QUASI-PROJECTION FOR A CLASS OF UNINORMS (2-UNINORMS)

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In 2021, Jayaram et al. demonstrated that a property called *Quasi-Projectivity* (QP) is a necessary condition for Clifford's relation to produce a partial order. Furthermore, their research revealed that although all triangular norms and triangular conorms satisfy (QP) and thus can generate posets, their generalized operator, uninorms, does not always possess this property, resulting in not all uninorms being able to generate a poset. In this work, we first investigate the satisfaction of (QP) for uninorms with continuous underlying operators, concluding that such uninorms are capable of yielding partial orders if and only if they are locally internal in $A(e)$, and the resulting partially ordered set is a chain. Based on this, we further explore the performance of inducing partial orders within the framework of 2-uninorms, and the results show that it is entirely determined by the underlying uninorms.

Keywords: uninorms, triangular norms, triangular conorms, Quasi-Projectivity

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1. INTRODUCTION

Following the pioneering works of Nambooripad [25], Mitsch [24] and others on obtaining orders over semigroups, a substantial body of literature proposing orders based on fuzzy connectives has emerged in recent years [3, 12, 15, 20, 26, 27]. Inspired by Clifford's work [4], it was Karaçal and Kesicioğlu in [17] first defined the order based on triangular norm T on a bounded lattice L in the following way:

$$x \preceq_T y \Longleftrightarrow T(\ell, y) = x, \qquad \text{for some } \ell \in L. \tag{1}$$

Notice that uninorms are a generalization of triangular norms and triangular conorms. Inspired by the aforementioned work and aiming to elucidate the dual behavior of the conjunction and disjunction of a unninorm U in different parts, Ertuğrul et al. made an appropriate modification based on Eg. (1) and introduced the following order on a bounded lattice L [7],

$$x \sqsubseteq_U y \Leftrightarrow \begin{cases} \text{if} \quad x, y \in [0, e] \text{ and } U(\ell, y) = x \text{ for some } \ell \in [0, e] \text{ or,} \\ \text{if} \quad x, y \in [e, 1] \text{ and } U(\zeta, x) = y \text{ for some } \zeta \in [e, 1] \text{ or,} \\ \text{if} \quad (x, y) \in L^2 \setminus \{[0, e]^2 \cup [e, 1]^2\} \text{ and } x \leq y, \end{cases}$$

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where e is the neutral element of U. Although the above relation can generate a partially ordered set for any U, it has two limitations. Firstly, the definition of the relation lacks uniformity and relies on the operation of uninorms in different regions. Secondly, Proposition 3 in [7] shows that if $x \sqsubseteq_U y$ then $x \leq y$, which prevents us from obtaining a more abundant ordered theoretical structure from U than the initial bounded lattice L.

In order to overcome the above awkward situations, it is noted that among the various definitions of order proposed within the framework of associative aggregation operators, only the order given in Eg. (1) or its dual form do not depend on the subdomain of its arguments. Based on this fact, in 2021, Gupta and Jayaram [9] explored the order-theoretic behavior of associative operator F by substituting T in Eg. (1) with the following method:

$$x \preceq_F y \Longleftrightarrow F(\ell, y) = x, \qquad \text{for some } \ell \in L. \tag{2}$$

In literature the above relation is well-known as the Clifford's relation, and the resulting partially ordered set is abbreviated as F-poset to reflect the relation with F. To ensure the relation given in Eg. (2) is a partial order, the authors introduced two technical properties called *Local Left Identity* (LLI) and *Quasi-Projection* (QP) respectively, and conducted their research primarily focusing on common types of uninorms defined on $[0,1]^2$ [9] and uninorms $U_{\min} \cup U_{\max}$ constructed on bounded lattices [11]. Essentially, for a uninorm, verifying the satisfaction of (QP) is a crucial step in making its underlying set to become a poset. It demonstrates that all triangular norms and triangular conorms satisfy (QP), thus enabling them to generate partially ordered sets, but their generalized operator uninorms are not always equipped with this property, resulting in not all uninorms being able to give a partial order. Specifically, in the research process within the framework of common families of uninorms on $[0,1]^2$, the authors in [9] found that uninorm U belongs to U_{lin} can successfully give rise to a partial order, but U_{rep} and $U_{\cos}$ class uninorms fail to possess this capability. Subsequently, in the concluding remarks of the article, the authors proposed a question that how is structure of the partially ordered set obtained by U is related to the posets obtained by its underlying operators.

Linking partially ordered sets with an algebra can help us intuitively explain algebraic concepts, much like how the graph of a function allows us to better understand algebraic expressions [26]. Furthermore, it can also aid us in gaining deeper insights into the essence of algebra and acquiring further valuable understanding. A well-known example is Clifford's influential result, which states that if the Clifford's relation produces a total order, then the commutative semigroup can be described as an ordinal sum of subsemigroups [4]. For this reason, following the theme, this work further investigates the satisfaction of (QP) within the framework of uninorms with continuous underlying operators. It is concluded that such uninorms are capable of producing partial orders if and only if they are locally internal in $A(e)$, and the resulting partial order is a chain. For those uninorms that are not locally internal in A(e), we provide a simple and intuitive modification method, ensuring that the modified operators are not only uninorms but also capable of generating posets. Based on this, we further explore the conditions under which 2-uninorms yield a poset, and the results indicate that it is entirely determined by the underlying uninorms.

The structure of this article is as follows. In Section 2, we introduce some basic concepts and notations related to order theory and uninorms (2-uninorms). Sections 3 and 4 focus on uninorms (2-uninorms) with continuous underlying operators, respectively exploring the order-theoretic behavior of these two types of operators. Finally, Section 5 presents the conclusion of the article and an outlook on future work.

2. PERLIMINARIES

In this section, to ensure the article remains as self-contained as possible, we briefly recall some relevant definitions and results utilized herein. For further details, readers are recommended to refer to [2, 5, 8, 13].

Definition 2.1. (Davey and Priestley [5]) Let $P \neq \varnothing$. A partial order on P is a binary relation on P such that, for all $p, q, r \in P$, the following properties hold:

(i) Reflexivity: $p \leq p$.

(ii) Antisymmetry: If $p \leq q$ and $q \leq p$, then $p = q$.

(iii) Transitivity: If $p \leq q$ and $q \leq r$, then $p \leq r$.

Definition 2.2. (Davey and Priestley [5]) Let $(P, \leq)$ be a poset.

(i) An element p in P is said to be

  (a) the greatest element if for every element q in P we have $p \geq q$.
  (b) the least element if for every element q in P we have $p \leq q$.

(ii) A pair of elements $p, q \in P$ is said to be *comparable*, denoted $p \sim q$, if either $p \leq q$ or $q \leq p$. Otherwise, it is denoted by $p \nsim q$.

(iii) P is said to be a *chain* if for any $p, q \in P$ either $p \leq q$ or $q \leq p$, i. e., $p \sim q$ for every $p, q \in P$.

Next, two crucial properties in the semigroup, abbreviated as (LLI) and (QP), are introduced, which play a pivotal role in determining whether $\preceq_F$ constitutes a partial order [9, 10].

Definition 2.3. (Gupta and Jayaram [9]) Let $P \neq \varnothing$ and $F : P \times P \to P$. F is said to satisfy the

(i) *Local Left Identity* property, if for every $x \in P$, there exists an $\ell \in P$ such that $F(\ell, x) = x$, i. e., for every $x \in P$,
$$A_x = \{\varsigma \in P | F(\varsigma, x) = x\} \neq \varnothing. \qquad (LLI)$$

(ii) *Quasi-Projection* property, if for any $x, y, z \in P$,
$$F(x, F(y, z)) = z \Rightarrow F(y, z) = z. \qquad (QP)$$

Theorem 2.4. (Gupta and Jayaram [9]) Let $P \neq \varnothing$ and $F : P \times P \to P$ be an associative operator. Let the relation $\preceq_F$ on P be defined by Eg. (2). Then the following are equivalent:

(i) $(P, \preceq_F)$ is a poset.

(ii) F satisfies both (LLI) and (QP).

Suppose that $F : P \times P \to P$ is associative and satisfies (LLI) and (QP). Then the relation $\preceq_F$ is a partial order on P. To emphasize that the partially ordered set $(P, \preceq_F)$ is derived from F, we usually refer to it as an F-poset.

Definition 2.5. A binary operator $F : [0,1]^2 \to [0,1]$ in region D is said to be locally internal if $F(x,y) \in \{x,y\}$ for each $(x,y) \in D$.

Definition 2.6. (Klement et al. [13]) A binary operator $T(S) : [0,1]^2 \to [0,1]$ is called a *triangular norm*, if it is associative, commutative, increasing in each variable and 1(0) is the neutral element.

Next, we will introduce the generalized operator of triangular norms and triangular conorms, known as uninorms, which has consistently captured the attention of scholars within the fuzzy community [16, 18, 28].

Definition 2.7. (Yager and Rybalov [29]) A binary operator $U : [0,1]^2 \to [0,1]$ is called a *uninorm* if it is associative, commutative, increasing in each variable and there exists an element $e \in [0,1]$ called *neutral element*, such that $U(e,x) = x$ for all $x \in [0,1]$.

