KYBERNETIKA — VOLUME *61* (2025), NUMBER 4, PAGES 537–553

ADAPTIVE INVERSE OPTIMAL CONTROL FOR UNSTABLE REACTION-DIFFUSION PDE SYSTEM

Xiushan Cai, Yue Li, Ping Wang, Yuhang Lin, Le Zhang, and Leipo Liu

We study adaptive inverse optimal boundary control for reaction-diffusion PDE system with unknown coefficient. First, an adaptive boundary control with parameter update rule is designed which no attempt is made to force parameter convergence. Next, it is proven through a non quadratic Lyapunov function that the closed-loop system is globally asymptotically stable. Further, it indicates that adaptive boundary control with parameter update law is optimal for a meaningful functional. Finally, the effectiveness of the proposed control design is demonstrated through an example.

Keywords: inverse optimality, adaptive boundary control, reaction-diffusion PDE, parameter update law, unknown coefficient

Classification: 93Cxx, 93Dxx

1. INTRODUCTION

Reaction-diffusion partial differential equations (PDEs) can describe a variety of practical phenomena, such as, heat transfer processes, biological, chemical reactions, and traffic flow in [21, 22] and the references therein.

For a class of reaction-diffusion PDEs, an adaptation mechanism is developed and the performance bounds are established in [18]. The first adaptive controllers for unstable reaction-diffusion PDEs without relative degree limitations, open-loop stability assumptions is [14]. Adaptive control based on Lyapunov method is provided for a class of reaction-diffusion PDEs, whereas the parameter estimation error is penalized through an exponential of its square in [11]. An output feedback stabilization of reaction-diffusion PDEs with a non-collocated boundary condition is studied in [15]. Based on modal decomposition method, a feedback control is proposed to stabilize a cascaded heat-heat system with different reaction coefficients in [20].

For a reaction-diffusion PDE with a constant delay, a finite-dimensional delayed controller is computed for the unstable part, and it is shown that this controller stabilizes the whole PDE in [17]. Intermittent static output feedback control is presented for stochastic delayed-switched positive systems with only partially measurable information in [8]. Controls of unstable reaction-diffusion PDEs with long input delays are also appeared based on backstepping technique in [12].

DOI: 10.14736/kyb-2025-4-0537

Inverse optimization method is to design a control law and prove that it is optimal for a meaningful function [2], [3], [6], [9], [19].

Inverse optimization has practical significance because it can design optimal control laws that minimize/maximize the physical quantity of interest, and it may have a certain robustness margin without solving a Hamilton–Jacobi–Isaacs PDE in [13]. Inverse optimal control is established for linear systems [1], [7], [23], strict-feedforward nonlinear system [5], robotic manipulators with compliant actuators [16], Burgers' equation [10], Korteweg–de Vries–Burgers equation [4]. However, to the best of the authors' knowledge, there are only limited results for inverse optimal control of unstable reaction-diffusion PDEs.

In this paper, we consider adaptive inverse optimal boundary control for an unstable reaction-diffusion PDE with unknown coefficient. Main contributions are as follows.

1. An adaptive boundary control with parameter update law is designed for this type of system. The control process does not force the parameters to converge, but the estimation transient is penalized simultaneously with the state and control.
2. Globally asymptotical stability of the closed-loop system is analyzed using a suitable Lyapunov functional.
3. It is shown that the adaptive boundary control with parameter update law is optimal to a meaningful functional.

This paper is organized as follows: System description and some lemmas are in Section 2. An adaptive boundary control and stability analysis are achieved in Section 3. Adaptive inverse optimal boundary control is shown in Section 4. Simulation is provided in Section 5. Concluding remarks are in Section 6.

Notation. A continuous function α is called a class $\mathcal{K}$ function, if it is strictly increasing, and $\alpha(0) = 0$. For an n-vector, $|\cdot|$ denotes the usual Euclidean norm. The spatial $L_2(0,1)$ norm is denoted by $|\cdot|$. The symbols $J(\cdot)$, $I_1(\cdot)$ denote the first order Bessel function of the first kind and the first order modified Bessel functions of the first kind, respectively.

2. SYSTEM DESCRIPTION AND SOME LEMMAS

Consider a reaction-diffusion PDE system

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \tag{1}$$
$$u(0,t) = 0, \tag{2}$$
$$u(1,t) = U(t), \tag{3}$$

where $0 \le x \le 1$, $t \ge 0$, and u is the state, λ is an unknown coefficient, and $U(t)$ is a boundary control.

The objective of this paper is to design an adaptive boundary control that globally stabilizes system (1)–(3). Further, it is shown that the proposed control is optimal to a meaningful functional.

The following Lemma 2.1 is from [14].

Lemma 2.1. For any $W \in C^1[0,1]$, it holds

$$\left|\int_0^1 W(x)\left(\int_0^x \zeta W(\zeta)\,\mathrm{d}\zeta\right)\mathrm{d}x\right| \leq \frac{1}{2\sqrt{3}}|W|^2 \leq \frac{2}{\sqrt{3}}|W_x|^2. \tag{4}$$

The following Lemmas 2.2, 2.3 are from [10].

