A SURVEY AND COMPARATIVE ANALYSIS OF DIFFERENT APPROACHES TO FUZZY DIFFERENTIAL EQUATIONS MODELING DYNAMICS WITH UNCERTAIN PARAMETERS OF DETERMINISTIC CHARACTER

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Dynamics containing deterministic uncertainties can be modeled with fuzzy differential equations. Unlike classical differential equations, fuzzy differential equations lack a unified interpretation and theoretical foundation, as researchers adopt different approaches to fuzziness, solution concepts, and underlying mathematical structures. The main reason is whether the fuzzy function derivative is used in the equation in question and, if it is used, what meaning it carries. Researchers who do not involve a derivative of a fuzzy number-valued function either use the extension principle, an alternative concept of fuzzy function, or transform the problem into a differential inclusion. Various definitions have been used in studies involving the derivatives of fuzzy number-valued functions. The main reason is that none of the known derivatives can fully meet the requirements: either the fuzziness increases excessively, or it becomes impossible to solve higher-order equations, or unnatural assumptions must be made. In this study, we tried to classify almost all studies on fuzzy differential equations. We compared the results of studies conducted in relatively recent years, particularly in initial value and boundary value problems, using examples. We discussed the possible direction of future research on fuzzy differential equations.

Keywords: fuzzy differential equations, interval differential equations, initial value problem, boundary value problem, bunch of functions, linear differential equations

Classification: 34B05, 93B03, 65G40

1. INTRODUCTION

Many new scientific studies on fuzzy sets, systems, and logic have been published in various fields of science and engineering. It is impossible to list all of these studies, so we give a few examples [40, 60, 61, 162, 174, 198, 226, 227, 250, 273]. The starting point of all these studies is Zadeh's article [308]. This article, published in 1965, was a revolutionary study in science, but it took at least 10 more years to understand that this was so. There was a serious objection to the work by mathematicians, and from their point of view, their objections were justified; this paper was not a serious mathematical work. Zadeh did not make such a claim anyway; he rather explained a new theory to

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model uncertainties. Probability theory experts objected to this: Was there a need for a new theory when there was probability theory? The flow of life answered "yes" to this question, and today, it is accepted by almost everyone that Zadeh's fuzzy logic and fuzzy sets theory have made an undeniable contribution to the development of artificial intelligence. It is worth noting that the path to the acceptance of Zadeh's theory was not straightforward. Until about 1975, most journals avoided accepting research papers on fuzzy theory. Interestingly, Kybernetika, which turns 60 in 2024, was one of the first journals to start publishing research on the topic. The journal declared that it was taking fuzzy theory research seriously by publishing a paper by Kramosil and Michálek [179] in 1975. As in the early days, the journal continues to publish studies on fuzzy logic, fuzzy sets, and systems [12, 49, 73, 124, 125, 127, 138, 278, 311].

After Zadeh's theory was accepted, over the years, mathematicians have tried to transfer the achievements of classical mathematics to fuzzy logic and fuzzy set theory, sometimes as necessary and sometimes as unnecessary. This wasn't always easy to do. Before defining concepts such as limit, derivative, and integral, it was necessary to define arithmetic operations here. Scientists doing research in the field of fuzzy logic and fuzzy sets know well how many difficulties a simple subtraction operation creates. Of course, we need to respect any scientific study that does not contain errors, just like Zadeh's article; it is difficult to predict which study will be linked where. History is full of examples of this. Zadeh himself could not have known that the article he published with great difficulty would make such an impact years later. However, we have difficulty understanding the attempts to transfer subjects that are too theoretical in mathematics and far removed from real life to fuzzy logic and fuzzy set theory. Differential equations, on the other hand, are not disconnected from real life but are a tool that allows us to model almost all kinds of motions.

The motion to be modeled may contain uncertainties. Stochastic differential equations [199, 205, 206], interval differential equations [104, 116, 118], and fuzzy differential equations [145, 147] can be used depending on the type of problem being considered and the uncertainty present. If the uncertainty includes randomness, the motion considered can be modeled with a stochastic equation. On the contrary, if the uncertainty is deterministic, an interval or fuzzy differential equation model is used. If the possibilities of all values that the variable containing uncertainty can take are equal, the interval model is appropriate, and if they are different, the fuzzy differential equation model is appropriate.

In this study, we examine the studies done so far on fuzzy differential equations (FDEs), classify these studies, compare the proposed methods with each other, investigate the weaknesses and strengths of different approaches, and discuss future research directions on this subject. We can divide the studies published so far on fuzzy differential equations into two main groups:
1) Studies using only classical derivatives, and 2) Studies using a new derivative concept.

The studies in the first group can be divided into the following subgroups:

– studies using Zadeh's extension principle

– studies using differential inclusion

– studies using a fuzzy bunch of real-valued functions.

The studies in the second group can be divided into the following subgroups:

- studies using Hukuhara derivative
- studies using the strongly generalized derivative (Bede–Gal derivative) defined via the Hukuhara difference
- studies using the Bede–Gal derivative defined via the generalized Hukuhara difference
- studies using interactive derivative
- studies using granular derivative.

In this study, we have examined more than 300 works on fuzzy differential equations. These works cover almost all studies done on this subject so far. We can summarize the results we observed in these studies as follows:

- In almost all of the studies, the coefficients of the differential equations being considered are real numbers, and the fuzziness is in the initial or boundary values. The main reason for this is that existing fuzzy number-valued derivatives are not effective enough in solving differential equations with fuzzy coefficients.
- All the proposed methods are naturally based on solving real differential equations. In some of the methods, the authors do this at the beginning stage of the solution, and in others, at the final stage. However, it does not matter when this is done; if the derivations are correct, the solution to the same problem will be the same in both cases.
- Derivatives defined so far for fuzzy number-valued functions have their shortcomings, and proposing a new derivative concept or a new method can only make sense in two cases: with their help, either at least a class of problems that existing methods cannot solve becomes solvable, or some problems can be solved more quickly.

2. PRELIMINARIES

This section gives preliminary information about fuzzy sets and fuzzy numbers used in this study.

2.1. Basic concepts of fuzzy sets theory

In classical set theory, an element either belongs to a set or does not belong. This belonging is usually expressed with the help of the characteristic function

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & \text{otherwise.} \end{cases}$$

In the theory of fuzzy sets, every element of the universal set U belongs to every fuzzy set. This belonging has a membership degree specific to the considered fuzzy set. Thus, a **fuzzy set** $\widetilde{A}$ is determined by a pair of the **universal set** U and the **membership function** $\mu : U \to [0,\ 1]$. We denote the membership function as $\mu_{\widetilde{A}}$ to emphasize that the fuzzy set $\widetilde{A}$ is under consideration. For each $x \in U$, the number $\mu_{\widetilde{A}}(x)$ is called the **membership degree** of x in $\widetilde{A}$. The classical set $\operatorname{supp}(\widetilde{A}) = \{x \in U \mid \mu_{\widetilde{A}}(x) > 0\}$ determines the **support** of the fuzzy set $\widetilde{A}$.

For a given $\alpha \in (0,\ 1]$, the classical set $A_\alpha = \{x \mid \mu_{\widetilde{A}}(x) \geq \alpha\}$ is called the α-cut of the fuzzy set $\widetilde{A}$. (The α-cut is also denoted as $[\widetilde{A}]_\alpha$.) For $\alpha = 0$, the 0-cut is defined as the closure of the support of $\widetilde{A}$, that is, $A_0 = \operatorname{closure}(\operatorname{supp}(\widetilde{A}))$.

In the special case, if the universal set U is the set of real numbers, that is, if $U = \mathbb{R}$, then the fuzzy sets that satisfy certain conditions are called fuzzy numbers.

Definition 2.1. Let $\widetilde{A}$ be a fuzzy set on the set of real numbers $U = \mathbb{R}$. If $\widetilde{A}$ satisfies the following conditions, then this fuzzy set is called a fuzzy number:

- $\widetilde{A}$ is a fuzzy normal set, i. e., $\mu_{\widetilde{A}}(x_0) = 1$ for some $x_0 \in \mathbb{R}$.
- $\widetilde{A}$ is a fuzzy convex set, i. e., $\mu_{\widetilde{A}}(tx+(1-t)y) \geq \min\{\mu_{\widetilde{A}}(x),\ \mu_{\widetilde{A}}(y)\}$ for all $t \in [0,\ 1]$ and for all $x, y \in \mathbb{R}$.
- $\mu_{\widetilde{A}}(x)$ is a upper semicontinuous function on $\mathbb{R}$, i. e., if $x_0 \in \mathbb{R}$, then $\forall \varepsilon > 0,\ \exists \delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$ the inequality $\mu_{\widetilde{A}}(x) - \mu_{\widetilde{A}}(x_0) < \varepsilon$ satisfies.
- The set $\operatorname{closure}(\operatorname{supp}(\widetilde{A}))$ is a closed, bounded interval.

For example, the following membership function represents a fuzzy number:

$$\mu_{\widetilde{A}}(x) = \begin{cases} (x-1)^3, & \text{if } 1 \leq x \leq 2 \\ (3-x)^2, & \text{if } 2 < x < 3 \\ 0, & \text{otherwise.} \end{cases}$$

The α-cuts of the above fuzzy number $\widetilde{A}$ are $A_\alpha = [\underline{A_\alpha},\ \overline{A_\alpha}] = [1 + \sqrt[3]{\alpha},\ 3 - \sqrt{\alpha}]$, for all $\alpha \in [0,\ 1]$.