For a uninorm U with a neutral element $e \in (0,1)$, it acts as a triangular norm T on $[0,e]^2$ and as a triangular conorm S on $[e,1]^2$, while in the remaining area $A(e)$, its value lies between the minimum and maximum. Therefore, T and S are referred to as the underlying operators of U. Any uninorm U satisfies that $U(0,1) \in \{0,1\}$ and it is called *conjunctive* when $U(0,1) = 0$ and *disjunctive* when $U(0,1) = 1$. The different families of uninorms that will be used in the article are described as follows:

(i) $U_{\min}(U_{\max})$ – uninorms whose values in $A(e)$ are given by the smaller (larger) of two variables.

(ii) U_{lin} – locally interval uninorms, i. e., $U(x,y) \in \{x,y\}$ on $A(e)$.

(iii) U_{rep} – uninorms continuous on $[0,1]^2 \setminus \{(0,1),(1,0)\}$. Fodor et al. proved that [8] such uninorms are strictly increasing in $(0,1)^2$. For any $U \in U_{\text{rep}}$, it holds that $U(0,1) = 0$ for all $x \in [0,1)$ and $U(x,1) = 1$ for all $x \in (0,1]$.

(iv) $U_{\cos}$ – uninorms continuous on the open unit square $(0,1)^2$. Based on the structural characteristics of this type of uninorm, it can be concluded that any representable uninorm U is a particular form among the members of $U_{\cos}$ [21].

(v) U_{cts} – uninorms with continuous underlying operators. Mesiarová-Zemánková and Su et al. have achieved a series of profound research results in exploring the characteristic structures of such uninorms [22, 30, 31].

For the sake of convenience, the set of uninorms with continuous underlying operators and have neutral element e in this article is denoted as $U_{cts(e)}$. Suppose that $U \in U_{cts(e)}$ and α is an idempotent element of U, then it obtained in [22] that $U(\alpha, \cdot)$ is locally internal on $[0,1]$.

Definition 2.8. (Akella [1]) Let $k \in (0,1), e_1 \in [0,k]$ and $e_2 \in [k,1]$. A binary operator $G : [0,1]^2 \to [0,1]$ is called a 2-*uninorm* if it is associative, commutative, increasing and fulfills

(i) $G(e_1, x) = x$ for each $x \in [0,k]$.

(ii) $G(e_2, x) = x$ for each $x \in [k,1]$.

For any fixed 2-uninorm G with parameters e_1, k, e_2, it works as a uninorm U_1 with neutral element $\frac{e_1}{k}$ on $[0,k]^2$ and as a uninorm U_2 with neutral element $\frac{e_2-k}{1-k}$ on $[k,1]^2$. Therefore, U_1 and U_2 are referred to as the underlying uninorms of G. In this sense, in this article the 2-uninorm G is denoted as $G = \langle U_1, U_2 \rangle$. If the underlying operators of U_1 and U_2 are continuous, then we say G have continuous underlying operators and it is denoted by $G_{cts(e_1,k,e_2)}$ in this paper. Moreover, Akella and Zong et al. stated that 2-uninorms can be divided into the following five exhaustive and mutually exclusive families [1, 32].

(i) The family of 2-uninorms with $G(0,1) = k$, denoted by G_k.

(ii) The family of 2-uninorms with $G(0,1) = 0$ and $G(1,k) = k$, denoted by $G_{c,k}$.

(iii) The family of 2-uninorms with $G(0,1) = 1$ and $G(0,k) = k$, denoted by $G_{d,k}$.

(iv) The family of 2-uninorms with $G(0,1) = 0$ and $G(1,k) = 1$, denoted by $G_{c,1}$.

(v) The family of 2-uninorms with $G(0,1) = 1$ and $G(0,k) = 0$, denoted by $G_{d,0}$.

The results presented below illustrate the structures of the aforementioned five classes of 2-uninorms.

Proposition 2.9. (Akella [1]) A binary operaor $G : [0,1]^2 \to [0,1]$ is a 2-uninorm with $G \in G_k$ if and only if G is expressed as

$$G(x,y) = \begin{cases} kU_1(\frac{x}{k}, \frac{y}{k}), & \text{if } (x,y) \in [0,k]^2, \\ k + (1-k)U_2(\frac{x-k}{1-k}, \frac{y-k}{1-k}) & \text{if } (x,y) \in [k,1]^2, \\ k, & \text{otherwise,} \end{cases}$$

where U_1 is a disjunctive uninorm with neutral element $\frac{e_1}{k}$ and U_2 is a conjunctive uninorm with neutral element $\frac{e_2-k}{1-k}$.

Proposition 2.10. (Zong et al. [32]) A binary operator $G : [0,1]^2 \to [0,1]$ is a 2-uninorm with $G \in G_{c,k}$ if and only if G is expressed as

$$G(x,y) = \begin{cases} kU_1(\frac{x}{k}, \frac{y}{k}), & \text{if } (x,y) \in [0,k]^2, \\ k + (1-k)U_2(\frac{x-k}{1-k}, \frac{y-k}{1-k}), & \text{if } (x,y) \in [k,1]^2, \\ kU_1(\frac{\min\{x,y\}}{k}, 1), & \text{otherwise,} \end{cases}$$

where U_1 and U_2 are conjunctive uninorms with neutral element $\frac{e_1}{k}$ and $\frac{e_2-k}{1-k}$ respectively.

Proposition 2.11. (Zong et al. [32]) A binary operator $G : [0,1]^2 \to [0,1]$ is a 2-uninorm with $G \in G_{d,k}$ if and only if G is expressed as

$$G(x,y) = \begin{cases} kU_1(\frac{x}{k}, \frac{y}{k}), & \text{if } (x,y) \in [0,k]^2, \\ k + (1-k)U_2(\frac{x-k}{1-k}, \frac{y-k}{1-k}), & \text{if } (x,y) \in [k,1]^2, \\ k + (1-k)U_2(\frac{\max\{x,y\}-k}{1-k}, 0), & \text{otherwise,} \end{cases}$$

where U_1 and U_2 are disjunctive uninorms with neutral element $\frac{e_1}{k}$ and $\frac{e_2-k}{1-k}$ respectively.

Proposition 2.12. (Zong et al. [32]) A binary operator $G : [0,1]^2 \to [0,1]$ is a 2-uninorm with $G \in G_{c,1} \cup G_{d,0}$, then there exist uninorms U_1 and U_2 with neutral element $\frac{e_1}{k}$ and $\frac{e_2-k}{1-k}$ respectively, such that

$$G(x,y) = \begin{cases} kU_1(\frac{x}{k}, \frac{y}{k}), & \text{if } (x,y) \in [0,k]^2, \\ k + (1-k)U_2(\frac{x-k}{1-k}, \frac{y-k}{1-k}), & \text{if } (x,y) \in [k,1]^2, \\ kU_1(\frac{\min\{x,y\}}{k}, 1), & \text{if } \min\{x,y\} \le k \le \max\{x,y\} \le e_2, \\ k + (1-k)U_2(\frac{\max\{x,y\}-k}{1-k}, 0), & \text{if } e_1 \le \min\{x,y\} \le k \le e_2 \le \max\{x,y\}, \end{cases}$$

and if $\min\{x,y\} \le e_1 \le e_2 \le max\{x,y\}$, then $G(x,y) \in [0,e_1] \cup \{k\} \cup [e_2,1]$.

Remark 2.13. (Zong et al. [32]) For the structures of 2-uninorms described above, the following observations are valid.

(i) If $G \in G_{c,k}$, then $G(x,y) = G(x,k)$ for all $(x,y) \in [0,k) \times [k,1]$, and $G(x,y) = G(k,y)$ for all $(x,y) \in [k,1] \times [0,k)$.

(ii) If $G \in G_{d,k}$, then $G(x,y) = G(k,y)$ for all $(x,y) \in [0,k) \times [k,1]$, and $G(x,y) = G(x,k)$ for all $(x,y) \in [k,1] \times [0,k)$.

(iii) If $G \in G_{c,1} \cup G_{d,0}$, then we have

  (a) $G(x,y) = G(x,k)$ for all $(x,y) \in [0,k) \times [k,e_2]$, and $G(x,y) = G(k,y)$ for all $(x,y) \in [k,e_2] \times [0,k)$.

  (b) $G(x,y) = G(k,y)$ for all $(x,y) \in [e_1,k) \times [e_2,1]$, and $G(x,y) = G(x,k)$ for all $(x,y) \in [e_2,1] \times [e_1,k)$.

3. QUASI-PROJECTION FOR UNINORMS WITH CONTINUOUS UNDERLYING OPERATORS

Given a fixed uninorm $U \in U_{cts(e)}$, the (LLI) property naturally holds. Therefore, as described in theorem 2.4, the key to making $(P, \preceq_U)$ a partially ordered set lies in verifying the validity of (QP) Based on the structural characteristics of U in $A(e)$, the discussions can be divided into two situations as follows.

(i) U is locally internal in $A(e)$.

(ii) U is not locally internal in $A(e)$.

For the case where U is locally internal in $A(e)$, the satisfaction of (QP) has been discussed in [10], and a positive result has been obtained. Due to the necessity of the context, we present the conclusion here.

Proposition 3.1. (Gupta and Jayaram [9]) Suppose that $U \in U_{\text{lin}}$, then U satisfies (QP).

