \in [e,t] \cup (L\setminus\,[0,t]) \cup I_e^t$.
3.1. $y_1 \in [e,t]$
3.1.1. $z_1 \in [0,e) \cup I_e^t$
$\mathbb{U}_1(x_1,z_1) = \mathbb{F}\,(x_1,z_1) \leqslant z_1 = \mathbb{U}_1(y_1,z_1)$.
3.1.2. $z_1 \in [e,t]$
$\mathbb{U}_1(x_1,z_1) = x_1 < y_1 \leqslant \mathbb{S}\,(y_1,z_1) = \mathbb{U}_1(y_1,z_1)$.
3.1.3. $z_1 \in L\setminus\,[0,t]$
$\mathbb{U}_1(x_1,z_1) = z_1 = \mathbb{U}_1(y_1,z_1)$.
3.2. $y_1 \in L\setminus\,[0,t]$
3.2.1. $z_1 \in [0,e) \cup I_e^t$
$\mathbb{U}_1(x_1,z_1) = \mathbb{F}\,(x_1,z_1) \leqslant x_1 < y_1 = \mathbb{U}_1(y_1,z_1)$.
3.2.2. $z_1 \in [e,t]$
$\mathbb{U}_1(x_1,z_1) = x_1 < y_1 = \mathbb{U}_1(y_1,z_1)$.
3.2.3. $z_1 \in L\setminus\,[0,t]$
$\mathbb{U}_1(x_1,z_1) = z_1 \leqslant \mathbb{R}\,(y_1,z_1) = \mathbb{U}_1(y_1,z_1)$.
3.3. $y_1 \in I_e^t$
3.3.1. $z_1 \in [0,e)\cup I_e^t$
$\mathbb{U}_1(x_1,z_1) = \mathbb{F}\,(x_1,z_1) \leqslant \mathbb{F}\,(y_1,z_1) = \mathbb{U}_1(y_1,z_1)$.
3.3.2. $z_1 \in [e,t]$
$\mathbb{U}_1(x_1,z_1) = x_1 < y_1 = \mathbb{U}_1(y_1,z_1)$.
3.3.3. $z_1 \in L\setminus\,[0,t]$
$\mathbb{U}_1(x_1,z_1) = z_1 = \mathbb{U}_1(y_1,z_1)$.

4. $x_1 \in I_{e,t}$. Then $y_1 \in [t,1] \cup I_{e,t}$.
4.1. $z_1 \in [0,e) \cup I_e^t \cup [e,t]$
$\mathbb{U}_1(x_1,z_1) = x_1 < y_1 = \mathbb{U}_1(y_1,z_1)$.
4.2. $z_1 \in L\setminus\,[0,t]$
$\mathbb{U}_1(x_1,z_1) = \mathbb{R}\,(x_1,z_1) \leqslant \mathbb{R}\,(y_1,z_1) = \mathbb{U}_1(y_1,z_1)$.

5. $x_1 \in (t,1]$. Then $y_1 \in (t,1]$.
5.1. $z_1 \in [0,e) \cup I_e^t \cup [e,t]$
$\mathbb{U}_1(x_1,z_1) = x_1 < y_1 = \mathbb{U}_1(y_1,z_1)$.
5.2. $z_1 \in L\setminus\,[0,t]$
$\mathbb{U}_1(x_1,z_1) = \mathbb{R}\,(x_1,z_1) \leqslant \mathbb{R}\,(y_1,z_1) = \mathbb{U}_1(y_1,z_1)$.

(*ii*) Associativity. Let $\mathbb{G}_1 = ([0,e) \cup I_e^t, \mathbb{F})$ and $\mathbb{G}_2 = ([e,t]\,, \mathbb{S})$ and $\mathbb{G}_3 = (L \setminus [0,t]\,, \mathbb{R})$. Obviously, $\mathbb{G}_1$, $\mathbb{G}_2$, $\mathbb{G}_3$ are semigroups and $(L, \mathbb{U}_1)$ is the ordinal sum of $\{\mathbb{G}_i\}_{i\in I}$, where $I = \{1,2,3\}$ is equipped with the order $3 < 1 < 2$. Hence according to Theorem 2.11, the associativity of $\mathbb{U}_1$ holds.
Consequently, we have the fact that $\mathbb{U}_1$ is a uninorm on L. □

Remark 3.3. If, in Theorem 3.2, we take $t = 1$, we can derive $\mathbb{U}_{\min}$, whose structure corresponds to the previously mentioned (1).

