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A NOVEL STUDY OF PROPERTIES, FUNCTIONAL EQUATIONS AND FAMILIES OF FUZZY IMPLICATIONS THROUGH STRICT MONOTONICITY

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It is well known that monotonicity has been an important defining criterion for fuzzy logic connectives, such as fuzzy negations, t-norms, t-conorms and fuzzy implications. Also, a stronger version of monotonicity, namely strict monotonicity, establishes some significant representation theorems of continuous fuzzy negations, continuous t-norms and continuous t-conorms. In this work, we propose the strict monotonicity for fuzzy implications and investigate some necessary conditions on fuzzy implications to fulfill the same. Also, the relationship between the basic properties, functional equations of fuzzy implications and the strict monotonicity will be investigated. Further, we examine the strict monotonicity for fuzzy implications that do come from different families of fuzzy implications and show that the strict monotonicity is a necessary condition for fuzzy polynomial implications, fuzzy rational implications and some subclasses of (S,N) and f-generated fuzzy implications.

Keywords: fuzzy implications, the law of importation, the law of contra-positive symmetry, $(S;N)$-implications, R-implications

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1. INTRODUCTION

Triangular norms (or shortly, t-norms), triangular conorms (or shortly, t-conorms), fuzzy negations and fuzzy implications are some of the important fuzzy logic connectives that play significant role in *fuzzy logic, fuzzy control, artificial intelligence, approximate reasoning, decision making* etc. They are defined as follows:

Definition 1.1. (Baczynski and Jaygram [2], Klement et al. [16])

(i) A function $N : [0,1] \to [0,1]$ is called a *fuzzy negation* if it is decreasing and satisfying $N(0) = 1$ and $N(1) = 0$.

(ii) A binary operation $T(S)\colon [0,1]^2 \to [0,1]$ is called a *t-norm (t-conorm)*, if it is increasing in both variables, commutative, associative and has $1(0)$ as the neutral element.

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(iii) A function $I\colon [0,1]^2 \to [0,1]$ is called a *fuzzy implication* if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0,1]$, the following conditions:

$$\text{if } x_1 \le x_2, \text{ then } I(x_1,y) \ge I(x_2,y), \text{ i.e., } I(\,\cdot\,, y) \text{ is decreasing,} \tag{I1}$$

$$\text{if } y_1 \le y_2, \text{ then } I(x,y_1) \le I(x,y_2), \text{ i.e., } I(x,\,\cdot\,) \text{ is increasing,} \tag{I2}$$

$$I(0,0) = 1,\ I(1,1) = 1,\ I(1,0) = 0. \tag{I3}$$

Let $\mathbb{I}$ denote the set of all fuzzy implications defined on $[0,1]$.

Note that, fuzzy negations, t-norms, t-conorms and fuzzy implications are well established over the decades in various aspects. For more details about these operators, please refer to [1, 2, 5, 13, 16, 17, 27].

Definition 1.2. (Klement et al. [16])

(i) A fuzzy negation is said to be strict if it is strictly decreasing and continuous.

(ii) A t-norm (t-conorm) is said to be strict if it is strictly increasing in both variables on $(0,1] \times (0,1]$($[0,1) \times [0,1)$) and continuous.

(iii) A t-norm T is called continuous Archimedean if it is continuous and satisfies the diagonal inequality $T(x,x) < x$, for all $x \in (0,1)$.

More details about strict t-norms, strict t-conorms and strict negations can be found in [6, 7, 16, 29, 31, 33].

1.1. Motivation for this paper

With respect to the strict monotonicity of t-norms, t-conorms, t-subnorms and overlap functions, we have the following observations:

(i) From Definitions 1.4.2, 2.1.2 in [2], observe that strict monotonicity is defined for fuzzy negations, t-norms and t-conorms only. Further, from Theorems 1.4.12, 2.1.8 and 2.2.8 in [2], note that strict monotonicity of fuzzy negations, t-norms and t-conorms along with continuity yields some representations of special classes of fuzzy negations, t-norms and t-conorms. Though, fuzzy implications are also an important class of fuzzy logic connectives and closely related with fuzzy negations, t-norms and t-conorms, unfortunately, till date, the strict monotonicity is not yet defined and studied for fuzzy implications, and consequently, representations results of fuzzy implications satisfying strict monotonicity are unavailable.

(ii) Theorems 2.4.10 - 2.4.12 and 2.5.17 in [2] prove that monotonicity is one of the demanding criterion in the characterization of some sub-classes of (S,N)- and R-implications. Also, the recent works [19, 20] show that strict monotonicity of fuzzy implications plays an important role in fuzzy inference systems. Due to the theoretical demand and applicational significance, it is essential to investigate the strict monotonicity for fuzzy implications. However, there exists, so far, no systematic study of fuzzy implications that fulfill strict monotonicity.

(iii) Cancellation property (equivalently, strict monotonicity for t-norms and t-conorms) has been extensively studied for fuzzy logic connectives such as t-norms, t-conorms, t-subnorms and overlap functions, see for instance, [8, 9, 18, 30, 32, 34, 36] over the decades and characterizations of some classes of cancellative pre t-norms are investigated in [10, 11]. Nevertheless, a comprehensive study of strict monotonicity for fuzzy implications was never done before, and thus lead to the lack of some insights of fuzzy implications.

Due to the above facts, it is clear that there is a significant necessity to explore the strict monotonicity for fuzzy implications. This forms the main motivation for this paper.

1.2. Objectives of the paper

From the motivation of the paper, it is clear that we are interested in a comprehensive study of strict monotonicity of fuzzy implications. Specifically, we have the following objectives:

(i) Formulate strict monotonicity for fuzzy implications and investigate basic necessary conditions on fuzzy implications that have strict monotonicity.

(ii) Study the relationship between strict monotonicity and other basic properties, functional equations of fuzzy implications.

(iii) Investigate the fuzzy implications that satisfy strict monotonicity from different families of fuzzy implications that are available in the literature.

1.3. Organization of the paper

In Section 2, we formulate the strict monotonicity for fuzzy implications and investigate some necessary conditions on fuzzy implications to fulfill (SM1) or (SM2) or both. Then, we analyze the relationship between the strict monotonicity and the basic desirable properties of fuzzy implications in Section 3. Further, we investigate the relationship between strict monotonicity and the functional equations of fuzzy implications in Section 4. In Section 5, we study the strict monotonicity for fuzzy implications that do come from different families. We study the strict monotonicity for fuzzy polynomial implications and some generalizations in Section 6. Some concluding remarks are discussed in Section 7.

2. STRICT MONOTONICITY OF FUZZY IMPLICATIONS: FORMULATION AND BASIC OBSERVATIONS

In this section, we formulate strict monotonicity for fuzzy implications and investigate some important characteristics of fuzzy implications satisfying strict monotonicity. Towards this, we start with a remark that motivates the formulation of strict monotonicity for fuzzy implications.

Remark 2.1. From the truth table of classical implication $\to$, we observe the following regarding strict monotonicity of $\to$, for all $x, y, z \in \{0, 1\}$:

(i) Let $z < 1$. Then $x < y$ implies $(x \to z) > (y \to z)$.

(ii) Let $x > 0$. Then $y < z$ implies $(x \to y) < (x \to z)$.

Now, we propose a natural generalization of strict monotonicity of classical implication $\to$, given in Remark 2.1, as follows.

Definition 2.2. A fuzzy implication I on $[0, 1]$ is called

(i) strictly decreasing in the first variable if for all $x, y \in [0, 1]$ and $z \in [0, 1)$,

$$x < y \text{ implies } I(x, z) > I(y, z) \tag{SM1}$$

(ii) strictly increasing in the second variable if for all $x \in (0, 1]$ and $y, z \in [0, 1]$,

$$y < z \text{ implies } I(x, y) < I(x, z) \tag{SM2}$$

(iii) strictly monotone (SM) if it is strictly decreasing in the first variable (SM1) and strictly increasing in the second variable (SM2).

Table 1 presents various examples of fuzzy implications w.r.t. (SM1) and (SM2).

Fuzzy implication	(SM1)	(SM2)
$I_{\mathbf{N}}(x, y) = \begin{cases} 1 - x, & \text{if } y < 1 \\ 1, & \text{if } y = 1 \end{cases}$	$\checkmark$	$\times$
$I_{\mathbf{D}}(x, y) = \begin{cases} 1, & \text{if } x = 0 \\ y, & \text{if } x > 0 \end{cases}$	$\times$	$\checkmark$
$I_{\mathbf{RC}}(x, y) = 1 - x + xy$	$\checkmark$	$\checkmark$
$I_{\mathbf{GD}}(x, y) = \begin{cases} 1, & \text{if } x \le y \\ y, & \text{if } x > y \end{cases}$	$\times$	$\times$

Tab. 1. Examples of fuzzy implications w.r.t. (SM1) and (SM2).

2.1. Strict monotonicity of fuzzy implications: Some necessary conditions

In the following, we investigate some necessary conditions on fuzzy implications that satisfy strict monotonicity.

Lemma 2.3. Let I be a fuzzy implication on $[0, 1]$. Then the following statements hold true:

(i) If I satisfies (SM1) (or (SM2)) then I has trivial one region. i.e., for any $x, y \in [0, 1]$, $I(x, y) = 1$ if and only if either $x = 0$ or $y = 1$.

