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LMI-BASED NONLINEAR OBSERVER DESIGN FOR A CLASS OF NONLINEAR SYSTEMS MODELED WITH DIFFERENTIAL ALGEBRAIC EQUATIONS

Miguel Bernal, Antonio Sala, and Antonio González

This work presents a novel methodology to design nonlinear observers for a class of systems modeled as differential algebraic equations. The proposal is based on writing both the system and the observer as nonlinear descriptor redundancy representations subject to algebraic restrictions; then the nonlinear observation error system is written in an explicit incremental form via suitable factorization techniques. A redundant Lyapunov function is then employed to guarantee asymptotic stability of the estimation error; linearity of the Lyapunov function and its time derivative with respect to the observer gains and Lyapunov function terms, allows gridding or convex treatment of expressions via linear matrix inequalities. Physical examples are presented to illustrate the proposal effectiveness against former methodologies.

Keywords: descriptor redundancy, differential algebraic equations, linear matrix inequality, nonlinear observer

Classification: 93B53, 93B50,93C10,93C15,93D05

1. INTRODUCTION

Differential algebraic equations (DAEs) are, broadly speaking, sets of differential and non-differential equations that are assumed to operate simultaneously [24]. They arise in a variety of engineering fields, e. g., network problems [25], constrained mass-point systems [23], constrained rigid-body systems [17], singular perturbations [8], discretization of partial differential equations modeling transport reaction systems [6], and transient analysis [21].

In general, DAEs may have indistinguishable differential and non-differential parts as they might be implicitly defined. However, it is understood that for a DAE to be well-posed, it must be transformable into a set of ordinary differential equations (ODEs) restricted to a manifold.

An effective and systematic way to perform this transformation is the Pantelides algorithm [20], whose number of steps is known as differential index. Index reduction has important numerical implications as it suggests that, ideally, a DAE can be simulated as an ODE under consistent initialization by any robust algorithm preventing the equations from drifting off the manifold, i. e., preventing them from failing to meet the

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algebraic constraints. The MATLAB DAE Toolbox, used in this work, provides routines to implement these steps [18].

A suitable choice for writing a DAE model due to its resemblance with state-space descriptions is that of descriptor models $\tilde{E}(\chi)\dot{\chi} = \tilde{F}(\chi)$ [9]. Quasi-linear DAEs are descriptors where $\tilde{E}(\chi)$ has constant rank (not necessarily full) [24], which means that the states are restricted to manifolds of invariant dimension. Most DAE models in engineering applications belong to this class, e. g., parallel robots and certain types of electrical circuits; hence, this note focuses on this class.

Descriptor models with full-rank $\tilde{E}(\chi)$ have been extensively studied by means of the direct Lyapunov method [15] and the sector nonlinearity approach [19], also called linear-parameter-varying (LPV) embedding, quasi-LPV modeling, or convex treatment of expressions [5]; sufficient conditions in the form of linear matrix inequalities (LMIs) [7] have been found for stability analysis [32], controller synthesis [34] and observer design [13]. Despite the fact that in this case $\tilde{E}(\chi)$ can be inverted as to directly obtain an ODE representation, it has been proved that avoiding such inversion significantly increases the feasibility of a variety of problems. Studies of the same sort for rank-deficient $\tilde{E}(\chi)$, on the other hand, are scarce, as a variety of singular phenomena may arise, e. g., inconsistency and impasse points [3]; quasi-linear DAEs considered in this work are rank-deficient but do not exhibit these phenomena.

For observer design, a common strategy is to rewrite the nonlinear system in a convex form, then propose an observer with the same structure, then deal with the resulting difficulties of handling an observation error system required to fit the convex form while splitting measurable from unmeasurable premises [16]. A more convenient strategy was adopted in [1] by first writing the nonlinear error model with the methodology in [22] to factorize error signals, then performing its convex rewriting to obtain LMI conditions; this enabled handling descriptors with constant but not full rank $\tilde{E}(\chi)$ resulting from modeling closed kinematic chains [2].

Contribution: Sufficient LMI conditions for nonlinear observer design of quasi-linear DAEs are provided; in contrast with [13], they can handle descriptors with algebraic restrictions; in comparison to [1], they are not limited to constrained rigid-body systems. Suitable factorization of the error signals allows taking advantage of available signals for gain scheduling while treating the rest of components in a robust way; this is a finer-grained description of nonlinearities than, say, assuming Lipschitz constants as in [4].

Methodology: Sufficient conditions for observer design are obtained from an explicit factorization in differences of the form $f(\hat{x}) - f(x)$ and the direct Lyapunov method. LMI tests based on convex rewriting of error systems are then deduced.

Organization: Section 2 is concerned with formulating the problem statement, which implies mathematical descriptions of the class of DAE systems under consideration and the corresponding observer proposal. Section 3 develops the main contribution of this paper, namely, a novel factorization technique to obtain a suitable observer DAE error system and sufficient LMI conditions to obtain nonlinear observer gains from a gridding approach or convex modeling of nonlinearities. Section 4 applies the proposed methodology to examples that former methodologies are unable to tackle. Finally, Section 5 discusses conclusions and future work.

2. PROBLEM STATEMENT

Consider a system of DAEs in the following descriptor form

$$\begin{aligned} &E(\chi_d, \chi_a)\dot{\chi}_d(t) = f(\chi_d, \chi_a, u), \\ &g(\chi_d, \chi_a, u) = 0, \\ &y(t) = h(\chi_d, \chi_a, u), \end{aligned} \tag{1}$$

where the first equation represents the system dynamics with $\chi_d \in \mathbb{R}^n$ as the vector of *differential state variables*, $u \in \mathbb{R}^m$ the input vector, and $\chi_a \in \mathbb{R}^q$ the vector of *algebraic variables*; $y \in \mathbb{R}^p$ is the output vector; $f : \mathbb{R}^{n \times q \times m} \mapsto \mathbb{R}^n$, $g : \mathbb{R}^{n \times q \times m} \mapsto \mathbb{R}^q$, $h : \mathbb{R}^{n \times q \times m} \mapsto \mathbb{R}^p$, and $E : \mathbb{R}^{n \times q} \mapsto \mathbb{R}^{n \times n}$ are $\mathscr{C}^1$ functions of their arguments.

Assumption 1. Let $\Omega_u \subset \mathbb{R}^m$ and $\Omega_\chi \subset \mathbb{R}^{n+q}$ be user-defined compact regions of interest such that $u(t) \in \Omega_u$ and the corresponding trajectories $(\chi_d(t), \chi_a(t)) \in \Omega_\chi$, $\forall t \geq 0$. Also, $E(\cdot)$ and $\partial g / \partial \chi_a(\cdot)$ are full-rank for every $(\chi_d, \chi_a, u) \in \Omega_\chi \times \Omega_u$.

Under the previous assumption, if an ODE representation were wished, then χ_a can be solved from χ_d via the algebraic constraints; then, χ_d would be the true state in the ODE representation, and $\mathscr{C}^1$ continuity guarantees existence and uniqueness of the solutions, subject to consistent initialization [24]. The number of recursive steps needed to explicitly formulate a DAE as an ODE is called its *differential index*. DAE systems may be expressed in a form different to (1), with higher differential index, but index reduction can be carried out by, say, Pantelides algorithm [20]. Thus, we will directly work with the representation (1), which comprises a class of systems usually referred to as *quasi-linear DAEs*.

As customary in descriptor analysis, (1) can be put in the *descriptor redundancy form* [5, Section 6.2.2]; indeed, by defining $x_1 \equiv \chi_d$, $x_2 \equiv \dot{\chi}_d$, and $x_3 \equiv \chi_a$, $x \equiv [x_1^T \; x_2^T \; x_3^T]^T$, we can write

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ f(x_1, x_3, u) - E(x_1, x_3)x_2 \\ g(x_1, x_3, u) \end{bmatrix}, \tag{2}$$

$$y = h(x_1, x_3, u).$$

From now on, it will be assumed that $(x_1, x_2, x_3) \in \Omega_x \subset \mathbb{R}^{2n+q}$, which is derived from (1), Ω_u, and Ω_χ.

Based on this form and defining $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$, as vectors of the same dimension as x_1, x_2, and x_3, respectively, $\hat{x} \equiv [\hat{x}_1^T \; \hat{x}_2^T \; \hat{x}_3^T]^T$, $\hat{y} \equiv h(\hat{x}_1, \hat{x}_2, \hat{x}_3, u)$, the following DAE observer is proposed:

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} \hat{x}_2 \\ f(\hat{x}_1, \hat{x}_3, u) - E(\hat{x}_1, \hat{x}_3)\hat{x}_2 \\ g(\hat{x}_1, \hat{x}_2, \hat{x}_3, u) \end{bmatrix} + \begin{bmatrix} L_1(\hat{x}, y, u) \\ L_2(\hat{x}, y, u) \\ L_3(\hat{x}, y, u) \end{bmatrix} (\hat{y} - y), \tag{3}$$

where $L_1(\hat{x}, y, u) \in \mathbb{R}^{n \times p}$, $L_2(\hat{x}, y, u) \in \mathbb{R}^{n \times p}$, and $L_3(\hat{x}, y, u) \in \mathbb{R}^{q \times p}$ are nonlinear observer gains to be designed.