Remark 3.4. If, in Theorem 3.2, we take $t = e$, then $I_e^t = I_t^e = \emptyset$, $I_{e,t} = I_e$, and $\mathbb{F}$ is a t-subnorm defined on $[0,e)^2$. Combined with the neutral element property, $\mathbb{U}_1 \mid_{[0,e]^2}$ is a t-norm $\mathbb{T}$. So we can derive $\mathbb{U}_{\max}$, whose structure corresponds to the previously mentioned (2).

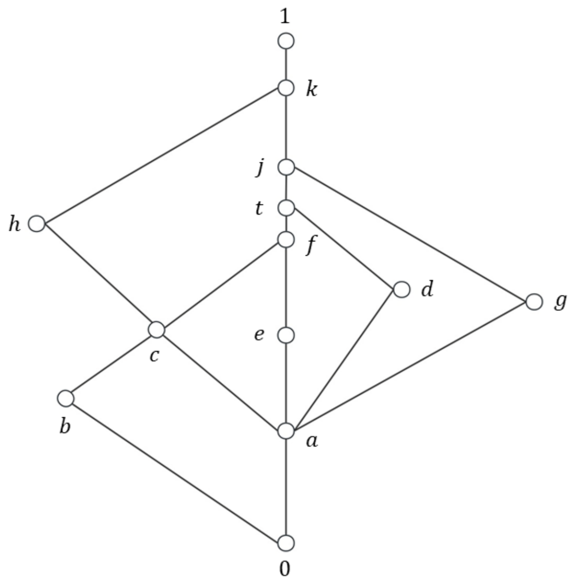

Fig. 2. The bounded lattice L_1.

Example 3.5. Let us discuss the bounded lattice $L_1 = \{0,a,b,c,d,e,f,t,g,h,j,k,1\}$ depicted in Figure 2, the t-subnorm $\mathbb{F} : \left([0,e) \cup I_e^t\right)^2 \to \left([0,e) \cup I_e^t\right)$ be given by $\mathbb{F}(x,y) = x \wedge y \wedge a$, the t-conorm $\mathbb{S} : [e,t]^2 \to [e,t]$ be given by $\mathbb{S}(x,y) = x \vee y$, and $\mathbb{R}(x,y) = x \vee y \vee k$ on $L \setminus [0,t]$. Based on Theorem 3.2, the construction of the uninorm $\mathbb{U}_1$ is as shown in Table 1.

Theorem 3.6. Let $e \in L \setminus \{0,1\}$ and $t \in [e,1]$. Suppose that $\mathbb{F}$ is a t-subnorm on $\left([0,e) \cup I_e^t\right)^2$, $\mathbb{S}$ is a positive t-conorm on $[e,t]$ and $\mathbb{R}$ is a t-superconorm on $L \setminus [0,t]$. If $I_t^e = \emptyset$, then the function $\mathbb{U}_2 : L^2 \to L$ defined as follows is a uninorm on L with the neutral or unit element e, where

$$\mathbb{U}_2(x,y) = \begin{cases} \mathbb{S}(x,y) & (x,y) \in [e,t]^2, \\ \mathbb{F}(x,y) & (x,y) \in \left([0,e) \cup I_e^t\right)^2, \\ y & (x,y) \in [e,t) \times \left([0,e) \cup I_e^t\right), \\ x & (x,y) \in \left([0,e) \cup I_e^t\right) \times [e,t), \\ y & (x,y) \in [0,t] \times (L \setminus [0,t)), \\ x & (x,y) \in (L \setminus [0,t)) \times [0,t], \\ \mathbb{R}(x,y) & (x,y) \in (L \setminus [0,t])^2. \end{cases} \tag{6}$$

$\mathbb{U}_1$	0	a	b	c	d	e	f	t	g	h	j	k	1
0	0	0	0	0	0	0	0	0	g	h	j	k	1
a	0	a	0	a	a	a	a	a	g	h	j	k	1
b	0	0	b	0	0	b	b	b	g	h	j	k	1
c	0	a	0	c	a	c	c	c	g	h	j	k	1
d	0	a	0	a	d	d	d	d	g	h	j	k	1
e	0	a	b	c	d	e	f	t	g	h	j	k	1
f	0	a	b	c	d	f	f	t	g	h	j	k	1
t	0	a	b	c	d	t	t	t	g	h	j	k	1
g	g	g	g	g	g	g	g	g	g	k	k	k	1
h	h	h	h	h	h	h	h	h	k	h	k	k	1
j	j	j	j	j	j	j	j	j	k	k	j	k	1
k	k	k	k	k	k	k	k	k	k	k	k	k	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1

Tab. 1. The uninorm $\mathbb{U}_1$ in Example 3.5.