(ii) If I is strictly monotone then I has trivial zero region. i.e., for any $x, y \in [0, 1]$, $I(x, y) = 0$ if and only if $x = 1$ and $y = 0$.

Proof. Follows directly. □

Remark 2.4. In general, the converse statements of Lemma 2.3 need not be true always. To see this,

(i) firstly, consider $I_{\mathbf{0}} \in \mathbb{I}$ given as follows:

$$I_{\mathbf{0}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } \ y = 1 \\ 0, & \text{if } x > 0 \text{ and } \ y < 1 \end{cases}, \qquad x, y \in [0,1].$$

Then, we have

$$I_{\mathbf{0}}(0.5, 0.6) = 0 = I_{\mathbf{0}}(0.5, 0.8) \text{ but } 0.6 \neq 0.8,$$
$$I_{\mathbf{0}}(0.5, 0.8) = 0 = I_{\mathbf{0}}(0.7, 0.8) \text{ but } 0.5 \neq 0.7.$$

Thus, $I_{\mathbf{0}}$ satisfies neither (SM1) nor (SM2). However, $I_{\mathbf{0}}$ has trivial one region, following by its definition.

(ii) Secondly, consider $I_{\mathbf{KD}} \in \mathbb{I}$ given by $I_{\mathbf{KD}}(x,y) = \max(1-x, y)$, for all $x, y \in [0,1]$. Once again, note that $I_{\mathbf{KD}}(x,y) = 0$ if and only if $x = 1$ and $y = 0$. i. e., $I_{\mathbf{KD}}$ has trivial zero region. However, from the statements

$$I_{\mathbf{KD}}(0.4, 0.7) = 0.7 = I_{\mathbf{KD}}(0.5, 0.7) \text{ but } 0.4 \neq 0.5,$$
$$I_{\mathbf{KD}}(0.7, 0.2) = 0.3 = I_{\mathbf{KD}}(0.7, 0.1) \text{ but } 0.2 \neq 0.1,$$

$I_{\mathbf{KD}}$ satisfies neither (SM1) nor (SM2).

Now, we show in the following that trivial range fuzzy implications satisfy neither (SM1) nor (SM2).

Lemma 2.5. If I is a fuzzy implication on $[0,1]$ such that $I(x,y) \in \{0,1\}$ for all $x, y \in [0,1]$ then I satisfies neither (SM1) nor (SM2).

Proof. Let $I \in \mathbb{I}$ be such that $I(x,y) \in \{0,1\}$ for all $x, y \in [0,1]$.

(i) Let I satisfy (SM1) and $a, b \in (0,1)$ be such that $a < b$. Since I satisfies (SM1), we must have $1 = I(0,a) > I(a,a) > I(b,a) > I(1,a) \geq 0$. Since, $I(x,y) \in \{0,1\}$ for all $x, y \in [0,1]$, it must follow that some of the values $I(0,a), I(a,a), I(b,a), I(1,a)$ must be equal and hence, due to (SM1) of I, we get a contradiction to the fact $a < b$. Thus, I cannot satisfy (SM1).

(ii) The proof of (SM2) of I is similar to the previous case.

□

Remark 2.6. (i) In general, the converse of Lemma 2.5 need not be true always. To see this, consider $I_{\mathbf{GD}}$ given in Table 1. It is clear that $I_{\mathbf{GD}}$ satisfies neither (SM1) nor (SM2). However, $I_{\mathbf{GD}}$ has range equal to $[0,1]$, by its definition.

(ii) From Lemma 2.5, it may appear that for a fuzzy implication satisfying strict monotonicity (SM), the range is entire unit interval $[0,1]$. However, this need not be true always. For instance, let $I \in \mathbb{I}$ be defined as follows: for all $x, y \in [0,1]$,

$$I(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } \ y = 1, \\ \frac{1}{3}(1 - x + xy), & \text{if } x > 0 \text{ and } \ y < 1. \end{cases} \tag{1}$$

Then clearly, I satisfies (SM). However, the range of I is equal to $[0, 1/3] \cup \{1\}$.

Among the partial functions $I(\cdot, y)$ and $I(x, \cdot)$ of fuzzy implication I, the function $I(x, 0)$ plays an important role in the structure of I and also on the satisfiablity of the basic properties. Recall, from [2], that if I is a fuzzy implication then the function $N_I \colon [0,1] \to [0,1]$ defined by $N_I(x) = I(x,0)$, for all $x \in [0,1]$, is always a fuzzy negation and it is called the natural negation of I.

Proposition 2.7. Let I be a fuzzy implication fulfilling (SM1). Then the following statements hold true:

(i) N_I is a strictly decreasing negation.

(ii) $\{0,1\} \subset$ Range of N_I.

P r o o f. Let $I \in \mathbb{I}$ satisfy (SM1).

(i) Since $I(\cdot, y)$ is strictly decreasing for all $y \in [0,1]$, it follows directly $N_I(x) = I(x, 0)$, for all $x \in [0,1]$, is also strictly decreasing.

(ii) Since N_I is a fuzzy negation, it is clear that $\{0,1\} \subseteq$ Range of N_I. Suppose that $\{0,1\} =$ Range of N_I. Let $a, b \in (0,1)$ be such that $a < b$. Since I satisfies (SM1), we must have $1 = I(0,0) > I(a,0) > I(b,0) > I(1,0) = 0$. Since, $I(x,0) \in \{0,1\}$ for all $x \in [0,1]$, some of the values $I(0,1), I(a,0), I(b,0), I(1,0)$ must be equal and hence, from (SM1) of I, we have either $a = 0$ or $a = b$ or $b = 1$,which is a contradiction to the fact $0 < a < b < 1$. Thus, $\{0,1\} \subset$ Range of N_I.

□

Remark 2.8. Once again observe that the converse of Proposition 2.7 need not be true always. To see this, note that for $I_{\mathbf{KD}}$, it follows that $N_{I_{\mathbf{KD}}}(x) = 1 - x$, for all $x \in [0,1]$, a strict fuzzy negation. However, from Remark 2.4, we get that $I_{\mathbf{KD}}$ satisfies neither (SM1) nor (SM2).

3. STRICT MONOTONICITY W.R.T. THE BASIC PROPERTIES OF FUZZY IMPLICATIONS

In this section, we investigate the relationship between the basic desirable properties and strict monotonicity of fuzzy implications. Towards this, we recall the basic desirable properties of fuzzy implications in the following.

Definition 3.1. (Baczynski and Jayaram [2]) A fuzzy implication I is said to satisfy

(i) the left neutrality property (NP), if

$$I(1,y)=y, \qquad y\in[0,1]. \tag{NP}$$

(ii) the ordering property (OP), if

$$x\leq y \Longleftrightarrow I(x,y)=1 \qquad x,y\in[0,1]. \tag{OP}$$

(iii) the identity principle (IP), if

$$I(x,x)=1, \qquad x\in[0,1]. \tag{IP}$$

(iv) the exchange principle (EP), if

$$I(x,I(y,z))=I(y,I(x,z)), \qquad x,y,z\in[0,1]. \tag{EP}$$

Lemma 3.2. A fuzzy implication satisfying either (SM1) or (SM2) does not satisfy (IP) and hence (OP).

Proof. Note that if $I\in\mathbb{I}$ satisfies (OP) then I satisfies also (IP). Thus, we prove the result only for (IP). For a contradiction, we assume that I satisfies (IP).

(i) Let I satisfy (SM1) and $a\in(0,1)$. Since I satisfies (SM1), we have $1=I(0,a)>I(a,a)>I(1,a)>0$. Then, due the fact I satisfies (IP), we get $I(a,a)=1$. Thus, we have $I(0,a)=1=I(a,a)$. Now, from (SM1) of I, it follows that $a=0$, a contradiction to our assumption $a\in(0,1)$. Thus, I cannot satisfy (IP) and consequently, (OP).

(ii) The case of (SM2) is similar to the previous case.

Thus, if $I\in\mathbb{I}$ satisfies either (SM1) or (SM2) then it does not satisfy both (IP) and (OP). □

Remark 3.3. (i) For Lemma 3.2, one can also present an alternative proof based on Lemma 2.3 as follows: Let $I\in\mathbb{I}$ satisfy (SM2) and (IP). Then, for any $x\in(0,1)$, we have $I(x,x)=1$. Then from Lemma 2.3(i), it follows either $x=0$ or $x=1$, which is a contradiction.

(ii) The converse of Lemma 3.2 need not be true always. For example, $I_{\mathbf{KD}}$ does not satisfy both (OP) and (IP), see Table 1.4 in [2]. However, from Remark 2.4(ii), it follows that $I_{\mathbf{KD}}$ satisfies neither (SM1) nor (SM2).

(iii) Interestingly, there are some fuzzy implications satisfying (OP) but also satisfy (SM) on some proper subset of $[0,1]^2$. For instance, fuzzy implications $I_{\mathbf{LK}}, I_{\mathbf{GG}}$ (see, Table 1.3 in [2]) do satisfy (OP). However, they also satisfy (SM) on the set $\{x,y\in[0,1]|x>y\}$. More details of fuzzy implications satisfying (SM) and (OP) on some subsets of $[0,1]$ are provided in Section 5.2 and also in Section 6.