Next, let us focus on situation (ii) that U is not locally internal in $A(e)$. According to Lemma 3.1 in [19], it is obtained that for any $U \in U_{_{cts(e)}}$, either there exists a representable uninorm centered around point (e, e) (namely, there exist $a \in [0, e)$ and $b \in (e, 1]$ such that U is a representable uninorm $R^{\maltese}$ on $[a, b]^2$.), or there does not exist a representable uninorm centered around point (e, e). Within this framework, there are two considerations regarding the observations on the satisfaction of (QP).

Lemma 3.2. Suppose that $U \in U_{_{cts(e)}}$ with a representable uninorm centered around point (e, e), then U does not satisfy (QP).

P r o o f. Given that $U \in U_{_{cts(e)}}$ with a representable uninorm centered around point (e, e), then we know there exist $a \in [0, e)$ and $b \in (e, 1]$ such that U is a representable uninorm $R^{\maltese}$ on $[a, b]^2$. That is, $U(x, y) = a + (b - a)R^{\maltese}(\frac{x-a}{b-a}, \frac{y-a}{b-a})$ for all $x, y \in [a, b]^2$. Taking $y_1 \in (a, e)$, then according to the fact that

$$U(y_1, a) = a + (b - a)R^{\maltese}(\frac{y_1 - a}{b - a}, \frac{a - a}{b - a}) = a,$$

and

$$U(y_1, b) = a + (b - a)R^{\maltese}(\frac{y_1 - a}{b - a}, \frac{b - a}{b - a}) = b,$$

one obtains that there exists some $z_1 \in (a, b)$ such that $U(y_1, z_1) = e$. Now, let us take $x = z_1$, then we have $U(x, U(y_1, z_1)) = U(x, e) = z_1$. That is, the antecedent of (QP) is established. However, according to $R^{\maltese}(\frac{y_1-a}{b-a}, \frac{z_1-a}{b-a}) < R^{\maltese}(\frac{e-a}{b-a}, \frac{z_1-a}{b-a}) = \frac{z_1-a}{b-a}$ we obtain that $U(y_1, z_1) = a + (b - a)R^{\maltese}(\frac{y_1-a}{b-a}, \frac{z_1-a}{b-a}) < z_1$, which means that the consequent of (QP) is not valid. Therefore, U does not satisfy (QP). □

The following conclusion already obtained in [9] can be directly achieved through Lemma 3.2.

Corollary 3.3. Let $U \in U_{\cos} \cup U_{\text{rep}}$, then U does not satisfy the (QP).

Next, we consider the case that there is no representable uninorm centered around point (e, e). In this situation, according to the Theorem 3.9 in [19], it obtains that there is a rectangular region $D = (a, b) \times (c, d)$ in $A(e)$ with $b < c$ such that for any $(x, y) \in D$ we obtain $x < U(x, y) < y$. Based on this viewpoint, to verify the validity of (QP), we need to introduce the concept of maximum non-locally internal rectangular region of U.

Definition 3.4. Let $U \in U_{cts}$, then we call a rectangular region $D = (a, b) \times (c, d)$ as a *maximum non-locally internal rectangular region* of U if the following two conditions are satisfied.

(i) $D \subseteq A(e)$ and $\min\{x, y\} < U(x, y) < \max\{x, y\}$ for each point $(x, y) \in D$.

(ii) For any rectangular region $\hat{D}$ satisfies $D \subset \hat{D}$, there is at least one point $(x_0, y_0) \in \hat{D} \backslash D$ such that $U(x_0, y_0) \in \{x_0, y_0\}$.

Note that if $\tilde{D} = (a, b) \times (c, d)$ is a maximum non-locally internal rectangular region of U, then the commutativity of U implies that $\tilde{\tilde{D}} = (c, d) \times (a, b)$ is also a maximum non-locally internal rectangular region. Therefore, in this sense, we refer to $(a, b) \times (c, d) \cup (c, d) \times (a, b)$ as the *maximum non-locally internal rectangular region pair* of U. Obviously, the non-locally internal region of a uninorm $U \in U_{cts}$ in $A(e)$ is the union of all maximum non-locally internal rectangular region pairs. Indeed, suppose that $(a, b) \times (c, d)$ with $b \leq c$ is a maximum non-locally internal rectangular region of $U \in U_{cts(e)}$, then according to the results in [19] we know U restricted on $[a, b]^2$ and $[c, d]^2$ is a strict triangular norm and a strict triangular conorm, respectively.

Proposition 3.5. (Drygaś [6], Li et al. [19], Su et al. [30]) Suppose that $D = (a, b) \times (c, d)$ with $b \leq c$ is a maximum non-locally internal rectangular region of $U \in U_{cts(e)}$, then the following conditions are valid.

(i) The set $([a, b] \cup [c, d])^2$ is closed under U and U is strictly increasing on the region of $(a, b) \times (c, d)$.

(ii) For each $x \in (a, b)$ there exists a $\lambda_x \in (c, d)$ such that $U(x, y) > e$ for all $y \in (\lambda_x, d)$ and $U(x, y) < e$ for all $y \in (c, \lambda_x)$.

Lemma 3.6. Let $U \in U_{cts(e)}$ is not locally internal in $A(e)$ and there is no representable uninorm centered around point (e, e), then U does not satisfies (QP).

P r o o f. According to the assumption that U belongs to $U_{cts(e)}$ and is non-locally interval in $A(e)$, we know that there is a point (x_0, y_0) in $A(e)$ that satisfies $x_0 < U(x_0, y_0) < y_0$. Therefore, there exists a maximum non-locally internal rectangular region $D_0 = (a_0, b_0) \times (c_0, d_0)$ in $A(e)$ such that $(x_0, y_0) \in D_0$. Further, by using Theorem 3.9 in [19] and the fact that there is no representable uninorm centered around point (e, e), it obtains that $b_0 < c_0$ and U restricted on $[a_0, b_0]^2$ and $[c_0, d_0]^2$ is a strict triangular norm and a strict triangular conorm, respectively.

Notice that by Proposition 3.5 (i) we know the set $([a_0, b_0] \cup [c_0, d_0])^2$ is closed under U and the section $U(x_0, \cdot)$ restricted on $(a_0, b_0) \times (c_0, d_0)$ is strictly increasing for each fixed $x_0 \in (a_0, b_0)$. Moreover, Proposition 3.5 (ii) further implies that there is some $\lambda_{x_0} \in (c_0, d_0)$ such that $U(x_0, y) > e$ for all $y \in (\lambda_{x_0}, d_0)$ and $U(x_0, y) < e$ for all $y \in (c_0, \lambda_{x_0})$. Based on these materials, we can see that for any fixed $y_1 \in (a_0, b_0)$, there always exists some $z_1 \in (\lambda_{y_1}, d_0)$ such that $c_0 < U(y_1, z_1) < d_0$. According to U restricted on $[c_0, d_0]^2$ is a strict triangular conorm, one obtains that $U(c_0, U(y_1, z_1)) = U(y_1, z_1) < z_1$ and $U(d_0, U(y_1, z_1)) = d_0 > z_1$. Therefore, it holds that there exists some $x_1 \in (c_0, d_0)$ such that $U(x_1, U(y_1, z_1)) = z_1$. However, from the fact that the point (y_1, z_1) comes from the region D_0, we also have that $U(y_1, z_1) < z_1$.

To summarize the above discussions, we have found a triplet (x_1, y_1, z_1) satisfies $U(x_1, U(y_1, z_1)) = z_1$ and $U(y_1, z_1) < z_1$. As a result, U does not satisfies (QP). $\square$

Combining the analyses of two situations above, the following conclusion can be reached.

Theorem 3.7. If $U \in U_{_{cts(e)}}$, then U satisfies the (QP) if and only if U is locally interval in $A(e)$.

Proof. Suppose that $U \in U_{c_{cts(e)}}$, then by Lemma 3.2 and Lemma 3.6, we know there exists no point $(x_0, y_0) \in A(e)$ satisfies $\min\{x_0, y_0\} < U(x_0, y_0) < \max\{x_0, y_0\}$. Therefore, U is locally interval in $A(e)$. The validity of necessity is guaranteed by Proposition 3.1. □

Theorem 3.8. If $U \in U_{_{cts(e)}}$, then $([0,1], \preceq_U)$ is a U-poset if and only if U is locally interval in $A(e)$. In this case, $([0,1], \preceq_U, U(0,1), e)$ is a bounded chain.

Proof. Since uninorm U has neutral element e, it is known that U naturally satisfies (LLI) property. Therefore, to make $([0,1], \preceq_U)$ a partially ordered set, by Theorem 2.4 it is only necessary to check that U satisfies (QP), which is equivalent to U being locally internal in $A(e)$ as shown in Theorem 3.7. Therefore, it obtains that $([0,1], \preceq_U)$ is a U-poset if and only if U is locally internal in $A(e)$.