Proof. Assume that $\mathbb{S}(x,y)$ is not positive, then there exists $(x_1,y_1) \in [e,t)^2$ such that $\mathbb{S}(x_1,y_1) = t$. Take $z_1 \in [0,t] \setminus [e,t]$. Then $\mathbb{U}_2(z_1,\mathbb{U}_2(x_1,y_1) = \mathbb{U}_2(z_1,\mathbb{S}(x_1,y_1)) = \mathbb{U}_2(z_1,t) = t$ and $\mathbb{U}_2(\mathbb{U}_2(x_1,z_1),y_1) = \mathbb{U}_2(z_1,y_1) = z_1$. Since $z_1 \neq t$, the associativity property of $\mathbb{U}_2(x_1,y_1)$ is violated. Thus $\mathbb{S}$ is required to be a positive t-conorm on $[e,t]$. The remainder of the proof shares similarities with the proof of Theorem 3.2. □

Remark 3.7. (i) In Theorem 3.6, it must hold that $t \neq e$. Otherwise, $t = e$, then $([0,t] \setminus [e,t]) = [0,e)$, $L \setminus [0,t) = [e,1] \cup I_e$. Take $x_1 \in [0,e)$ and $y_1 = e \in [e,1] \cup I_e$. Then $\mathbb{U}_2(x,e) = e$. So the neutral element property of $\mathbb{U}_2(x,y)$ is violated.

(ii) The big difference between $\mathbb{U}_1(x,y)$ and $\mathbb{U}_2(x,y)$ is that $\mathbb{U}_1(x_1,t) = x_1$ and $\mathbb{U}_2(x_1,t) = t$ for all $x_1 \in [0,e) \cup I_e^t$. Obviously, both $\mathbb{U}_1$ and $\mathbb{U}_2$ belong to the first class of mixed uninorms $\mathbb{U}^1_{\text{mix}}$.

(iii) If, in Theorem 3.6, we take $t = 1$, we can derive $\mathbb{U}^1_{\text{min}}$, whose structure corresponds to the previously mentioned (3).

Example 3.8. Let us discuss the bounded lattice L_1 in Figure 2 and the t-subnorm $\mathbb{F} : ([0,e) \cup I_e^t)^2 \to ([0,e) \cup I_e^t)$ be given by $\mathbb{F}(x,y) = x \wedge y \wedge a$ and the t-conorm $\mathbb{S} : [e,t]^2 \to [e,t]$ be be given by $\mathbb{S}(x,y) = x \vee y$ and taking $\mathbb{R}(x,y) = x \vee y \vee k$ on $L \setminus [0,t)$. According to Theorem 3.6, the construction of the uninorm $\mathbb{U}_2$ is as shown in Table 2.

Definition 3.9. Let $\mathbb{U} : L^2 \to L$ be a uninorm with neutral or unit element $e \in L \setminus \{0,1\}$ on L. We call $\mathbb{U}$ a uninorm of the second class of mixed type uninorms if there exists $t \in [0,e]$, such that the following conditions are satisfied:

$\mathbb{U}_2$	0	a	b	c	d	e	f	t	g	h	j	k	1
0	0	0	0	0	0	0	0	t	g	h	j	k	1
a	0	a	0	a	a	a	a	t	g	h	j	k	1
b	0	0	b	0	0	b	b	t	g	h	j	k	1
c	0	a	0	c	a	c	c	t	g	h	j	k	1
d	0	a	0	a	d	d	d	t	g	h	j	k	1
e	0	a	b	c	d	e	f	t	g	h	j	k	1
f	0	a	b	c	d	f	f	t	g	h	j	k	1
t	t	t	t	t	t	t	t	t	g	h	j	k	1
g	g	g	g	g	g	g	g	g	g	k	k	k	1
h	h	h	h	h	h	h	h	h	k	h	k	k	1
j	j	j	j	j	j	j	j	j	k	k	j	k	1
k	k	k	k	k	k	k	k	k	k	k	k	k	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1

Tab. 2. The uninorm $\mathbb{U}_2$ in Example 3.8.