Lemma 2.2. (Poincare's inequality) For any $W \in C^1[0,1]$, it holds

$$\int_0^1 W(x)^2\,\mathrm{d}x \leq 2W(0)^2 + 4\int_0^1 W_x(x)^2\,\mathrm{d}x, \tag{5}$$

$$\int_0^1 W(x)^2\,\mathrm{d}x \leq 2W(1)^2 + 4\int_0^1 W_x(x)^2\,\mathrm{d}x. \tag{6}$$

Lemma 2.3. (Agmon's inequality). For any $W \in C^1[0,1]$, it holds

$$\max_{x\in[0,1]} W(x)^2 \leq W(0)^2 + 2\sqrt{\int_0^1 W(x)^2 dx \int_0^1 W_x(x)^2\,\mathrm{d}x}, \tag{7}$$

$$\max_{x\in[0,1]} W(x)^2 \leq W(1)^2 + 2\sqrt{\int_0^1 W(x)^2 dx \int_0^1 W_x(x)^2\,\mathrm{d}x}. \tag{8}$$

3. ADAPTIVE BOUNDARY CONTROL DESIGN

First, system (1)–(3) is transferred to the following system

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}w(x,t,\widehat{\lambda}(t)) &= w_{xx}(x,t,\widehat{\lambda}(t)) \\ &\quad + \dot{\widehat{\lambda}}(t)\int_0^x \frac{y}{2}w(y,t,\widehat{\lambda}(t))\,\mathrm{d}y \\ &\quad + (\lambda - \widehat{\lambda}(t))w(x,t,\widehat{\lambda}(t)),\end{aligned} \tag{9}$$

$$w(0,t,\widehat{\lambda}(t)) = 0, \tag{10}$$

$$w(1,t,\widehat{\lambda}(t)) = U(t) - \int_0^1 k(1,y,\widehat{\lambda}(t))u(y,t)\,\mathrm{d}y, \tag{11}$$

by the backstepping transformation

$$w(x,t,\widehat{\lambda}(t)) = u(x,t) - \int_0^x k(x,y,\widehat{\lambda}(t))u(y,t)\,\mathrm{d}y, \tag{12}$$

where the kernel satisfies the following PDE equations

$$k_{xx}(x,y,\widehat{\lambda}(t)) = k_{yy}(x,y,\widehat{\lambda}(t)) + \widehat{\lambda}(t)k(x,y,\widehat{\lambda}(t)), \tag{13}$$

$$k(x,0,\widehat{\lambda}(t)) = 0, \tag{14}$$

$$k(x,x,\widehat{\lambda}(t)) = -\frac{\widehat{\lambda}(t)}{2}x, \tag{15}$$

with $0 \le y \le x \le 1$. The solution of PDE equations (13)–(15) is

$$k(x,y,\widehat{\lambda}(t)) = -\widehat{\lambda}(t)y\frac{I_1\left(\sqrt{\widehat{\lambda}(t)(x^2-y^2)}\right)}{\sqrt{\widehat{\lambda}(t)(x^2-y^2)}}, \tag{16}$$

where $\widehat{\lambda}(t)$ is a real-time estimate of λ. The inverse backstepping transformation of (12) is given by

$$u(x,t) = w(x,t,\widehat{\lambda}(t)) + \int_0^x l(x,y,\widehat{\lambda}(t))w(y,t,\widehat{\lambda}(t))\,\mathrm{d}y, \tag{17}$$

where the kernel satisfies

$$l_{xx}(x,y,\widehat{\lambda}(t)) = l_{yy}(x,y,\widehat{\lambda}(t)) - \widehat{\lambda}(t)l(x,y,\widehat{\lambda}(t)), \tag{18}$$

$$l(x,0,\widehat{\lambda}(t)) = 0, \tag{19}$$

$$l(x,x,\widehat{\lambda}(t)) = -\frac{\widehat{\lambda}(t)}{2}x, \tag{20}$$

with $0 \le x \le y \le 1$, and the solution of PDE equations (18)–(20) is

$$l(x,y,\widehat{\lambda}(t)) = -\widehat{\lambda}(t)y\frac{J_1\left(\sqrt{\widehat{\lambda}(t)(x^2-y^2)}\right)}{\sqrt{\widehat{\lambda}(t)(x^2-y^2)}}. \tag{21}$$

The inverse backstepping transformation (17) transfers system (9)–(11) to system (1)–(3). An adaptive boundary control is designed as

$$U(t) = \int_0^1 k(1,y,\widehat{\lambda}(t))u(y,t)\,\mathrm{d}y, \tag{22}$$

with the parameter update law

$$\dot{\widehat{\lambda}}(t) = \frac{\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x}{1+\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x}. \tag{23}$$

Remark 1. From (11), in order to get $w(1,t,\widehat{\lambda}(t)) = 0$, it leads to the control law (22). the parameter update law (23) is chosen such that $0 \le \dot{\widehat{\lambda}}(t) < 1$, and $\dot{\widehat{\lambda}}(t) \to 0$ when the target system (9)–(11) converges to zero in the L_2 sense.

We first consider system (9)–(11) under adaptive boundary control law (22), with parameter update law (23), we have the following result.

Lemma 3.1. Consider system (9)–(11), with control law (22), and the parameter update law (23), for any $\widehat{\lambda}(0) \in R$, it holds

$$\begin{aligned} & w(x,t,\widehat{\lambda}(t))^2 \\ & \leq 4\int_0^1 w_x(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x \\ & + \frac{12e^{(\widehat{\lambda}(0)-\lambda)^2}}{3-\sqrt{3}}\left(e^{\int_0^1 w(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x+(\widehat{\lambda}(0)-\lambda)^2}-1\right)^{1/2} \\ & \times\left(1+\int_0^1 w(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x\right), \end{aligned} \tag{24}$$

for all $t \geq 0$, $0 \leq x \leq 1$. Furthermore, $\widehat{\lambda}(t)$ and the solution $w(x,t,\widehat{\lambda}(t))$ are uniformly bounded and $w(x,t,\widehat{\lambda}(t))^2 \to 0$, as $t \to \infty$, uniformly in $x \in [0,1]$, and $\widehat{\lambda}(t) \to \lambda_\infty$, as $t \to \infty$, for some λ_∞.