Next, let us prove that $([0,1], \preceq_U)$ is actually a bounded chain takes $U(0,1)$ and e as the bottom and top elements, respectively. For any $x, y \in [0, e]$ satisfies $x \leq y$, the continuity of underlying triangular norm of U implies that there exists some $\ell \in [0, e]$ such that $U(\ell, y) = x$. Therefore, we get $x \preceq_U y$, which means that any two elements in $[0, e]$ are comparable with respect to $\preceq_U$, and the order is consistent with the natural order. Similarly, by utilizing the continuity of the underlying triangular conorm of U, it can be inferred that any two elements in $[e, 1]$ are also comparable, and the order is the reverse of the natural order. As for the case $(x, y) \in A(e)$, if $U(x, y) = x$, then $x \preceq_U y$. Similarly, if $U(x, y) = y$, then $y \preceq_U x$. Therefore, it has been shown that any two elements in $[0,1]$ are comparable with respect to $\preceq_U$, that is, $([0,1], \preceq_U)$ is a chain. Finally, by $(x, e) = x$ for all $x \in [0,1]$, we have e is the top element. Using the same approach, the fact that $U(U(0,1), x) = U(0,1)$ for all $x \in [0,1]$ leads to $U(0,1)$ is the bottom element. □

Note that the results of Theorem 3.8 answer the question raised in Ref. [9] concern the relationship between the partially ordered set induced by U and the partially ordered sets generated by the underlying operators T_U and S_U within the framework of U comes from U_{cts}. The results indicate that the partially ordered set $([0,1], \preceq_U)$ inherits the orders generated by the underlying operators and ultimately forms a chain. Meanwhile, it also emphasises the value of order-theoretic exploration of algebraic structures. Whether a given uninorm $U \in U_{_{cts(e)}}$ is locally interval in $A(e)$ or not is characterised by whether it is equipped with the ability to yield a partial order under the relation $\preceq_U$. It shows that $([0,1], \preceq_U)$ is a U-poset if and only if $U \in U_{\text{lin}}$. To illustrate this viewpoint, the following example is provided.

Example 3.9. Let $a_n = \frac{3}{8} - \frac{3}{7+n}$, $b_n = \frac{3}{8} - \frac{3}{8+n}$, $c_n = \frac{5}{8} + \frac{3}{8+n}$ and $d_n = \frac{5}{8} + \frac{3}{7+n}$, where $n \in N^+$ and N^+ stands for the set of all positive integers. Define the following mapping,

$$g_n(x) = \begin{cases} a_n + (b_n - a_n)\frac{x}{e}, & \text{if } x < \frac{1}{2}, \\ c_n, & \text{if } x = \frac{1}{2}, \\ d_n - \frac{(1-x)(d_n - c_n)}{1-e}, & \text{if } x > \frac{1}{2}, \end{cases}$$

where $e = \frac{1}{2}$. Then g_n is a linear isomorphism from $[0,1]$ to $[a_n, b_n) \cup [c_n, d_n]$. For each $n \in N^+$, consider $U_n^{\maltese} : ([a_n, b_n) \cup [c_n, d_n])^2 \to ([a_n, b_n) \cup [c_n, d_n])$ given by $U_n^{\maltese}(x,y) = g_n(\tilde{U}_n(g_n^{-1}(x), g_n^{-1}(y)))$, where $\tilde{U}_n$ is a conjunctive representable uninorm with neutral element $\frac{1}{2}$. Then $U_n^{\maltese}$ is a commutative, associative, non-decreasing operator for each $n \in N^+$. Next, we define the following operator $U(x,y) =$

$$\begin{cases} U_n^{\maltese}(x,y), & \text{if } (x,y) \in \bigcup\limits_{n\in N^+}([a_n,b_n) \cup [c_n,d_n])^2, \\ \frac{3}{8} + \frac{2}{8}\tilde{U}(\frac{x-\frac{3}{8}}{\frac{5}{8}-\frac{3}{8}}, \frac{y-\frac{3}{8}}{\frac{5}{8}-\frac{3}{8}}), & \text{if } (x,y) \in [\frac{3}{8}, \frac{5}{8}]^2, \\ \min\{x,y\}, & \text{if } (x,y) \in [0,1]^2 \setminus \{\bigcup\limits_{n\in N^+}([a_n,b_n) \cup [c_n,d_n])^2 \cup [\frac{3}{8}, \frac{5}{8}]^2\} \\ & \text{and } x + y < 1, \\ \max\{x,y\}, & \text{otherwise}, \end{cases}$$

where $\tilde{U}$ is an idempotent uninorm with neutral element $\frac{1}{2}$. Then U is a semigroup $G_n = (([a_n, b_n) \cup [c_n, d_n])^2, U_n^{\maltese})$ on $([a_n, b_n) \cup [c_n, d_n])^2$ and a semigroup $G_\beta = ([\frac{3}{8}, \frac{5}{8}]^2, \tilde{U})$ on $[\frac{3}{8}, \frac{5}{8}]^2$. Let $(N^+ \cup \{\beta\}, \preceq)$ be a linearly ordered set with $n \preceq n+1$ and $n \preceq \beta$ for each $n \in N^+$. Based on these materials, Theorem 3.42 in [13] indicates that U is a semigroup. Therefore, it obtains that U is a uninorm from U_{cts} with neutral element $\frac{1}{2}$ and is not locally internal in $A(\frac{1}{2})$.

Obviously, $g_n^{-1}(y_n) \in (0, \frac{1}{2})$ for each $y_n \in (a_n, b_n)$. Therefore, by $\tilde{U}_n$ is a representable uninorm, there is some $z_n \in (c_n, d_n)$ such that $\frac{1}{2} < \tilde{U}_n(g_n^{-1}(y_n), g_n^{-1}(z_n)) < 1$, which leads to $c_n < g_n(\tilde{U}_n(g_n^{-1}(y_n), g_n^{-1}(z_n))) < g_n(g_n^{-1}(z_n)) = z_n$. Notice that

$$\begin{aligned} U(c_n, U(y_n, z_n)) &= U(c_n, g_n(\tilde{U}_n(g_n^{-1}(y_n), g_n^{-1}(z_n)))) \\ &= g_n(\tilde{U}_n(g_n^{-1}(c_n), g_n^{-1}(g_n(\tilde{U}_n(g_n^{-1}(y_n), g_n^{-1}(z_n)))))) \\ &= g_n(\tilde{U}_n(\frac{1}{2}, \tilde{U}_n(g_n^{-1}(y_n), g_n^{-1}(z_n)))) \\ &= g_n(\tilde{U}_n(g_n^{-1}(y_n), g_n^{-1}(z_n))) \end{aligned}$$

and

$$\begin{aligned} U(d_n, U(y_n, z_n)) &= U(d_n, g_n(\tilde{U}_n(g_n^{-1}(y_n), g_n^{-1}(z_n)))) \\ &= g_n(\tilde{U}_n(g_n^{-1}(d_n), g_n^{-1}(g_n(\tilde{U}_n(g_n^{-1}(y_n), g_n^{-1}(z_n)))))) \\ &= g_n(\tilde{U}_n(1, \tilde{U}_n(g_n^{-1}(y_n), g_n^{-1}(z_n)))) \\ &= g_n(1) \\ &= d_n. \end{aligned}$$

Therefore, the continuity of U in $[c_n, d_n]^2$ implies that there is some $x_n \in (c_n, d_n)$ such that

$$U(x_n, g_n(\tilde{U}_n(g_n^{-1}(y_n), g_n^{-1}(z_n)))) = U(x_n, U(y_n, z_n)) = z_n.$$

However, $U(y_n, z_n) = g_n(\tilde{U}_n(g_n^{-1}(y_n), g_n^{-1}(z_n))) < z_n$. Therefore, we obtain that U does not satisfies (QP), and $([0,1], \preceq_U)$ is naturally not a U-poset by Theorem 2.4. Structures of U is visualized in Figure 1.

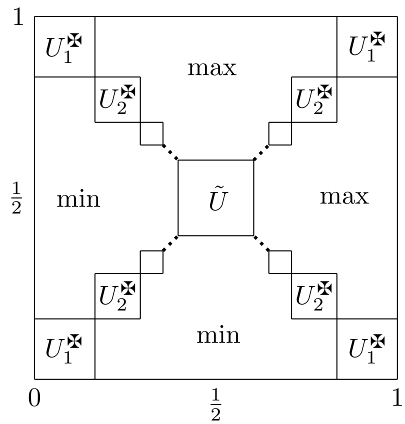

Fig. 1. Structure of U in Example 3.9.

Theorem 3.8 states that for a uninorm $U \in \mathcal{U}_{cts(e)}$, which is not locally internal in $A(e)$, the relation $\preceq_U$ can not induce a U-poset. Therefore, on the topic of whether such a uninorm is capable of generating a partially ordered set, our first reaction is to exclude it, which is naturally correct. However, the following results show that as long as it is subjected to a simple modification, the resulting operator is not only a uninorm but also equipped with the ability to satisfy (QP).

Lemma 3.10. Suppose that $U \in \mathcal{U}_{cts(e)}$ and $D = (a,b) \times (c,d)$ with $b \leq c$ is one of its maximum non-locally internal rectangular regions, then the following conditions are satisfied.

(i) $U(x,y) = \min\{x,y\}$ for all $(x,y) \in (a,b) \times [e,c] \cup [0,a] \times (c,d)$.

(ii) $U(x,y) = \max\{x,y\}$ for all $(x,y) \in (a,b) \times [d,1] \cup [b,e] \times (c,d)$.