The set of fuzzy numbers will be denoted as $\mathcal{F}_{\mathbb{R}}$. We note that $\mathbb{R} \subset \mathcal{F}_{\mathbb{R}}$. We also note that for a given fuzzy number $\widetilde{A}$, the α-cuts are closed, bounded intervals in $\mathbb{R}$.

Let a, c and b be real numbers such that $a \leq c \leq b$. A fuzzy number $\widetilde{A}$ with membership function

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{x-a}{c-a}, & a < x < c \\ 1, & x = c \\ \frac{b-x}{b-c}, & c < x < b \\ 0, & \text{otherwise} \end{cases}$$

is called a **triangular fuzzy number** and is denoted as $\widetilde{A} = (a,\ c,\ b)$.

We can express a triangular fuzzy number $\widetilde{A}$ as $\widetilde{A} = a_{cr} + \widetilde{A}_{un}$ (crisp part + uncertain part). Here $a_{cr} = c$ and $\widetilde{A}_{un} = (a-c,\ 0,\ b-c)$.

For a triangular fuzzy number $\widetilde{A} = (a,\ c,\ b)$ the α-cuts are intervals $A_\alpha = [\underline{A_\alpha},\ \overline{A_\alpha}]$, where $\underline{A_\alpha} = a + \alpha(c-a)$ and $\overline{A_\alpha} = b + \alpha(c-b)$.

In this paper, we use the usual arithmetic operations (i. e., Minkowski operations) of addition, subtraction, multiplication, and division on fuzzy numbers. It is worth emphasizing that these operations can also be considered as Zadeh's extensions of the corresponding operations on real numbers.

2.2. Triangular fuzzy function

Fuzzy functions can be defined in two different ways: as fuzzy number-valued functions or as fuzzy sets (bunches) of real-valued functions. In the second case, every real-valued function belongs to the fuzzy set with a certain membership degree. When we refer to the value $\widetilde{F}(t)$ of such a fuzzy function $\widetilde{F}$ at time t, we mean a fuzzy set (i. e., a fuzzy number) consisting of values of the real functions at t. If the same value takes place for different functions, the higher membership degree of the functions is assigned to be the membership degree of the value. More formally, we put

$$\mu_{\widetilde{F}(t)}(x) = \alpha, \quad \text{if } \exists y(\cdot) : (\mu_{\widetilde{F}}(y) = \alpha \wedge y(t) = x) \text{ and } \forall z(\cdot) : (\mu_{\widetilde{F}}(z) > \alpha \to z(t) \neq x).$$

Besides, we use the concept of triangular fuzzy functions (a practical case of the fuzzy bunch) introduced by Gasilov et al. [117].

Definition 2.2. (Triangular fuzzy function) Let $f_a(\cdot),\ f_c(\cdot),\ f_b(\cdot)$ be continuous functions on an interval I. The fuzzy set $\widetilde{F}$, determined by the membership function

$$\mu_{\widetilde{F}}(y(\cdot)) = \begin{cases} \alpha, & y = f_a + \alpha(f_c - f_a) \text{ and } 0 < \alpha \leq 1 \\ \alpha, & y = f_b + \alpha(f_c - f_b) \text{ and } 0 < \alpha \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

is called a triangular fuzzy function and it is denoted as $\widetilde{F} = \langle f_a, f_c, f_b \rangle$.

According to this definition, a triangular fuzzy function is a fuzzy set (or, fuzzy bunch) of real functions. Among them only two functions have the membership degree α: the functions $y_1 = f_a + \alpha(f_c - f_a)$ and $y_2 = f_b + \alpha(f_c - f_b)$ (if f_a, f_c, f_b are pairwise distinct functions).

For each time $t \in I$, the value of a triangular fuzzy function is a triangular fuzzy number and can be expressed by the following formula:

$$\widetilde{F}(t) = (\min\{f_a(t), f_c(t), f_b(t)\},\ f_c(t),\ \max\{f_a(t), f_c(t), f_b(t)\}).$$

2.3. Zadeh's extension principle

We first define Zadeh's extension principle and then briefly explain how it can be applied to solve fuzzy differential equations.

Let $\mathcal{F}_X$ denote the set of all fuzzy subsets of X.

Definition 2.3. (Zadeh's extension principle [266, 308]) Zadeh's extension of a given function $f : X \to Z$ is the function $\widetilde{F} : \mathcal{F}_X \to \mathcal{F}_Z$ defined for each fuzzy set $\widetilde{A} \in \mathcal{F}_X$ as the fuzzy value $\widetilde{F}(\widetilde{A}) \in \mathcal{F}_Z$, whose membership function is determined as

$$\mu_{\widetilde{F}(\widetilde{A})}(z) = \begin{cases} \sup\limits_{x \in f^{-1}(z)} \mu_{\widetilde{A}}(x), & \text{if } f^{-1}(z) \neq \emptyset \\ 0, & \text{if } f^{-1}(z) = \emptyset \end{cases}$$

for all z, where $f^{-1}(z) = \{ x \in X \mid f(x) = z \}$.

For Zadeh's extension of a continuous function f, the following equality holds:

$$\left[\widetilde{F}(\widetilde{A})\right]_\alpha = f\left(A_\alpha\right), \tag{1}$$

where $f\left(A_\alpha\right) = \{ f(x) \mid x \in A_\alpha \}$ (the image of A_α under f).

Zadeh's extension principle can be effectively implemented for solving fuzzy differential equations as follows. Let a differential equation with fuzzy inputs be given. For clarity, let them be fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$. If we substitute $\widetilde{A}$ and $\widetilde{B}$ with real parameters a and b, respectively, we have a real differential equation with two parameters. Assume that we can solve this differential equation analytically, and the solution is $y = y(t, a, b)$. Then, the solution of the given fuzzy differential equation by Zadeh's extension principle is determined as

$$\left[\widetilde{Y}\left(t, \widetilde{A}, \widetilde{B}\right)\right]_\alpha = y\left(t, A_\alpha, B_\alpha\right), \tag{2}$$

where $y\left(t, A_\alpha, B_\alpha\right) = \{ y(t, a, b) \mid a \in A_\alpha,\ b \in B_\alpha \}$.

2.4. Different fuzzy derivative concepts

Chang and Zadeh [66] provide the first formal definition of the derivative for fuzzy functions. In their approach, a fuzzy function is treated as a family of real-valued functions, each associated with a specific membership degree. They then define how the derivatives of these real-valued functions contribute to constructing a new fuzzy set. Importantly, their definition does not represent the derivative of a fuzzy number-valued function in the conventional sense, but rather focuses on the behavior of its level sets.

Dubois and Prade give the next derivative definition [87]. Their derivative is made over α-cuts, so in this definition, the derivative of a real-valued function was used.

The first derivative definition for fuzzy number-valued functions is proposed by Puri and Ralescu in 1983 [249]. It is an adaptation of the set-valued function derivative definition proposed by Hukuhara [134]. The definition suggested by Puri and Ralescu is as follows:

Definition 2.4. (Hukuhara derivative) Let $\widetilde{F} : I \to \mathcal{F}_\mathbb{R}$ be a fuzzy number-valued function. We say that $\widetilde{F}$ has a Hukuhara derivative $\widetilde{F}'(t) \in \mathcal{F}_\mathbb{R}$ at $t \in I$, if for all $h > 0$ that are sufficiently close to 0, the H-differences and the limits exist in the following expression:

$$\lim_{h \to 0^+} \frac{\widetilde{F}(t+h) \ominus \widetilde{F}(t)}{h} = \lim_{h \to 0^+} \frac{\widetilde{F}(t) \ominus \widetilde{F}(t-h)}{h} = \widetilde{F}'(t).$$

In the above definition, $\ominus$ denotes the H-difference (i. e., Hukuhara difference).

Seikkala [271] represents a fuzzy function by its α-cuts and defines its derivative as follows:

Definition 2.5. (Seikkala derivative) Let a fuzzy function $\widetilde{F}: I \to \mathcal{F}_{\mathbb{R}}$ be given via its α-cuts: $F_\alpha(t) = (\underline{F_\alpha}(t), \overline{F_\alpha}(t))$. Then, $F'_\alpha(t) = (\underline{F_\alpha}'(t), \overline{F_\alpha}'(t))$, provided that $\underline{F_\alpha}'(t) \le \overline{F_\alpha}'(t)$, and $\underline{F_\alpha}'(t) \le \underline{F_\beta}'(t)$ and $\overline{F_\beta}'(t) \le \overline{F_\alpha}'(t)$ whenever $\beta \ge \alpha$.

Puri and Ralescu's definition of the derivative and Seikkala's definition are equivalent. In other words, if a function is differentiable in the sense of Definition 2.4, it is also differentiable in the sense of Definition 2.5 and vice versa. One can find the proof of this in [62]. In other words, the Seikkala derivative is nothing but the Hukuhara derivative, and all the studies done under the name of the Seikkala derivative are repetitions of the previous studies done with the Hukuhara derivative.

The support of a Hukuhara differentiable fuzzy function increases over time, so this derivative is not suitable for modeling many uncertain dynamics. Moreover, even some linear functions, namely, linear functions with decreasing uncertainty, do not have derivatives in the Hukuhara sense. To overcome these shortcomings, Bede and Gal propose the strongly generalized Hukuhara derivative [45]. After this proposal, the theory of fuzzy differential equations gained momentum. Despite all the disadvantages that we will explain below, this derivative definition has marked at least 10 years of the theory of fuzzy differential equations. We believe that it should bear the name of Bede and Gal, and throughout this article, we will refer to it as the Bede–Gal derivative.