(i) $\mathbb{U}(x_0, y_0) = x_0$ for all $x_0 \in (e, 1]$ and $y_0 \in (t, e]$;

(ii) $\mathbb{U}(x_0, y_0) = y_0$ for all $x_0 \in (e, 1]$ and $y_0 \in [0, t)$;

(iii) $\mathbb{U}(t, y_0) \in \{t, y_0\}$ for all $y_0 \in (e, 1]$.

Here, we use $\mathbb{U}^2_{\text{mix}}$ to denote the set of all the second class of mixed uninorms on bounded lattices.

Similarly, we can take $t \in [0, e]$ to obtain a uninorm in the second class of mixed uninorms.

Theorem 3.10. Let $e \in L \setminus \{0, 1\}$ and $t \in [0, e]$. Suppose that $\mathbb{F}$ is a t-subnorm on $L \setminus [t, 1]$, $\mathbb{T}$ is a t-norm on $[t, e]$ and $\mathbb{R}$ is a t-superconorm on $[t, 1] \setminus [t, e]$. If $I_t^e = \emptyset$, then the function $\mathbb{U}_3 : L^2 \to$ L is a uninorm on L with neutral or unit element e, where

$$\mathbb{U}_3(x, y) = \begin{cases} \mathbb{T}(x, y) & (x, y) \in [t, e]^2, \\ \mathbb{R}(x, y) & (x, y) \in ([t, 1] \setminus [t, e])^2, \\ y & (x, y) \in [t, e] \times (L \setminus [t, e]), \\ x & (x, y) \in (L \setminus [t, e]) \times [t, e], \\ y & (x, y) \in ([t, 1] \setminus [t, e]) \times ((L \setminus [t, 1])), \\ x & (x, y) \in ((L \setminus [t, 1])) \times ([t, 1] \setminus [t, e]), \\ \mathbb{F}(x, y) & (x, y) \in (L \setminus [t, 1])^2. \end{cases} \tag{7}$$

Proof. It can be demonstrated using a similar method to that employed in the proof of Theorem 3.2. □

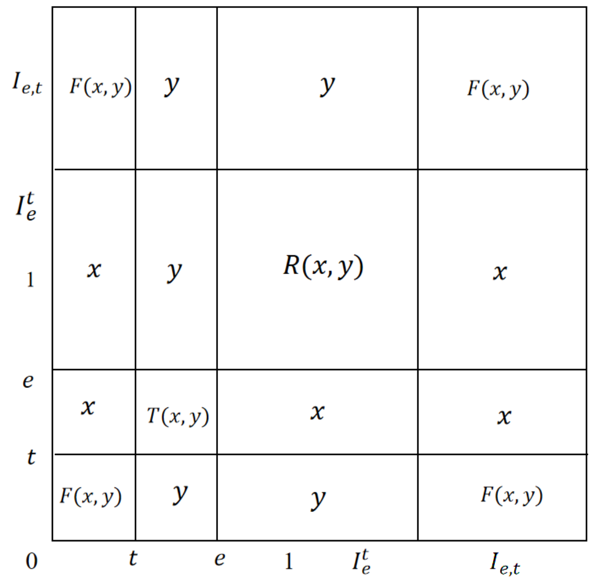

Fig. 3. The uninorm $\mathbb{U}_3$ in Theorem 3.10.

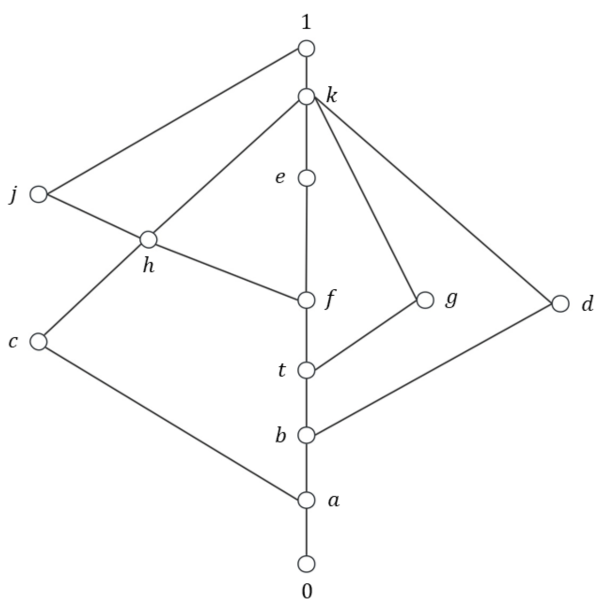

Fig. 4. The bounded lattice L_2.