Proof. Let

$$V(t) = \left(1+\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\right)e^{(\widehat{\lambda}(t)-\lambda)^2}-1. \tag{25}$$

The derivative of $V(t)$ along the trajectory of the closed-loop system (9)–(11) and (22), (23) is

$$\begin{aligned} & \dot{V}(t) \\ & = 2e^{(\widehat{\lambda}(t)-\lambda)^2}w(1,t,\widehat{\lambda}(t))w_x(1,t,\widehat{\lambda}(t)) \\ & - 2e^{(\widehat{\lambda}(t)-\lambda)^2}\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x \\ & + e^{(\widehat{\lambda}(t)-\lambda)^2}\dot{\widehat{\lambda}}(t)\int_0^1 w(x,t,\widehat{\lambda}(t))\int_0^x yw(y,t,\widehat{\lambda}(t))\,\mathrm{d}y\mathrm{d}x \\ & + 2e^{(\widehat{\lambda}(t)-\lambda)^2}(\lambda-\widehat{\lambda}(t))\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x \\ & + 2e^{(\widehat{\lambda}(t)-\lambda)^2}\left(1+\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\right)(\widehat{\lambda}(t)-\lambda)\dot{\widehat{\lambda}}(t). \end{aligned} \tag{26}$$

By Lemma 2.1, it holds

$$\left|\int_0^1 w(x,t,\widehat{\lambda}(t))\int_0^x yw(y,t,\widehat{\lambda}(t))\,\mathrm{d}y\mathrm{d}x\right| \leq \frac{2\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x}{\sqrt{3}}. \tag{27}$$

Using (22), (23) and (27), from (26), one has

$$\dot{V}(t) \leq -2e^{(\widehat{\lambda}(t)-\lambda)^2}\left(1-\frac{\sqrt{3}}{3}\right)\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x. \tag{28}$$

From (25), (28), we can deduce that

$$\begin{aligned}
&\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x + (\widehat{\lambda}(t)-\lambda)^2\\
&\le V(t)\\
&\le V(0)\\
&= \left(1+\int_0^1 w(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x\right) e^{(\widehat{\lambda}(0)-\lambda)^2} - 1\\
&\le e^{\int_0^1 w(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x + (\widehat{\lambda}(0)-\lambda)^2} - 1. \qquad (29)
\end{aligned}$$

From (29), it follows that $\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x$, and $\widehat{\lambda}(t)$ are bounded for all $t \ge 0$.

In what follows, we show that $w(x,t,\widehat{\lambda}(t))$ is bounded for all $t \ge 0$ and $0 \le x \le 1$. Using (9)–(11), (22) and (23), we know $w(0,t,\widehat{\lambda}(t)) = 0$, $w(1,t,\widehat{\lambda}(t)) = 0$, and $0 \le \dot{\widehat{\lambda}}(t) < 1$, and consider

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}(\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x)}{\mathrm{d}t}\\
&= \int_0^1 w_x(x,t,\widehat{\lambda}(t))\frac{\mathrm{d}w_x(x,t,\widehat{\lambda}(t))}{\mathrm{d}t}\,\mathrm{d}x\\
&= w_x(1,t,\widehat{\lambda}(t))\frac{\mathrm{d}w(1,t,\widehat{\lambda}(t))}{\mathrm{d}t}\\
&\quad - w_x(0,t,\widehat{\lambda}(t))\frac{\mathrm{d}w(0,t,\widehat{\lambda}(t))}{\mathrm{d}t}\\
&\quad - \int_0^1 w_{xx}(x,t,\widehat{\lambda}(t))\frac{\mathrm{d}w(x,t,\widehat{\lambda}(t))}{\mathrm{d}t}\,\mathrm{d}x\\
&= -\int_0^1 w_{xx}(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\\
&\quad - \dot{\widehat{\lambda}}(t)\int_0^1 w_{xx}(x,t,\widehat{\lambda}(t))\int_0^x \frac{y}{2}w(y,t,\widehat{\lambda}(t))\,\mathrm{d}y\mathrm{d}x\\
&\quad - (\lambda-\widehat{\lambda}(t))\int_0^1 w_{xx}(x,t,\widehat{\lambda}(t))w(x,t,\widehat{\lambda}(t))\,\mathrm{d}x\\
&= -\int_0^1 w_{xx}(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\\
&\quad + \frac{\dot{\widehat{\lambda}}(t)}{2}\int_0^1 xw(x,t,\widehat{\lambda}(t))w_x(x,t,\widehat{\lambda}(t))\,\mathrm{d}x\\
&\quad + (\lambda-\widehat{\lambda}(t))\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\\
&= -\int_0^1 w_{xx}(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\dot{\widehat{\lambda}}(t)}{4}\left(w(1,t,\widehat{\lambda}(t))^2 - \int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\right)\\
&+ (\lambda - \widehat{\lambda}(t))\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\\
&= -\int_0^1 w_{xx}(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x - \frac{\dot{\widehat{\lambda}}(t)}{4}\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\\
&+ (\lambda - \widehat{\lambda}(t))\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\\
&\le (\lambda - \widehat{\lambda}(t))\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x. \qquad (30)
\end{aligned}
$$

Denote

$$
\widetilde{\lambda}(t) = \widehat{\lambda}(t) - \lambda, \qquad (31)
$$

Integrating the last inequality of (30), we have

$$
\begin{aligned}
&\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\\
&\le \int_0^1 w_x(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x\\
&+ 2\sup_{0\le\tau\le t}\widetilde{\lambda}(\tau)\int_0^t\int_0^1 w_x(x,\tau,\widehat{\lambda}(\tau))^2\,\mathrm{d}x\mathrm{d}\tau. \qquad (32)
\end{aligned}
$$