Definition 2.6. (Bede–Gal derivative) Let $\widetilde{F}: I \to \mathcal{F}_{\mathbb{R}}$ be a fuzzy number-valued function. We say that $\widetilde{F}$ has a Bede–Gal derivative $\widetilde{F}'(t) \in \mathcal{F}_{\mathbb{R}}$ at $t \in I$, if for all $h > 0$ that are sufficiently close to 0, the H-differences and the limits exist in at least one of the following items:

$$(i) \qquad \lim_{h\to 0^+} \frac{\widetilde{F}(t+h) \ominus \widetilde{F}(t)}{h} = \lim_{h\to 0^+} \frac{\widetilde{F}(t) \ominus \widetilde{F}(t-h)}{h} = \widetilde{F}'(t)$$

or

$$(ii) \qquad \lim_{h\to 0^+} \frac{\widetilde{F}(t) \ominus \widetilde{F}(t+h)}{-h} = \lim_{h\to 0^+} \frac{\widetilde{F}(t-h) \ominus \widetilde{F}(t)}{-h} = \widetilde{F}'(t),$$

or

$$(iii) \qquad \lim_{h\to 0^+} \frac{\widetilde{F}(t+h) \ominus \widetilde{F}(t)}{h} = \lim_{h\to 0^+} \frac{\widetilde{F}(t-h) \ominus \widetilde{F}(t)}{-h} = \widetilde{F}'(t),$$

or

$$(iv) \qquad \lim_{h\to 0^+} \frac{\widetilde{F}(t) \ominus \widetilde{F}(t+h)}{-h} = \lim_{h\to 0^+} \frac{\widetilde{F}(t) \ominus \widetilde{F}(t-h)}{h} = \widetilde{F}'(t).$$

If, for example, item (i) takes place, we say that $\widetilde{F}$ is (i)-differentiable (or 1-differentiable). Also, if, for example, $\widetilde{F}$ is (ii)-differentiable and $\widetilde{F}'$ is (i)-differentiable, $\widetilde{F}$ is called (2, 1)-differentiable.

The only assumption in the Bede–Gal derivative is the existence of the H-difference, but it is known that this condition is not satisfied for many functions. Therefore, after introducing the generalized H-difference by Stefanini [283], the Bede–Gal derivative is used with this difference as well [48]. This expands the class of differentiable fuzzy number-valued functions.

Chalco-Cano et al. [65] adapt the concept of π-derivative, proposed by Bank and Jacobs [37] for set-valued functions, to fuzzy number-valued functions and show that this derivative has the same meaning as the Hukuhara derivative. The authors examine the relationship between the Bede–Gal derivative and the π-derivative. The main disadvantage of the Bede–Gal derivative is the difficulty in computing higher-order derivatives, which makes it almost impossible to use for higher-order equations. Therefore, some studies have tried to overcome this difficulty. Perfilieva and Kreinovich [244] propose using a fuzzy transform to compute higher-order derivatives.

Similar to Epuganti and Tenali [91] investigated the relationship between Hukuhara, Bede–Gal, and Plotnikov–Skripnik derivatives for set-valued functions, the relationship between the different derivatives for fuzzy number-valued functions can also be investigated in a special study.

We will discuss the relatively recently proposed concepts of interactive and granular derivatives in Sections 5 and 6, where we make comparisons regarding these concepts.

3. SURVEY OF THE STUDIES

The behavior of a dynamical system is described mainly by a differential equation. In a real-world problem, some parameters in the equation are determined from measurements (or observations) and may contain uncertainties. Often, these parameters are adequately modeled using fuzzy set theory [308]. This gives rise to fuzzy differential equations. The theory of fuzzy differential equations has not progressed as smoothly as the classical theory. In addition to the many difficulties encountered, articles containing minor or major errors have also been published. Moreover, these studies have been published in serious journals specializing in fuzzy sets and systems, and even the researchers who laid the foundations of the theory and whom we respect very much have made errors. Most of the mistakes made were due to the theory not being fully established and the rules of classical mathematics not always being valid here. Fortunately, most serious errors were detected either by the researchers themselves or by other researchers after a while, and today, a certain level has been reached. In this study, we aim to review the chronological development of the theory. We also express some reservations about the latest derivative concepts. We plan to share our more serious thoughts about these concepts in another study with our readers.

As in the classical case, initial and boundary value problems are studied for differential equations or systems of equations that contain uncertainty. Apart from this, methods for finding an approximate solution in cases where an analytical solution cannot be obtained and problems involving delays are also among the research topics. In this study, we focus on the methods proposed to solve initial and boundary value problems rather than any problem for differential equations containing uncertainty. However, this section reminds the readers of almost all the studies on fuzzy differential equations. In the next sections, we will discuss the methods proposed by various researchers.

In some studies, although the authors solve fuzzy differential equations, they assume that each derivative in the equations is a classical derivative. However, most researchers consider each derivative to be a derivative of a fuzzy-valued function, in some sense.

As in real calculus, the path to fuzzy calculus begins with the concepts of differentiability and integrability. Kaleva [154] deals with set-valued mappings of a real variable, where the mapping values are fuzzy numbers. He studies the differentiability and integrability properties of such functions and gives an existence and uniqueness theorem for the solutions of fuzzy differential equations. In another work, Kaleva [155] tries to find a class of fuzzy differential equations for which Peano's theorem is valid. One year after this study, Kloeden [176] provides some different results for the same problem. Ding et al. [84] study the existence of solutions of fuzzy differential equations with parameters by the topological degree method. Friedman et al. [102] question the situations in which Peano's theorem applies. Kaleva [156] answers and dispels doubts raised by Friedman et al. [102] regarding Peano's theorem. Hüllermeier proposes a method based on transforming a fuzzy differential equation into crisp differential inclusions [135]. Lakshmikantham and Leela [182] publish a study on the stability theory of fuzzy differential equations. Ma et al. [196] adapt the Euler method to find an approximate solution to a fuzzy differential equation. Diamond [82] investigates stability and periodicity in fuzzy differential equations. Vorobiev and Seikkala [290] make a comparative analysis of studies that do not use the fuzzy derivative concept. Xue and Fu [303] prove an existence theorem when the right-hand side of the fuzzy differential equation satisfies the Caratheodory condition. Diamond [83] develops a fuzzy version of the classical variation of constants method. Lakshmikantham and Nieto [184] introduce the concept of differential equations in a metric space. Agarwal et al. [2] present the existence results for fuzzy differential equations by the stacking theorem. Bhaskar et al. [51], by giving examples, explain that the theory of fuzzy differential equations is important and needs to be developed. Buckley and Feuring [55] propose first to find the solution of the associated crisp differential equation and then fuzzify it. Babolian et al. [35] suggest finding an approximate solution of the associated crisp differential equation by the Adomian method and then fuzzifying this approximate solution. Bede and Gal [44] investigate almost periodic functions and apply them to fuzzy differential equations. Abbasbandy et al. [1] propose a numerical method to find an approximate solution of fuzzy differential inclusion.

Lakshmikantham [181] is the first to realize that starting to study set differential equations in a metric space would be important for investigating fuzzy differential equations and would also provide various advantages. There were studies on set differential equations before Lakshmikantham realized this (See, for example, [53]). Amrahov et al. [31] investigate the connection between old studies on the subject and fuzzy differential equations about 10 years after Lakshmikantham. Important studies in this direction are being done by Komleva et al. [178], Malinowski [202], Plotnikov et al. [246], Plotnikov and Skripnik [247]. An important subsequent contribution to the theory of fuzzy differential equations, developed independently from the aforementioned line of research, is made by Kaleva [157], who introduces a novel approach based on the extension of the classical differential equation framework to fuzzy set-valued functions. In this study, Kaleva includes the forcing term in the equation. Nieto and Rodríguez-López [230] find sufficient conditions for the boundness of every solution of first-order fuzzy differential equations. Allahviranloo et al. [19] propose a predictor-corrector method for the numerical solution of fuzzy differential equations. However, Bede questions the examples solved in this study and presents new examples [43]. Allahviranloo et al. [15] improve their method and select examples more accurately. Rodríguez-López [259] provides a comparative analysis of the results of the studies on fuzzy differential equations under the Hukuhara derivative. Rodríguez-López [260] develops the monotone iterative technique to approximate the extremal solutions for the fuzzy initial value problem. Chalco-Cano and Román-Flores [62] show that fuzzy differential equations have different solutions when viewed in terms of the Bede–Gal derivative than in the case of the Hukuhara derivative. In their next work [63], the authors compare the solutions obtained by different derivative concepts.

Later, the study of FDEs was carried out mainly in five directions:
1) Analytical and numerical methods to solve FDEs; 2) Application of various transformations; 3) Introducing new concepts of solutions; 4) Existence-uniqueness and stability issues for uncertain dynamic systems; 5) New types of FDEs.