Let us define $e \equiv \hat{x} - x$ as the *observation error* with $e_i \equiv \hat{x}_i - x_i$, $i \in \{1, 2, 3\}$. Our goal is to find, via sufficient LMI conditions, possibly nonlinear gains $L_1(\cdot)$, $L_2(\cdot)$, and $L_3(\cdot)$, exclusively depending on measurable signals, such that $\lim_{t\to\infty} e = 0$ for some level set, provided $x(t)$ and $\hat{x}(t)$ remain within some compact regions of interest. The referred LMI conditions are found in the next section by means of quadratic Lyapunov analysis and sector nonlinearity approach.

3. MAIN RESULTS

From this point on, arguments are omitted when convenient. The nonlinear observation error system resulting from subtracting (3) to (2) is:

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} e_2 \\ f_e(x_1, x_3, \hat{x}_1, \hat{x}_3, u) - e_e(x, \hat{x}) \\ g_e(x, \hat{x}, u) \end{bmatrix} + \begin{bmatrix} L_1(\cdot) \\ L_2(\cdot) \\ L_3(\cdot) \end{bmatrix} h_e(x, \hat{x}, u), \qquad (4)$$

where differences $f_e(x_1, x_3, \hat{x}_1, \hat{x}_3, u) \equiv f(\hat{x}_1, \hat{x}_3, u) - f(x_1, x_3, u)$, $e_e(x, \hat{x}) \equiv E(\hat{x}_1, \hat{x}_3)\hat{x}_2 - E(x_1, x_3)x_2$, $g_e(x, \hat{x}, u) \equiv g(\hat{x}, u) - g(x, u)$, and $h_e(x, \hat{x}, u) \equiv h(\hat{x}, u) - h(x, u)$, require the error vector e to be factorized at their right-hand side to ease Lyapunov stability analysis for later polytopic bounds and LMIs to apply. This is done via the next result:

Lemma 3.1. Consider the vector field $\phi(\cdot) : \mathbb{R}^s \mapsto \mathbb{R}^s$, $\phi \in \mathscr{C}^1$; there exists an *explicit* mapping $\bar{\phi}(\hat{x}, e) : \mathbb{R}^s \times \mathbb{R}^s \mapsto \mathbb{R}^{s\times s}$ such that $\phi(\hat{x}) - \phi(x) = \bar{\phi}(\hat{x}, e)e$, where $e = \hat{x} - x$.

Proof. Since $\phi(\hat{x}) - \phi(x) = \phi(\hat{x}) - \phi(\hat{x} - e) \equiv \phi_e(\hat{x}, e)$, it follows that $\lim_{e\to 0} \phi_e(\hat{x}, e) = 0$. Therefore, if the error vector is split as $e = [e_1\ \bar{e}_2^T]^T$, with $e_1 \in \mathbb{R}$ and $\bar{e}_2 \in \mathbb{R}^{s-1}$, abusing notation $\phi_e(\hat{x}, e) = \phi_e(\hat{x}, e_1, \bar{e}_2)$ we have:

$$\begin{aligned} \phi_e(\hat{x}, e) &= \phi_e(\hat{x}, e_1, \bar{e}_2) - \phi_e(\hat{x}, 0, \bar{e}_2) + \phi_e(\hat{x}, 0, \bar{e}_2) - \phi_e(\hat{x}, 0) \\ &= \bar{\phi}_1(\hat{x}, e)e_1 + \phi_{\bar{e}_2}(\hat{x}, \bar{e}_2), \end{aligned}$$

where $\phi_{\bar{e}_2} : \mathbb{R}^s \times \mathbb{R}^{s-1} \mapsto \mathbb{R}^s$ is defined as $\phi_{\bar{e}_2}(\hat{x}, \bar{e}_2) \equiv \phi_e(\hat{x}, 0, \bar{e}_2) - \phi_e(\hat{x}, 0) = \phi_e(\hat{x}, 0, \bar{e}_2)$ and

$$\bar{\phi}_1(\hat{x}, e) \equiv \begin{cases} \dfrac{\phi_e(\hat{x}, e_1, \bar{e}_2) - \phi_e(\hat{x}, 0, \bar{e}_2)}{e_1}, & e_1 \neq 0 \\ \lim_{e_1\to 0} \dfrac{\phi_e(\hat{x}, e_1, \bar{e}_2) - \phi_e(\hat{x}, 0, \bar{e}_2)}{e_1}, & e_1 = 0. \end{cases}$$

All limits involved above do exist because of the $\mathscr{C}^1$ assumption. The expression $\phi_{\bar{e}_2}(\hat{x}, \bar{e}_2)$ can now be treated as just done for $\phi_e(\hat{x}, e)$ by splitting $\bar{e}_2 = [e_2\ \bar{e}_3^T]^T$, with $e_2 \in \mathbb{R}$ and $\bar{e}_3 \in \mathbb{R}^{s-2}$. If this process is recursively repeated until the last error is processed and $\bar{\phi}_s(\hat{x}, e)$ is thus defined, then

$$\phi(\hat{x}) - \phi(x) = \sum_{i=1}^{s} \bar{\phi}_i(\hat{x}, e)e_i = \bar{\phi}(\hat{x}, e)e,$$

where $\bar{\phi}(\hat{x}, e) = [\bar{\phi}_1(\hat{x}, e)\ \bar{\phi}_2(\hat{x}, e)\ \cdots\ \bar{\phi}_s(\hat{x}, e)]^T$, which concludes the proof. □

Remark 3.2. The factorization of the error signal in Lemma 3.1 is not unique: there is indeed an infinite number of choices for $\bar{\phi}(\hat{x}, e)$, as any reordering or, in general, any invertible transformation $\tilde{e} = T(e)$ would allow carrying out the exact same operations with $\tilde{e}$. The interested reader is referred to [27, 31] for more details. Note also that if $\phi(\cdot)$ fulfills a Lipschitz condition then $\phi(\hat{x}) - \phi(x)$ can be expressed as $\bar{\phi}(\hat{x}, e)e$ with $\|\bar{\phi}(\hat{x}, e)\| \leq L$ for some $L > 0$; which means Lipschitz bounding as in [4] is included in our framework.

Based on the factorization above, we can find explicit expressions $F(\hat{x}_1, \hat{x}_3, e_1, e_3, u) \in \mathbb{R}^{n\times(n+q)}$, $G(\hat{x}, e, u) \in \mathbb{R}^{q\times(2n+q)}$, $H(\hat{x}, e, u) \in \mathbb{R}^{p\times(2n+q)}$ for (4) such that $f_e(x_1, x_3, \hat{x}_1, \hat{x}_3, u) = F(\hat{x}_1, \hat{x}_3, e_1, e_3, u)[e_1^T \; e_3^T]^T$, $g_e(x, \hat{x}, u) = G(\hat{x}, e, u)e$, and $h_e(x, \hat{x}, u) = H(\hat{x}, e, u)e$; similarly,

$$\begin{aligned} &E(\hat{x}_1, \hat{x}_3)\hat{x}_2 - E(x_1, x_3)x_2 \\ &= E(\hat{x}_1, \hat{x}_3)\hat{x}_2 - E(\hat{x}_1, \hat{x}_3)x_2 + E(\hat{x}_1, \hat{x}_3)x_2 - E(x_1, x_3)x_2 \\ &= E(\hat{x}_1, \hat{x}_3)e_2 + \mathcal{E}(\hat{x}_1, \hat{x}_3, e)[e_1^T \; e_3^T]^T, \end{aligned}$$

for some $\mathcal{E}(\hat{x}_1, \hat{x}_3, e) \in \mathbb{R}^{n\times n+q}$. Note that some of the arguments of $F(\cdot)$, $G(\cdot)$, and $H(\cdot)$, are measurable or computable (estimated states, inputs) but others are not (errors are unknown); thus, further factorizations will be carried out later on, as a product of the former terms, which will be referred to as "measurable" from now on, times the latter non-available ones.