Using (29), it holds

$$
\widetilde{\lambda}(t) \le (e^{\int_0^1 w(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x + \widetilde{\lambda}(0)^2} - 1)^{1/2}, \qquad (33)
$$

for all $t \ge 0$. Integrating (28), we obtain

$$
\begin{aligned}
&\int_0^t e^{\widetilde{\lambda}(\tau)^2}\int_0^1 w_x(x,\tau,\widehat{\lambda}(\tau))^2\,\mathrm{d}x\mathrm{d}\tau\\
&\le \frac{3}{2(3-\sqrt{3})}\left(1 + \int_0^1 w(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x\right)e^{\widetilde{\lambda}(0)^2}. \qquad (34)
\end{aligned}
$$

Substituting (33), (34) into (32), we get

$$
\begin{aligned}
&\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\\
&\le \int_0^1 w_x(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x\\
&+ \frac{3}{(3-\sqrt{3})}(e^{\int_0^1 w(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x + \widetilde{\lambda}(0)^2} - 1)^{1/2}
\end{aligned}
$$

$$\times\left(1+\int_0^1 w(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x\right)e^{\widetilde{\lambda}(0)^2}. \tag{35}$$

Using the fact that $w(0,t,\widehat{\lambda}(t))=0$, by Agmon's and Poincare's inequalities, we get

$$\max_{x\in[0,1]} w(x,t,\widehat{\lambda}(t))^2 \le 4\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x, \tag{36}$$

thus $w(x,t,\widehat{\lambda}(t))$ is uniformly bounded for $t\ge 0$, $0\le x\le 1$, and we get (24). Next, we prove $\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\to 0$, as $t\to\infty$. By (9)–(11), and (22), (23), and using Lemma 2.1, noting that $0\le \dot{\widehat{\lambda}}(t)<1$, we have

$$\begin{aligned}
&\left|\frac{1}{2}\frac{\mathrm{d}(\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x)}{\mathrm{d}t}\right|\\
&=\frac{1}{2}\left|2\int_0^1 w(x,t,\widehat{\lambda}(t))\frac{\mathrm{d}w(x,t,\widehat{\lambda}(t))}{\mathrm{d}t}\,\mathrm{d}x\right|\\
&=\left|\int_0^1 w(x,t,\widehat{\lambda}(t))\left(w_{xx}(x,t,\widehat{\lambda}(t))+\dot{\widehat{\lambda}}(t)\int_0^x \frac{y}{2}w(y,t,\widehat{\lambda}(t))\,\mathrm{d}y\right.\right.\\
&\left.\left.+(\lambda-\widehat{\lambda}(t))w(x,t,\widehat{\lambda}(t)\right)dx\right|\\
&\le\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x+\left(\frac{1}{4\sqrt{3}}+|\widetilde{\lambda}(t)|\right)\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x.
\end{aligned} \tag{37}$$

Using (29), (35), (37), we know that $\left|\frac{\mathrm{d}(\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x)}{\mathrm{d}t}\right|$ is bounded. Thus $\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x$ is uniformly continuous. Using (34), it is easy to know

$$\begin{aligned}
&\int_0^t\int_0^1 w_x(x,\tau,\widehat{\lambda}(\tau))^2\,\mathrm{d}x\mathrm{d}\tau\\
&\le\frac{3}{2(3-\sqrt{3})}\left(1+\int_0^1 w(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x\right)e^{\widetilde{\lambda}(0)^2},
\end{aligned} \tag{38}$$

for any $t\ge 0$, that is, $\int_0^1 w_x(x,\tau,\widehat{\lambda}(\tau))^2\,\mathrm{d}x$ is integrable in time over the infinite time interval. By Barbarlat's lemma it follows that $\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\to 0$, as $t\to\infty$. Note $w(0,t,\widehat{\lambda}(t))=0$, using Agmon's inequality, it holds

$$w(x,t,\widehat{\lambda}(t))^2\le 2\left(\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{39}$$

Since $\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x$ is bounded from (35), and $\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\to 0$ as $t\to\infty$, then $w(x,t,\widehat{\lambda}(t))\to 0$ as $t\to\infty$, in view of (23), it yields $\dot{\widehat{\lambda}}(t)\to 0$ as $t\to\infty$, then $\widehat{\lambda}(t)\to\lambda_\infty$, as $t\to\infty$ for some λ_∞, which completes the proof. □

Remark 2. One can see that no attempt is made to force parameter convergence, but the estimation transient, the parameter is penalized simultaneously with the state and control under the control law (22) and parameter update law (23).

Remark 3. The well-posedness of the closed-loop system (9)–(11), together with control law (22) and parameter update law (23) can be followed from [14].

In what follows, we prove stability of the closed-loop system (1)–(3) under control law (22) with parameter update law (23).

Theorem 3.2. Consider system (1)–(3), with an adaptive boundary controller (22) and parameter update law (23), then for any initial condition $u_0(x) \in H_2(0,1)$ (where $u_0(x) = u(x,0)$) compatible with boundary conditions, and any $\widehat{\lambda}(0) \in R$, it holds

$$\begin{aligned}
& u(x,t)^2 \\
& \leq 80(e^{\int_0^1 w(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x + (\widehat{\lambda}(0)-\lambda)^2} + \lambda^2) \\
& \times \int_0^1 w_x(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x \\
& + \frac{240 e^{(\widehat{\lambda}(0)-\lambda)^2}}{3-\sqrt{3}} (e^{\int_0^1 w(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x + (\widehat{\lambda}(0)-\lambda)^2} + \lambda^2)^{3/2} \\
& \times \left(1 + \int_0^1 w(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x\right),
\end{aligned} \tag{40}$$

for all $t \geq 0$, $0 \leq x \leq 1$. Furthermore, $\lim_{t\to\infty} u(x,t) = 0$ uniformly in $x \in [0,1]$, and $\widehat{\lambda}(t) \to \lambda_\infty$ as $t \to \infty$, for some λ_∞.