The main studies in the first direction can be listed chronologically as follows. Nieto et al. [229] show that any suitable numerical method for ordinary differential equations can solve numerically fuzzy differential equations under Bede–Gal differentiability. Palligkinis et al. [237] adapt the Runge–Kutta method to find approximate solutions to fuzzy differential equations. Khastan and Ivaz [164] solve fuzzy differential equations numerically by the Nystrom method. Effati and Pakdaman [88] introduce an artificial neural network approach for solving fuzzy differential equations. Khastan et al. [167] develop a variation of the constants formula for first-order fuzzy differential equations under the Bede-.Gal derivative. Allahviranloo et al. [17] propose a novel operator method for solving fuzzy linear differential equations under the Bede–Gal derivative. Mosleh and Otadi [221] simulate fuzzy differential equations with fuzzy neural networks. Ghazanfari and Shakerami [121] apply a numerical algorithm for solving fuzzy first-order initial value problems based on extended Runge–Kutta-like formulae of order 4. Dizicheh et al. [85] note that the example given in [121] does not reflect what is described in the study itself. Shang and Guo [272] introduce Adams predictor-corrector systems for solving fuzzy differential equations. Chalco-Cano and Román-Flores [64] give some remarks on numerical algorithms for solving fuzzy differential equations via differential inclusions. Kloeden and Lorenz [177] show that removing the convexity condition in the differential inclusion approach can give a more general reachable set. Rabiei et al. [254] develop the Fuzzy Improved Runge–Kutta–Nystrom (FIRKN) method for solving second-order fuzzy differential equations. Allahviranloo et al. [16] propose solving nonlinear fuzzy differential equations using the fuzzy variational iteration method. Darabi et al. [81] apply the Euler method to solve fully fuzzy differential equations under the Bede–Gal differentiability. Jameela et al. [149] present a numerical approach based on the 5th-order Runge–Kutta method to solve fuzzy differential equations. Garg [103] proposes a numerical method using Runge–Kutta and Biogeography-based optimization for solving fuzzy differential equations. Khastan and Rodríguez-López [171] examine different formulations of first-order linear fuzzy differential equations under Bede–Gal differentiability. They provide sufficient conditions for the existence and obtain a general expression for the solutions. Hosseini et al. [133] solve linear and nonlinear fuzzy differential equations by the variational iteration method. They obtain a sequence of functions that converge to the exact solution. Abu Arqub et al. [28] propose a method based on the reproducing kernel theory to solve fuzzy differential equations under Bede–Gal differentiability. Mondal and Roy [219] provide a solution procedure using the Lagrange multiplier method and extension principle for the second-order fuzzy differential equations. Ashraf et al. [33] solve fuzzy differential equations with the average extension principle. Dai et al. [80] propose a mathematical model for universal oscillators described by fuzzy differential equations. Cabral and Barros [59] examine differential equations with interactive fuzzy parameters via t-norms. Khastan and Rodríguez-López [173] study first-order linear fuzzy differential equations under differential inclusions and the Bede–Gal differentiability approaches. They reveal some relationships between the solutions obtained using these approaches. Harir et al. [130] present a new solution to the Susceptible-Infected-Recovered (SIR) epidemic model with fuzzy initial value. Shen [274] considers first-order linear fuzzy differential equations where the differentiability of fuzzy functions is defined under the assumption of linear correlation between fuzzy numbers. Many other studies propose solving fuzzy differential equations using various numerical methods [5, 6, 75, 101, 137, 141, 150, 158, 161, 191, 235, 255, 270, 287].

In the second direction, where the authors use different transformations, the following studies attract attention. Allahviranloo et al. [26] extend the differential transformation method to solve fuzzy differential equations under the Bede–Gal derivative. Allahviranloo and Ahmady [18] propose a fuzzy Laplace transform under the Bede–Gal differentiability concept. Jafari et al. [140] apply a variational iteration method to solve higher-order fuzzy differential equations. Rahman and Ahmad [256] study the fuzzy Simudu transform and apply it to first-order fuzzy differential equations. ElJaoui et al. [89] extend the fuzzy Laplace transform to solve second-order fuzzy differential equations. By using fuzzy Mellin transform, Sun and Yang [285] solve some fuzzy differential equations under the Bede–Gal differentiability. Jafari and Razvarz [139] approximate fuzzy differential equations with fuzzy Sumudu transform. Jameel et al. [148] develop the differential transformation method to find semi-analytical solutions for high-order fuzzy differential equations. Salamat et al. [263] find the switching points for the solutions of second-order fuzzy differential equations by a differential transformation method. Salgado et al. [265] model a type of harmonic oscillator with fuzzy differential equations and solve them via the Laplace transform.

The work of Khastan et al. [163] can be considered as one of the first studies belonging to the third direction. In this study, the authors introduce $(1,1)$, $(1,2$, $(2,1)$ and $(2,2)$-solutions to fuzzy differential equations. Qiu et al. [253] examine fuzzy differential equations in the quotient space of fuzzy numbers. Khastan and Rodríguez-López [172] find periodic solutions to first-order linear fuzzy differential equations via differential inclusions.

In the fourth direction, related to the issues of existence-uniqueness and stability, the following works stand out. Mizukoshi et al. [214] investigate the stability of fuzzy dynamic systems. Zhu [313] provides a stability analysis of the solutions of fuzzy differential equations. Mazandarani and Najariyan [208] note that the method of Xu et al. [301] can produce unstable solutions to stable systems. Shen and Wang [277] investigate the Ulam stability of fuzzy differential equations under Bede–Gal differentiability. Qiu et al. [251] examine the stability of the solutions in the Lyapunov sense. There are many other works [78, 79, 131, 146, 153, 189, 190, 207, 257, 276, 304] on stability, but we do not focus on this topic in this study. Chehlabi [67] shows the existence of a class of first-order fuzzy differential equations with discontinuous coefficients that have continuous solutions. Malinowski and Michta [206] present the existence and uniqueness of solutions to stochastic fuzzy differential equations driven by Brownian motion. Zhang et al. [310] prove the existence theorem for second-order fuzzy differential equations with initial conditions under the Bede–Gal derivative. Qui et al. [252] introduce a metric on the quotient space of fuzzy numbers, study the differentiability and integrability properties of fuzzy functions, and give an existence and uniqueness theorem for a solution to a fuzzy differential equation.

In the fifth direction, the authors propose new types of FDEs. Malinowski [199] introduces the concept of random fuzzy differential equations. In his next work [200], he proves the existence theorem for random fuzzy equations. Malinowski continues to examine this issue under different conditions [201, 203, 204, 205]. Vu [291] is another researcher who studies random fuzzy differential equations.

3.1. Studies on initial value problems

As with other problems, studies investigating the fuzzy initial value problem can be divided into two groups. While in the first group, no specifically defined derivative for fuzzy functions is involved, in the second group, a fuzzy derivative in some sense is used. The studies given in Table 1 investigated the initial value problem for fuzzy differential equations. In most of these studies, the coefficients are real numbers, and in all of them, the initial value is given with a fuzzy number. For the studies that do not use a fuzzy number-valued function's derivative, in the table, the derivative is indicated as classical. In the homogeneity column, it is stated whether the equation is homogeneous or not.

After Kandel and Byatt [159] first proposed the term "fuzzy differential equation", fuzzy versions of all the problems studied for classical differential equations naturally began to be discussed. One of these problems, the initial value problem, was first studied by Seikkala [271] and Kaleva [154, 155]. The initial value problem for fuzzy differential equations may differ depending on the type of derivatives used in the equation and whether the initial values and force function are fuzzy or not [7, 38, 45, 46, 55, 56, 57, 63, 114, 119, 123, 135, 167, 184, 243].

Park and Han [238] prove the existence and uniqueness of fuzzy solutions for the initial value problem using the properties of Hasegawa's functionals and successive approximation. Buckley and Feuring [55, 56] propose different methods to solve the fuzzy initial value problem. In the first method, they use the left and right endpoints of the given fuzzy number as the initial value and solve two real problems. Unfortunately, with this method, there is no guarantee that the solutions obtained will be the left and right endpoints of a fuzzy number for each t. The authors themselves demonstrate this situation with some examples. In the second method suggested by the authors, they find the solution using Zadeh's extension principle. According to this principle, they first solve the corresponding crisp problem. Then, they replace the initial value in the found solution of the crisp problem with the fuzzy one to transform the solution into a fuzzy function. After this, the authors check whether the resulting fuzzy function satisfies the differential equation and fuzzy initial conditions. Although this second method works

Studies	Coefficients	Homogeneity	Derivative
Kaleva [154, 155]	real	homogeneous	classical
Seikkala [271]	real	homogeneous	Hukuhara
Wu et al. [299]	real	homogeneous	Hukuhara
Congxin and Shiji [74]	real	homogeneous	Hukuhara
Nieto [228]	real	homogeneous	Hukuhara
Buckley and Feuring [55, 56]	real	homogeneous	classical
Hüllermeier [136]	real	homogeneous	classical
Song and Wu [281]	real	homogeneous	Hukuhara
Park and Han [238]	real	homogeneous	Hukuhara
O'Regan et al. [236]	real	homogeneous	Hukuhara
Georgiou et al. [119]	real	homogeneous	Hukuhara
Bede and Gal [45]	real	homogeneous	Bede–Gal
Song et al. [282]	real	homogeneous	Hukuhara
Nieto et al. [232]	real	homogeneous	classical
Xiaoping and Yongqiang [300]	real	homogeneous	Hukuhara
Mizukoshi et al. [213]	real	homogeneous	Hukuhara
Bede et al. [46]	real	homogeneous	Bede–Gal
Lupulescu [193]	real	homogeneous	Hukuhara
Allahviranloo et al. [20]	real	homogeneous	classical
Perfilieva et al. [243]	real	homogeneous	classical
Allahviranloo et al. [25]	real	homogeneous	Bede–Gal
Allahviranloo et al. [21]	real	homogeneous	classical
Khastan et al. [167]	real	non-homogeneous	Bede–Gal
Li et al. [186]	real	homogeneous	Hukuhara
Gasilov et al. [117]	real	non-homogeneous	classical
Tapaswini and Chakraverty [286]	real	homogeneous	Hukuhara
Akın et al. [7]	fuzzy	non-homogeneous	classical
Gasilov et al. [114]	real	homogeneous	classical
Cabral and Barros [58]	real	homogeneous	interactive
Villamizar-Roa et al. [289]	real	homogeneous	Bede–Gal
Khastan et al. [170]	real	homogeneous	classical
de Barros and Santo Pedro [39]	real	homogeneous	interactive
Esmi et al. [92]	real	homogeneous	interactive
Salgado et al. [264]	real	homogeneous	interactive
Alikhani and Mostafazadeh [14]	fuzzy	homogeneous	Bede–Gal
Akram et al. [8]	fuzzy	homogeneous	Hukuhara

Tab. 1. Studies investigating the fuzzy initial value problem, and the properties of differential equations in them.

for linear fuzzy differential equations, it will not find solutions to nonlinear equations or even systems of linear equations ([57]).