$\mathbb{U}_3$	0	a	b	c	d	t	f	e	g	h	j	k	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	a	a	a	a	a	a	a	a	a	a	a	a
b	0	a	b	a	a	b	b	b	b	b	b	b	b
c	0	a	a	c	a	c	c	c	c	c	c	c	c
d	0	a	a	a	d	d	d	d	d	d	d	d	d
t	0	a	b	c	d	t	t	t	g	h	j	k	1
f	0	a	b	c	d	t	f	f	g	h	j	k	1
e	0	a	b	c	d	t	f	e	g	h	j	k	1
g	0	a	b	c	d	g	g	g	g	k	1	k	1
h	0	a	b	c	d	h	h	h	k	k	1	k	1
j	0	a	b	c	d	j	j	j	1	1	j	1	1
k	0	a	b	c	d	k	k	k	k	k	1	k	1
1	0	a	b	c	d	1	1	1	1	1	1	1	1

Tab. 3. The uninorm $\mathbb{U}_3$ in Example 3.12.

Remark 3.11. *(i)* If, in Theorem 3.10, we take $t = e$, then $I_e^t = \emptyset$, $I_{e,t} = I_e$, and $\mathbb{U}_3(x_1, e) = x_1$ for all $x_1 \in L$ and $\mathbb{U}_3(e, y_1) = y_1$ for all $y_1 \in L$. In this case, $\mathbb{U}_3$ limited on $[e, 1]^2$ is a t-conorm. So in this way, we can derive the uninorm $\mathbb{U}_{\min}$, whose structure corresponds to the previously mentioned (1).

(ii) If, in Theorem 3.10 we take $t = 0$, then we can derive the uninorm $\mathbb{U}_{\max}$, whose structure corresponds to the previously mentioned (2).

Example 3.12. Let $L_2 = \{0, a, b, c, d, e, f, t, g, h, j, k, 1\}$ be depicted in Figure 4, t-subnorm $\mathbb{F} : (L \setminus [t, 1])^2 \to (L \setminus [t, 1])$ be given by $\mathbb{F}(x, y) = x \wedge y \wedge a$, and t-norm $\mathbb{T} : [t, e]^2 \to [t, e]$ be given by $\mathbb{T}(x, y) = x \wedge y$, and $\mathbb{R}(x, y) = x \vee y \vee k$ on $([t, 1] \setminus [t, e])$. Based on Theorem 3.10, the construction of the uninorm $\mathbb{U}_3$ is as shown in Table 3.

Theorem 3.13. Let $e \in L \setminus \{0, 1\}$ and $t \in [0, e)$. Suppose that $\mathbb{F}$ is a t-subnorm on $L \setminus [t, 1]$, $\mathbb{T}$ is a positive t-norm on $[t, e]$ and $\mathbb{R}$ is a t-superconorm on $((e, 1] \cup I_e^t)^2$. If $I_t^e = \emptyset$, then the function $\mathbb{U}_4 : L^2 \to L$ defined as follows is a uninorm on L with neutral or unit element e, where $[t, 1] = [t, e] \cup (e, 1] \cup I_e^t$ and the structure is as follows:

$$\mathbb{U}_4(x, y) = \begin{cases} \mathbb{T}(x, y) & (x, y) \in [t, e]^2, \\ \mathbb{R}(x, y) & (x, y) \in ((e, 1] \cup I_e^t)^2, \\ y & (x, y) \in (t, e] \times ((e, 1] \cup I_e^t), \\ x & (x, y) \in ((e, 1] \cup I_e^t) \times (t, e], \\ y & (x, y) \in [t, 1] \times (L \setminus (t, 1]), \\ x & (x, y) \in (L \setminus (t, 1]) \times [t, 1], \\ \mathbb{F}(x, y) & (x, y) \in (L \setminus [t, 1])^2. \end{cases} \tag{8}$$

Proof. It can be demonstrated using a similar method to that employed in the proof of Theorem 3.6. □

Remark 3.14. The big difference between $\mathbb{U}_3(x,y)$ and $\mathbb{U}_4(x,y)$ is that $\mathbb{U}_3(x_1,t)=x_1$ and $\mathbb{U}_4(x_1,t)=t$ for all $x_1 \in [t,1]\setminus[t,e]$. Obviously, both $\mathbb{U}_3$ and $\mathbb{U}_4$ are uninorms of the second class of mixed uninorms $\mathbb{U}^2_{\text{mix}}$.