Considering partitions of adequate dimensions for matrices $F(\cdot)$, $G(\cdot)$, $H(\cdot)$, and $\mathcal{E}(\cdot)$, the nonlinear observation error dynamics (4) can thus be rewritten as:

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \left(\begin{bmatrix} 0 & I & 0 \\ A_1(\cdot) & -E(\cdot) & A_3(\cdot) \\ G_1(\cdot) & G_2(\cdot) & G_3(\cdot) \end{bmatrix} + \begin{bmatrix} L_1(\cdot) \\ L_2(\cdot) \\ L_3(\cdot) \end{bmatrix}\begin{bmatrix} H_1(\cdot) & H_2(\cdot) & H_3(\cdot) \end{bmatrix}\right)\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \tag{5}$$

where $[A_1(\cdot) \; A_3(\cdot)] \equiv F(\cdot) - \mathcal{E}(\cdot)$, $[G_1(\cdot) \; G_2(\cdot) \; G_3(\cdot)] \equiv G(\cdot)$, and $[H_1(\cdot) \; H_2(\cdot) \; H_3(\cdot)] \equiv H(\cdot)$.

Remark 3.3. The right-hand side of the above expression can be viewed as a quasi-LPV model which has matrices multiplying the error e that depend on estimated state $\hat{x}$, error e and input u; we will study it in some modeling regions $\Omega_{\hat{x}}$ and Ω_e to obtain polytopic bounds to prove robust stability w.r.t. unmeasurable signals such as e and gain-scheduled stability w.r.t. measurable ones such as $\hat{x}$, y, and u.

Consider the Lyapunov function candidate $V(e_1) = e_1^T P_1 e_1$, which can be rewritten as:

$$V(e) = e^T \bar{E}^T P(\hat{x}, y, u)e, \tag{6}$$

where $\bar{E} \equiv \text{block-diag}(I_n, 0_n, 0_q)$,

$$P(\hat{x}, y, u) \equiv \begin{bmatrix} P_1 & 0 & 0 \\ P_{21}(\hat{x}, y, u) & P_{22}(\hat{x}, y, u) & P_{23}(\hat{x}, y, u) \\ P_{31}(\hat{x}, y, u) & P_{32}(\hat{x}, y, u) & P_{33}(\hat{x}, y, u) \end{bmatrix},$$

with $P_1 \in \mathbb{R}^{n\times n}$, $P_1 = P_1^T > 0$, $P_{21}(\hat{x}, y, u) \in \mathbb{R}^{n\times n}$, $P_{22}(\hat{x}, y, u) \in \mathbb{R}^{n\times n}$, $P_{23}(\hat{x}, y, u) \in \mathbb{R}^{n\times q}$, $P_{31}(\hat{x}, y, u) \in \mathbb{R}^{q\times n}$, $P_{32}(\hat{x}, y, u) \in \mathbb{R}^{q\times n}$, and $P_{33}(\hat{x}, y, u) \in \mathbb{R}^{q\times q}$. Clearly, $\bar{E} = \bar{E}^T$, $\bar{E}^T P(\cdot) = P^T(\cdot)\bar{E} \geq 0$.

Taking the time derivative of (6) and substituting (5), yields:

$$\begin{aligned}\dot{V}(\hat{x}, e, u) =& e^T\bar{E}^T P(\cdot)\dot{e} + \dot{e}^T\bar{E}^T P(\cdot)e = e^T P^T(\cdot)\bar{E}\dot{e} + \dot{e}^T\bar{E}^T P(\cdot)e \\ =& \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^T \begin{bmatrix} P_1 & P_{21}^T(\cdot) & P_{31}^T(\cdot) \\ 0 & P_{22}^T(\cdot) & P_{32}^T(\cdot) \\ 0 & P_{23}^T(\cdot) & P_{33}^T(\cdot) \end{bmatrix} \left(\begin{bmatrix} 0 & I & 0 \\ A_1(\cdot) & -E(\cdot) & A_3(\cdot) \\ G_1(\cdot) & G_2(\cdot) & G_3(\cdot) \end{bmatrix} \right. \\ &\left. + \begin{bmatrix} L_1(\cdot) \\ L_2(\cdot) \\ L_3(\cdot) \end{bmatrix} \begin{bmatrix} H_1(\cdot) & H_2(\cdot) & H_3(\cdot) \end{bmatrix} \right) \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + (*) \end{aligned} \tag{7}$$

Consider observer gains $L_i(\hat{x}, y, u)$, $i \in \{1, 2, 3\}$, to be defined mimicking [13], i. e.:

$$\begin{bmatrix} L_1(\cdot) \\ L_2(\cdot) \\ L_3(\cdot) \end{bmatrix} = \begin{bmatrix} P_1 & 0 & 0 \\ P_{21}(\cdot) & P_{22}(\cdot) & P_{23}(\cdot) \\ P_{31}(\cdot) & P_{32}(\cdot) & P_{33}(\cdot) \end{bmatrix}^{-T} \begin{bmatrix} N_1(\hat{x}, y, u) \\ N_2(\hat{x}, y, u) \\ N_3(\hat{x}, y, u) \end{bmatrix}, \tag{8}$$

with $N_1(\cdot) \in \mathbb{R}^{n\times p}$, $N_2(\cdot) \in \mathbb{R}^{n\times p}$, and $N_3(\cdot) \in \mathbb{R}^{q\times p}$, to be determined later.

This choice makes $\dot{V}(\hat{x}, e)$ in (7) negative if

$$\begin{bmatrix} \Gamma_{11}(\cdot) & (*) & (*) \\ \Gamma_{21}(\cdot) & \Gamma_{22}(\cdot) & (*) \\ \Gamma_{31}(\cdot) & \Gamma_{32}(\cdot) & \Gamma_{33}(\cdot) \end{bmatrix} < 0, \tag{9}$$

where the expressions that follow can be obtained substituting the observer gains (8) in (7) and performing operations:

$$\begin{aligned}
\Gamma_{11}(\cdot) \equiv& P_{21}^T(\cdot)A_1(\cdot) + P_{31}^T(\cdot)G_1(\cdot) + N_1(\cdot)H_1(\cdot) + (*), \\
\Gamma_{21}(\cdot) \equiv& P_{22}^T(\cdot)A_1(\cdot) + P_{32}^T(\cdot)G_1(\cdot) + N_2(\cdot)H_1(\cdot) + P_1 - E^T(\cdot)P_{21}(\cdot) + G_2^T(\cdot)P_{31}(\cdot) + H_2^T(\cdot)N_1^T(\cdot), \\
\Gamma_{22}(\cdot) \equiv& -P_{22}^T(\cdot)E(\cdot) + P_{32}^T(\cdot)G_2(\cdot) + N_2(\cdot)H_2(\cdot) + (*), \\
\Gamma_{31}(\cdot) \equiv& P_{23}^T(\cdot)A_1(\cdot) + P_{33}^T(\cdot)G_1(\cdot) + N_3(\cdot)H_1(\cdot) + A_3^T(\cdot)P_{21}(\cdot) + G_3^T(\cdot)P_{31}(\cdot) + H_3^T(\cdot)N_1^T(\cdot), \\
\Gamma_{32}(\cdot) \equiv& -P_{23}^T(\cdot)E(\cdot) + P_{33}^T(\cdot)G_2(\cdot) + N_3(\cdot)H_2(\cdot) + A_3^T(\cdot)P_{22}(\cdot) + G_3^T(\cdot)P_{32}(\cdot) + H_3^T(\cdot)N_2^T(\cdot), \\
\Gamma_{33}(\cdot) \equiv& P_{23}^T(\cdot)A_3(\cdot) + P_{33}^T(\cdot)G_3(\cdot) + N_3(\cdot)H_3(\cdot) + (*).
\end{aligned}$$

Theorem 3.4. Let Ω_e be a compact region of interest for the error signal e such that $0 \in \Omega_e$. The origin of the nonlinear error system (5) resulting from the implementation of observer (3) on system (2) is asymptotically stable if there exists matrices $P_1 \in \mathbb{R}^{n\times n}$, $P_1 = P_1^T > 0$, $P^{21}(\hat{x}, y, u) \in \mathbb{R}^{n\times n}$, $P^{22}(\hat{x}, y, u) \in \mathbb{R}^{n\times n}$, $P^{23}(\hat{x}, y, u) \in \mathbb{R}^{n\times q}$, $P^{31}(\hat{x}, y, u) \in \mathbb{R}^{q\times n}$, $P^{32}(\hat{x}, y, u) \in \mathbb{R}^{q\times n}$, $P^{33}(\hat{x}, y, u) \in \mathbb{R}^{q\times q}$, $N^1(\hat{x}, y, u) \in \mathbb{R}^{n\times p}$, $N^2(\hat{x}, y, u) \in \mathbb{R}^{n\times p}$, and $N^3(\hat{x}, y, u) \in \mathbb{R}^{q\times p}$, such that (9) holds $\forall \hat{x} \in \Omega_{\hat{x}}$, $e \in \Omega_e$, $u \in \Omega_u$, where modeling regions Ω_e and $\Omega_{\hat{x}}$ are chosen such that $\Omega_x + \Omega_e \subset \Omega_{\hat{x}}$. In that case, the observer gains are calculated as in (8).