In the study [243], Perfilieva et al. develop an approach that resembles the Buckley–Feuring method in structure. However, unlike similar studies, their work uniquely characterizes the solution's degree of fuzziness as a function of the independent variable. This is achieved by applying fuzzy transformations and inference rules based on the Łukasiewicz logic framework.

Lakshmikantham and Nieto [184] consider differential equations in metric spaces and examine the initial value problem.

In some studies, the solution of the fuzzy initial value problem is sought as a fuzzy set of real functions. Studies such as Gomes and Barros [123], Barros et al. [38], and Gasilov et al. [114] can be given as examples. Gomes and Barros [123] and Barros et al. [38] propose fuzzy calculus concepts similar to classical calculus and solve fuzzy differential equations in terms of this calculus. They prove the existence of a solution to the first-order fuzzy initial value problem under certain conditions. Gasilov et al. [114] propose a method that can find the fuzzy solution by using linear transformations to solve the fuzzy initial value problem. Although the authors explain their proposed method on second-order equations for easy understanding, it can also be applied to higher-order linear differential equations. It has also been shown that the fuzzy solution obtained in the study coincides with the results of the extension principle.

Among the studies in the second group, Chehlabi and Allahviranloo [68] address the initial value problem for first-order fully fuzzy linear differential equations under Bede–Gal differentiability. The fuzzy solution can be found in the proposed method based on the solutions of crisp ordinary differential equation systems. The authors also obtain the necessary and sufficient conditions for the existence of the fuzzy solution.

Santo Pedro et al. [268] investigate the Fuzzy Initial Value Problem (FIVP), which describes an autocorrelated evolution process. To do this, the authors use the concept of correlated derivatives for fuzzy-valued functions. They establish a relationship between the diameter of the solution and its derivative. Moreover, they observe how the derivative affects the interactivity of the process by calculating a measure of the probabilistic interactivity of the solution. Additionally, the authors analyze a population growth model, namely the Logistic Model.

Esmi et al. [92] develop a generalization of Zadeh's extension principle and solve higher-order linear differential equations with initial conditions given by interactive fuzzy numbers.

Akram et al. [8] consider the initial value problem for linear third-order fuzzy differential equations. In methods that do not use derivatives of fuzzy-valued functions, the order of the equation is unimportant. Generally, the methods are explained on second-order equations but are valid for equations of any order. However, the order of the equation is important because many situations arise when derivatives of fuzzy-valued functions are used. In the study, the authors consider the third-order fuzzy differential equation under the first and second Hukuhara derivatives. They also obtain a relationship between the Laplace transform of the fuzzy-valued function and the third-order derivative. Using the Laplace transform technique, they propose an algorithm to determine the potential solution of the linear third-order fuzzy initial value problem. Mohapatra and Chakraverty [215] investigate type-2 fuzzy initial value problems under granular differentiability.

3.2. Studies on boundary value problems

Fuzzy boundary value problems (FBVPs) arise naturally in various real-world and engineering applications [52, 143, 309]. The problems have been extensively investigated in many studies due to their practical significance and mathematical complexity. These studies are shown in Table 2. In all of these studies, the coefficients of the equation considered are real numbers, and the boundary values are fuzzy numbers. As in Table 1, the homogeneity column shows whether the equation is homogeneous, and the derivative column shows which derivative is used.

FBVPs are important to investigate because they can occur in many application problems. Examining HIV infection, Zarei et al. [309] propose a fuzzy mathematical model described by a linear fuzzy differential equation. The authors consider a fuzzy optimal control problem that minimizes both viral load and drug costs. Based on the proposed model, the authors derive an optimality condition in the form of a fuzzy boundary value problem (FBVP). Jafelice et al. [143] propose a mathematical model for the evolution of the positive HIV population and the emergence of AIDS. In the study, the authors evaluate the transmission rate of HIV to AIDS as a fuzzy number. They hypothesize that this transmission rate depends on infected individuals' viral load and CD4+ level. Salahshour and Haghi [262] naturally encounter FBVP when they consider the fuzzy heat equation under Bede–Gal differentiability. In the work, the authors transform the fuzzy heat equation into the corresponding fuzzy two-point boundary value problem using the fuzzy Laplace transform.

FBVPs were first described by Lakshmikantham et al. [183]. They, as well as O'Regan et al. [236] propose a solution method by assuming that a two-point boundary value problem for a fuzzy differential equation is equivalent to a fuzzy integral equation. However, Bede [42] proves by a counterexample that a two-point boundary value problem cannot always be transformed into a fuzzy integral equation. This circumstance shows that FBVPs do not have a solution in most cases. Chen et al. [70, 71] investigate the necessary and sufficient conditions for the two-point boundary value problem to be equivalent to the fuzzy integral equation and prove the existence of the solution under these conditions.

Murty and Kumar [224] propose existence and uniqueness conditions for a class of boundary value problems for third-order nonlinear fuzzy differential equations. They use Green's functions and the contraction mapping principle in their work. Prakash et al. [248] present a three-point boundary value problem for another type of boundary condition, but still use Green's functions. Khastan and Nieto [165] propose a new solution method for the two-point boundary value problem for the second-order fuzzy differential equation using Bede–Gal differentiability. Li et al. [185] study two-point boundary value problems involving undamped and uncertain dynamical systems. They define the concept of big solutions and prove that such solutions exist and are unique. Nieto, Rodríguez-López and Villanueva-Pesqueira [233] prove the existence and uniqueness of a solution for a first-order linear fuzzy differential equation with impulses subject to boundary value conditions. Liu [188] investigates the support of solutions for two-point fuzzy boundary value problems. Fard et al. [95] present some sufficient conditions for the existence and uniqueness of a solution using Hukuhara differentiability. Rodríguez-López [258] provides sufficient conditions for the existence of solutions of pe-

Studies	Homogeneity	Derivative
Lakshmikantham et al. [183]	homogeneous	Hukuhara
O'Regan et al. [236]	homogeneous	Hukuhara
Bede [42]	homogeneous	Hukuhara
Prakash et al. [248]	homogeneous	Hukuhara
Khastan and Nieto [165]	homogeneous	Bede–Gal
Nieto et al. [233]	non-homogeneous	Hukuhara
Li et al. [185]	homogeneous	Hukuhara
Gasilov et al. [108]	homogeneous	classical
Fatullayev and Köroglu [97]	homogeneous	Bede–Gal
Fatullayev and Köroglu [98]	homogeneous	Bede–Gal
Rodríguez-López [258]	homogeneous	Bede–Gal
Gasilov et al. [110]	homogeneous	classical
Gasilov et al. [111]	non-homogeneous	classical
Wang [294]	non-homogeneous	Bede–Gal
Wang [295]	non-homogeneous	Bede–Gal
Wang [296]	non-homogeneous	Bede–Gal
Citil [76]	homogeneous	Hukuhara
Gholami et al. [122]	homogeneous	Bede–Gal
Esmi et al. [92]	homogeneous	interactive
Farajzadeh et al. [94]	homogeneous	Hukuhara
Sánchez et al. [267]	homogeneous	interactive
Yang et al. [305]	homogeneous	granular
Soma et al. [280]	homogeneous	granular
Sarvestani and Chehlabi [269]	homogeneous	Bede–Gal
Yang et al. [306]	homogeneous	granular
Alavi [11]	homogeneous	Hukuhara
Yang and Wu [307]	homogeneous	granular

Tab. 2. Studies investigating the fuzzy boundary value problem, and the properties of differential equations in them.

riodic boundary value problems for first-order linear fuzzy differential equations under Bede–Gal differentiability and switching points.

Khastan et al. [168] investigate the existence of solutions for a class of FBVPs under Bede–Gal differentiability. Nieto and Rodríguez-López [231] calculate the exact solution for a class of FBVPs for first-order fuzzy linear differential equations with impulses under Hukuhara differentiability. Ahmadi et al. [4] use the Laplace transform to solve a fuzzy second-order differential equation involving the Bede–Gal derivative. Allahviranloo and Chehlabi [22] consider fuzzy differential equations based on the concept of the length function.

Numerous publications exist regarding numerical methods for FBVPs. Fatullayev and Köroglu [97] propose an algorithm that uses the finite difference method to solve FBVPs numerically. Bede and Rudas [47] propose a shooting algorithm for numerically solving fuzzy, two-point boundary value problems such as the fuzzy elastica problem. Jamshidi and Avazpour [151] apply the shooting method to solve second-order, fuzzy boundary value problems under Bede–Gal differentiability. Dahalan et al. [77] study numerical solutions to FBVPs using the Gauss-Seidel and successive over-relaxation iterative methods.