Proof. Notice that U is a strict triangular norm and a strict triangular conorm on $[a,b]^2$ and $[c,d]^2$, respectively. Then, using Remark 3.7 in [19] and Lemma 10 in [22], it can be concluded that the results are valid. □

Proposition 3.11. Suppose that $U \in \mathcal{U}_{cts(e)}$ is not locally internal in $A(e)$, then the operator defined as

$$\tilde{U}(x,y) = \begin{cases} \min\{x,y\}, & \text{if } (x,y) \in \tilde{D}, \\ U(x,y), & \text{otherwise}, \end{cases}$$

is a uninorm and satisfies (QP), where $\tilde{D}$ is the non-locally internal region of U in $A(e)$.

P r o o f. To obtain the conclusion, it is sufficient to prove that the defined operator $\tilde{U}$ is a uninorm. Obviously, $\tilde{U}$ satisfies commutativity and e is its neutral element. Notice that $\tilde{D} = \underset{i\in I}{\cup} D_i$, where $D_i = (a_i, b_i) \times (c_i, d_i) \cup (c_i, d_i) \times (a_i, b_i)$ with $b_i \leq c_i$ is the maximum non-locally internal rectangular region pair of U, and I is an at most countable index set. By using Lemma 3.10, one can easily obtain that $\tilde{U}$ is increasing. Therefore, to ensure that $\tilde{U}$ is a uninorm, it is sufficient to verify the associativity. According to the expression of $\tilde{U}$, it is enough for us to make the following two verifications.

(i) Checking the equality $\tilde{U}(\tilde{U}(x, y), z) = \tilde{U}(x, \tilde{U}(y, z))$ on sets $\{x, y, z\}$ such that none of the pairs in the set $\{x, y, z\} \times \{x, y, z\}$ comes from $\tilde{D}$. We have the following considerations.

 (a) If $(x, y, z) \in [0, e]^3$ or $(x, y, z) \in [e, 1]^3$, then the associativity of $\tilde{U}$ is guaranteed by U.

 (b) Suppose that there are two elements in the set $\{x, y, z\}$ belong to $[0, e]$, and the remaining element comes from $[e, 1]$. We only verify the situation that $(x, y) \in [0, e]^2$ and $z \in [e, 1]$ as other cases can be discussed similarly. In this case, we have $\tilde{U}(y, z) = U(y, z) \in \{y, z\}$. Thus, it holds that $\tilde{U}(x, \tilde{U}(y, z)) = U(x, U(y, z))$. As for the value of $\tilde{U}(\tilde{U}(x, y), z))$, we get $\tilde{U}(\tilde{U}(x, y), z)) = \tilde{U}(U(x, y), z))$. If $U(x, y) = \min\{x, y\}$, then by assumption, it is obtained that $\tilde{U}(U(x, y), z)) = U(U(x, y), z))$. If $U(x, y) < \min\{x, y\}$, then we know that there exist $a, b \in [0, e]$ such that $(x, y) \in (a, b)^2$ and U restricted on $[a, b]^2$ is a continuous Archimedean triangular norm. By the assumption that none of the pairs in the set $\{x, y, z\} \times \{x, y, z\}$ belongs to $\tilde{D}$, one concludes that $(\omega, z) \notin \tilde{D}$ for any $\omega \in [a, b]$. Consequently, we have $\tilde{U}(U(x, y), z)) = U(U(x, y), z))$ by the fact that $U(x, y) \in [a, b)$. Therefore, one concludes that $\tilde{U}(\tilde{U}(x, y), z) = \tilde{U}(x, \tilde{U}(y, z))$.

 (c) Suppose that there are two elements in the set $\{x, y, z\}$ belong to $[e, 1]$, and the remaining element comes from $[0, e]$. Then similar to the verifications of situation (i) (b), the associativity is established.

(ii) Verifying the equality $\tilde{U}(\tilde{U}(x, y), z) = \tilde{U}(x, \tilde{U}(y, z))$ on sets $\{x, y, z\}$ such that at least one of the pairs in the set $\{x, y, z\} \times \{x, y, z\}$ belongs to $\tilde{D}$. We will only verify the case of $(x, y) \in \tilde{D}$ here, as the certifications for other cases can be provided in the same way. In this situation, there exists some $i \in I$ such that $(x, y) \in D_i$.

 (a) Assume that $(x, y) \in (a_i, b_i) \times (c_i, d_i)$. Then, $\tilde{U}(\tilde{U}(x, y), z) = \tilde{U}(x, z)$. As for the value of $\tilde{U}(x, \tilde{U}(y, z))$, we have the following considerations.

  i. If $z \in [0, b_i) \cup [d_i, 1]$, then $\tilde{U}(x, \tilde{U}(y, z)) = \tilde{U}(x, \tilde{U}(z, y))$. Further, by Lemma 3.10 we have $\tilde{U}(x, \tilde{U}(z, y)) = \tilde{U}(x, z)$.

  ii. If $z \in [b_i, c_i]$, then we get $\tilde{U}(x, \tilde{U}(y, z)) = \tilde{U}(x, \tilde{U}(z, y)) = \tilde{U}(x, U(z, y))$. By Lemma 3.10 it holds that $\tilde{U}(x, U(z, y)) = \tilde{U}(x, y) = x$ and $\tilde{U}(x, z) = x$.

  iii. If $z \in (c_i, d_i)$, then $\tilde{U}(x, \tilde{U}(y, z)) = \tilde{U}(x, U(y, z)) = U(y, z) \wedge x = x$ by using $c_i < U(y, z) < d_i$. Meanwhile, it also holds that $\tilde{U}(x, z) = x$.

(b) Assume that $(x, y) \in (c_i, d_i) \times (a_i, b_i)$. Then, $\tilde{U}(\tilde{U}(x, y), z) = \tilde{U}(y, z)$. As for the value of $\tilde{U}(x, \tilde{U}(y, z))$, the following verifications appear.
   i. If $z \in [0, a_i] \cup [d_i, 1]$, then by Lemma 3.10 it holds that $\tilde{U}(x, \tilde{U}(y, z)) = \tilde{U}(x, U(y, z)) = \tilde{U}(x, z) = z = \tilde{U}(y, z)$.
   ii. If $z \in [b_i, d_i)$, then $\tilde{U}(x, \tilde{U}(y, z)) = \tilde{U}(x, y) = y$ and $\tilde{U}(y, z) = y$.
   iii. If $z \in (a_i, b_i)$, then $\tilde{U}(x, \tilde{U}(y, z)) = \tilde{U}(x, U(y, z)) = U(y, z) \wedge x = U(y, z)$ by using $a_i < U(y, z) < b_i$. Meanwhile, we also have that $\tilde{U}(y, z) = U(y, z)$.

Taking all the above discussions together, one concludes that $\tilde{U}$ is a uninorm. □

Proposition 3.11 shows that if the operation corresponding to the non-locally internal region in $A(e)$ is replaced with the smaller of two variables, the resulting operator remains a uninorm and satisfies (QP). The following result indicates that on the basis of Proposition 3.11, changing the operation corresponding to the maximum non-locally internal rectangular region pairs to the larger of two variables is also a viable approach.

Proposition 3.12. Suppose that $U \in U_{cts(e)}$ is not locally internal in $A(e)$, $\tilde{D} = \bigcup_{i \in I} D_i$, where $D_i = (a_i, b_i) \times (c_i, d_i) \cup (c_i, d_i) \times (a_i, b_i)$ is the maximum non-locally internal rectangular region pair of U. Let $\tilde{I}$ be a subset if I, then the operator defined as

$$\tilde{\tilde{U}}(x, y) = \begin{cases} \max\{x, y\}, & \text{if } (x, y) \in D_i \text{ and } i \in \tilde{I}, \\ \tilde{U}(x, y), & \text{otherwise}, \end{cases} \tag{3}$$

is a uninorm and satisfies (QP), where $\tilde{U}$ is defined in Proposition 3.11.

Proof. From Proposition 3.11 one obtains that $\tilde{U}$ is a uninorm. Due to the fact that operator $\tilde{\tilde{U}}$ is obtained by replacing the value of $\tilde{U}$ in region D_i $(i \in \tilde{I})$ with the larger of two variables, it can be concluded that $\tilde{\tilde{U}}$ is a uninorm by using the same proof method as Proposition 3.11. □

The results of Proposition 3.11 and Proposition 3.12 demonstrate that for a uninorm $U \in U_{cts(e)}$ that is not locally internal in $A(e)$, as long as the operation corresponding to the maximum non-locally internal region pair is replaced with the smaller (larger) of two variables, then the resulting operator is not only a uninorm but also satisfies (QP).

Example 3.13. For the uninorm U presented in Example 3.9, it has been verified that it cannot generate a U-poset. However, according to the results of Proposition 3.12, for each region $D_n = (a_n, b_n) \times (c_n, d_n) \cup (c_n, d_n) \times (a_n, b_n)$ with $n \in N^+$, as long as the values of U in this region are all replaced with the smaller of two variables, or all are changed to the larger of two variables, then the resulting uninorm is able to yield a U-poset. Further, assuming that the function value of U on each $(a_n, b_n) \times (c_n, d_n) \cup (c_n, d_n) \times (a_n, b_n)$ with $n \in N^+$ is replaced by the smaller of the two variables, and that $\tilde{U}$ is a member of $U_{\min}$, then we can obtain the following order relationship between elements. For the convenience of expression, we denote the set C is equal to $A(\frac{1}{2}) \backslash \{\bigcup_{n \in N^+} \{(a_n, b_n) \times (c_n, d_n) \cup (c_n, d_n) \times (a_n, b_n)\} \cup [\frac{3}{8}, \frac{5}{8}]^2\}$.