P r o o f. The algebraic manipulations prior to the theorem statement allow us to deduce that LMI $P_1 = P_1^T > 0$ guarantees (6) is a Lyapunov function candidate for the nonlinear DAE error system (5). In addition, since (9) holds in a vicinity of $e = 0$, it means $\dot{V} < 0$ therein. Therefore, V is a Lyapunov function establishing asymptotic stability of $e_1 = 0$. □

Remark 3.5. Asymptotic stability of $e_1 = 0$ guarantees the existence of a small enough Lyapunov level set $\mathcal{E}_c \equiv \{e \in \Omega_e : e_1^T P_1 e_1 \leq c\}$, $c > 0$, such that $e(0) \in \mathcal{E}_c \Rightarrow \lim_{t\to\infty} e_1(t) = 0$.

Remark 3.6. Note that (9) is an LMI in decision variables P_1, $P_{ij}(\cdot)$, $i \in \{2,3\}$, $j \in \{1,2,3\}$, $N_k(\cdot)$, $k \in \{1,2,3\}$, for fixed matrices $A_1(\cdot)$, $A_3(\cdot)$, $E(\cdot)$, $G_k(\cdot)$, $H_k(\cdot)$, $k \in \{1,2,3\}$. However, it cannot be posed as a finite set of LMIs unless a gridding approach $\forall \hat{x} \in \Omega_{\hat{x}}$, $e \in \Omega_e$, and $u \in \Omega_u$, is adopted (which of course may lead to wrong results if the grid is not dense enough [28]) or some polytopic bounds can be crafted, which will be discussed next.

In order to find a finite set of LMIs guaranteeing (9), define $\Omega \equiv \{(\hat{x}, e, u) \in \Omega_{\hat{x}} \times \Omega_e \times \Omega_u : \Omega_x + \Omega_e \subset \Omega_{\hat{x}}\}$. Then, if $A_1(\cdot)$, $A_3(\cdot)$, $E(\cdot)$, $G_k(\cdot)$, $H_k(\cdot)$, $k \in \{1,2,3\}$, are expressed as *multilinear* polynomial functions of *measurable* signals $z_i(\hat{x}, y)$, $i \in \{1,2,\ldots,r\}$, and possibly non-measurable terms $\zeta_j(\cdot)$, $j \in \{1,2,\ldots,\rho\}$ (which may depend on any signal in (9)), we can define the following set of functions that are positive and add up to one in Ω:

$$w_0^i(\cdot) \equiv \frac{z_i^1 - z_i(\cdot)}{z_i^1 - z_i^0},\ w_1^i(\cdot) \equiv 1 - w_0^i(\cdot),\ i \in \{1,2,\ldots,r\},$$
$$\omega_0^j(\cdot) \equiv \frac{\zeta_j^1 - \zeta_j(\cdot)}{\zeta_j^1 - \zeta_j^0},\ \omega_1^j(\cdot) \equiv 1 - \omega_0^j(\cdot),\ j \in \{1,2,\ldots,\rho\},$$

where the bounds

$$z_i^1 = \sup_{(\hat{x},e,u)\in\Omega} z_i(\cdot),\ z_i^0 = \inf_{(\hat{x},e,u)\in\Omega} z_i(\cdot),\ i \in \{1,2,\ldots,r\},$$
$$\zeta_j^1 = \sup_{(\hat{x},e,u)\in\Omega} \zeta_j(\cdot),\ \zeta_j^0 = \inf_{(\hat{x},e,u)\in\Omega} \zeta_j(\cdot),\ j \in \{1,2,\ldots,\rho\},$$

are guaranteed to exist due to Assumption 1. Hence, it is clear that each $z_i(\cdot)$ and $\zeta_j(\cdot)$ can be written as a convex sum of its minima and maxima within Ω, i. e.

$$z_i(\cdot) = w_0^i(\cdot) z_i^0 + w_1^i(\cdot) z_i^1, \quad \zeta_j(\cdot) = \omega_0^j(\cdot)\zeta_j^0 + \omega_1^j(\cdot)\zeta_j^1.$$

Straightforward generalizations to fully (not necessarily multilinear) polynomial expressions of $z_i(\cdot)$ and $\zeta_j(\cdot)$ can be made based on repeated interpolating functions [5][1].

Let $\mathbb{B} \equiv \{0,1\}$. Since $A_1(\cdot)$, $A_3(\cdot)$, $E(\cdot)$, $G_k(\cdot)$, $H_k(\cdot)$, $k \in \{1,2,3\}$, are multilinear polynomial functions of $z_i(\cdot)$ and $\zeta_j(\cdot)$, we can write

$$A_i(\cdot) = \sum_{\mathbf{i}\in\mathbb{B}^r} \sum_{\mathbf{j}\in\mathbb{B}^\rho} \mathbf{w}_\mathbf{i}(\cdot) \boldsymbol{\omega}_\mathbf{j}(\cdot) A_{\mathbf{ij}}^i,\ E(\hat{x}) = \sum_{\mathbf{i}\in\mathbb{B}^r} \mathbf{w}_\mathbf{i}(\cdot) E_\mathbf{i},$$

[1]For instance, if $z(\cdot) = \sum_{i=0}^1 w_i(\cdot) z^i$ for $z(\cdot) \in [z^0, z^1]$ with $w_0(\cdot) + w_1(\cdot) = 1$, $w_i(\cdot) \geq 0$ therein, then $z^2(\cdot) = \sum_{i=0}^1 \sum_{j=0}^1 w_i(\cdot) w_j(\cdot) z^i z^j$.

$$H_k(\cdot)=\sum_{\mathbf{i}\in\mathbb{B}^r}\sum_{\mathbf{j}\in\mathbb{B}^\rho}\mathbf{w_i}(\cdot)\boldsymbol{\omega}_\mathbf{j}(\cdot)H^k_{\mathbf{ij}},\ G_k(\cdot)=\sum_{\mathbf{i}\in\mathbb{B}^r}\sum_{\mathbf{j}\in\mathbb{B}^\rho}\mathbf{w_i}(\cdot)\boldsymbol{\omega}_\mathbf{j}(\cdot)G^k_{\mathbf{ij}},$$

for $i\in\{1,3\}$, $k\in\{1,2,3\}$, where functions $\mathbf{w_i}(\cdot)$ and $\boldsymbol{\omega}_\mathbf{j}(\cdot)$ are defined as $\mathbf{w_i}(\cdot)=w^1_{i_1}(\cdot)w^2_{i_2}(\cdot)\cdots w^r_{i_r}(\cdot)$ and $\boldsymbol{\omega}_\mathbf{j}(\cdot)=\omega^1_{j_1}(\cdot)\omega^1_{j_2}(\cdot)\cdots\omega^\rho_{j_\rho}(\cdot)$, and matrices $A^i_{\mathbf{ij}}$, $i\in\{1,3\}$, $E_\mathbf{i}$, $G^k_{\mathbf{ij}}$, and $H^k_{\mathbf{ij}}$, $k\in\{1,2,3\}$, $\mathbf{i}\in\mathbb{B}^r$, $\mathbf{j}\in\mathbb{B}^\rho$, are (abusing notation)

$$A^i_{\mathbf{ij}}=A_i(z^{i_1}_1,\ldots,z^{i_r}_r,\zeta^{j_1}_1,\ldots,\zeta^{j_\rho}_\rho),\ E_\mathbf{i}=E(z^{i_1}_1,\ldots,z^{i_r}_r)$$
$$G^k_{\mathbf{ij}}=G_k(z^{i_1}_1,\ldots,z^{i_r}_r,\zeta^{j_1}_1,\ldots,\zeta^{j_\rho}_\rho),\ H^k_{\mathbf{ij}}=H_k(z^{i_1}_1,\ldots,z^{i_r}_r,\zeta^{j_1}_1,\ldots,\zeta^{j_\rho}_\rho).$$

Now that the known expressions on the left-hand side of (9) have been split into measurable and unmeasurable signals, we can use the former to propose specific convex structures for $P_{ij}(\cdot)$, $i\in\{2,3\}$, $j\in\{1,2,3\}$, $N_k(\cdot)$, $k\in\{1,2,3\}$, i. e.:

$$P_{ij}(\hat{x},y,u)=\sum_{\mathbf{k}\in\mathbb{B}^r}\mathbf{w_k}(\cdot)P^{ij}_\mathbf{k},\ N_k(\hat{x},y,u)=\sum_{\mathbf{k}\in\mathbb{B}^r}\mathbf{w_k}(\cdot)N^k_\mathbf{k}, \tag{10}$$

where matrices $P^{ij}_\mathbf{k}$, $N^k_\mathbf{k}$, $\mathbf{k}\in\mathbb{B}^r$, will be found via LMIs to satisfy (9); once found, the observer gains $L_k(\cdot)$, $k\in\{1,2,3\}$, in (3) will be found by means of (8).