Khastan et al. [163, 169] and Khastan and Nieto [165] investigate fuzzy differential equations under Bede–Gal differentiability. However, their examples show that the obtained solutions are difficult to interpret because the four different problems generated using the Bede–Gal first and second derivatives often do not reflect the nature of the problem. Liu reduces these four problems to two when the right-hand side function is monotonic [188]. In [3], Ahmad et al. study analytical and numerical solutions of fuzzy differential equations based on the extension principle. By considering the dependency problem in fuzzy interval arithmetic, the authors propose a new fuzzification of Euler's method. Recently, Akın et al. [7] present a new algorithm for solving fuzzy differential equations with the Bede–Gal derivative. They first solve the corresponding classical problem, and then construct the fuzzy solution by assuming that it is "close" to the classical solution.

To find an exact solution to the periodic boundary value problem for a first-order linear fuzzy differential equation with impulses, Nieto et al. use the crisp solution [233].

In another approach, the fuzzy problem is transformed into a crisp problem. There are two ways to realize this approach. The first, suggested by Hüllermeier [135], uses the concept of differential inclusion. In this way, by taking an α-cut of the initial value and the solution, the given differential equation is converted to a differential inclusion, and its solution is accepted as the α-cut of the fuzzy solution function. Li et al. use this approach in [185]. In the paper, the concept of big solutions is introduced, and some existence and uniqueness theorems are established. The second way is offered by Gasilov et al. [108, 110]. In this way, the fuzzy problem is considered to be a set of crisp problems. The authors investigate a differential equation with fuzzy boundary values. They interpret the problem as a set of crisp problems. For linear equations, the authors propose a method based on the properties of linear transformations. They show that if the solution of the corresponding crisp problem exists and is unique, the fuzzy problem also has a unique solution. Moreover, the authors prove that if the boundary values are triangular fuzzy numbers, then the solution value is a triangular fuzzy number at each time. The authors explain the proposed method with examples. They find analytical expressions for the solution of a second-order linear differential equation with constant coefficients. In an example, they demonstrate the advantages of the proposed method compared to the method that uses the Bede–Gal derivative.

Khastan et al. demonstrate that a class of first-order linear differential equations subject to periodic boundary conditions can be solved by alternating two types of Bede–Gal derivatives at the switching points [168]. Rodríguez-López improves this result for equations whose coefficient may change sign a finite number of times [258].

To avoid difficulties with fuzzy derivatives, some researchers propose solving an equivalent fuzzy integral equation instead of the fuzzy differential equation [17, 70, 236].

In almost all above-mentioned studies, the forcing and solution functions are assumed to be fuzzy number-valued functions. However, this assumption leads to some difficulties.

If to assume that the solution is a fuzzy number-valued function, it is natural to use the Bede–Gal derivative, which has four types: $(1,1)$, $(1,2)$, $(2,1)$, and $(2,2)$-derivatives (in the second-order case). In later studies, it is assumed that each of these four derivatives has an essentially local nature. This means that the requirement for a global (i. e., defined on the entire time interval) $(1,1)$, $(1,2)$, $(2,1)$, or $(2,2)$-Bede–Gal differentiable solution is not fully justified and, in most cases, does not exist. Consequently, we should consider switching between Bede–Gal differentiability cases at some time moments [47, 48, 165]. For example, to solve the problem, we could start with a $(1,1)$ derivative and then switch to a $(2,2)$ derivative, and so on. This approach raises some questions regarding how to choose the switching points, what type of derivative to start with, what type of derivative to switch to, and whether the solution is unique. Further investigations are needed to answer these questions [110]. Another difficulty related to the Bede–Gal derivative is that it may give results far from the associated classical solution.

To overcome the above-mentioned difficulties, the main idea of Gasilov et al. [110] is that fuzzy-valued functions are not the only tool for modeling uncertainties that change with time. They interpret a fuzzy function as a fuzzy set of real functions. In other words, as a fuzzy bunch of real functions. Each of these real functions has a certain membership degree. This approach is useful when avoiding the difficulties of fuzzy derivatives. The authors consider the forcing function also to be a fuzzy set of real functions. Since the triangular membership function became a useful tool in many engineering applications [242], they assume this set to be triangular.

In [111] an FBVP for a second-order linear differential equation with a fuzzy forcing function is considered. The authors develop the approach proposed in [108, 110] for an FBVP where only the boundary values were fuzzy. They represent the forcing function as a triangular fuzzy function defined in [117]. The proposed method is the first application for FBVPs with a fuzzy forcing function and boundary values. The authors also show that different solutions can be obtained using various t-norms.

Wang [294, 295, 296, 297] solves two-point boundary value problems for first-order nonlinear fuzzy differential equations under the Bede–Gal derivative. He proposes a numerical monotone iterative method to solve the problem. The author proves the existence of solutions to some boundary value problems for second-order fuzzy differential equations.

3.3. Studies on fuzzy systems of differential equations

Studies on fuzzy systems of differential equations are shown in Table 3. Buckley and Feuring [55] and Buckley et al. [57] give a very general formulation of the fuzzy first-order initial value problem. They first find the crisp solution, fuzzify it, and then verify whether it satisfies the fuzzy system of differential equations (FSDEs). Rodriguez-Lopez [259] considers several comparison results for the solutions of FSDEs obtained through different methods using the Hukuhara derivative. Mizukoshi et al. [213] show that the solutions of the Cauchy problem obtained by Zadeh's extension principle and by using a family of differential inclusions are the same. However, Allahviranloo et al. [27] demonstrate with an example that the main result of [213] is incorrect. Xu et al. [301] use complex number representation for α-level sets to solve a fuzzy system and prove theorems that provide the solutions in this representation. Chalco-Cano and Román-

Flores [63] study the class of fuzzy differential equations where the dynamics are given by a continuous fuzzy mapping which is obtained via Zadeh's extension principle.

Gasilov et al. [107] apply a geometric approach to a fuzzy linear system of differential equations (FLSDEs) with real coefficients and with the initial conditions described by a vector of fuzzy numbers. The authors interpret a vector of fuzzy numbers as a rectangular prism in n-dimensional space and show that at any time the solution corresponds to an n-dimensional parallelepiped. Unlike earlier research, they are not looking for solutions for FLSDE in the form of a vector of fuzzy functions. Instead, their solutions constitute a fuzzy set of real vector functions. Each member in the solution set satisfies the system with a certain possibility.

In articles [29, 105, 106, 112], using the same geometric approach, an algorithm is proposed to solve linear systems of algebraic equations with crisp coefficients and fuzzy numbers on the right-hand side. Alamin et al. [9, 10] propose using the geometric approach, developed by Gasilov et al. [105], to solve fuzzy linear difference equations.

One of the first works on studying fuzzy differential equation systems belongs to Oberguggenberger and Pittschmann [234]. They apply Zadeh's extension principle to the system of differential equations with fuzzy parameters and introduce the notions of fuzzy solutions and component-wise fuzzy solutions. Buckley, Feuring, and Hayashi [57] propose two close methods for the solution of linear systems of first-order differential equations with fuzzy initial conditions. In the first method, the authors fuzzify the crisp solution and then check to see if its α-cuts satisfy the differential equations. In the second method, they solve the level-wise system and then check whether the solution always (i. e., for all t) defines a valid fuzzy number or not. Unfortunately, a solution of type 1 or type 2, defined in such a way, exists only for specific systems. Xu, Liao, and Hu [301] also investigate linear first-order fuzzy differential equation systems with fuzzy initial values. They use the complex number representation for the α-level sets, which was first proposed by Pearson [240], and prove existence theorems for solutions. The authors also describe phase portraits of two-dimensional fuzzy dynamical systems. Xu, Liao, and Nieto [302] study the properties of first-order linear dynamical systems with fuzzy matrices. They construct the fuzzy solution from the solutions of the classical differential equations, obtained by using the α-level representation for the fuzzy system. Fard and Ghal-Eh [96] propose an iterative method to obtain approximate solutions for linear systems of first-order differential equations with fuzzy constant coefficients. Ghazanfari, Niazi, and Ghazanfari [120] investigate linear first-order fuzzy matrix differential equation systems using the complex number representation for the α-level sets. Mosleh and Otadi [222] propose a method for finding a minimal solution of a system of fuzzy linear differential equations in the form $A\dot{x}(t) = B\dot{x}(t) + Cx(t)$. Hashemi, Malekinagad, and Marasi [129] apply the homotopy analysis method to derive an approximate analytical solution for the system of fuzzy differential equations. Mosleh [220] presents a neural network to solve a system of fuzzy differential equations with fuzzy initial values.

As can be seen from Table 3, the Bede–Gal derivative [45] has rarely been used to solve FSDEs. The reason is that the concept of the Bede–Gal derivative leads to some difficulties:

1) The Bede–Gal derivative is mainly a combination of the 1-derivative (or Hukuhara derivative) and the 2-derivative (or, second type Hukuhara derivative). Under the 1-derivative, we have a solution, the uncertainty (fuzziness) of which increases with time. In contrast, under the 2-derivative, we have a solution with decreasing uncertainty. To describe a solution, the uncertainty of which alternates, we have to alternate 1- and 2-derivatives. Consequently, we have a priori to divide the time domain into subintervals. How many subintervals should one divide the time domain? How to choose the length of each subinterval? Which derivative should be used in each subinterval? How to address these questions remains an open problem.