Proof. From (16), (21), we know that functions $k(x,y,\lambda)$ and $l(x,y,\lambda)$ are continuous and zero at $\lambda = 0$, thus there exist class $\mathcal{K}$ functions L and M such that

$$\sup_{0\leq y\leq x\leq 1} |k(x,y,\lambda)| \leq L(|\lambda|), \tag{41}$$

and

$$\sup_{0\leq y\leq x\leq 1} |l(x,y,\lambda)| \leq M(|\lambda|). \tag{42}$$

Using (17), (21), and Poincare's inequality, it holds

$$\begin{aligned}
& \int_0^1 u_x(x,t)^2\,\mathrm{d}x \\
& \leq 2\left(1 + \widehat{\lambda}(t)^2 + 4S\right) \int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x,
\end{aligned} \tag{43}$$

where

$$S = \int_0^1 \left(\int_0^1 |l_x(x,y,\widehat{\lambda}(t))|\,\mathrm{d}y \right)^2 \mathrm{d}x, \tag{44}$$

and

$$l_x(x,y,\widehat{\lambda}(t))=\widehat{\lambda}(t)xy\frac{J_2\left(\sqrt{\widehat{\lambda}(t)(x^2-y^2)}\right)}{x^2-y^2}. \tag{45}$$

Follow [14], it can be deduced

$$\int_0^1 |l_x(x,y,\widehat{\lambda}(t)|\,\mathrm{d}y \le |\widehat{\lambda}(t)|x+1, \tag{46}$$

thus

$$S \le \frac{1}{2}\widehat{\lambda}(t)^2+\frac{5}{2}. \tag{47}$$

Using (47), from (43), we have

$$\begin{aligned}
&\int_0^1 u_x(x,t)^2\,\mathrm{d}x \\
&\le 2\left(3\widehat{\lambda}(t)^2+10\right)\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x \\
&\le 20\left(\widetilde{\lambda}(t)^2+\lambda^2+1\right)\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x.
\end{aligned} \tag{48}$$

With the help of (33), (35), and using Poincare's inequality and Agmon's inequality, and note $u(0,t)=0$, we obtain

$$\begin{aligned}
&u(x,t)^2 \\
&\le 4\int_0^1 u_x(x,t)^2\,\mathrm{d}x \\
&\le 80\left(\widetilde{\lambda}(t)^2+\lambda^2+1\right)\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x \\
&\le 80(e^{\int_0^1 w(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x+(\widehat{\lambda}(0)-\lambda)^2}+\lambda^2) \\
&\times\int_0^1 w_x(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x \\
&+\frac{240}{3-\sqrt{3}}(e^{\int_0^1 w(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x+(\widehat{\lambda}(0)-\lambda)^2}+\lambda^2)^{3/2} \\
&\times\left(1+\int_0^1 w(x,0,\widehat{\lambda}(0))^2\,\mathrm{d}x\right)e^{(\widehat{\lambda}(0)-\lambda)^2},
\end{aligned} \tag{49}$$

for all $t\ge 0$, $0\le x\le 1$. Thus $u(x,t)$ is uniform bounded for all $t\ge 0$, $0\le x\le 1$. From (17) and (42), it holds

$$\int_0^1 u(x,t)^2\,\mathrm{d}x \le 2\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x$$

$$
\begin{aligned}
&+2\int_0^1\left(\int_0^x l(x,y,\widehat{\lambda}(t))w(y,t,\widehat{\lambda}(t))\,\mathrm{d}y\right)^2 dx\\
&\le 2\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\\
&+2\int_0^1\left(\int_0^x l(x,y,\widehat{\lambda}(t))^2\,\mathrm{d}y\int_0^x w(y,t,\widehat{\lambda}(t))^2\,\mathrm{d}y\right)\mathrm{d}x\\
&\le 2(1+(M(\widehat{\lambda}(t)))^2\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x. \qquad (50)
\end{aligned}
$$

By (33), it is known that $\widehat{\lambda}(t)$ is finite, so $M(\widehat{\lambda}(t))$ is finite. From Lemma 3.1, we know $\int_0^1 w(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x \to 0$, as $t\to\infty$. Thus $\int_0^1 u(x,t)^2\,\mathrm{d}x \to 0$, as $t\to\infty$. Note that $u(0,t)=0$, so it holds

$$
u(x,t)^2 \le 2\left(\int_0^1 u(x,t)^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^1 u_x(x,t)^2\,\mathrm{d}x\right)^{\frac{1}{2}}, \qquad (51)
$$

where $\int_0^1 u_x(x,t)^2\,\mathrm{d}x$ is bounded from (48), (35). Thus $u(x,t)\to 0$, as $t\to\infty$. This completes the proof. □

Remark 4. The state $u(x,t)$ of the closed-loop system (1)–(3) with adaptive boundary control (22) and parameter update law (23) uniformly converges to zero in $x\in[0,1]$, and $\widehat{\lambda}(t)\to\lambda_\infty$ as $t\to\infty$, for some λ_∞.