Convex rewriting of $A_1(\cdot)$, $A_3(\cdot)$, $E(\cdot)$, $G_k(\cdot)$, $H_k(\cdot)$, $P_{ij}(\cdot)$, $N_k(\cdot)$, $i\in\{2,3\}$, $j\in\{1,2,3\}$, $k\in\{1,2,3\}$, allows writing (9) as

$$\sum_{\mathbf{i}\in\mathbb{B}^r}\sum_{\mathbf{j}\in\mathbb{B}^\rho}\sum_{\mathbf{k}\in\mathbb{B}^r}\mathbf{w_i}(\cdot)\boldsymbol{\omega}_\mathbf{j}(\cdot)\mathbf{w_k}(\cdot)\begin{bmatrix}\Upsilon^{11}_{\mathbf{ijk}} & (*) & (*)\\ \Upsilon^{21}_{\mathbf{ijk}} & \Upsilon^{22}_{\mathbf{ijk}} & (*)\\ \Upsilon^{31}_{\mathbf{ijk}} & \Upsilon^{32}_{\mathbf{ijk}} & \Upsilon^{33}_{\mathbf{ijk}}\end{bmatrix}<0, \tag{11}$$

where

$$\begin{aligned}
\Upsilon^{11}_{\mathbf{ijk}}&\equiv(P^{21}_\mathbf{k})^TA^1_{\mathbf{ij}}+(P^{31}_\mathbf{k})^TG^1_{\mathbf{ij}}+N^1_\mathbf{k}H^1_{\mathbf{ij}}+(*),\\
\Upsilon^{21}_{\mathbf{ijk}}&\equiv(P^{22}_\mathbf{k})^TA^1_{\mathbf{ij}}+(P^{32}_\mathbf{k})^TG^1_{\mathbf{ij}}+N^2_\mathbf{k}H^1_{\mathbf{ij}}+P_1-E^T_\mathbf{i}P^{21}_\mathbf{k}+(G^2_{\mathbf{ij}})^TP^{31}_\mathbf{k}+(H^2_{\mathbf{ij}})^T(N^1_\mathbf{k})^T,\\
\Upsilon^{22}_{\mathbf{ijk}}&\equiv-(P^{22}_\mathbf{k})^TE_\mathbf{i}+(P^{32}_\mathbf{k})^TG^2_{\mathbf{ij}}+N^2_\mathbf{k}H^2_{\mathbf{ij}}+(*),\\
\Upsilon^{31}_{\mathbf{ijk}}&\equiv(P^{23}_\mathbf{k})^TA^1_{\mathbf{ij}}+(P^{33}_\mathbf{k})^TG^1_{\mathbf{ij}}+N^3_\mathbf{k}H^1_{\mathbf{ij}}+(A^3_{\mathbf{ij}})^TP^{21}_\mathbf{k}+(G^3_{\mathbf{ij}})^TP^{31}_\mathbf{k}+(H^3_{\mathbf{ij}})^T(N^1_\mathbf{k})^T,\\
\Upsilon^{32}_{\mathbf{ijk}}&\equiv-(P^{23}_\mathbf{k})^TE_\mathbf{i}+(P^{33}_\mathbf{k})^TG^2_{\mathbf{ij}}+N^3_\mathbf{k}H^2_{\mathbf{ij}}+(A^3_{\mathbf{ij}})^TP^{22}_\mathbf{k}+(G^3_{\mathbf{ij}})^TP^{32}_\mathbf{k}+(H^3_{\mathbf{ij}})^T(N^2_\mathbf{k})^T,\\
\Upsilon^{33}_{\mathbf{ijk}}&\equiv(P^{23}_\mathbf{k})^TA^3_{\mathbf{ij}}+(P^{33}_\mathbf{k})^TG^3_{\mathbf{ij}}+N^3_\mathbf{k}H^3_{\mathbf{ij}}+(*).
\end{aligned}$$

A variety of LMI conditions are available in the literature to guarantee (11), see [29, 30], which in turn guarantees $\dot{V}<0$, thus establishing the validity of $V(e)$ in (6) as a Lyapunov function for the nonlinear observer error system (5), i. e., establishing asymptotic stability of $e=0$ in (5), which means observation takes place. One of the possible choices for sufficient LMI conditions guaranteeing (11) is

$$\sum_{(\mathbf{i},\mathbf{j},\mathbf{k})\in\mathcal{P}(\mathbf{i}',\mathbf{j}',\mathbf{k}')}\begin{bmatrix}\Upsilon^{11}_{\mathbf{ijk}} & (*) & (*)\\ \Upsilon^{21}_{\mathbf{ijk}} & \Upsilon^{22}_{\mathbf{ijk}} & (*)\\ \Upsilon^{31}_{\mathbf{ijk}} & \Upsilon^{32}_{\mathbf{ijk}} & \Upsilon^{33}_{\mathbf{ijk}}\end{bmatrix}<0, \tag{12}$$

for all $\mathbf{i}' \in \mathbb{B}^r$, $\mathbf{k}' \in \mathbb{B}^r$, $\mathbf{j}' \in \mathbb{B}^\rho$, with $\mathcal{P}(\mathbf{i}', \mathbf{j}', \mathbf{k}')$ being the set of indices $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ whose associated products $\mathbf{w_i}(x)\boldsymbol{\omega}_\mathbf{j}(x,u)\mathbf{w_k}(x)$ are algebraically identical to $\mathbf{w_{i'}}(x)\boldsymbol{\omega}_\mathbf{j'}(x,u)\mathbf{w_{k'}}(x)$. Details on multiple-sum relaxation are omitted for brevity, see the above-cited works for details on less conservative (assymptotically exact) versions of (12) involving additional LMIs and decision variables.

Theorem 3.7. The origin of the nonlinear error system (5) resulting from the implementation of observer (3) on system (2) is asymptotically stable if there exists matrices $P_1 \in \mathbb{R}^{n\times n}$, $P_1 = P_1^T > 0$, $P_\mathbf{k}^{21} \in \mathbb{R}^{n\times n}$, $P_\mathbf{k}^{22} \in \mathbb{R}^{n\times n}$, $P_\mathbf{k}^{23} \in \mathbb{R}^{n\times q}$, $P_\mathbf{k}^{31} \in \mathbb{R}^{q\times n}$, $P_\mathbf{k}^{32} \in \mathbb{R}^{q\times n}$, $P_\mathbf{k}^{33} \in \mathbb{R}^{q\times q}$, $N_\mathbf{k}^1 \in \mathbb{R}^{n\times p}$, $N_\mathbf{k}^2 \in \mathbb{R}^{n\times p}$, and $N_\mathbf{k}^3 \in \mathbb{R}^{q\times p}$, $\mathbf{k} \in \mathbb{B}^p$, such that LMIs (12) hold for $\Upsilon_\mathbf{ijk}^{ij}$, $i, j \in \{1, 2, 3\}$, $i \geq j$, as defined above. In that case, the observer gains are calculated as in (8) and there exists a small enough Lyapunov level set $\mathcal{E}_c \equiv \{e \in \Omega_e : e_1^T P_1 e_1 \leq c\}$, $c > 0$, which is a subset of the domain of attraction of the origin.

Proof. It follows the same lines as proof of Theorem 3.4 since LMIs (12) guarantee (11) which is equivalent to (9) for $(\hat{x}, e, u) \in \Omega$. □

Remark 3.8. Numerical complexity of LMI conditions can be estimated as $\log_{10}\left(n_d^3 n_l\right)$, where n_d is the number of decision variables and n_l is the number of LMI rows [10]; therefore, numerical complexity of Theorem 3.7 can be calculated from the fact that there are $n_d = n(n+1)/2 + 2^r(2n^2 + 3nq + q^2 + 2np + qp)$ decision variables and $n_l = n + 2^{2r+p}(2n+q)$ LMI rows.

Remark 3.9. The choice of the modeling region in which the domain of attraction will be included may not be an easy task as, on the one hand, a large modeling region opens up the sector-nonlinearity bounds thus hindering the ability to find a feasible solution [33]; on the other hand, a small modeling region will approach the linearized model, easing LMI feasibility, but, in such a case, initial error conditions in which observer convergence can be proved might be difficult to be verified in practice [12]. This is an issue common to all sector-nonlinearity LPV observation and control which is, actually, dependent on the application, so we did not delve further into it. Other sources of conservatism are the non-uniqueness of decoupling (5) [26] and the choice of Lyapunov function (6) [11], among others.