Remark 3.15. In Theorem 3.13, it must hold that $t \neq e$. Otherwise, $t=e$, then $([t,1]\setminus[t,e]) = (e,1]$, $L\setminus(t,1] = [0,e]\cup I_e$. Take $x_1 \in (e,1]$ and $y_1 = e \in [0,e]\cup I_e$. Then $\mathbb{U}_4(x_1,e)=e$. So the neutral element property of $\mathbb{U}_4(x,y)$ is violated.

Remark 3.16. If, in Theorem 3.13, we take $t=0$, then we can derive the uninorm $\mathbb{U}^0_{\max}$, whose structure corresponds to the previously mentioned (4).

Example 3.17. Let us discuss the bounded lattice L_2 in Figure 4 and the t-subnorm $\mathbb{F} : (L\setminus(t,1])^2 \to (L\setminus(t,1])$ be given by $\mathbb{F}(x,y) = x\wedge y\wedge a$ and the t-norm $\mathbb{T} : [t,e]^2 \to [t,e]$ be be given by $\mathbb{T}(x,y) = x\wedge y$ and taking $\mathbb{R}(x,y) = x\vee y\vee k$ on $([t,1]\setminus[t,e])$. Based on Theorem 3.13, the construction of the uninorm $\mathbb{U}_4$ is as shown in Table 4.

$\mathbb{U}_4$	0	a	b	c	d	t	f	e	g	h	j	k	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	a	a	a	a	a	a	a	a	a	a	a	a
b	0	a	b	a	a	a	b	b	b	b	b	b	b
c	0	a	a	c	a	a	c	c	c	c	c	c	c
d	0	a	a	a	d	a	d	d	d	d	d	d	d
t	0	a	a	a	a	t	t	t	t	t	t	t	t
f	0	a	b	c	d	t	f	f	g	h	j	k	1
e	0	a	b	c	d	t	f	e	g	h	j	k	1
g	0	a	b	c	d	t	g	g	g	k	1	k	1
h	0	a	b	c	d	t	h	h	k	k	1	k	1
j	0	a	b	c	d	t	j	j	1	1	j	1	1
k	0	a	b	c	d	t	k	k	k	k	1	k	1
1	0	a	b	c	d	t	1	1	1	1	1	1	1

Tab. 4. The uninorm $\mathbb{U}_4$ in Example 3.17.

4. CONCLUSION

In this paper, we commence by presenting precise and apt definitions for the first and second types of mixed uninorms (*i. e.*, $\mathbb{U}^1_{\text{mix}}$, $\mathbb{U}^2_{\text{mix}}$) within the context of bounded lattices. Building upon these definitions, we delve into innovative construction methods for these two types of mixed uninorms under specific bounded lattices, The first type of mixed uninorms, denoted as $\mathbb{U}^1_{\text{mix}}$, is constructed primarily using t-conorm and t-subnorm as its building blocks, while the second type, $\mathbb{U}^2_{\text{mix}}$, is constructed using t-norm and t-superconorm as its building blocks. Intriguingly, we discover that these construction methods can generalize $\mathbb{U}_{\max}$, $\mathbb{U}_{\min}$, $\mathbb{U}^0_{\max}$ and $\mathbb{U}^1_{\min}$ on bounded lattices, which have been previously explored in the literature. To illustrate the practicality of our methods, we provide several examples demonstrating how to construct mixed uninorms on bounded lattices. For future research endeavors, our primary objectives are twofold: Firstly, we aim to delve deeply into and explore innovative construction methods for mixed uninorms within the broader context of general bounded lattices. Secondly, we intend to focus on the characterization of mixed uninorms on bounded lattices. By pursuing these objectives, we hope to contribute significantly to the advancement of this field and to foster new insights and discoveries in the process.

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