Now, regarding the strict monotonicity (SM) and (NP), (EP) of fuzzy implications, we have the following observations:

Example 3.4. Let I and J be two fuzzy implications defined by $I(x,y) = 1 - x + xy^2$ and $J(x,y) = I_{\mathbf{RC}}(x,y) = 1 - x + xy$, respectively, for all $x, y \in [0,1]$. Then I and J are strictly monotone.

(i) Since $I(1,y) = y^2$, clearly I does not satisfy (NP), while $J(1,y) = y$ implies that J satisfies (NP).

(ii) Note that, when $x = 0.4, y = 0.6$ and $z = 0$, we have $I(x, I(y,z)) = 0.664 \neq 0.616 = I(y, I(x,z))$ which implies that I does not satisfy (EP), while J satisfies (EP), follows from Table 1.4 in [2].

From Example 3.4, it follows that there exist some strictly monotone fuzzy implications that satisfy (NP) ((EP)). However, it is to be noted that the strict monotonicity of fuzzy implication I alone is not sufficient for verifying whether I satisfies (NP) or (EP), as the case may be.

In the following, let us examine the relationship between the continuity and strict monotonicity of fuzzy implications. For this purpose, let us recall the following definition.

Definition 3.5. (Baczynski and Jayaram [2]) A fuzzy implication I is said to be continuous if it is continuous in both variables.

Lemma 3.6. Let I be a fuzzy implication satisfying (SM1). If I is continuous in the first variable, then N_I is strict.

Proof. Let $I \in \mathbb{I}$ be satisfy (SM1) and continuous in the first variable. Then, from Proposition 2.7, it follows that N_I is strictly decreasing. Since I is continuous in the first variable, it follows also that $N_I = I(\cdot, 0)$ is continuous. Thus, N_I is a strict negation. □

Remark 3.7. In the following, we show the conditions in Lemma 3.6, in general, are independent to each other.

(i) Let $I : [0,1]^2 \to [0,1]$ be defined, for all $x, y \in [0,1]$, by

$$I(x,y) = \begin{cases} 1 - x + xy, & \text{if } x \leq y, \\ \frac{1}{2}(1 - x + xy), & \text{if } x > y. \end{cases} \tag{2}$$

Then, I is a fuzzy implication satisfying (SM). However, the negation N_I of I given, for all $x \in [0,1]$, by

$$N_I(x) = \begin{cases} 1, & \text{if } x = 0, \\ \frac{1}{2}(1 - x), & \text{if } x > 0, \end{cases}$$

is strictly decreasing but not continuous. Thus, N_I is not a strict negation.

(ii) From Remark 2.4(ii), $I_{\mathbf{KD}}$ satisfies neither (SM1) nor (SM2). Thus, it follows that $I_{\mathbf{FD}} \in \mathbb{I}$ given by

$$I_{\mathbf{FD}}(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ \max(1-x,y), & \text{if } x > y \end{cases}, \qquad x,y \in [0,1]$$

does not satisfy (SM). Also, $I_{\mathbf{FD}}$ is not continuous, see Table 1.4 in [2]. However, $N_{I_{\mathbf{FD}}} = N_{\mathbf{C}}$ is a strict (in fact strong) negation.

Note that $I_{\mathbf{RC}}$ is an example of fuzzy implication that is both strictly monotone and continuous, see Table 1.4 in [2]. However, in the following remark, we present some examples of fuzzy implications that show that the strictly monotone and continuity are two independent properties for fuzzy implications.

Remark 3.8. (i) Note that every strictly monotone fuzzy implication need not be continuous, in general. For instance, I defined in Eq. (2) is strictly monotone but is discontinuous at point $(1, 0.5)$.

(ii) Also, note that continuous fuzzy implication need not be strictly monotone, in general. For instance, $I_{\mathbf{KD}}$ satisfies neither (SM1) nor (SM2), due to Remark 2.4(ii) but $I_{\mathbf{KD}}$ is a continuous fuzzy implication, due to Example 1.2.3 in [2].

4. STRICT MONOTONICITY W.R.T. FUNCTIONAL EQUATIONS OF FUZZY IMPLICATIONS

In this section, we investigate the relationship between the strict monotonicity and three important functional equations of fuzzy implications.

4.1. Strict monotonicity and the law of importation

The law of importation is one of the significant functional equations which plays a key role in characterizing some important families of fuzzy implications, see [2, 3, 15, 21, 26]. The law of importation is defined as follows:

Definition 4.1. (Baczynski and Jayaram [2]) A fuzzy implication I is said to satisfy the law of importation (LI) w.r.t. a t-norm T, if

$$I(x, I(y,z)) = I(T(x,y), z), \qquad x,y,z \in [0,1]. \tag{LI}$$

In this section, we investigate the relationship between the strict monotonicity (SM) and the law of importation (LI) of fuzzy implications. Towards this, we have the following example.

Example 4.2. (i) It is well known that $I_{\mathbf{LK}} \in \mathbb{I}$ satisfies the law of importation (LI) w.r.t. $T_{\mathbf{LK}}$, see Theorem 7.3.5 in [2]. However, $I_{\mathbf{LK}}$ does not satisfy strict monotonicity, as it satisfies (OP), see Lemma 3.2. Thus, it follows that a fuzzy implication satisfying (LI) need not satisfy (SM) always.

(ii) In Example 3.4(ii), note that I does not satisfy (EP). Then, I does not satisfy the law of importation (LI) w.r.t. any t-norm T, due to Remark 7.3.1 in [2] and hence (SM) need not imply (LI) always.

(iii) Finally, $I_{\mathbf{RC}}$ satisfies both the strict monotonicity (SM) and the law of importation (LI) (with $T_{\mathbf{P}}$, see Table 7.1 in [2]).

From Example 4.2, it is clear that there are some fuzzy implications that satisfy both (SM) and (LI). The following result establishes representations of such fuzzy implications.

Proposition 4.3. Let $I \in \mathbb{I}$ satisfy (LI) w.r.t. a t-norm T and N_I a strict negation. Then I is strictly monotone if and only if T is strictly monotone.

P r o o f. Let $I \in \mathbb{I}$ satisfy (LI) w.r.t. a t-norm T and N_I a strict negation. Then, from [28], I is given by

$$I(x, y) = N_I(T(x, N_I^{-1}(y))), \qquad x, y \in [0, 1]. \tag{3}$$

Since, N_I is strict, we obtain directly that I is strictly monotone if and only if T is strictly monotone. □

Remark 4.4. Note that, in Proposition 4.3, N_I of a fuzzy implication I should necessarily be a strict negation. To see this, $I_{\mathbf{WB}} \in \mathbb{I}$ has the natural fuzzy negation $N_{I_{\mathbf{WB}}} = N_{\mathbf{D2}}$, which is not strict. However, $I_{\mathbf{WB}}$ satisfies (LI) w.r.t. every t-norm T, see Example 7.3.6 in [2]. Finally, $I_{\mathbf{WB}}$ is not strict monotone, due to Lemma 3.2 and the fact that $I_{\mathbf{WB}}$ satisfies (IP).

4.2. Strict monotonicity and the law of contrapositive symmetry

The law of contrapositive symmetry is a functional equation involving a fuzzy negation and a fuzzy implication and it has been studied in [2, 4, 12]. In the literature, the law of contrapositive symmetry is defined as follows:

Definition 4.5. (Baczynski and Jayaram [2] and Fodor [12]) Let I be a fuzzy implication and N a fuzzy negation. We say that I satisfies

(i) the *law of contraposition* (or in other words, the *contrapositive symmetry*) with respect to N, if

$$I(x, y) = I(N(y), N(x)), \qquad x, y \in [0, 1]. \tag{CP}$$

(ii) the *law of left contraposition* with respect to N, if

$$I(N(x), y) = I(N(y), x), \qquad x, y \in [0, 1]. \tag{L-CP}$$

(iii) the *law of right contraposition* with respect to N, if

$$I(x, N(y)) = I(y, N(x)), \qquad x, y \in [0, 1]. \tag{R-CP}$$

If I satisfies the (left, right) contrapositive symmetry with respect to N, then we denote this by CP(N) (respectively, by L-CP(N), R-CP(N)).

With respect to the strict monotonicity and the law of contrapositive symmetry, we have the following example.

Example 4.6. (i) Recall that $I_{\mathbf{RC}} \in \mathbb{I}$ satisfies (SM). Also, note that $I_{\mathbf{RC}}$ satisfies (CP) w.r.t the fuzzy negation $N(x) = 1 - x$, for all $x \in [0, 1]$.

(ii) Also, recall that $I_{\mathbf{KD}} \in \mathbb{I}$ satisfies (CP) w.r.t the fuzzy negation $N(x) = 1 - x$, for all $x \in [0, 1]$. However, it is already known that $I_{\mathbf{KD}}$ satisfies neither (SM1) nor (SM2).

In the following, we investigate the relationship between strict monotonicity and contrapositive symmetry of fuzzy implications. Before this, since CP(N) is a functional equation involving two functions, namely, a fuzzy implication I and a fuzzy negation N, note that it is not possible to find the functions I and N simultaneously such that the pair (I, N) satisfies CP(N) and strict monotonicity. However, we assume some conditions on I and N to establish some relationship between (SM) and CP(N).