4. EXAMPLES

Two examples are presented: the first one is a nonlinear RLC circuit from literature; the second one is a 2-bar kinematic chain. Simulations were carried out using routines of the Symbolic Math Toolbox of MATLAB 2024b for solving DAEs; these routines guarantee consistent initialization and avoidance of drifting of variables off the manifold induced by algebraic restrictions [14].

Example 4.1. Consider the task of observing the states of the following DAE model of a nonlinear RLC circuit, taken from [35]:

$$\dot{q} = (1/L)\phi,$$

$$\dot{\phi} = -(R/L)\phi - v_c + u,$$
$$0 = v_c - q - 0.5\sin(q^2),$$
$$y = v_c,$$

where q denotes the capacitor charge, ϕ is the flux through the inductor, v_c is the capacitor voltage, u is the voltage input, and y denotes the output. This system can be put in the descriptor redundancy form (2) by defining the differential state as $x_1 = [q\ \phi]^T$, the redundant state as $x_2 = [\dot{q}\ \dot{\phi}]^T$, and the non-differential state as $x_3 = v_c$; $\hat{x}_i$, e_i, $i \in \{1,2,3\}$ can be right away defined. This means that

$$A_1(\cdot) = \begin{bmatrix} 0 & 1/L \\ 0 & -R/L \end{bmatrix},\ A_3(\cdot) = \begin{bmatrix} 0 \\ -1 \end{bmatrix},\ H_1(\cdot) = H_2(\cdot) = 0_{1\times 2},\ H_3(\cdot) = 1,\ E(\cdot) = I_2,$$

in the error system (5), whereas $G_1(\cdot)$, $G_2(\cdot)$, and $G_3(\cdot)$ should be obtained via Lemma 3.1 in such a way that the following holds with $e_q = \hat{q} - q$ and $e_{v_c} = \hat{v}_c - v_c$:

$$[G_1(\cdot)\ G_2(\cdot)\ G_3(\cdot)][e_1^T\ e_2^T\ e_3]^T = -e_q - 0.5(\sin(\hat{q}^2) - \sin(q^2)) + e_{v_c},$$

where the latter arises from the subtraction of the algebraic restrictions in the observer and system.

Clearly, $G_2(\cdot) = 0_{1\times 2}$ and $G_3(\cdot) = 1$; last, $G_1(\cdot) = [-1 - 0.5\bar{g}(\hat{q}, e_q)\ \ 0]$, where $\bar{g}(\hat{q}, e_q)$ results from the explicit factorization detailed below:

$$\begin{aligned}
&\sin(\hat{q}^2) - \sin(q^2) = \sin(\hat{q}^2) - \sin((\hat{q} - e_q)^2) = \sin(\hat{q}^2) - \sin(\hat{q}^2 - 2\hat{q}e_q + e_q^2) \\
&= \sin(\hat{q}^2) - \sin(\hat{q}^2)\cos(e_q^2 - 2\hat{q}e_q) - \cos(\hat{q}^2)\sin(e_q^2 - 2\hat{q}e_q) \\
&= (z_1(\hat{q})\zeta_1(\hat{q}, e_q) - z_2(\hat{q})\zeta_2(\hat{q}, e_q))e_q,
\end{aligned}$$

where $z_1(\hat{q}) = \sin(\hat{q}^2)$, $z_2(\hat{q}) = \cos(\hat{q}^2)$, and

$$\zeta_1(\hat{q}, e_q) = \begin{cases} (1 - \cos(e_q^2 - 2\hat{q}e_q))/e_q, & e_q \neq 0, \\ 0, & e_q = 0, \end{cases}$$

$$\zeta_2(\hat{q}, e_q) = \begin{cases} \sin(e_q^2 - 2\hat{q}e_q)/e_q, & e_q \neq 0, \\ 1, & e_q = 0, \end{cases}$$

thus allowing to define $\bar{g}(\hat{q}, e_q) \equiv z_1(\hat{q})\zeta_1(\hat{q}, e_q) - z_2(\hat{q})\zeta_2(\hat{q}, e_q)$. Values at $e_q = 0$ have been defined by suitable limits.

The factorization above consists in two measurable terms $z_1(\hat{q})$ and $z_2(\hat{q})$, depending on the available state estimate $\hat{q}$, and two unmeasurable terms $\zeta_1(\hat{q}, e_q)$ and $\zeta_2(\hat{q}, e_q)$, further depending on the estimation error e_q. In order to employ Theorem 3.7, matrices $G^1_{\mathbf{ij}}$, $\mathbf{i} \in \mathbb{B}^2$, $\mathbf{j} \in \mathbb{B}^2$, should be found according to some bounds (matrices $A^i_{\mathbf{ij}} = A_i(\cdot)$, $E_{\mathbf{i}} = I_2$, $H^k_{\mathbf{ij}} = H_k(\cdot)$, $G^2_{\mathbf{ij}} = G_2(\cdot)$, $G^3_{\mathbf{ij}} = G_3(\cdot)$, being constant). Suppose that we are interested in considering $u(t) = 0.5 + 2\sin(3t) - 2\cos(7t)$; this means that the state belongs to $\Omega_x = \{x : 0.3 \leq q \leq 0.5\}$; if error bounds $\Omega_e = \{e : -0.4 \leq e_q \leq 0.4\}$ are considered, this yields $\Omega_{\hat{x}} = \{\hat{x} : -0.1 \leq \hat{q} \leq 0.9\}$ in order for condition $\Omega_x + \Omega_e \subset \Omega_{\hat{x}}$ to hold. Within these regions we have that $z_i \in [z_i^0, z_i^1]$, $\zeta_j \in [\zeta_j^0, \zeta_j^1]$, $i, j \in \{1, 2\}$, where ($z_1^0 = 0$, $z_1^1 = 0.7243$), ($z_2^0 = 0.6895$, $z_2^1 = 1$), ($\zeta_1^0 = -0.9071$, $\zeta_1^1 = 0.3819$(, ($\zeta_2^0 = -1.9611$, $\zeta_2^1 = 0.5943$). Therefore, $G^1_{\mathbf{ij}}$ are given by

$$G^1_{\mathbf{ij}} = \begin{bmatrix} -1 - 0.5 z_1^{i_1}\zeta_1^{j_1} - z_2^{i_2}\zeta_2^{j_1} & 0 \end{bmatrix}.$$

Theorem 3.7 is now invoked with two modifications: the simplicity of the model makes unnecessary to include the redundancy part associated with x_2, and an exponential decay rate bound of $\alpha = 0.3$ has been added. A feasible solution has been found with 4 vertex matrices for $P(\hat{x}, y, u)$ in (6) and 4 vertex matrices for each $N_i(\hat{x}, y, u)$, $i \in \{1, 2, 3\}$, in (8). For illustration purposes, P_1, $N_{\mathbf{i}}$, $\mathbf{i} \in \mathbb{B}^2$, are shown below; the remaining decision variables are omitted for brevity:

$$P_1 = \begin{bmatrix} 1.1869 & 0.4101 \\ 0.4101 & 1.9000 \end{bmatrix}, \quad \begin{matrix} N_{00} = \begin{bmatrix} -0.2932 & 0.1353 & 0 \end{bmatrix}^T, \\ N_{01} = \begin{bmatrix} -0.2909 & 0.1440 & 0 \end{bmatrix}^T, \end{matrix}$$
$$N_{10} = \begin{bmatrix} -0.2975 & 0.1271 & 0 \end{bmatrix}^T, N_{11} = \begin{bmatrix} -0.3204 & 0.0701 & 0 \end{bmatrix}^T.$$

Recall that the final nonlinear observer gains $L_i(\hat{x}, y, u)$, $i \in \{1, 2, 3\}$, are calculated with (8) once nonlinear expressions for $P(\hat{x}, y, u)$ and $N_i(\hat{x}, y, u)$, $i \in \{1, 2, 3\}$, are obtained via (10). Therefore, in this case, the final expression of the observer gains will be a nonlinear expression of measurable factors $z_1(\hat{q}) = \sin(\hat{q}^2)$ and $z_2(\hat{q}) = \cos(\hat{q}^2)$.

Figure 1 shows signals $q(t)$, $\phi(t)$, and $v_c(t)$ (bold lines) along with their estimates (dashed lines) for $t \in [0, 10]$, from initial conditions $q(0) = 0.3$, $\phi(0) = 0$, $v_c(0) = 0.3449$, $\hat{q}(0) = 0.1832$, $\hat{\phi}(0) = 5$, and $\hat{v}_c(0) = 0.2$.