4. ADAPTIVE INVERSE OPTIMAL BOUNDARY CONTROL

Theorem 4.1. Consider system (1)–(3), the control law (22) with the parameter update law (23) minimizes the cost functional

$$
J=\int_0^\infty S(t,\widehat{\lambda}(t))\,\mathrm{d}t, \qquad (52)
$$

where

$$
\begin{aligned}
&S(t,\widehat{\lambda}(t))\\
&=-2e^{(\widehat{\lambda}(t)-\lambda)^2}w(1,t,\widehat{\lambda}(t))w_x(1,t,\widehat{\lambda}(t))\\
&+e^{(\widehat{\lambda}(t)-\lambda)^2}w(1,t,\widehat{\lambda}(t))^2\\
&+2e^{(\widehat{\lambda}(t)-\lambda)^2}\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\\
&-e^{(\widehat{\lambda}(t)-\lambda)^2}\dot{\widehat{\lambda}}(t)\int_0^1 w(x,t,\widehat{\lambda}(t))\int_0^x yw(y,t,\widehat{\lambda}(t))\,\mathrm{d}y\mathrm{d}x. \qquad (53)
\end{aligned}
$$

Proof. From (53), with the help of (23), (27), it holds

$$
S(t,\widehat{\lambda}(t))
$$

$$\begin{aligned}
&\geq e^{(\widehat{\lambda}(t)-\lambda)^2}(w(1,t,\widehat{\lambda}(t))-w_x(1,t,\widehat{\lambda}(t)))^2\\
&-e^{(\widehat{\lambda}(t)-\lambda)^2}w_x(1,t,\widehat{\lambda}(t))^2\\
&+2e^{(\widehat{\lambda}(t)-\lambda)^2}\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\\
&-e^{(\widehat{\lambda}(t)-\lambda)^2}\frac{2\sqrt{3}\|w_x(t)\|^2}{3\pi^2}\\
&\geq e^{(\widehat{\lambda}(t)-\lambda)^2}\left(1-\frac{2\sqrt{3}}{3\pi^2}\right)\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x.\\
&\geq e^{(\widehat{\lambda}(t)-\lambda)^2}\frac{\pi^2}{4}\left(1-\frac{2\sqrt{3}}{3\pi^2}\right)\\
&\times\frac{\int_0^1 u(x,t)^2\,\mathrm{d}x}{2(1+\sup\limits_{0\leq y\leq x\leq 1}|l(x,y,\widehat{\lambda}(t))|^2)}.
\end{aligned} \tag{54}$$

Remark 5. The last inequality is achieved through the use of Poincare's inequality and inverse backstepping transformation (17).

Let $V(t)$ be given by (25). Calculate the derivative of $V(t)$ along the trajectory of the closed-loop system (9)–(11) together with (22), it holds (26). Note that the parameter update law is given by (23), from (26), we deduce that

$$\begin{aligned}
&\dot{V}(t)\\
&=2e^{(\widehat{\lambda}(t)-\lambda)^2}w(1,t,\widehat{\lambda}(t))w_x(1,t,\widehat{\lambda}(t))\\
&-2e^{(\widehat{\lambda}(t)-\lambda)^2}\int_0^1 w_x(x,t,\widehat{\lambda}(t))^2\,\mathrm{d}x\\
&+e^{(\widehat{\lambda}(t)-\lambda)^2}\dot{\widehat{\lambda}}(t)\int_0^1 w(x,t,\widehat{\lambda}(t))\int_0^x yw(y,t,\widehat{\lambda}(t))\,\mathrm{d}y\mathrm{d}x.
\end{aligned} \tag{55}$$

From (53), (55), it is easy to know

$$S(t,\widehat{\lambda}(t))=-\dot{V}(t)+e^{(\widehat{\lambda}(t)-\lambda)^2}w(1,t,\widehat{\lambda}(t))^2. \tag{56}$$

By (52), we get

$$\begin{aligned}
J&=\int_0^\infty S(t,\widehat{\lambda}(t))\,\mathrm{d}t\\
&=V(0)-V(\infty)+\int_0^\infty e^{(\widehat{\lambda}(t)-\lambda)^2}w(1,t,\widehat{\lambda}(t))^2\,\mathrm{d}t.
\end{aligned} \tag{57}$$

Using Lemma 3.1, the control law (22) with the the parameter update law (23) globally stabilizes system (9)–(11), so $V(\infty)=0$. From (57), it is clear that $w(1,t,\widehat{\lambda}(t))=0$, that is, the control law (22) ensures that the cost function J in (52) is equal to $V(0)$. Thus the minimum of (52) is

$$V(0) = \left(1 + \int_0^1 w(x, 0, \widehat{\lambda}(0))^2 \, \mathrm{d}x\right) e^{(\widehat{\lambda}(0) - \lambda)^2} - 1, \tag{58}$$

which completes the proof. □

5. SIMULATION

In this section, we show that control law (22) with parameter update law (23) globally stabilizes system (1)–(3), and minimizes functional (52).

Example 1. Consider the reaction-diffusion PDE system

$$u_t(x, t) = u_{xx}(x, t) + \lambda u(x, t), \tag{59}$$

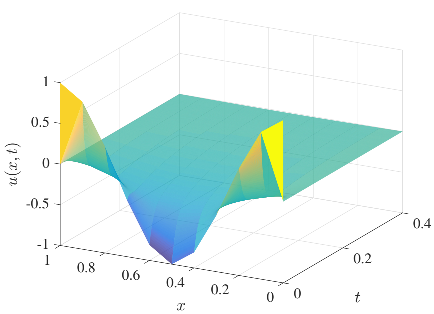
$$u(0, t) = 0, \tag{60}$$

$$u(1, t) = U(t), \tag{61}$$

where λ is an unknown real coefficient.

Using Theorem 4.1, control law is designed as (22), the parameter update law is (23). We conducted two simulation studies, one with $\widehat{\lambda}(0) = 0.1$, $u_0(x) = \cos(2\pi x)$; the other is $\widehat{\lambda}(0) = 0.1$, $u_0(x) = rand(1)$ ($rand(1)$ represents a random number with a value within the interval of $(0, 1)$). Responses of the PDE state $u(x, t)$ together with the state norm $\|u(\cdot, t)\|$ under control law (22) with the parameter update law (23) are in Figure 1, and Figure 3, respectively, and the control law (22) and parameter update law (23) are given in Figure 2, and Figure 4, respectively.