- Since U in region $[0, \frac{1}{2}]^2$ is a continuous triangular norm, we know that in this area, $\preceq_U$ is consistent with the natural order.
- By U in region $[\frac{1}{2}, 1]^2$ is a continuous triangular conorm, one obtains that in this area, $\preceq_U$ is consistent with the reverse natural order.
- According to $U(x, y) = y$ for each $(x, y) \in C$ satisfies $x + y > 1$ and $y > x$, we have $y \preceq_U x$ in this case.
- From $U(x, y) = y$ for each $(x, y) \in C$ such that $x + y < 1$ and $x > y$, we obtain $y \preceq_U x$ in this situation.
- In the remaining area of the unit region, we have $x \preceq_U y$ by $U(x, y) = x$.

After the order equivalence relation for triangular norms was investigated in the framework of Clifford's relation [14], it is natural to consider the order equivalence relation of uninorms, which serve as their generalized operators. In the following context, an equivalence relation for the class of uninorms will be defined and explored.

Definition 3.14. Uninorms U_1 and U_2 are said to be *order equivalent*, denoted as $U_1 \sim U_2$, if and only if U_1-poset coincides with U_2-poset.

Obviously, the relation $\sim$ presented in Definition 3.14 is an equivalence relation.

Lemma 3.15. Suppose that $U_1 \in U_{cts(e_1)}$ and $U_2 \in U_{cts(e_2)}$. If $U_1 \sim U_2$, then $e_1 = e_2$.

Proof. Suppose that $e_1 < e_2$, then let us take $x, y \in (e_1, e_2)$ such that $x < y$. In this case, there is some $\zeta \in (e_1, 1]$ satisfies $U_1(\zeta, x) = y$. So we obtain $y \preceq_{U_1} x$. On the other hand, there also exists some $\gamma \in [0, e_2)$ satisfies $U_2(\gamma, y) = x$, which implies that $x \preceq_{U_2} y$. According to $U_1 \sim U_2$ and the antisymmetry of partial order, we have $x = y$. A contradiction is reached. Through a similar argument, the case of $e_1 > e_2$ is also impossible. Aa a result, $e_1 = e_2$. □

Lemma 3.16. Suppose that U_1 and U_2 are two uninorms, then $U_1 \sim U_2$ if and only if $Ran(U_1(\cdot, x)) = Ran(U_2(\cdot, x))$ for all $x \in [0, 1]$.

Proof. Taking $y \in Ran(U_1(\cdot, x))$, then there exists some $\ell_1 \in [0, 1]$ such that $y = U_1(\ell_1, x)$. By using the definition of $\preceq_{U_1}$ we have $y \preceq_{U_1} x$. Therefore, by $U_1 \sim U_2$ we have $y \preceq_{U_2} x$, which indicates that there is some $\ell_2 \in [0, 1]$ such that $U_2(\ell_2, x) = y$. That is, $y \in Ran(U_2(\cdot, x))$. Thus, we have proven that $Ran(U_1(\cdot, x)) \subseteq Ran(U_2(\cdot, x))$. Through a similar verification, the reverse inclusion $Ran(U_2(\cdot, x)) \subseteq Ran(U_1(\cdot, x))$ is also established. Therefore, one concludes that $Ran(U_1(\cdot, x)) = Ran(U_2(\cdot, x))$.

On the contrary, suppose that $x \preceq_{U_1} y$, then there exists some $\ell_3 \in [0, 1]$ such that $U_1(\ell_3, y) = x$. Therefore, the fact that $Ran(U_1(\cdot, x)) = Ran(U_2(\cdot, x))$ implies there is some $\ell_4 \in [0, 1]$ such that $U_2(\ell_4, y) = x$. As a result, $x \preceq_{U_2} y$. By the same method, it is also valid that $x \preceq_{U_1} y$ can be derived by $x \preceq_{U_2} y$. Therefore, it is true that $U_1 \sim U_2$. □

Lemma 3.16 shows that $U_1 \sim U_2$ if and only if the range of values of their horizontal-sections remain consistent. The following result further shows that their values in $A(e)$ are exactly the same.

Theorem 3.17. If $U_1 \in U_{c_{cts(e)}}$ and $U_2 \in U_{_{cts(e)}}$. Then $U_1 \sim U_2$ if and only if the following statements hold simultaneously.

(i) U_1 and U_2 are locally interval in $A(e)$.

(ii) $U_1(x, y) = U_2(x, y)$ for all $(x, y) \in A(e)$.

Proof. To ensure that both U_1 and U_2 are capable of generating partially ordered sets, Theorem 3.7 illustrates that U_1 and U_2 must be locally interval in $A(e)$. That is, item (ii) is established. Based on this fact, by Lemma 3.16 and the fact that U_1 and U_2 have the same neutral element, it can be seen that $U_1(x, y) = U_2(x, y)$ for all $(x, y) \in A(e)$.

The reverse implication is guaranteed by Lemma 3.16. □

Corollary 3.18. Suppose that U_1 and U_2 belong to $U_{_{cts(e)}}$ such that $U_1 \sim U_2$. The following two statements are valid.

(i) $U_1 \in U_{min}$ if and only if $U_2 \in U_{min}$.

(ii) $U_1 \in U_{max}$ if and only if $U_2 \in U_{max}$.

Proof. It can be directly obtained by Proposition 3.17. □

4. QUASI-PROJECTION FOR 2-UNINORMS WITH CONTINUOUS UNDERLYING OPERATORS

In this section, we explore the performance of generating partial orders in the framework of 2-uninorms, which are generalizations of uninorms. Due to the fact that 2-uninorms are equipped with two local neutral elements, the property of (LLI) is naturally satisfied. Therefore, whether a 2-uninorm G can induce a G-poset depends entirely on the satisfaction of (QP).

Lemma 4.1. Suppose that $G = \langle U_1, U_2 \rangle$ is a 2-uninorm. If G satisfies (QP) then U_1 and U_2 satisfy (QP).

Proof. Suppose that $U_1(\frac{x_0}{k}, U_1(\frac{y_0}{k}, \frac{z_0}{k})) = \frac{z_0}{k}$ for some $x_0, y_0, z_0 \in [0, k]$, then we have $kU_1(\frac{x_0}{k}, U_1(\frac{y_0}{k}, \frac{z_0}{k})) = z_0$. Therefore, it holds that $G(x_0, G(y_0, z_0)) = z_0$. According to G satisfies (QP) one has that $G(y_0, z_0) = z_0$. Therefore, we have that $kU_1(\frac{y_0}{k}, \frac{z_0}{k}) = z_0$. That is, $U_1(\frac{y_0}{k}, \frac{z_0}{k}) = \frac{z_0}{k}$. Consequently, it is obtained that U_1 satisfies (QP). Through a similar argument, it is also true that U_2 satisfies (QP). □

Lemma 4.2. Suppose that $G = \langle U_1, U_2 \rangle \in G_k$, then G satisfies (QP) if and only if U_1 and U_2 satisfy (QP).

Proof. The sufficiency has been guaranteed by Lemma 4.1. Suppose that $G(x, G(y, z)) = z$. To verify that $G(y, z) = z$, we need to consider the following situations.

(i) If $y, z \in [0, k]$, then according to the structure of the 2-uninorm we have $G(y, z) \leq k$.

(a) If $x \in [0,k]$, then by $G(x,G(y,z)) = z$ we have $U_1(\frac{x}{k}, U_1(\frac{y}{k}, \frac{z}{k})) = \frac{z}{k}$. According to the fact that U_1 satisfies (QP), one has $U_1(\frac{y}{k}, \frac{z}{k}) = \frac{z}{k}$, that is, $G(y,z) = z$.

(b) If $x \in (k,1]$, then by $G(x,G(y,z)) = z$ we get $z = k$. Therefore, $G(y,z) = k$ is established.

(ii) If $y, z \in [k,1]$, in this case, it holds that $G(y,z) \geq k$.

(a) If $x \in [0,k)$, then from $G(x,G(y,z)) = z$, we know that $z = k$. Thus, we have $G(y,z) = k$.

(b) If $x \in [k,1]$, then $z = G(x,G(y,z)) = k + (1-k)U_2(\frac{x-k}{1-k}, U_2(\frac{y-k}{1-k}, \frac{z-k}{1-k}))$, which means $U_2(\frac{x-k}{1-k}, U_2(\frac{y-k}{1-k}, \frac{z-k}{1-k})) = \frac{z-k}{1-k}$. By using U_2 satisfies (QP), we have $U_2(\frac{y-k}{1-k}, \frac{z-k}{1-k}) = \frac{z-k}{1-k}$, which implies that $G(y,z) = z$.

(iii) If $(y,z) \in [0,k) \times (k,1] \cup (k,1] \times [0,k)$, then we have $G(y,z) = k$. For any $x \in [0,1]$ it holds that $G(x,G(y,z)) = G(x,k) = k$, which means the antecedent of (QP) is not satisfied.

Above discussions show that G satisfies (QP) if and only if U_1 and U_2 satisfy (QP). □

Lemma 4.3. Suppose that $G = \langle U_1, U_2 \rangle \in G_{c,k}$, then G satisfies (QP) if and only if U_1 and U_2 satisfy (QP).