Proposition 4.7. Let I be a fuzzy implication that satisfies (SM1) and $I(\cdot, 0)$, $I(1, \cdot)$ be two bijections on $[0, 1]$ with N_I continuous.

(i) If I satisfies contrapositive symmetry CP(N) w.r.t. some fuzzy negation N then N is strong.

(ii) If further, I satisfies (NP) then $N_I = N$.

Proof. Let $I \in \mathbb{I}$ satisfy (SM1) and $I(\cdot, 0), I(1, \cdot)$ be bijections on $[0, 1]$.

(i) Let $I \in \mathbb{I}$ satisfy CP(N) w.r.t. some fuzzy negation N. First, we show that N is strict and then strong. When $x = 1$, CP(N) for I becomes $I(N(y), N(1)) = I(1, y)$, or equivalently, $I(N(y), 0) = I(1, y)$. i. e., $N_I(N(y)) = I(1, y)$. Then from Lemma 3.6, it follows that N_I is strict, and hence, $N(y) = N_I^{-1}(I(1, y))$, for all $y \in [0, 1]$. Thus, N being a composition of strict negation and an increasing bijection, becomes strict. Let $x \in (0, 1)$. Since N is strict, $N(x) = y \in (0, 1)$. Now, due to CP(N), we have $I(x, N(x)) = I(N(N(x)), N(x))$ and hence, $N(N(x)) = x$, due to (SM1) of I. Since $x \in (0, 1)$ is chosen arbitrarily, $N(N(x)) = x$, for all $x \in [0, 1]$ and thus, N is strong.

(ii) Follows obviously.

□

Remark 4.8. Note that the converse of Proposition 4.7 need not be true, in general. i. e., if $I \in \mathbb{I}$ satisfies (SM) then I need not satisfy CP(N), in general, even N is a strict negation. For instance, let $I(x, y) = 1 - x^2 + x^2y$, for all $x, y \in [0, 1]$ and $N(x) = 1 - x$, for all $x \in [0, 1]$. Then, clearly I is strictly monotone and N is a strict negation. Also, for all $x, y \in [0, 1]$, we get $I(N(y), N(x)) = 1 - x + 2xy - xy^2$, which is different from I.

Lemma 4.9. Let $I \in \mathbb{I}$ satisfy (SM1), (NP) and N_I be strict on $[0, 1]$. Then the following statements are equivalent:

(i) I satisfies (CP) w.r.t. N.

(ii) I satisfies (L-CP) w.r.t. N.

(iii) I satisfies (R-CP) w.r.t. N.

Proof. Follows from Proposition 4.7 and Proposition 1.5.3 in [2]. □

4.3. Strict monotonicity and T-conditionality of fuzzy implications

In the following, we investigate the relationship between the strict monotonicity and T-conditionality of fuzzy implications. We recall the following definition.

Definition 4.10. (Baczynski and Jayaram [2]) Let I be a fuzzy implication and T be a t-norm. We say that the pair (I, T) satisfies T-conditionality if for all $x, y \in [0, 1]$,

$$T(x, I(x, y)) \leq y. \tag{TC}$$

Example 4.11. Let $I = I_{\mathbf{RC}} \in \mathbb{I}$ which is strictly monotone.

(i) Let $T(x, y) = \max(x + y - 1, 0)$ be a t-norm. Then, we have $T(x, I(x, y)) = \max(x + 1 - x + xy - 1, 0) = xy \leq y$, for all $x, y \in [0, 1]$. Thus the pair (I, T) satisfies (TC).

(ii) Let $T(x, y) = x \cdot y$ be a t-norm. Then,

$$T(0.5, I(0.5, 0)) = 0.25 > 0$$

implies that the pair (I, T) does not satisfy (TC).

From Example 4.11, it is clear that every pair (I, T) of fuzzy implication and t-norm need not satisfy (TC). Since it is impossible to find fuzzy implications I and t-norms T simultaneously such that the pair (I, T) satisfies (TC), in the following, we consider some special t-norms and strictly monotone fuzzy implications and validate (TC) for them.

Proposition 4.12. Let I be a fuzzy implication satisfying (SM1). Then the following statements hold true.

(i) The pair (I, T) never satisfies (TC) for any t-norm T with fuzzy negation $N_T = N_{D1}$.

(ii) The pair (I, T) never satisfies (TC) for any strict t-norm T.

Proof. Let I be a fuzzy implication satisfying (SM1).

(i) Let T be a t-norm with fuzzy negation $N_T = N_{D1}$. Suppose that the pair (I,T) satisfies (TC). Then from (TC), it follows that $T(x, I(x,0)) = 0$, for all $x \in [0,1]$. i. e., $N_I(x) \leq N_T(x)$, for all $x \in [0,1]$. Since $N_T = N_{D1}$, we get $N_I = N_{D1}$, a contradiction due to Proposition 2.7.

(ii) Let T be a strict t-norm. Then clearly $N_T = N_{\mathbf{D1}}$, the least fuzzy negation. Now, the remaining proof follows from the previous case.

□

5. STRICT MONOTONICITY OF FUZZY IMPLICATIONS FROM DIFFERENT FAMILIES

In this section, we study the strict monotonicity for fuzzy implications that do come from well-established families. We do this mainly for the families of $(S,N), R, QL$, Yager's f-g-, h, generalized h and k-generated fuzzy implications. For a comprehensive reading of these families of fuzzy implications, please refer to [2, 14, 27, 37, 38].

5.1. (S,N)-implications

In the literature, (S,N)-implications are defined as follows.

Definition 5.1. (Baczynski and Jayaram [2]) A function $I : [0,1]^2 \to [0,1]$ is called an *(S,N)-implication* if there exist a t-conorm S and a fuzzy negation N such that

$$I(x,y) = S(N(x), y), \qquad x, y \in [0,1].$$

In the following, we discuss the strict monotonicity for some (S,N)-implications.

Example 5.2. (i) Note that $I_{\mathbf{RC}}(x,y) = S_{\mathbf{p}}(N_{\mathbf{C}}(x), y)$, for all $x, y \in [0,1]$ and hence, $I_{\mathbf{RC}}$ is an (S,N)-implication. Already, we know that $I_{\mathbf{RC}}$ is strictly monotone.

(ii) Note that $I_{\mathbf{LK}}(x,y) = S_{\mathbf{LK}}(N_{\mathbf{C}}(x), y)$, for all $x, y \in [0,1]$ and hence, $I_{\mathbf{LK}}$ is also an (S,N)-implication. Since $I_{\mathbf{LK}}$ satisfies (OP), $I_{\mathbf{LK}}$ is not strictly monotone, due to Lemma 3.2.

From Example 5.2, it follows that all (S,N)-implications need not satisfy strict monotonicity. In the following, we investigate (S,N)-implications that satisfy strict monotonicity. Before doing this, note that, since (S,N)-implications involve two functions S and N, it is not possible to find S and N simultaneously such that the (S,N)-implications satisfying the strict monotonicity. Due to this, we restrict our investigations to some well established subclasses of (S,N)-implications.

Theorem 5.3. Let I be an (S,N)-implication generated from some t-conorm S and continuous fuzzy negation N. Then the following statements are equivalent:

(i) I is strictly monotone.

(ii) S is strictly monotone and N is strict.

P r o o f . Let I be an (S, N)-implication generated from some t-conorm S and continuous fuzzy negation N.

(i) $\Rightarrow$ (ii). Let I be strictly monotone. Then, from Proposition 2.7, we have $N_I = N$ is strict. Now, it is enough to show that S is strictly monotone. Let $x < 1$ and $y_1, y_2 \in [0, 1]$ be such that $S(x, y_1) = S(x, y_2)$. Since N is strict, there exists some $x' \in (0, 1]$ such that $N(x') = x$. Thus, we have

$$\begin{aligned} S(x, y_1) = S(x, y_2) &\Longleftrightarrow S(N(x'), y_1) = S(N(x'), y_2) \\ &\Longleftrightarrow I(x', y_1) = I(x', y_2) \Longleftrightarrow y_1 = y_2. \end{aligned}$$

This shows that S is cancellative or equivalently strictly monotone.

(ii) $\Rightarrow$ (i). Let S be strictly monotone and N strict. We show that I satisfies (SM2) only since the proof for other case can be obtained similarly. For this purpose, let $x > 0$ and $y_1, y_2 \in [0, 1]$ be such that $I(x, y_1) = I(x, y_2)$. This implies that $S(N(x), y_1) = S(N(x), y_2)$. Since N is strict and $x > 0$, we have $N(x) < 1$. From the strict monotonicity of S, we get $y_1 = y_2$, and hence, I satisfies (SM2).

This completes the proof. □

Remark 5.4. Theorem 5.3 presents a characterization of (S, N)-implications that satisfy strict monotonicity but only for the case of (S, N)-implications with the continuous negations N. However, the (S, N)-implications with non-continuous negations need not satisfy the strict monotonicity. To see this, consider $I_{\mathbf{WB}}$, which is also an (S, N)-implication. Clearly, the natural negation of $I_{\mathbf{WB}}$ is $N_{\mathbf{D2}}$, which is non-continuous. Then, $I_{\mathbf{WB}}(x, y) = 1 = I_{\mathbf{WB}}(z, y)$, for any $x, z \in [0, 1)$ and $y \in [0, 1]$. Thus, $I_{\mathbf{WB}}$ does not satisfy (SM1) and hence, it is not strictly monotone.