One can see that control law (22) with parameter update law (23) is such that the closed-loop system converges to zero quick, and the parameter update law (23) tends to zero. In addition, control law (22) with parameter update law (23) exhibits robustness to initial values of random disturbances.



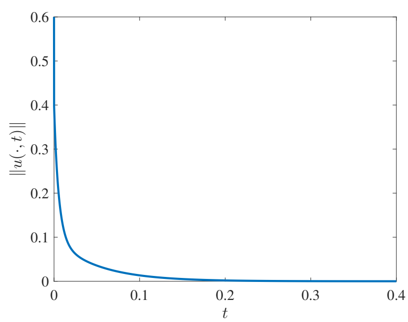


Fig. 1. State $u(x, t)$ and norm $\|u(\cdot, t)\|$ under control law (22), parameter update law (23) with $\widehat{\lambda}(0) = 0.1$, $u_0(x) = \cos(2\pi x)$.

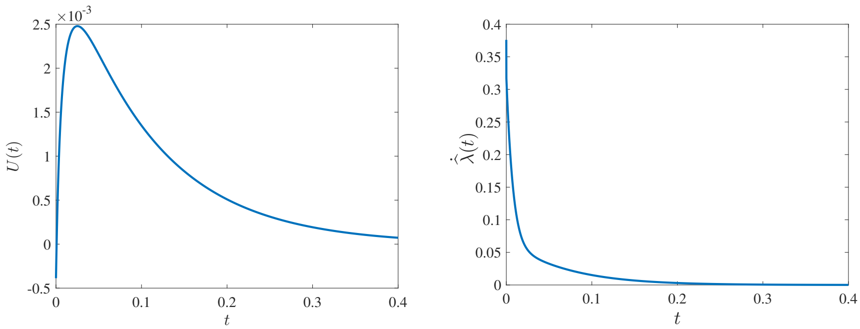


Fig. 2. Control law (22), parameter update law (23) with $\widehat{\lambda}(0) = 0.1$, $u_0(x) = \cos(2\pi x)$.

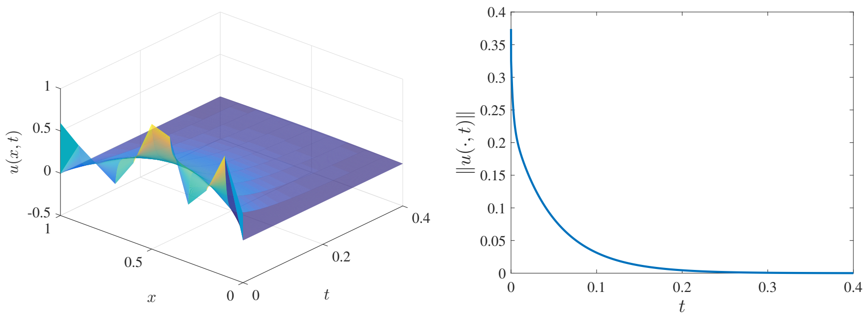


Fig. 3. State $u(x,t)$ and norm $\|u(\cdot,t)\|$ under control law (22), parameter update law (23) with $\widehat{\lambda}(0) = 0.1$, $u_0(x) = rand(1)$.

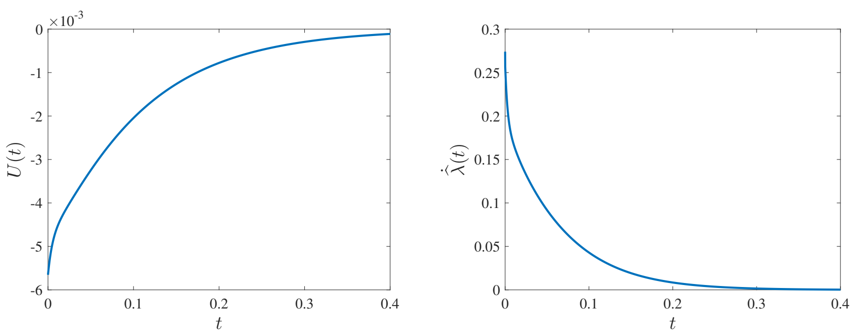


Fig. 4. Control law (22), parameter update law (23) with $\widehat{\lambda}(0) = 0.1$, $u_0(x) = rand(1)$.

6. CONCLUSION

We consider adaptive inverse optimal control for reaction-diffusion PDE system with unknown coefficient. First, an adaptive boundary control with parameter update law has been designed. We don't try to force parameter convergence, but the estimated transients are penalized simultaneously with the state and control. Next, it has been proven that the closed-loop system is globally asymptotically stable through a non quadratic Lyapunov function. Further, it has been shown that adaptive boundary control with parameter update law is optimal for a meaningful functional. Finally, the effectiveness of the proposed control design is demonstrated through an example.

(Received October 30, 2024)

REFERENCES

[1] X. Cai, N. Bekiaris-Liberis, and M. Krstic: Input-to-state stability and inverse optimality of linear time-varying-delay predictor feedbacks. IEEE Trans. Automat. Control *63* (2018), 233–240. DOI:10.1109/TAC.2017.2722104

[2] X. Cai, N. Bekiaris-Liberis, and M. Krstic: Input-to-state stability and inverse optimality of predictor feedback for multi-input linear systems. Automatica *103* (2019), 549–557. DOI:10.1016/j.automatica.2019.02.038

[3] X. Cai, L. Liao, J. Zhang, and W. Zhang: Observer design for a class of nonlinear system in cascade with counter-conveting transport dynamics. Kybernetika *52* (2016), 76–88. DOI:10.14736/kyb-2016-1-0076

[4] X. Cai, Y. Lin, C. Lin, and L. Liu: Inverse optimality of adaptive control for Korteweg–de Vries–Burgers equation. Int. J. Dynamics Control *12* (2024), 486–493. DOI:10.1007/s40435-023-01195-5