2) When the Bede–Gal derivative is used, for an n-dimensional fuzzy system, it is necessary to examine 2^n classical systems. This circumstance limits the number of researchers who use the Bede–Gal derivative.

3) In general, the solution of a fuzzy differential equation under the Bede–Gal differentiability is not unique. What to do if we have 2 or more solutions? This question has not been answered yet.

Studying fuzzy differential equation systems is also very important for solving fuzzy optimal control problems. There are only a few studies on this issue due to the reasons described above [13, 30, 100, 217, 225, 245, 312].

Studies	Coefficients	Homogeneity	Derivative
Xu et al. [301]	real	non-homogeneous	classical
Xu et al. [302]	fuzzy	homogeneous	classical
Gasilov et al. [107]	real	homogeneous	classical
Hashemi et al. [129]	fuzzy	homogeneous	classical
Mosleh and Otadi [223]	fuzzy	homogeneous	Bede–Gal
Gasilov et al. [113]	real	non-homogeneous	classical
Mondal et al. [218]	real	homogeneous	classical

Tab. 3. Studies investigating systems of fuzzy differential equations, and the properties of differential equations in these studies.

3.4. Studies on fuzzy partial differential equations

Since knowledge about dynamic systems modeled by differential equations is often incomplete or vague, many mathematical models of physical, chemical, and biological phenomena are described by fuzzy partial differential equations (FPDEs) [216]. Studies on fuzzy partial differential equations are shown in Table 4. In all studies, the considered equation is homogeneous, the coefficients are real, and the initial and boundary values are fuzzy numbers. The first study on FPDEs belongs to Buckley and Feuring [54]. The authors, as they do in their other later works on the fuzzy partial differential equation, try to obtain a solution by the extension principle, and if they cannot get a solution, they follow Seikkala's procedure [271]. However, this approach works only in some simple cases. Jafelice et al. consider an application of PDEs with fuzzy parameters obtained through fuzzy rule-based systems [142]. Chen et al. present a new inference method with applications to FPDEs [72].

Studies	Derivative
Buckley and Feuring [54]	classical
Chen et al. [72]	classical
Chen and Han [69]	Hukuhara
Gasilov et al. [109]	classical
Allahviranloo et al. [23]	Bede–Gal
Gasilov et al. [115]	classical
Mirzaee and Yari [212]	Bede–Gal
Macias-Diaz and Tomasiello [197]	Hukuhara
Long et al. [192]	Bede–Gal
Wasques [298]	Interactive
Shen [275]	Interactive

Tab. 4. Studies investigating fuzzy partial differential equations, and the derivatives used in these studies.

Salahshour and Haghi [262] solve the fuzzy heat equation under the Bede–Gal differentiability. Based on the fuzzy Laplace transform, they convert the original fuzzy heat equation to the corresponding two-point fuzzy boundary value problem.

Karami et al. [160] use fuzzy logic to predict the heat transfer in an air-cooled heat exchanger equipped with tube inserts of three types (butterfly, classic, and jagged twisted tape). The results show that the fuzzy technique has a low error rate: the average error was found to be 0.68% as compared with experimental data.

Bertone et al. [50] investigate heat, wave, and Poisson equations as classical models of partial differential equations with uncertainty. In each problem, only one parameter (diffusion coefficient in heat equation, speed coefficient in wave equation, and permittivity coefficient in Poisson equation) is taken to be uncertain, considering it as a fuzzy number. The authors build the fuzzy solution from the deterministic solution using Zadeh's extension principle. They prove the stability of the fuzzy solution concerning the initial-boundary data and show that as time goes to zero, the diameter of the uncertainty converges to zero.

Some researchers develop numerical methods for solving FPDEs. Allahviranloo and Kermani [24] use the finite difference method for the numerical solution of FPDEs. Mikaeilvand and Khakrangin [210] propose a transform method to solve FPDEs. They use the fuzzy two-dimensional differential transform method of fixed grid size to find approximate solutions. Štěpnička and Valášek [284] apply the fuzzy transform technique to find numerical solutions of crisp PDEs.

In the study [115], to solve FPDEs the authors develop the method proposed by Gasilov et al. [107, 109, 110, 111, 114, 117]. The main difference from all other studies that investigate FPDEs is that they look for the solution as a fuzzy set (bunch) of real functions, not as a fuzzy-valued function. The novelties of the above-mentioned study are:

1) The proposed method is applied to FPDEs for the first time;

2) A more general concept for triangular fuzzy functions is considered, which includes non-regular ones;

3) The concept of triangular fuzzy function is extended for the case of two variables;
4) The existence and uniqueness theorem is established for the fuzzy heat equation, in commonly accepted conditions.

Arif et al. [32] modify the exponential time integrator with an explicit scheme to solve fuzzy partial differential equations and analyze its stability and convergence. Shen [275] proposes an approach to solve fuzzy heat and fuzzy wave equations with crisp and fuzzy coefficients in the space of strongly linearly correlated fuzzy numbers using the separation of variables method.

3.5. Studies on fuzzy delay differential equations

Many processes can be modeled by delay differential equations (DDEs), also known as functional equations [34, 128, 180]. In particular, we meet such models in chemistry and biology [90, 144, 152, 261]. The delays may appear because of different reasons, such as the physical properties of equipment used in the system, signal transmission, or measurement of system variables.

Often, when we model a process by deterministic ordinary differential equations, we ignore some uncertainties or vagueness. In many cases, input values in the model cannot be measured exactly, and there is a significant amount of uncertainty that we have to take into account. In other words, unfortunately, in application problems, some of the parameters carry uncertainty. In these cases, to construct a mathematical model, we can use stochastic analysis, interval analysis, or fuzzy logic and fuzzy sets, depending on the type of uncertainty. Many authors have dealt with fuzzy delay differential equations (FDDEs) from various points of view. Studies on fuzzy delay differential equations are shown in Table 5. In all studies, the considered equation is homogeneous, the coefficients are real, and the initial and boundary values are fuzzy numbers. Jafelice, Barros, and Bassanezi [144] investigate HIV dynamics and propose an FDDE model. To solve the FDDE, they use Zadeh's extension principle. Lupulescu and Abbas [195] consider a model of FDDE under the Liu process [187]. They prove the existence and uniqueness theorem and investigate the continuous dependence of the solution on initial data. Malinowski introduces stochastic FDDE and proves the existence and uniqueness theorem [203]. Guo, Peng, and Xu give oscillation criteria for a class of second-order FDDEs [126], formulated as a family of differential inclusions [135]. Barzinji, Maan, and Aris investigate stability analysis of linear fuzzy delay differential equation systems [41]. Khastan, Nieto, and Rodriguez-Lopez consider FDDE under Bede–Gal differentiability and prove the existence of two fuzzy solutions (one for the first type, the other for the second type of differentiability) [166].

Min, Huang, and Zhang extend some known results of fuzzy differential equations to fuzzy differential inclusions and prove local and global existence theorems for fuzzy delay differential inclusions [211].

Some authors investigate FDDEs under the terminology "fuzzy functional differential equations" (FFDEs). Balasubramaniam and Muralisankar [36] prove the existence and uniqueness of fuzzy solutions for the nonlinear fuzzy neutral functional differential equation via the Banach fixed-point approach. Park and Jeong [239] get the existence and uniqueness result for random fuzzy functional differential equations by using the method of successive approximations. Donchev and Nosheen [86] study FFDEs with continuous right-hand sides and prove the existence and uniqueness of a solution under dissipative-type conditions. They also show the continuous dependence of the solution on the initial conditions and analyze the existence of the solution on an infinite interval and its stability. Tri et al. [288] obtained different types of solutions to FFDEs and sheaf FFDEs generated by the use of the Bede–Gal derivative.

Hoa et al. [132] investigate stability for FFDEs by defining a new Lyapunov-like function. Vu and Hoa [292] establish the existence and uniqueness of the solution to impulsive fuzzy functional differential equations under Bede–Gal differentiability via the principle of contraction mappings.

In the study [99], Fatullayev et al. investigate inhomogeneous FDDE in which the initial function and the source function are fuzzy. To solve the FDDE they develop the method proposed by Gasilov et al. [111, 117] for fuzzy differential equations. The initial function and source function they represent in the form of a triangular fuzzy function (TFF) defined in [111, 117]. Under this approach, the authors show the existence and uniqueness of the solution for the considered problem. They demonstrate the developed method on inhomogeneous FDDEs of the first order. In the literature, it is a usual practice that solution methods, proposed for fuzzy differential equations without delay, are generally explained on equations of second order for clarity. However, when differential equations contain delay, explanations are given for equations of the first order. This circumstance also occurs in all the above-cited studies. The reason can be explained as follows. (1) Delay differential equations are more general (consequently, harder to solve) than the ones without delay. As a result of this, for linear differential equations without delay, there are strong theories and effective solution methods, while for delay differential equations, the findings are modest. Therefore, to develop a method for delay differential equations, inspired by an idea for usual differential equations, always, specific derivations and additional efforts are always required. (2) When one solves a fuzzy problem, the types of fuzzy input are more critical than their numbers. For example, by applying Zadeh's extension principle, one can easily solve nth order homogeneous linear differential equations with fuzzy initial values (i. e., with n fuzzy inputs in the form of fuzzy numbers). But the appearance of a fuzzy forcing function leads to that, even for the differential equation of first order (i. e., for the problem with 2 fuzzy inputs in the form of fuzzy functions), it becomes unclear how to use the principle. (3) One can see that the proposed method can be adapted for FDDEs of higher order.