5.2. R-implications

In the following, we study the strict monotonicity for R-implications.

Definition 5.5. (Baczynski and Jayaram [2]) A function $I : [0, 1]^2 \to [0, 1]$ is called an *R- implication* if there exists a t-norm T such that

$$I(x, y) = \sup\{t \in [0, 1] | T(x, t) \le y\}, \qquad x, y \in [0, 1].$$

Lemma 5.6. No R-implication is strictly monotone.

P r o o f . Let I be an R-implication. Then I satisfies (IP), due to Theorem 2.5.4 in [2]. Thus, I is not strictly monotone, follows from Lemma 3.2. □

Remark 5.7. (i) From Lemma 5.6, it follows that, if $I \in \mathbb{I}$ is strictly monotone then it cannot be an R-implication I_T for any t-norm T. Thus, strict monotonicity has become a basic criterion to characterize whether a fuzzy implication is an R-implication or not.

(ii) Recall, from Theorem 2.5.4 in [2], that every R-implication I satisfies (IP). That is, at least for the region $S = \{(x,y) \in [0,1]^2 | x \leq y\}$, I takes value 1, and hence it is not strictly monotone on S and consequently, on $[0,1]^2$ also. However, sometimes it may happen that R-implications may fulfill either (SM1) or (SM2) or both on the remaining part of the unit square i. e., $\{(x,y) \in [0,1]^2 | x > y\}$. For instance, consider an R-implication $I_{\mathbf{LK}}$ which is generated by the t-norm $T_{\mathbf{LK}}$. It is clear that $I_{\mathbf{LK}}$ does not satisfy strict monotonicity on $\{(x,y) \in [0,1]^2 | x \leq y\}$, as it takes the constant 1. However, for $x,y \in [0,1]$ with $x > y$, it is easy to see that $I_{\mathbf{LK}}(x,y) = 1-x+y$ satisfies both (SM1) and (SM2), and hence strictly monotone on the set $\{(x,y) \in [0,1]^2 | x > y\}$, i. e., the lower triangle of $[0,1]^2$.

In view of Remark 5.7(ii), we propose the following definition.

Definition 5.8. If a fuzzy implication I satisfies (SM1) ((SM2)) on a subset S of $[0,1]^2$ then we say I is locally left strictly monotone(locally right strictly monotone) on S. If I satisfies both (SM1) and (SM2) on S, then we say that I is locally strict monotone (SM) on S.

In the following, we investigate the local strict monotonicity for R-implications. For this purpose, we consider only the class of R-implications that are generated from continuous t-norms, due to the availability of characterization and representation results, see [2, 16].

Now, in view of Remark 5.7(ii) and Definition 5.8, we investigate the strict monotonicity of I_T, generated from continuous t-norm T, on the set $\{(x,y) \in [0,1]^2 | x > y\}$. For a better readability, we use the following notation: $S_1 = \{(x,y) \in [0,1]^2 | x \leq y\}$, $S_2 = \{(x,y) \in [a_\alpha, e_\alpha]^2 \text{ for some } \alpha \,|x > y\}$ and $S_3 = \{(x,y) \notin [a_\alpha, e_\alpha]^2 \text{ for any } \alpha \,|x > y\}$. Clearly, $S_2 \cup S_3 = \{(x,y) \in [0,1]^2 | x > y\}$ and $S_1 \cup S_2 \cup S_3 = [0,1]^2$.

Theorem 5.9. Let I_T be an R-implication generated from a continuous t-norm T given as in Theorem 2.1.10 in [2]. Then,

(i) I_T is locally left strictly monotone on S_2 if and only if each I_{T_α} is locally left strictly monotone on $[0,1]$.

(ii) I_T is locally right strictly monotone on S_2 if and only if each I_{T_α} is locally right strictly monotone on $[0,1]$.

(iii) I_T is only locally left strictly monotone on S_3 but not locally right strictly monotone on S_3.

Proof. Let I_T be an R-implication generated from a continuous t-norm T given as in Theorem 2.1.10 in [2]. Then I_T is given for all $x,y \in [0,1]$ by

$$I_T(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ a_\alpha + (e_\alpha - a_\alpha) \cdot I_{T_\alpha}\left(\frac{x-a_\alpha}{e_\alpha - a_\alpha}, \frac{y-a_\alpha}{e_\alpha - a_\alpha}\right), & \text{if } x,y \in [a_\alpha, e_\alpha] \text{ and } x > y, \\ y, & \text{otherwise.} \end{cases} \tag{4}$$

Now, it is clear that I_T does not satisfy (SM), due to (OP).

(i) Let $(x,y) \in S_2$. Then from Eq.(4), it follows that

$$I_T(x,y) = a_\alpha + (e_\alpha - a_\alpha) \cdot I_{T_\alpha}\left(\frac{x - a_\alpha}{e_\alpha - a_\alpha}, \frac{y - a_\alpha}{e_\alpha - a_\alpha}\right).$$

Now, the proof of I_T is locally left strictly monotone on S_2 if and only if each I_{T_α} is locally left strictly monotone on $[0,1]$ follows obviously.

(ii) This proof is similar to the previous case.

(iii) This proof follows directly from the definition of I_T.

□

Remark 5.10. Let $T = T_{\mathbf{M}}$, a continuous t-norm on $[0,1]$. Then $I_T = I_{\mathbf{GD}}$ satisfies neither (SM1) nor (SM2). However, $I_{\mathbf{GD}}$ satisfies locally right strictly monotonicity on the set $\{(x,y) \in [0,1]^2 | x > y\}$.

Now, in the following, we discuss the locally strict monotonicity of R-implication I_T in the case T is continuous Archimedean.

Theorem 5.11. Let I_T be an R-implication generated from a continuous Archimedean t-norm. Then the following statements hold true:

(i) I_T is locally left strictly monotone on $S_2 \cup S_3$.

(ii) I_T is locally right strictly monotone on $S_2 \cup S_3 \setminus \{(x,y) \in [0,1]^2 | y \neq 0\}$.

(iii) I_T is locally strictly monotone on $S_2 \cup S_3 \setminus \{(x,y) \in [0,1]^2 | y \neq 0\}$. Moreover, if T is nilpotent, I_T is locally strictly monotone on $S_2 \cup S_3$.

Proof. Follows from Theorem 2.5.21 in [2] directly. □

5.3. QL-implications

In the literature QL-operations are defined as follows.

Definition 5.12. (Baczynski and Jayaram [2]) A function $I\colon [0,1]^2 \to [0,1]$ is called a *QL-operation* if there exist a t-norm T, a t-conorm S and a fuzzy negation N such that

$$I(x,y) = S(N(x), T(x,y)), \qquad x,y \in [0,1].$$

If I is a QL-operation generated from the triple (T,S,N), then it is denoted by $I_{T,S,N}$.

From Remark 2.6.3 in [2], note that every QL-operation need not be a fuzzy implication. In case a QL-operation is a fuzzy implication, it is called a QL-implication. Regarding the strict monotonicity of QL-implications, we have the following example.

Example 5.13. (i) Note, from Table 2.8 in [2], that $I_{\mathbf{RC}}$ is a QL-implication. Also from Table 1, it follows that $I_{\mathbf{RC}}$ is strictly monotone.

(ii) Note, also from Table 2.8 in [2], that $I_{\mathbf{LK}}$ is a QL-implication. However, since $I_{\mathbf{LK}}$ satisfies (OP) and due to Lemma 3.2, $I_{\mathbf{LK}}$ is not strictly monotone.

From Example 5.13, note that there do exist some QL-implications that are also strictly monotone. In the following, we investigate conditions on QL-implications for satisfying the strict monotonicity.

Theorem 5.14. If a QL-implication $I_{T,S,N}$ is strictly monotone then fuzzy negation N and t-norm T are strictly monotone.

Proof. Let a QL-implication $I_{T,S,N}$ be strictly monotone. Then from Proposition 2.7, it follows that N is strictly monotone. Suppose T is not strictly monotone. Then there exist some $x, y_1, y_2 \in (0,1]$ with $y_1 < y_2$ and satisfying $T(x,y_1) = T(x,y_2)$. Then, it follows that $S(N(x), T(x,y_1)) = S(N(x), T(x,y_2))$, which implies that $I_{T,S,N}(x,y_1) = I_{T,S,N}(x,y_2)$ with $y_1 < y_2$. This is a contradiction to the fact that $I_{T,S,N}$ is strictly monotone. Thus, T is strictly monotone. □

Remark 5.15. (i) In Theorem 5.14, the t-conorm S need not be strictly monotone always. To see this, let $T = T_{\mathbf{P}}, S = S_{\mathbf{LK}}$ and $N = N_{\mathbf{C}}$. Then, from the duality between t-norms and t-conorms and Theorem 2.18 in [16], it follows that $S = S_{\mathbf{LK}}$ is not strictly monotone. However, from Table 2.8 in [2], we get that the QL-implication $I_{T,S,N} = I_{\mathbf{RC}}$ is strictly monotone.