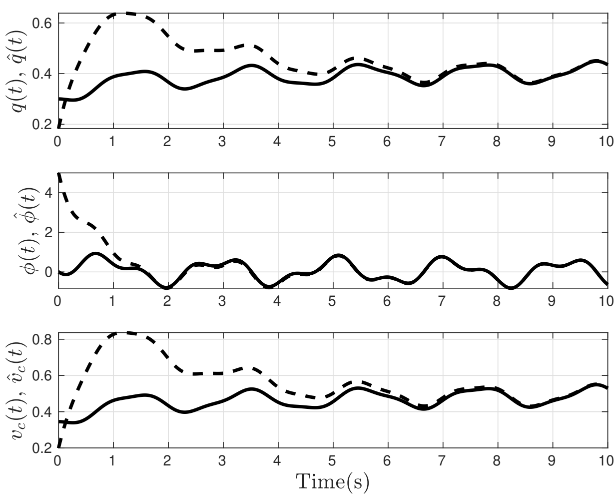

Fig. 1. Time evolution of the states and their estimates in Example 4.1.

Comparisons: The approach in [13] seems to be applicable to this example because it is aimed to descriptor models with invertible $E(x)$, something achievable by substituting v_c in $\dot{\phi}$; however, the system is required to be written in a convex form with measurable premises, which is hindered by the presence of unmeasurable q in the model. On the other hand, [4] requires using global Lipschitz constants and solving BMIs, which is out of the scope of numerically efficient convex optimization techniques in this work.

Example 4.2. Let us consider the 2-bar mechanism shown in Figure 2. The bar labeled l_1 has an actuated rotational joint at coordinates $(0, 0)$; it is linked by another rotational joint with the bar labelled l_2, which in turn is constrained at its right end by the vertical coordinate $y_r = 0.5$, being able to slide horizontally. The system is subject to torque $\tau = 0$N·m on the left and linear force $\nu = 1.5$N on the right; it is therefore a 1-degree-of-freedom system whose differential states, according to (1), may be chosen, for instance, $\chi_1 = [\theta_1\ \omega_1]^T$, where θ_1 denotes the angular position of the first bar and ω_1 its angular velocity. If these states and input $u = [\tau\ \nu]^T$ are known at any time, the rest of the system variables is completely determined, namely, $\chi_2 = [\theta_2\ \omega_2\ \alpha_1\ \alpha_2\ \lambda]^T$, where θ_i are angular positions, ω_i are angular velocities, α_i are angular accelerations, and λ is a Lagrange multiplier (constraint force).

The position, velocity, and acceleration restrictions are given, respectively, by

$$\begin{aligned}
0 &= l_1 \sin\theta_1 + l_2 \sin\theta_2 - y_r, \\
0 &= l_1\omega_1 \cos\theta_1 + l_2\omega_2 \cos\theta_2, \text{ and} \\
0 &= -l_1 \sin\theta_1\omega_1^2 - l_2 \sin\theta_2\omega_2^2 + l_1\alpha_1 \cos\theta_1 + l_2\alpha_2 \cos\theta_2.
\end{aligned}$$

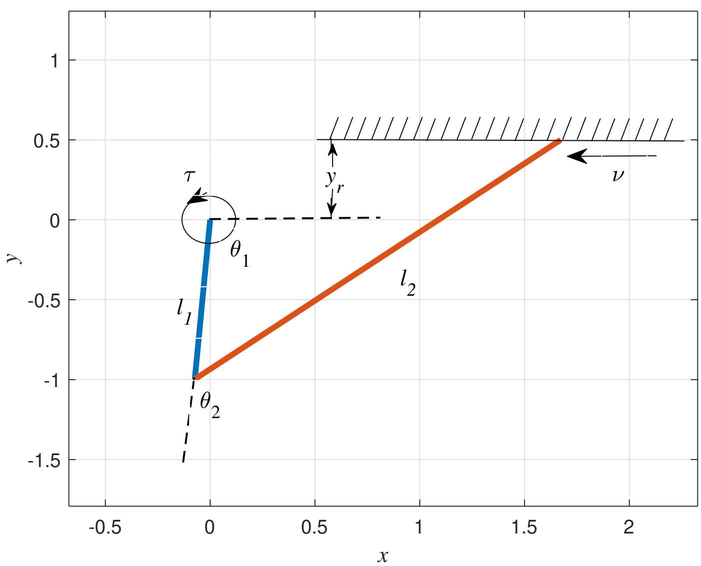

Fig. 2. Two-bar mechanism in Example 4.2.

The Euler–Lagrange dynamics equations are

$$\begin{aligned}0=&10(c_1m_1+l_1m_2)\cos\theta_1+0.4\omega_1+c_2l_1m_2\sin(\theta_1-\theta_2)\omega_2^2+(m_1c_1^2+m_2l_1^2+J_1)\alpha_1\\&+c_2l_1m_2\cos(\theta_1-\theta_2)\alpha_2-l_1\cos\theta_1\lambda+\tau-l_1\sin\theta_1\nu,\\0=&10c_2m_2\cos\theta_2-c_2l_1m_2\sin(\theta_1-\theta_2)\omega_1^2+0.4\omega_2-l_2\cos\theta_2\lambda\\&+c_2l_1m_2\cos(\theta_1-\theta_2)\alpha_1+(m_2c_2^2+J_2)\alpha_2-l_2\sin\theta_2\nu.\end{aligned}$$

For simulation purposes, the parameters given in Table 1 will be considered.

Parameter	Symbol	Value
Left-bar length	l_1	1 m
Right-bar length	l_2	2.3 m
Left-bar mass	m_1	1 kg
Right-bar mass	m_2	2.5 kg
Left-bar moment of inertia	J_1	1 kg·m^2
Right-bar moment of inertia	J_2	3 kg·m^2
Left-bar center of gravity (from the left)	c_1	0.5 m
Right-bar center of gravity (from the left)	c_2	1 m
Gravity	g	10 ms^{-2}

Tab. 1. Parameters of the 2-bar mechanism.

In order to write the model as (1), the dynamical part will consist simply in $\dot\theta_1=\omega_1$ and $\dot\omega_1=\alpha_1$. Therefore, $n=2$ and $q=5$ in (1). Based on $\nu=1.5$ and $\tau=0$, and the initial conditions desired for simulation, the region of interest Ω_x is defined with $\theta_1\in[-1.75,-1.45]$, $\omega_1\in[-0.3,0.3]$. Also, error bounds are set to $e_{\theta_1}\in[-0.1,0.1]$, and $e_{\omega_1}\in[-0.3,0.3]$, and by means of the algebraic constraints, error bounds for all variables yield suitable regions Ω_e and $\Omega_{\hat x}$.

Let $y=\theta_1$ be the only measurable signal. Positions θ_i, velocities ω_i, accelerations α_i, $i\in\{1,2\}$, and the Lagrange multiplier λ, are the signals to be estimated by the proposed observer (3), which is to be designed by means of Theorem 3.7. To this end, the error dynamics (5) must be found by subtracting the system equations to the observer ones. Observer states are $\hat\theta_i$, $\hat\omega_i$, $\hat\alpha_i$, $i\in\{1,2\}$, and $\hat\lambda$, from which the error signals $e_{\theta_i}=\hat\theta_i-\theta_i$, $e_{\omega_i}=\hat\omega_i-\omega_i$, $e_{\alpha_i}=\hat\alpha_i-\alpha_i$, $i\in\{1,2\}$, and $e_\lambda=\hat\lambda-\lambda$ can be defined. It is clear that

$$A_1(\cdot)=\begin{bmatrix}0&1\\0&0\end{bmatrix},\ A_3(\cdot)=\begin{bmatrix}0&0&0&0&0\\0&0&1&0&0\end{bmatrix},\ E=\begin{bmatrix}1&0\\0&1\end{bmatrix},$$
$$H_1(\cdot)=\begin{bmatrix}1&0\end{bmatrix},\ H_2(\cdot)=\begin{bmatrix}0&0\end{bmatrix},\ H_3(\cdot)=\begin{bmatrix}0&0&0&0&0\end{bmatrix}.$$

On the other hand, due to the complexity of the algebraic restrictions above, entries of $G_1(\cdot)\in\mathbb{R}^{5\times2}$, $G_2(\cdot)\in\mathbb{R}^{5\times2}$, and $G_3(\cdot)\in\mathbb{R}^{5\times5}$ in (5), can only be found by the repeated application of Lemma 3.1. The resulting expressions are too long to fit in this note, but consider, for illustration purposes, entry $(1,1)$ of $G_1(\cdot)$, which is $\sin\hat\theta_1(1-\cos e_{\theta_1})/e_{\theta_1}+\cos\hat\theta_1\sin e_{\theta_1}/e_{\theta_1}$, where $e_{\theta_1}=\hat\theta_1-\theta_1$.