[5] X. Cai, C. Lin, L. Liu, and X. Zhan: Inverse optimal control for strict-feedforward nonlinear systems with input delays. Int. J. Robust Nonlinear Control *28* (2018), 2976–2995. DOI:10.1002/rnc.4062

[6] X. Cai, Y. Lin, J. Zhang, and C. Lin: Predictor control for wave PDE/nonlinear ODE cascaded system with boundary value-dependent propagation speed. Kybernetika *58* (2022), 400–425. DOI:10.14736/kyb-2022-3-0400

[7] X. Cai, J. Wu, X. Zhan, and X. Zhang: Inverse optimal control for linearizable nonlinear systems with input delays. Kybernetika *55* (2019), 727–739. DOI:10.14736/kyb-2019-4-0727

[8] K. Ding and Q. Zhu: Intermittent static output feedback control for stochastic delayed-switched positive systems with only partially measurable information. IEEE Trans. Automat. Control *68* (2023), 8150–8157. DOI:10.1109/TAC.2023.3293012

[9] A. Freeman and P. V. Kokotovic: Inverse optimality in robust stabilization. SIAM J. Control Optim. *34* (1996), 1365–1391. DOI:10.1137/S0363012993258732

[10] M. Krstic: On global stabilization of Burgers' equation by boundary control. Systems Control Lett. *37* (1999), 123–141. DOI:10.1016/S0167-6911(99)00013-4

[11] M. Krstic: Optimal adaptive control-contradiction in terms or a matter of choosing the right cost functional? IEEE Trans. Automat. Control *53*(2008), 1942–1947. DOI:10.1109/TAC.2008.929464

[12] M. Krstic: Control of an unstable reaction-diffusion PDE with long input delay. System Control Lett. *58* (2009), 773–782. DOI:10.1016/j.sysconle.2009.08.006

[13] M. Krstic and Z. Li: Inverse optimal design of input-to-state stabilizing nonlinear controllers. IEEE Tran. Automat. Control *43* (1998), 336–350. DOI:10.1109/9.661589

[14] M. Krstic and A. Smyshlyaev: Adaptive boundary control for unstable parabolic PDEs-part I: Lyapunov design. IEEE Trans. Automat. Control *53* (2008), 1575–1591. DOI:10.1109/TAC.2008.927798

[15] H. Lhachemi and C. Prieur: Output feedback stabilization of reaction-diffusion PDEs with a non-collocated boundary condition. Systems Control Lett. *164* (2022), 105238. DOI:10.1016/j.sysconle.2022.105238

[16] K. Lu, Z. Liu, H. Yu, C. L. Philip Chen, and Y. Zhang: Adaptive fuzzy inverse optimal fixed-time control of uncertain nonlinear systems. IEEE Trans. Fuzzy Systems *30* (2022), 3857–3868. DOI:10.1109/TFUZZ.2021.3132151

[17] C. Prieur and E. Trelat: Feedback stabilization of a 1-D linear reaction-diffusion equation with delay boundary control. IEEE Trans. Automat. Control *64* (2019), 1415–1425. DOI:10.1109/TAC.2018.2849560

[18] A. Smyshlyaev and M. Krstic: Closed form boundary state feedbacks for a class of paratial integro-differential equations. IEEE Trans. Automat. Control *49* (2004), 2185–2202. DOI:10.1109/TAC.2004.838495

[19] E. D. Sontag: A 'universal' construction of Artsteins theorem on nonlinear stabilization. SIAM J. Control Optim. *13* (1989), 117–123. DOI:10.1016/0167-6911(89)90028-5

[20] J. Tang, J. Wang and W. Kang: Boundary feedback stabilization of an unstable cascaded heat-heat system with different reaction coefficients. Systems Control Lett. *183* (2024), 105684. DOI:10.1016/j.sysconle.2023.105684

[21] J. Wang and M. Krstic: Output feedback boundary control of a heat PDE sandwiched between two ODEs. IEEE Trans. Automat. Control *64* (2019), 4653–4660. DOI:10.1109/TAC.2019.2901704

[22] J. S. Wettlaufer: Heat flux at the ice-ocean interface. J. Geophysical Res.: Oceans *96*(1991), 7215–7236. DOI:10.1029/90JC00081

[23] W. Xue, P. Kolaric, J. Fan, B. Lian, T. Chai, and F. L. Lewis: Inverse reinforcement learning in tracking control based on inverse optimal control. IEEE Trans. Cybernetics *52* (2022), 10570–10581. DOI:10.1109/TCYB.2021.3062856

Xiushan Cai, School of Artificial Intelligence, Xiamen Institute of Technology, Xiamen, 361021, P. R. China, and College of Electrical Engineering and Automation, Hubei Normal University, Huangshi, 435002, China, and College of Physics and Electronic Information Engineering, Zhejiang Normal University, Jinhua, 321004. P. R. China.

e-mail: xiushancai@163.com

Yue Li, Department of Computer Science, Purdue University, West Lafayette, IN 47907. USA.

e-mail: lyfttm@gmail.com

Ping Wang, School of Artificial Intelligence, Xiamen Institute of Technology, Xiamen, 361021. P. R. China.
e-mail: wangping@xit.edu.cn

Yuhang Lin, Meta Platforms, Menlo Park, CA 94025. USA.
e-mail: dryuhanglin@gmail.com

Le Zhang, School of Artificial Intelligence, Xiamen Institute of Technology, Xiamen, 361021. P. R. China.
e-mail: zhangle@xit.edu.cn

Leipo Liu, College of Electric and Information Engineering, Henan University of Science and Technology, Luoyang, 471000. P. R. China.
e-mail: liuleipo123@163.com