Proof. From Lemma 4.1, it can be inferred that sufficiency holds. Now, let us verify $G(y,z) = z$ is valid under the condition that $G(x,G(y,z)) = z$.

(i) Assume that $y, z \in [0,k]$, then according to the structure of the G, we have $G(y,z) \leq k$.

(a) If $x \in [0,k]$, then through a verification similar to Lemma 4.2, one has that $G(y,z) = z$.

(b) If $x \in (k,1]$, then we get $z = G(x,G(y,z)) = G(k,G(y,z)) = kU_1(\frac{k}{k}, U_1(\frac{y}{k}, \frac{z}{k}))$. According to the fact that U_1 satisfies (QP), we have $U_1(\frac{y}{k}, \frac{z}{k}) = \frac{z}{k}$, which means that $G(y,z) = z$.

(ii) Suppose that $y, z \in [k,1]$, then we have $G(y,z) \geq k$.

(a) If $x \in [0,k]$, in this case, we have $G(x,G(y,z)) = G(x,k) \leq G(k,k) = k$. To ensure the antecedent of (QP) is valid, it must be that $z = k$. Therefore, $G(y,z) = G(y,k) = k$.

(b) If $x \in [k,1]$, then by using the same verification as Lemma 4.2, one has that $G(y,z) = z$.

(iii) If $(y,z) \in [0,k) \times (k,1]$, then according to the structure of the G, we have $G(x,G(y,z)) = G(x,G(y,k)) = G(G(x,y),k) \leq G(G(1,y),k) = G(G(k,y),k) = G(G(k,k),y) = G(k,y) \leq G(k,k) = k$. Therefore, in this case, the antecedent of (QP) is not valid.

(iv) If $(y,z) \in (k,1] \times [0,k)$, then we have the following considerations.

(a) If $x \in [0,k]$, then it obtains that $z = G(x, G(y,z)) = G(x, G(k,z))$. By using U_1 satisfies (QP), one concludes that $G(k,z) = z$. As a result, $G(y,z) = G(k,z) = z$.

(b) If $x \in [k,1]$, then we have $z = G(x, G(y,z)) = G(x, G(k,z)) = G(G(x,z),k) = G(G(k,z),k) = G(z, G(k,k)) = G(z,k)$. Therefore, we obtain that $G(y,z) = G(k,z) = z$.

The above analyses indicate that G satisfies (QP) if and only if U_1 and U_2 satisfy (QP). □

Next, we consider the case where $G(0,1) = 1$ and $G(0,k) = k$.

Lemma 4.4. Let $G = \langle U_1, U_2 \rangle \in G_{d,k}$, then G satisfies the (QP) if and only if U_1 and U_2 satisfy (QP).

Proof. The sufficiency has been guaranteed by Lemma 4.1. Suppose that $G(x, G(y,z)) = z$. To verify that $G(y,z) = z$, we need to consider the following situations.

(i) If $y, z \in [0,k]$, then it holds that $G(y,z) \le k$.

(a) If $x \in [0,k]$, then by using U_1 satisfies (QP), one has that $G(y,z) = z$.

(b) If $x \in (k,1]$, then according to the structure of the G, we have $G(x, G(y,z)) = G(x,k) \ge k$. To ensure the antecedent of (QP) is valid, it must be that $z = k$. Therefore, it holds that $G(y,z) = G(y,k) = k$.

(ii) If $y, z \in [k,1]$, then we have $G(y,z) \ge k$.

(a) Let $x \in [0,k]$, then according to the structure of the G, we have $z = G(x, G(y,z)) = G(k, G(y,z)) = k + (1-k)U_2(\frac{k-k}{1-k}, U_2(\frac{y-k}{1-k}, \frac{z-k}{1-k}))$, which means that $U_2(\frac{k-k}{1-k}, U_2(\frac{y-k}{1-k}, \frac{z-k}{1-k})) = \frac{z-k}{1-k}$. According to the fact that U_2 satisfies (QP), we have $U_2(\frac{y-k}{1-k}, \frac{z-k}{1-k}) = \frac{z-k}{1-k}$, which implies that $G(y,z) = z$.

(b) If $x \in [k,1]$, then by using U_2 satisfies (QP), one has that $G(y,z) = z$.

(iii) If $(y,z) \in [0,k) \times (k,1]$, then according to the structure of the G, we have $z = G(x, G(y,z)) = G(x, G(k,z)) = G(k, G(x,z))$.

(a) If $x \in [0,k]$, then $G(k, G(x,z)) = G(k, G(k,z)) = G(G(k,k),z) = G(k,z)$, i. e., $z = G(k,z)$. Therefore, $G(y,z) = G(y, G(k,z)) = G(k, G(y,z)) = G(k, G(k,z)) = G(k,z) = z$.

(b) If $x \in [k,1]$, then $G(k, G(x,z)) = k + (1-k)U_2(\frac{k-k}{1-k}, U_2(\frac{x-k}{1-k}, \frac{z-k}{1-k}))$. Thus, if the antecedent of (QP) is valid, then one gets that $U_2(\frac{k-k}{1-k}, U_2(\frac{x-k}{1-k}, \frac{z-k}{1-k})) = \frac{z-k}{1-k}$. Further, by U_2 satisfies (QP) we have $U_2(\frac{x-k}{1-k}, \frac{z-k}{1-k}) = \frac{z-k}{1-k}$. That is, $G(x,z) = z$, which means that $G(y,z) = G(y, G(x,z)) = G(x, G(y,z))$.

(iv) If $(y,z) \in (k,1] \times [0,k)$. Then according to the structure of the G, we have $G(x,G(y,z)) = G(x,G(y,k)) = G(G(k,x),y) \geq G(G(k,0),y) = G(k,y) \geq k$, which means that the antecedent of (QP) is not valid.

Therefore, we conclude that G satisfies the (QP) if and only if U_1 and U_2 satisfy (QP). □

Theorem 4.5. If $G = \langle U_1, U_2 \rangle \in G_k \cup G_{c,k} \cup G_{d,k}$, then $([0,1], \preceq_G)$ is a G-poset if and only if $([0,1], \preceq_{U_1})$ and $([0,1], \preceq_{U_2})$ are respectively U_1-poset and U_2-poset.

Proof. Note that if $([0,1], \preceq_{U_1})$ and $([0,1], \preceq_{U_2})$ are respectively U_1-poset and U_2-poset, then by Theorem 2.4 we know that U_1 and U_2 satisfy (QP). Therefore, by using Lemma 4.2, Lemma 4.3 and Lemma 4.4, we know the result is valid. □

Corollary 4.6. Let $G = \langle U_1, U_2 \rangle \in G_{cts(e_1,k,e_2)}$ belongs to $G_k \cup G_{c,k} \cup G_{d,k}$, then $([0,1], \preceq_G)$ is a G-poset if and only if U_1 and U_2 are locally internal in $A(\frac{e_1}{k})$ and $A(\frac{e_2-k}{1-k})$ respectively.

Proof. By combining Theorem 4.5 and Theorem 3.8, the result is straightforward. □

Next, we consider the situation that $G \in G_{c,1} \cup G_{d,0}$. In this situation, as stated in Proposition 2.12, the value of G restricts to $[0,e_1) \times (e_2,1] \cup (e_2,1] \times [0,e_1)$ falls into $[0,e_1] \cup k \cup [e_2,1]$. To discuss of the satisfaction of (QP), the following observations are obtained when $(x,y,z) \in [0,e_1) \times [0,e_1) \times (e_2,1]$.

(i) If $G(y,z) \in [0,e_1] \cup \{k\}$, then we get $G(x,G(y,z)) \leq G(x,k) \leq k$. In this case, the antecedent of (QP) is not established.

(ii) Suppose that $G(y,z) \in [e_2,1]$. In this case, since there is no additional information available regarding the value of $G(y,z)$, we cannot confirm that the antecedent of (QP) is invalid. Furthermore, even if the antecedent $G(x,G(y,z)) = z$ is established, we still cannot determine the validity of the consequent $G(y,z) = z$.

Therefore, when G comes from $G_{c,1}$ or $G_{d,0}$, we will impose a reasonable condition that G is locally internal on $[0,e_1) \times (e_2,1] \cup (e_2,1] \times [0,e_1)$ to facilitate the discussion of the satisfaction of (QP).

Lemma 4.7. Suppose that $G = \langle U_1, U_2 \rangle$ comes from $G_{c,1} \cup G_{d,0}$ and is locally internal on $[0,e_1) \times (e_2,1] \cup (e_2,1] \times [0,e_1)$. Then G satisfies the (QP) if and only if U_1 and U_2 satisfy the (QP).

Proof. According to Lemma 4.1, sufficiency is established. To verify that $G(y,z) = z$ is true on the condition that $G(x,G(y,z)) = z$, the following considerations arise.

(i) If $x,y,z \in [0,e_2]$, then G restricted on $[0,e_2]^2$ can be expressed as $G(x,y) = e_2\tilde{G}(\frac{x}{e_2}, \frac{y}{e_2})$ with $\tilde{G} \in G_{c,k}$ and the underlying uninorms of $\tilde{G}$ satisfy (QP). Therefore, the condition $G(x,G(y,z)) = z$ is equivalent to that $\tilde{G}(\frac{x}{e_2}, \tilde{G}(\frac{y}{e_2}, \frac{z}{e_2})) = \frac{z}{e_2}$. By Lemma 4.3 it holds that $\tilde{G}(\frac{y}{e_2}, \frac{z}{e_2}) = \frac{z}{e_2}$, which means that $G(y,z) = z$.