Convex modeling of $G_1(\cdot)$, $G_2(\cdot)$, and $G_3(\cdot)$, require picking measurable signals for scheduling and leaving the rest for robust treatment; again, for illustrations purposes, consider entry $(1,1)$ of $G_1(\cdot)$, which can be written as $z_1(\hat{\theta}_1)\zeta_1(e_{\theta_1}) + z_2(\hat{\theta}_1)\zeta_2(e_{\theta_1})$, with $z_1(\hat{\theta}_1) = \sin\hat{\theta}_1$ and $z_2(\hat{\theta}_1) = \cos\hat{\theta}_1$ being measurable signals, and $\zeta_1(e_{\theta_1}) = (1 - \cos e_{\theta_1})/e_{\theta_1}$ and $\zeta_2(e_{\theta_1}) = \sin e_{\theta_1}/e_{\theta_1}$ being unmeasurable terms. The total number of measurable/unmeasurable expressions for this example is 3 and 8, respectively.

Theorem 3.7 can now be invoked. In this case, $N_2(\cdot)$ and $N_3(\cdot)$ have been chosen as zero matrices of adequate size, making the corresponding $L_2(\cdot)$ and $L_3(\cdot)$ observer gains zero too, which is equivalent to say that the observer itself behaves as a 2-bar mechanism along the observation process without breaking the algebraic constraints.

Since there are 3 measurable expressions, a total of $2^3 = 8$ different matrices $P_{\mathbf{k}}^{3i}$, $P_{\mathbf{k}}^{4ij}$, $i,j \in \{1,2\}$, and $N_{\mathbf{k}}^{k}$, $k \in \{1,2,3\}$, are obtained; their corresponding nonlinear expressions are obtained as in (10). Matrix P_1 in $P(\hat{x},y,u)$ along with some parts of the resulting $N_{\mathbf{k}}^{1}$, obtained by LMIs in Theorem 3.7, are shown below for illustration purposes:

$$P_1 = \begin{bmatrix} 0.0688 & -0.0222 \\ -0.0222 & 0.0151 \end{bmatrix}, N_{000}^1 = \begin{bmatrix} -0.5051 \\ -0.0005 \end{bmatrix},$$
$$N_{001}^1 = \begin{bmatrix} -0.4936 \\ -0.0001 \end{bmatrix}, N_{100}^1 = \begin{bmatrix} -0.4977 \\ -0.0003 \end{bmatrix}, N_{101}^1 = \begin{bmatrix} -0.5009 \\ -0.0004 \end{bmatrix}.$$

Recall that the actual nonlinear observer gain $L_1(\hat{x},y,u)$ is obtained as in (8), where $P(\hat{x},y,u)$ and $N_1(\hat{x},y,u)$ result from the convex sums in (10). Also, $L_2(\cdot)$ and $L_3(\cdot)$ are zero matrices of adequate size as a consequence of $N_2(\cdot)$ and $N_3(\cdot)$ being zero.

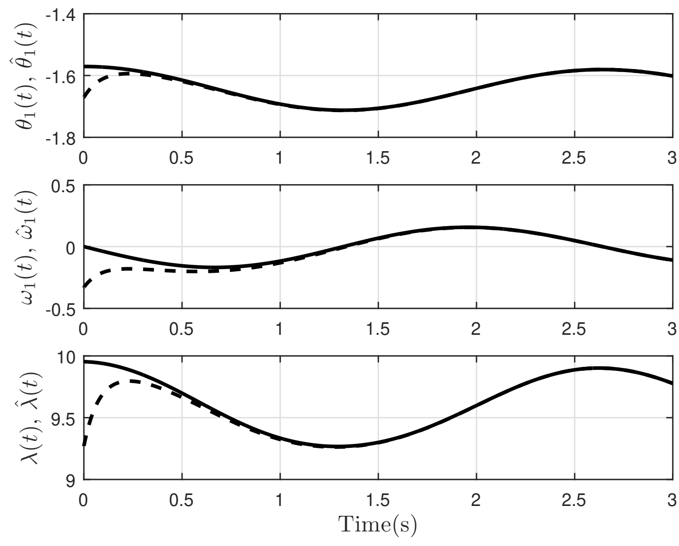

Fig. 3. Time evolution of θ_1, ω_1 and λ and their estimates in Example 4.2.

Figure 3 shows the dynamic states in $\chi_1(t) = [\theta_1(t)\ \omega_1(t)]^T$ and Lagrange multiplier $\lambda(t)$ (bold lines) along with their estimates $\hat{\theta}_1(t)$, $\hat{\omega}_1(t)$, and $\hat{\lambda}(t)$ (dashed lines) for $t \in [0,3]$. The plant initial conditions are $\theta_1(0) = -1.5708$, $\omega_1(0) = 0$, $\theta_2(0) = 0.7104$, $\omega_2(0) = 0$, $\alpha_1(0) = -0.4$, $\alpha_2(0) = 0$, $\lambda(0) = 9.9532$, and the observer initial conditions are $\hat{\theta}_1(0) = -1.6708$, $\hat{\omega}_1(0) = -0.3$, $\hat{\theta}_2(0) = 0.7076$, $\hat{\omega}_2(0) = -0.0188$, $\hat{\alpha}_1(0) = 0.1485$, $\hat{\alpha}_2(0) = -0.08454$, $\hat{\lambda}(0) = 9.2704$. Figure 4 shows the time evolution of the observation errors $e_{\theta_i} = \hat{\theta}_i - \theta_i$, $e_{\omega_i} = \hat{\omega}_i - \omega_i$, $e_{\alpha_i} = \hat{\alpha}_i - \alpha_i$, $i \in \{1,2\}$, and $e_\lambda = \hat{\lambda} - \lambda$, under the same conditions: expectedly, all these signals go asymptotically to 0.

Comparisons: In this case, both approaches in [13] and [4] are inapplicable: the first one because the system has a non-invertible $E(x)$ on the left-hand side; the second one because the restriction is not in the form $0 = Cx_1 + Dx_2 + f_2(u,y)$, with C and D constant matrices.

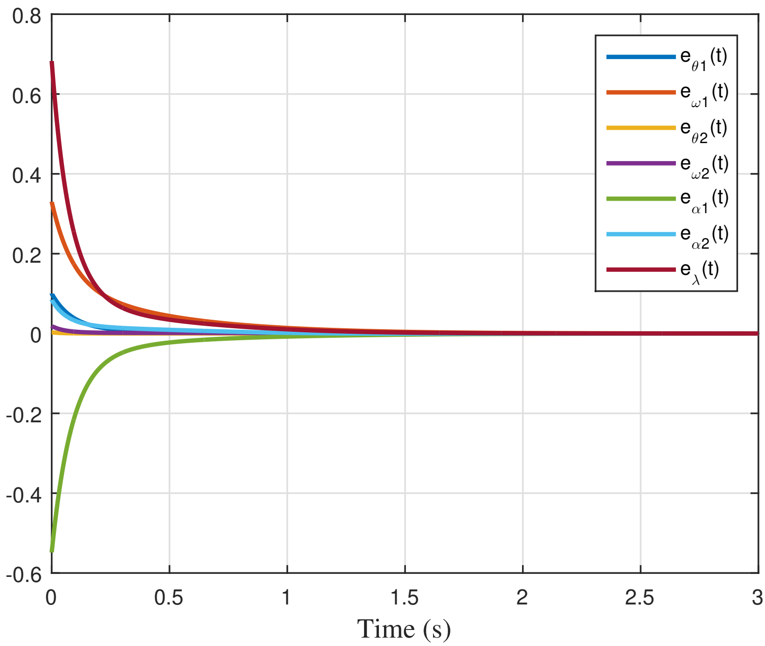

Fig. 4. Time evolution of the observation errors in Example 4.2.

5. CONCLUSIONS

A novel methodology for designing nonlinear observers for a class of DAE nonlinear systems has been presented. Stability analysis of the resulting observation error system, expressed via exact factorization of the error vector, has been conducted using a redundant Lyapunov function, allowing design conditions to be formulated as LMIs.

Computable factors depending on estimated states and measurements have been explicitly extracted and gain-scheduled, while robustness to unmeasurable process variables has been ensured by isolating non-computable factors. Polytopic bounds have been imposed to enable a quasi-LPV framework. The advantages of the proposed approach over existing methods have been demonstrated through examples; sources of conservativeness have also been discussed. Future work will focus on extending observer structures, ensuring well-posedness in user-defined regions, and addressing computational challenges related to convex modeling of nonlinearities.

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Miguel Bernal, Dept. of Electrical and Electronics Engineering of the Sonora Institute of Technology, 5 de Febrero 818 Sur, C.P. 85000, Cd. Obregon. Mexico.
e-mail: miguel.bernal@itson.edu.mx

Antonio Sala, Instituto de Automática e Informática Industrial, Universitat Politècnica de València, Camino de Vera s/n, 46022, Valencia. Spain.
e-mail: asala@isa.upv.es

Antonio González, Instituto de Automática e Informática Industrial, Universitat Politècnica de València, Camino de Vera s/n, 46022, Valencia. Spain.
e-mail: angonsor@upv.es