(ii) Let S be a t-conorm, T a t-norm and N a fuzzy negation such that $I_{T,S,N}$ is a fuzzy implication. Note that if S is strictly monotone (implies S is positive, see Definition 2.2.2 in [2]) then from Proposition 2.6.7 in [2], we get that $N = N_{\mathbf{D2}}$, which is not strict and hence, the QL-implication $I_{T,S,N} = I_{\mathbf{WB}}$ satisfies neither (SM1) nor (SM2).

Since there are no characterization results available for QL-implications, so far, in general, we content ourselves with the necessary conditions provided in Theorem 5.14 for the strict monotonicity of QL-implications.

5.4. f-generated implications

In [37], Yager proposed a new family of fuzzy implications from the additive generators of continuous Archimedean t-norms. They are defined as follows.

Definition 5.16. (Baczynski and Jayaram [2], Yager [37]) Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing and continuous function with $f(1) = 0$. The function $I : [0,1]^2 \to [0,1]$ defined by

$$I(x,y) = f^{-1}(x \cdot f(y)), \qquad x, y \in [0,1],$$

with the understanding $0 \cdot \infty = 0$, is called an *f-generated implication.*

If I is an f-generated implication then it is denoted by I_f. We use $\mathbb{I}_{\mathbb{F}}$ to denote the set of f-generated implications. Two subsets of $\mathbb{I}_{\mathbb{F}}$ are classified as follows:

- $\mathbb{I}_{\mathbb{F},\infty}$ – the family of f-generated implications such that $f(0) = \infty$.
- $\mathbb{I}_{\mathbb{F},1}$ – the family of f-generated implications such that $f(0) < \infty$.

With respect to the strict monotonicity for f-implications, we have the following example:

Example 5.17. (i) Recall that $I_{\mathbf{RC}}$ is an f-implication generated by $f(x) = 1 - x$, for all $x \in [0, 1]$. Then, from Table 1, it follows that $I_{\mathbf{RC}}$ is strictly monotone.

(ii) Also, from Example 3.1.3(i) in [2], recall that $I_{\mathbf{YG}} \in \mathbb{I}_{\mathbb{F}}$. Since $I(0.3, 0) = 0 = I(0.6, 0)$, it follows that $I_{\mathbf{YG}}$ does not satisfy (SM1) and hence strict monotonicity.

From Example 5.17, it follows that f-implications need not satisfy strict monotonicity, in general. In the following, we investigate the f-implications which fulfill the strict monotonicity.

Lemma 5.18. Let I_f be an f-generated fuzzy implication. Then the following statements hold true:

(i) I_f satisfies (SM2).

(ii) I_f satisfies (SM1) only when $f(0) < \infty$.

Proof. Let $I_f \in \mathbb{I}_{\mathbb{F}}$.

(i) Follows from Lemma 6.25 in [35].

(ii) Let $f(0) < \infty$. Then, from Lemma 3.1.8 in [2], (SM1) of I_f follows directly.

□

Corollary 5.19. Let I_f be an f-generated fuzzy implication. Then I_f is strictly monotone if and only if $f(0) < \infty$, i. e., $I_f \in \mathbb{I}_{\mathbb{F},1}$.

Remark 5.20. From Corollary 5.19, note that the strict monotonicity is a criterion to characterize whether an f-generated fuzzy implication belongs to the set $\mathbb{I}_{\mathbb{F},1}$ or not.

5.5. g-generated implications

In [37], Yager proposed also a new family of fuzzy implications using the multiplicative generators of continuous Archimedean t-norms. They are defined as follows.

Definition 5.21. (Baczynski and Jayaram [2], Yager [37]) Let $g : [0, 1] \to [0, \infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I : [0, 1]^2 \to [0, 1]$ defined by

$$I(x, y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right), \qquad x, y \in [0, 1],$$

with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$, is called a *g-generated implication*, where the function $g^{(-1)}$ is the pseudo inverse of g.

If I is a g-generated implication then it is denoted by I_g. Let $\mathbb{I}_{\mathbb{G}}$ denote the family of g-generated implications. Let us use the following notation for subclasses of $\mathbb{I}_{\mathbb{G}}$:

- $\mathbb{I}_{\mathbb{G},\infty}$ – the family of g-generated implications such that $g(1) = \infty$.
- $\mathbb{I}_{\mathbb{G},1}$ – the family of g-generated implications such that $g(1) < \infty$.

With respect to the strict monotonicity of g-generated implications, we have the following.

Theorem 5.22. Let I_g be a g-generated implication. Then the following statements are true.

(i) I_g satisfies (SM2) if and only if $g(1) = \infty$.

(ii) I_g satisfies (SM1) on $(0,1) \times (0,1)$ if and only if $g(1) = \infty$.

Proof. Let I_g be a g-generated implication.

(i) ($\Rightarrow$) Let I_g satisfy (SM2) and suppose $g(1) < \infty$. Then I_g is given, for all $x, y \in [0,1]$, by

$$I_g(x,y) = \begin{cases} 1, & \text{if } x \leq g(y), \\ g^{-1}\Big(\frac{1}{x} \cdot g(y)\Big), & \text{if } x > g(y). \end{cases} \tag{5}$$

Now, choose $x, y_1, y_2 \in [0,1]$ be such that $y_1 < y_2$ and $x \leq g(y_1), g(y_2)$. Then, from Eq.(5), it follows that $I_g(x, y_1) = 1 = I_g(x, y_2)$, while $y_1 < y_2$, a contradiction. Thus, $g(1) = \infty$.
($\Leftarrow$) Conversely, let I_g be a g-implication such that $g(1) = \infty$. Then, from Proposition 4.4.1 in [2], it follows that $I_g \in \mathbb{I}_{\mathbb{G},\infty} = \mathbb{I}_{\mathbb{F},\infty}$. Thus, from Lemma 5.18(i), it follows that I_g satisfies (SM2).

(ii) ($\Rightarrow$) Let I_g satisfy (SM1) on $(0,1)$ and suppose $g(1) < \infty$. Then, for any $y \in (0,1)$ choose $x_1 < x_2 \in (0,1]$ such that $x_1 \leq g(y)$ and $x_2 \leq g(y)$. Then, from Eq.(5), we get that $I_g(x_1, y) = 1 = I_g(x_2, y)$ with $x_1 < x_2$, a contradiction. Thus, we have $g(1) = \infty$.
($\Leftarrow$) Conversely, assume that $g(1) = \infty$. Then $I_g(x,y) = g^{-1}\Big(\frac{1}{x} \cdot g(y)\Big)$, for all $x, y \in [0,1]$. Since $g(0) = 0$, for any different $x_1, x_2 \in (0,1]$, we get $I_g(x_1, 0) = 0 = I_g(x_2, 0)$ and hence, I_g does not satisfy (SM1) at 0. On the other hand, for any $y \in (0,1)$ and $x_1, x_2 \in (0,1]$, the condition $I_g(x_1, y) = I_g(x_2, y)$ implies that $x_1 = x_2$. Thus, on $(0,1)$, g-implication I_g satisfies (SM1) whenever $g(1) = \infty$.

□

Corollary 5.23. g-generated implications are not strictly monotone.

Remark 5.24. From Corollary 5.23, it follows that if $I \in \mathbb{I}$ is strictly monotone then I cannot be a g-generated implication for any g-generator. Thus, strict monotonicity becomes a criterion to characterize whether a fuzzy implication is a g-generated implication or not.

5.6. h-generated fuzzy implications

In [27], Massanet and Torrens proposed a new family of fuzzy implications using the additive generators of representable uninorms. They are defined as follows.

Definition 5.25. (Massanet and Torrens [27]) Fix an $e \in (0,1)$ and let $h : [0,1] \to [-\infty, +\infty]$ be a strictly increasing and continuous function with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$. The function $I^h : [0,1]^2 \to [0,1]$ defined, for all $x, y \in [0,1]$, by

$$I^h(x,y) = \begin{cases} 1 & \text{if } x = 0, \\ h^{-1}\left(x \cdot h(y)\right) & \text{if } x > 0 \text{ and } y \le e, \\ h^{-1}\left(\frac{1}{x} \cdot h(y)\right) & \text{if } x > 0 \text{ and } y > e, \end{cases}$$

is called an h-generated implication.

Remark 5.26. Let I^h be an h-generated implication and $x_1, x_2 \in (0,1]$ be such that $x_1 < x_2$. Then, from Definition 5.25, it follows that $I^h(x_1, e) = e = I^h(x_2, e)$. Thus, I^h does not satisfy (SM1). However, except at $y = e$, the h-generated implication I^h satisfies (SM1) always, which is proved below.

Theorem 5.27. Let I be a h-generated fuzzy implication on $[0,1]$. Then,

(i) I is locally left strictly monotone on $(0,1] \times ([0,1) \setminus \{e\})$.

(ii) I satisfies (SM2) always.

P r o o f. Let I^h be an h-generated implication.

(i) Let $y \in [0,1) \setminus \{e\}$ and $x_1, x_2 \in (0,1]$ be such that $I^h(x_1, y) = I^h(x_2, y)$.