(ii) If $x, y, z \in [e_1, 1]$, then G restricted on $[e_1, 1]^2$ can be expressed as $G(x, y) = e_1 + (1 - e_1)\tilde{\tilde{G}}(\frac{x-e_1}{1-e_1}, \frac{y-e_1}{1-e_1})$ with $\tilde{\tilde{G}} \in G_{d,k}$ and the underlying uninorms of $\tilde{\tilde{G}}$ satisfy (QP). Similar to the arguments of (i), we obtain that $G(y, z) = z$ by Lemma 4.4.

Therefore, for the verifications of (QP), we only need to consider the case where at least one of the pairs in the set $\{x, y, z\} \times \{x, y, z\}$ falls into $[0, e_1) \times (e_2, 1] \cup (e_2, 1] \times [0, e_1)$. Here we only check the situation that at least one pair comes from $[0, e_1) \times (e_2, 1]$, as the other case can be addressed in a similar manner.

(i) Suppose that $(y, z) \in [0, e_1) \times (e_2, 1]$. If $G(y, z) = z$, then the consequent of (QP) is naturally valid. As for the case $G(y, z) = y$, we have $G(x, G(y, z)) = G(x, y)$ and the following considerations emerge.

  (a) If $x \in [0, e_2]$, then $G(x, y) \leq e_2 < z$, which means that the antecedent of (QP) is not valid.

  (b) If $x \in (e_2, 1]$, then $G(x, y) \in \{x, y\}$. If the antecedent of (QP) is valid, then it must be that $G(x, y) = x = z$. As a result, $G(y, z) = G(y, x) = G(x, G(y, z))$.

(ii) Suppose that $(z, y) \in [0, e_1) \times (e_2, 1]$. If $G(z, y) = z$, then the consequent of (QP) is automatically true. If $G(z, y) = y$, then we get $G(x, G(y, z)) = G(x, y)$ and the following discussions are needed.

  (a) If $x \in [0, e_1)$, then $G(x, y) \in \{x, y\}$. If the antecedent of (QP) is valid, it must be that $G(x, y) = x = z$. Therefore, $G(y, z) = (y, x) = G(x, G(y, z))$.

  (b) If $x \in [e_1, 1]$, then $G(x, y) \geq G(e_1, e_1) = e_1 > z$. That is, the antecedent of (QP) is not valid.

(iii) Suppose that $(x, z) \in [0, e_1) \times (e_2, 1]$. If $G(x, z) = z$, then $G(x, G(y, z)) = G(y, G(x, z)) = G(y, z)$. Therefore, we only need to consider the situation that $G(x, z) = x$. In this case, we have that $G(x, G(y, z)) = G(x, y)$.

  (a) Suppose that $y \in [0, e_2]$, then $G(x, y) \leq G(e_2, e_2) = e_2 < z$, which means that the antecedent of the (QP) is not valid.

  (b) If $y \in (e_2, 1]$, then $G(x, y) \in \{x, y\}$. If the antecedent of (QP) is valid, then it must be that $G(x, y) = y$. In this case, $G(y, z) = G(G(x, y), z) = G(x, G(y, z))$.

(iv) Assume that $(z, x) \in [0, e_1) \times (e_2, 1]$. If $G(z, x) = z$, then $G(x, G(y, z)) = G(y, G(z, x)) = G(y, z)$. As for the case of $G(z, x) = x$, it gets that $G(x, G(y, z)) = G(x, y)$.

  (a) If $y \in [0, e_1)$, then we have $G(x, y) \in \{x, y\}$. If the antecedent of (QP) is valid, then it must be that $G(x, y) = y$. Therefore, $G(y, z) = G(G(x, y), z) = G(x, G(y, z))$.

  (b) If $y \in [e_1, 1]$, then $G(x, y) \geq G(e_1, e_1) = e_1 > z$, which means that the antecedent of the (QP) is invalid.

(v) Assume that $(x, y) \in [0, e_1) \times (e_2, 1]$ or $(y, x) \in [0, e_1) \times (e_2, 1]$. In this case, if $G(x, y) = y$, then $G(x, G(y, z)) = G(G(x, y), z) = G(y, z)$. If $G(x, y) = x$, then $G(x, G(y, z)) = G(G(x, y), z) = G(x, z)$. If the antecedent of (QP) is valid, then it must be that $G(x, z) = z$. Therefore, $G(y, z) = G(y, G(x, z)) = G(x, G(y, z))$.

Based on the above discussions, it can be concluded that G satisfies the (QP) if and only if U_1 and U_2 satisfy (QP). □

Remark 4.8. Lemma 4.7 adds the condition that G is locally internal on $[0, e_1) \times (e_2, 1] \cup (e_2, 1] \times [0, e_1)$. In fact, as long as some conditions are attached to the boundary of G, then G possesses this property as described below.

(i) For each G comes from $G_{c,1}$, Theorem 7 in [32] shows that if $G(1, \cdot)$ is continuous on $[0, e_1)$, then G is given by the minimum of two variables on $[0, e_1) \times (e_1, 1] \cup (e_1, 1] \times [0, e_1)$.

(ii) For any G belongs to $G_{d,0}$, Theorem 8 in [32] indicates that as long as $G(0, \cdot)$ is continuous on $(e_2, 1]$, then G is represented as the maximum of two variables on $[0, e_2) \times (e_2, 1] \cup (e_2, 1] \times [0, e_2)$.

Theorem 4.9. If $G = \langle U_1, U_2 \rangle$ comes from $G_{c,1} \cup G_{d,0}$ and is locally internal on $[0, e_1) \times (e_2, 1] \cup (e_2, 1] \times [0, e_1)$, then $([0, 1], \preceq_G)$ is a G-poset if and only if $([0, 1], \preceq_{U_1})$ and $([0, 1], \preceq_{U_2})$ are respectively U_1-poset and U_2-poset.

Proof. Note that if $([0, 1], \preceq_{U_1})$ and $([0, 1], \preceq_{U_2})$ are respectively U_1-poset and U_2-poset, then we know that U_1 and U_2 satisfy (QP) by Theorem 2.4. Therefore, in view of the result obtained from Lemma 4.7, it can be concluded that the result is true. □

Corollary 4.10. Let $G = \langle U_1, U_2 \rangle \in G_{c,1} \cup G_{d,0}$ has continuous underlying operators and is locally internal on $[0, e_1) \times (e_2, 1] \cup (e_2, 1] \times [0, e_1)$, then $([0, 1], \preceq_G)$ is a G-poset if and only if U_1 and U_2 are locally internal in $A(\frac{e_1}{k})$ and $A(\frac{e_2 - k}{1 - k})$ respectively.

Proof. By combining Theorem 4.9 and Theorem 3.8, the result is immediate. □

Remark 4.11. Similar to the discussions on the order equivalence of uninorms in the third section, exploring the order equivalence classes of 2-uninorms under the framework of Clifford's order is also an interesting topic in order theory research. Clearly, analogous to Lemma 3.16, it can be equivalently transformed into verifying whether the range of values of the functions corresponding to the horizontal-sections are equal. For this reason it will not be discussed in detail here.

Remark 4.12. Although the focus of this article and Refs. [22, 23] is on uninorms (2-uninorms) with continuous underlying operators, their research themes are fundamentally different, which can be reflected in the following three aspects.

(i) This article primarily investigates whether the operators can generate partial order, while Refs. [22, 23] focus on the structural representation of operators. For example, Ref. [22] deals with the ordinal sum representation of such uninorms by decomposing the characterizing set-valued function into segments.

(ii) Our findings indicate that a fixed U belongs to $U_{cts(e)}$ can generate partial order if and only if it is locally internal in $A(e)$, which means that not every member of such operators possesses the ability to generate a partially ordered set. However, the results in Ref. [22] show that any member of such operators can be decomposed into an ordinal sum.

(iii) Although Refs. [22] and [23] also addressed expressions of order in their research processes, this order pertains to the index set of the decomposed semigroups. The primary focus of these two articles regarding the order aspect is how to assign order relations between the elements in the index set corresponding to the semigroups. In contrast, the order discussed in this article is the order between elements in the interval $[0, 1]$, derived within the framework of Eg. (2) and in order to prove the relation $\preceq_F$ can become a partial order, the core task of the article is to verify the validity of the (QP) property.

5. CONCLUSIONS

In this work, we first investigated the satisfaction of (QP) for uninorms with continuous underlying operators. It was found that such uninorms are capable of producing partial orders if and only if they are locally internal in $A(e)$, and the resulting partially ordered set is a chain. For non-locally internal uninorms, we provided a simple and intuitive reconstruction method, ensuring that the modified operators not only remain uninorms but also gain the ability to generate partial orders. Building on this, we further explored the performance of generating partial orders within the framework of 2-uninorms. The results revealed that this capability is entirely determined by the underlying uninorms. In the future, investigating the order theory behavior of n-uninorms will also be an intriguing task, and the following observation can be made.

Observation 5.1. Let G is a n-uninorm with continuous underlying operators. Then according to the results of Theorem 3.8 one obtains that if $([0, 1], \preceq_G)$ is a poset then each underlying uninorm of G is locally internal in $A(e)$.

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