- Let $y \in [0, e)$. Then $I^h(x_1, y) = I^h(x_2, y)$ implies that $h^{-1}(x_1 \cdot h(y)) = h^{-1}(x_2 \cdot h(y))$. Now, from the strictness of h, it follows directly that $x_1 = x_2$.
- Let $y \in (e, 1]$. Then $I^h(x_1, y) = I^h(x_2, y)$ implies that $h^{-1}\left(\frac{1}{x_1} \cdot h(y)\right) = h^{-1}\left(\frac{1}{x_2} \cdot h(y)\right)$. Now, from the strictness of h, it follows directly that $x_1 = x_2$.

(ii) Let $x > 0$ and $y_1, y_2 \in [0,1]$ be such that $I^h(x, y_1) = I^h(x, y_2)$. Note that, for y_1, y_2, there are three possibilities, namely, $y_1, y_2 \in [0, e]$ or $y_1, y_2 \in (e, 1]$ or y_1, y_2 belong to different intervals. As the proof for first two cases directly follows from the strictness of h, we discuss the third case. Let $y_1 \in [0, e]$ and $y_2 \in (e, 1]$. Then, from $I^h(x, y_1) = I^h(x, y_2)$ and strictness of h, we get that $x^2 \cdot h(y_1) = h(y_2)$. Since $h(y_1) < 0$ and $h(y_2) > 0$, it is not possible to have some $x \in (0,1]$ such that $x^2 \cdot h(y_1) = h(y_2)$. Thus, the third possibility does not occur and hence, I^h satisfies (SM2).

□

From Theorem 5.27, it follows that h-generated implications are locally strictly monotone on $(0,1] \times ([0,1) \setminus \{e\})$.

5.7. generalized (h, e)-generated fuzzy implications

In the following, we recall the definition of (h, e)-generated fuzzy implications.

Definition 5.28. (Hliněná et al. [14]) Fix an $e \in (0,1)$ and let $h : [0,1] \to [-\infty, +\infty]$ be a strictly increasing and continuous function with $h(e) = 0$ and $h(1) = +\infty$. The function $I^{h_g,e} : [0,1]^2 \to [0,1]$ defined, for all $x, y \in [0,1]$, by

$$I^{h_g,e}(x,y) = \begin{cases} 1 & \text{if } x = 0, \\ h^{(-1)}\left(\frac{x}{e} \cdot h(y)\right) & \text{if } x > 0 \text{ and } y \le e, \\ h^{-1}\left(\frac{e}{x} \cdot h(y)\right) & \text{if } x > 0 \text{ and } y > e, \end{cases} \tag{6}$$

where the function $h^{(-1)}$ is the pseudo-inverse of h.

Theorem 5.29. Let $I^{h_g,e}$ be a generalized (h, e)-generated implication. Then,

(i) $I^{h_g,e}$ is locally right strictly monotone if and only if the second variable of $I^{h_g,e}$ lies in $(e, 1]$.

(ii) $I^{h_g,e}$ is locally left strictly monotone when the second variable is in $(e, 1)$.

Proof. Let $I^{h_g,e}$ be any generalized (h, e)-generated implication.

(i) $(\Rightarrow)$ Let $I^{h_g,e}$ be locally right strictly monotone and $x > 0, y_1, y_2 \in [0, 1)$. Then choose $y_1, y_2 \in [0, e]$ be such that $y_1 < y_2$ and $-\infty < \dfrac{x}{e}h(y_1), \dfrac{x}{e}h(y_2) < h(0)$. Then, from Eq.(6), it follows that, $I^{h_g,e}(x_1, y) = 0 = I^{h_g,e}(x, y)$ while $y_1 < y_2$. Thus, $I^{h_g,e}$ does not satisfy (SM2) when the second variable lies in $[0, e]$.
$(\Leftarrow)$ Let $x > 0$ and $y_1, y_2 \in (e, 1]$. Then, from Eq.(6), it follows that,

$$\begin{gathered} I^{h_g,e}(x, y_1) = I^{h_g,e}(x, y_2) \Longleftrightarrow h^{-1}\left(\frac{e}{x} \cdot h(y_1)\right) = h^{-1}\left(\frac{e}{x} \cdot h(y_2)\right) \Longleftrightarrow \\ \frac{e}{x} \cdot h(y_1) = \frac{e}{x} \cdot h(y_2) \Longleftrightarrow h(y_1) = h(y_2) \Longleftrightarrow y_1 = y_2, \end{gathered}$$

due to the fact h is strictly increasing. Thus, $I^{h_g,e}$ is locally right strictly monotone when the second variable lies in $(e, 1]$.

(ii) Let $x, y \in (0, 1]$ and $z \in (e, 1]$ be such that $I(x, z) = I(y, z)$. Then from Eq.(6), due to the strictness of h, we get $x = y$ directly.

□

Corollary 5.30. Generalized (h, e)-generated implications are locally strictly monotone if and only if the second variable lies in $(e, 1)$.

Thus, in this case too, the strict monotonicity plays an important role in characterizing a fuzzy implication is a generalized (h, e)-implication or not.

5.8. k-generated fuzzy implications

In [38], Zhou proposed a new family of fuzzy implications using the continuous multiplicative generators of t-norms. They are defined as follows.

Definition 5.31. (Zhou [38]) Let $k : [0,1] \to [0,1]$ be a strictly increasing and continuous function with $k(1) = 1$. The function $I_k : [0,1]^2 \to [0,1]$ defined by

$$I_k(x,y) = k^{(-1)}\left(\frac{1}{x} \cdot k(y)\right), \quad x,y \in [0,1],$$

with the understanding $\frac{0}{0} = 1$ and $\frac{1}{0} = +\infty$ and where $k^{(-1)}$ is the pseudo-inverse of k, is called a k-generated implication.

In this case, I_k is given by, for all $x,y \in [0,1]$,

$$I_k(x,y) = k^{-1}\Big(\min(\frac{1}{x}k(y),1)\Big). \tag{7}$$

For a k-generated fuzzy implication I_k with generator k, let us denote a subset $\mathcal{D}_k$ of $[0,1]^2$ as follows:

$$\mathcal{D}_k = \{(x,y) \in (0,1] \times [0,1) | x > k(y)\}.$$

Theorem 5.32. Let I be a k-generated implication. Then,

(i) I satisfies the locally right strictly monotonicity on $\mathcal{D}_k$ of $[0,1]^2$.

(ii) I satisfies locally left strictly monotonicity on $\mathcal{D}_k$ of $[0,1]^2$.

Proof. Let I_k be a k-generated implication for some generator k.

(i) Let $x > 0$, $y_1, y_2 \in [0,1)$. We discuss locally right strict monotonicity of I_k in three cases.

- Let $x > 0$, $y_1 < y_2 \in [0,1)$ be such that $x \leq k(y_1), k(y_2)$. Then from Eq.(7), it follows that $I_k(x,y_1) = 1 = I_k(x,y_2)$, which shows that I_k need not satisfy (SM2).
- Let $x > 0$, $y_1, y_2 \in [0,1)$ be such that $x > k(y_1), k(y_2)$. Then from Eq.(7), it follows that

$$I_k(x,y_1) = I_k(x,y_2) \Longleftrightarrow k^{-1}\left(\frac{1}{x} \cdot k(y_1)\right) = k^{-1}\left(\frac{1}{x} \cdot k(y_2)\right) \Longleftrightarrow$$
$$\frac{1}{x} \cdot k(y_1) = \frac{1}{x} \cdot k(y_2) \Longleftrightarrow k(y_1) = k(y_2) \Longleftrightarrow y_1 = y_2,$$

due to the fact k is strictly increasing. Thus, in this case I_k satisfies locally right strict monotonicity on $\mathcal{D}_k$.

- Let $x > 0$, $y_1 < y_2 \in [0,1)$ be such that $k(y_1) < x \leq k(y_2)$. Then from Eq.(7), it follows that

$$I_k(x,y_1) = I_k(x,y_2) \Longleftrightarrow k^{-1}\left(\frac{1}{x}\cdot k(y_1)\right) = 1 \Longleftrightarrow \frac{1}{x}\cdot k(y_1) = 1 \Longleftrightarrow x = k(y_1),$$

  a contradiction. Thus, there exist no $x > 0$, $y_1 < y_2 \in [0,1)$ be such that $k(y_1) < x \leq k(y_2)$ and $I_k(x,y_1) = I_k(x,y_2)$. Thus, in this case also, I_k does not satisfy (SM2).

(ii) This proof is similar to the previous case.

□

Corollary 5.33. k-generated implication I_k is locally strict monotone on $\mathcal{D}_k$ of $[0,1]^2$.

Remark 5.34. From Corollary 5.33, note that the strict monotonicity is not fulfilled by k-generated fuzzy implications and hence, the strict monotonicity is a criterion to characterize k-generated implications.

6. FUZZY POLYNOMIAL IMPLICATIONS W.R.T. STRICT MONOTONICITY

In [23], Massanet et al proposed a class of fuzzy implications that are expressible as bivariate polynomials on $[0,1]$. They are defined as follows.

Definition 6.1. (Massanet et al. [22], Massanet et al. [23]) Let $n \in \mathbb{N}$. A binary operator $I\colon [0,1]^2 \to [0,1]$ is called a fuzzy polynomial implication of degree n if it is a fuzzy implication function and its expression is given by

$$I(x,y) = \sum_{\substack{0\leq i,j\leq n\\ i+j\leq n}} a_{ij}x^i y^j, \tag{8}$$

for all $x, y \in [0,1]$, $a_{ij} \in \mathbb{R}$ and there exist some $0 \leq i, j \leq n$ with $i + j = n$ such that $a_{ij} \neq 0$.

Theorem 6.2. Every fuzzy polynomial implication is strictly monotone.

Proof. Let I be a fuzzy polynomial implication given by (8).

- Let $x, y \in [0,1], z \in [0,1)$. Then, we get

$$\begin{aligned} I(x,z) = I(y,z) &\Longleftrightarrow \sum_{\substack{0\leq i,j\leq n\\ i+j\leq n}} a_{ij}x^i z^j = \sum_{\substack{0\leq i,j\leq n\\ i+j\leq n}} a_{ij}y^i z^j \\ &\Longleftrightarrow \sum_{\substack{0\leq i,j\leq n\\ i+j\leq n}} a_{ij}(x^i - y^i)z^j = 0 \\ &\Longleftrightarrow x^i - y^i = 0 \Longleftrightarrow x - y = 0 \qquad \lceil\text{due to } a_{ij} \neq 0. \end{aligned}$$

  Thus, I satisfies (SM1).

- I satisfies (SM2) can be proved similarly. □

Remark 6.3. (i) Note that every strictly monotone fuzzy implication need not be a fuzzy polynomial implication. For example, I defined in Eq. (2) is a strictly monotone fuzzy implication which is not a fuzzy polynomial implication.

(ii) Also note that, strict monotonicity is a necessary condition for a fuzzy implication I to be a fuzzy polynomial implication. Thus, strict monotonicity provides a partial characterization of fuzzy polynomial implications.

As a generalization to fuzzy polynomial implications, Massanet et al proposed fuzzy (OP)-polynomial implications in [24] as follows.

Definition 6.4. (Massanet et al. [24]) A binary operator $I : [0,1]^2 \to [0,1]$ is called a fuzzy (OP)-polynomial implication of degree n if it is a fuzzy implication and its expression is given by

$$I(x,y) = \begin{cases} 1, & \text{if } x \le y, \\ \displaystyle\sum_{\substack{0\le i,j\le n\\ i+j\le n}} a_{ij}x^i y^j, & \text{if } x > y, \end{cases} \tag{9}$$

for all $x, y \in [0,1]$, $a_{ij} \in \mathbb{R}$ and there exist some $0 \le i, j \le n$ with $i + j = n$ such that $a_{ij} \neq 0$.

Remark 6.5. If I is a fuzzy (OP)-polynomial implication, then from Definition 6.4, I satisfies (OP). Consequently, I does not satisfy the strict monotonicity (SM), due to Lemma 3.2. Strictly speaking, I does not satisfy (SM) on the set $\{(x,y) \in [0,1]^2 | x \le y\}$. For the strict monotonicity of I on the remaining region, we have the following result.

Lemma 6.6. Every fuzzy (OP)-polynomial implication is locally strict monotone on the set $S_2 \cup S_3$. i.e., $\{(x,y) \in [0,1]^2 | x > y\}$.

P r o o f. Follows from Theorem 6.2. □

As yet another generalization of fuzzy polynomial implications, Massanet et al proposed fuzzy rational implications in [25] as follows.

Definition 6.7. (Massanet et al. [25]) Consider $n, m \in \mathbb{N}$. A binary operator $I : [0,1]^2 \to [0,1]$ is called a fuzzy rational implication of degree (n,m) if it is a fuzzy implication function and its expression is given by

$$I(x,y) = \frac{\displaystyle\sum_{\substack{0\le i,j\le n\\ i+j\le n}} a_{ij}x^i y^j}{\displaystyle\sum_{\substack{0\le s,t\le m\\ s+t\le m}} b_{st}x^s y^t}, \tag{10}$$

for all $x, y \in [0,1]$ where

(i) $a_{ij} \in \mathbb{R}$ for all $0 \leq i, j \leq n$ and $i + j \leq n$ and there exist some $0 \leq i, j \leq n$ with $i + j = n$ such that $a_{ij} \neq 0$.

(ii) $b_{st} \in \mathbb{R}$ for all $0 \leq s, t \leq m$ and $s + t \leq m$ and there exist some $0 \leq s, t \leq m$ with $s + t = m$ such that $b_{st} \neq 0$.

(iii) the polynomials $p(x, y) = \sum\limits_{\substack{0 \leq i,j \leq n \\ i+j \leq n}} a_{ij} x^i y^j$ and $q(x, y) = \sum\limits_{\substack{0 \leq s,t \leq m \\ s+t \leq m}} b_{st} x^s y^t$ have no factors in common.

(iv) $q(x, y) \neq 0$, for all $x, y \in [0, 1]$.

Proposition 6.8. Every fuzzy rational implication is strictly monotone.

Proof. Let I be fuzzy rational implication of the form (10) for some polynomials p and q satisfying the conditions given in Definition 6.7. We prove that I satisfies (SM2) only since the proof for (SM1) follows similarly. Let $x, y, z \in [0, 1]$ be such that $x > 0$ and $I(x, y) = I(x, z)$. Then, we get

$$
\begin{aligned}
I(x, y) = I(x, z) &\iff \frac{\sum\limits_{\substack{0 \leq i,j \leq n \\ i+j \leq n}} a_{ij} x^i y^j}{\sum\limits_{\substack{0 \leq s,t \leq m \\ s+t \leq m}} b_{st} x^s y^t} = \frac{\sum\limits_{\substack{0 \leq i,j \leq n \\ i+j \leq n}} a_{ij} x^i z^j}{\sum\limits_{\substack{0 \leq s,t \leq m \\ s+t \leq m}} b_{st} x^s z^t} \\
&\iff \sum_{\substack{0 \leq i,j \leq n \\ i+j \leq n}} a_{ij} x^i y^j \cdot \sum_{\substack{0 \leq s,t \leq m \\ s+t \leq m}} b_{st} x^s z^t = \sum_{\substack{0 \leq s,t \leq m \\ s+t \leq m}} b_{st} x^s y^t \cdot \sum_{\substack{0 \leq i,j \leq n \\ i+j \leq n}} a_{ij} x^i z^j \\
&\iff \sum_{\substack{(0 \leq i,j \leq n) \ \& \ (0 \leq s,t \leq m) \\ (i+j \leq n) \& (s+t \leq m)}} c_{ijst} x^{i+s} y^j z^t = \sum_{\substack{(0 \leq i,j \leq n) \ \& \ (0 \leq s,t \leq m) \\ (i+j \leq n) \ \& \ (s+t \leq m)}} c_{ijst} x^{i+s} y^t z^j \\
&\iff \sum_{\substack{(0 \leq i,j \leq n) \ \& \ (0 \leq s,t \leq m) \\ (i+j \leq n) \ \& \ (s+t \leq m)}} c_{ijst} x^{i+s} (y^j z^t - y^t z^j) = 0,
\end{aligned}
$$

for some constants $c_{ijst} \in \mathbb{R}$. Since p, q are non-zero polynomials, clearly some constants $c_{ijst} \in \mathbb{R}$ are non-zero. Thus, for all $x > 0$, we get $x^{i+s}(y^j z^t - y^t z^j) = 0$, which obviously implies $y = z$. □

Note that Remark 6.3 holds true for fuzzy rational implications also.

7. CONCLUDING REMARKS

In this work, we recalled that monotonicity has been used to define the fuzzy logic connectives such as fuzzy negations, t-norms, t-conorms and fuzzy implications. However, it is pointed out that the strict monotonicity is not yet proposed and studied for fuzzy implications, in general. Since fuzzy implications are not commutative, we proposed strict monotonicity of fuzzy implications in each variable, as (SM1) and (SM2). Then, we investigated some necessary conditions on fuzzy implications to fulfill either (SM1) or (SM2) or both. Next, we explored the relationship between the basic properties, functional equations of fuzzy implications and the strict monotonicity. Finally, we have studied the strict monotonicity for fuzzy implications that do come from different families of fuzzy implications. The salient features of the work done are as follows:

- It was proved that the trivial range fuzzy implications do not satisfy strict monotonicity.
- It was shown that the strict monotonicity of fuzzy implications is independent of other basic properties and functional equations of fuzzy implications.
- The sub-classes of (S, N)-implications and f-implications that satisfy the strict monotonicity were investigated.
- Since none of R-implications, Yager's g-implications, h, generalized (h, e) and k-implications satisfy either (SM1) or (SM2), it is to be noted that strict monotonicity becomes a criterion to validate whether a fuzzy implication belongs to these families of fuzzy implications, whatever the case it is. Also, we have investigated the sub-regions of the unit square on which the aforementioned fuzzy implications satisfy either (SM1) or (SM2) or both locally, whatever the case it is.
- It was shown that the strict monotonicity is a necessary condition for fuzzy polynomial implications and fuzzy rational implications. Since fuzzy (OP)-implications do not satisfy the strict monotonicity, it was proved that they are locally strictly monotone on the set $\{(x, y) \in [0, 1]^2 | x > y\}$.

Finally observe that, in this work, we have obtained some insights of fuzzy implications satisfying the strict monotonicity without assuming any additional conditions/ properties of fuzzy implications. However, it is obvious that the strict monotonicity along with some properties of fuzzy implications will enable us to glean more perspectives of fuzzy implications in the future.

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DECLARATION OF CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

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