Studies	Derivative
Jafelice et al. [144]	classical
Kichmarenko and Skripnik [175]	Hukuhara
Lupulescu [194]	Hukuhara
Farahi and Barati [93]	classical
Vu et al. [293]	Hukuhara
Fatullayev et al. [99]	classical
Wang [297]	Bede–Gal

Tab. 5. Studies investigating fuzzy delay differential equations, and the derivatives used in these studies.

4. COMPARISON OF DIFFERENT APPROACHES

In the following part of the text, we compare our approach ([110, 111, 114, 117]) with benchmark approaches, namely, Zadeh's extension principle and the Bede–Gal derivative approach. To make the comparison clearer, we use numerical examples.

4.1. A comparison based on the initial value problem

To make it easy to understand the steps involved in solving the fuzzy problem presented below, we first examine a related real problem.

Example 4.1. Consider the second-order initial value problem

$$\begin{cases} y'' + 2y' = 3y, \\ y(0) = a, \\ y'(0) = b. \end{cases} \tag{3}$$

The solutions of the differential equation $y'' + 2y' = 3y$ corresponding to the initial conditions $y(0) = 1$, $y'(0) = 0$ and $y(0) = 0$, $y'(0) = 1$ are $y = w_1(t) = \frac{1}{4}\left(3e^t + e^{-3t}\right)$ and $y = w_2(t) = \frac{1}{4}\left(e^t - e^{-3t}\right)$, respectively. Then the solution of IVP (3) is

$$y = aw_1(t) + bw_2(t)$$

or

$$y = y_{ab}(t) = \frac{1}{4}\left[a\left(3e^t + e^{-3t}\right) + b\left(e^t - e^{-3t}\right)\right]. \tag{4}$$

Below, we consider the same problem (3) but with fuzzy inputs.

Example 4.2. Solve the following second-order fuzzy initial value problem:

$$\begin{cases} \widetilde{Y}'' + 2\widetilde{Y}' = 3\widetilde{Y}, \\ \widetilde{Y}(0) = \widetilde{A}, \\ \widetilde{Y}'(0) = \widetilde{B}, \end{cases} \tag{5}$$

where $\widetilde{A} = (0,\ 1,\ 2)$ and $\widetilde{B} = (-4,\ -3,\ -2)$.

First, we solve the problem using our approach [110, 111, 114, 117]. We see this as an opportunity to explain the approach once again.

We interpret FIVP (5) as a set of real IVPs, which are built by taking a number a from $\widetilde{A}$ and a number b from $\widetilde{B}$ (more precisely, $a \in \left[\widetilde{A}\right]_0 = [0,\ 2]$ and $b \in \left[\widetilde{B}\right]_0 = [-4,\ 2]$), i. e., as a set of IVPs (3). To each IVP (3) and its solution function, we assign the membership degree $\mu_{ab} = \min\left\{\mu_{\widetilde{A}}(a),\ \mu_{\widetilde{B}}(b)\right\}$. The bunch (set) of all these solution functions, together with their membership degrees, we define as the solution of FIVP (5). Then, for the upper and lower bounds of the fuzzy bunch, we have:

$$\begin{aligned} \overline{y}(t) &= \max_{a\in[0,\,2],\ b\in[-4,\,2]} y_{ab}(t), \\ \underline{y}(t) &= \min_{a\in[0,\,2],\ b\in[-4,\,2]} y_{ab}(t), \end{aligned}$$

where $y_{ab}(t)$ is determined by (4). Since $w_1(t) \geq 0$ and $w_2(t) \geq 0$ on $t \geq 0$, the max is attained at $a = \overline{a} = 2,\ b = \overline{b} = 2$, and the min at $a = \underline{a} = 0,\ b = \underline{b} = -4$. In the result, we have the upper bound, the crisp solution (the solution with membership degree 1), and the lower bound be

$$\begin{aligned}\overline{y}(t) &= e^t + e^{-3t},\\ y_{cr}(t) &= e^{-3t},\\ \underline{y}(t) &= -e^t + e^{-3t}.\end{aligned}$$

Since the differential equation is linear, and the initial values are triangular fuzzy numbers, the value of the fuzzy solution at a time t is determined as the triangular fuzzy number $\widetilde{Y}(t) = \left(-\left(e^t - e^{-3t}\right),\ e^{-3t},\ e^t + e^{-3t}\right)$. We present the obtained solution in Figure 1.

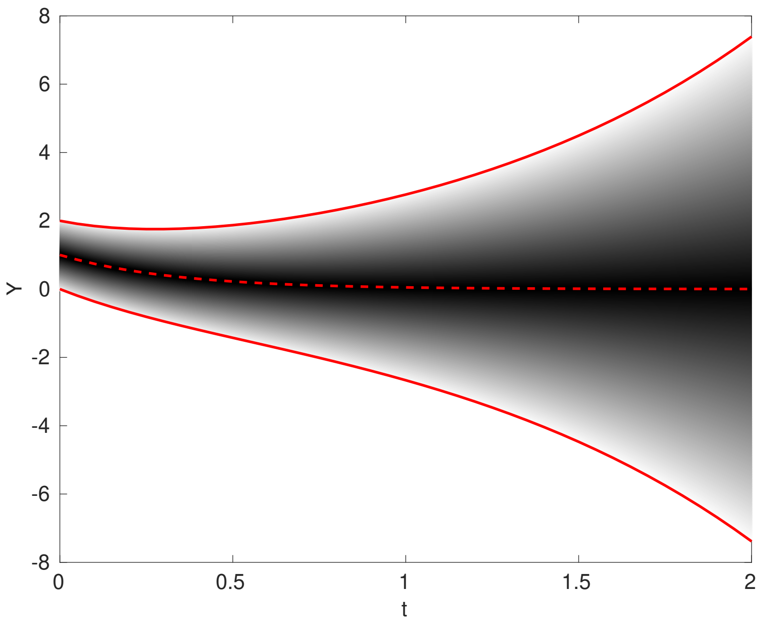

Fig. 1. The fuzzy solution, obtained by the proposed method, for Example 4.2. This solution coincides with the solutions obtained by Zadeh's extension principle and by the generalized derivative approach. The dashed line depicts the crisp solution, the continuous lines represent the lower and upper bounds of the fuzzy solution.

If in (4) we replace the real values with the corresponding fuzzy values, we obtain the solution by Zadeh's extension principle:

$\widetilde{Y} = \frac{1}{4}\left[\left(3e^t + e^{-3t}\right)\widetilde{A} + \left(e^t - e^{-3t}\right)\widetilde{B}\right]$, or

$\widetilde{Y} = \frac{1}{4}\left[\left(3e^t + e^{-3t}\right)(0,\ 1,\ 2) + \left(e^t - e^{-3t}\right)(-4,\ -3,\ -2)\right].$

Thus, the solution by Zadeh's extension principle is the fuzzy-valued function

$$\widetilde{Y} = \left(-\left(e^t - e^{-3t}\right),\ e^{-3t},\ e^t + e^{-3t}\right). \tag{6}$$

Graphically, this solution is the same as the one obtained by our approach (see Figure 1).

Now, we look for a solution under the Bede–Gal differentiability. If we search for the global $(1,1)$-solution for FIVP (5), we will have the same solution. To sum up, all three approaches (our approach, Zadeh's extension principle, and the Bede–Gal derivative approach) give the same solution for the example under consideration (expecting the differences in representations/interpretations). It is worth noting that such examples, where various approaches work and give solutions that coincide with other approaches, inspire researchers to insist on their favorite approach and hope that their approach will cover a wider class of problems after some improvements. Unfortunately, the reality is different. To see why, let us consider the following FIVP:

$$\begin{cases} \widetilde{Y}'' + 2\widetilde{Y}' - 3\widetilde{Y} = 0, \\ \widetilde{Y}(0) = \widetilde{A}, \\ \widetilde{Y}'(0) = \widetilde{B}. \end{cases} \tag{7}$$

The only difference of this FIVP from (5) is that the term $3\widetilde{Y}$ is translated to the left-hand side of the differential equation. This translation does not change the solution if we consider a problem under the real calculus. But, if we investigate (7) under fuzzy calculus, it does not have a solution. (The reason is that since the initial values are proper fuzzy numbers, the problem can have only a proper fuzzy solution, i. e., a solution with non-zero fuzziness. But in this case, the left-hand side of the equation will also be a proper fuzzy quantity. Therefore, it cannot be equal to 0, i. e., to the right-hand side.) It can be easily seen that by our approach and by Zadeh's extension principle, FIVP (7) has a solution, which coincides with the solution to (5). In summary, the considered example points to one of the main drawbacks of the fuzzy calculus approach: Even IVPs with well-behaved input data may not have a solution under this approach.

4.2. Example of application of Zadeh's extension principle

In this subsection, we consider an application of Zadeh's extension principle to solving fuzzy differential equations.

Example 4.3. Consider the FIVP

$$\begin{cases} \widetilde{Y}'' = \widetilde{C}^2\, \widetilde{Y} \\ \widetilde{Y}(0) = \widetilde{A} \\ \widetilde{Y}'(0) = 0 \end{cases} \tag{8}$$

where $\widetilde{C} = \left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$ and $\widetilde{A} = (2, 2.6, 3)$.

The associated real IVP is

$$\begin{cases} y'' = c^2\, y \\ y(0) = a \\ y'(0) = 0 \end{cases} \tag{9}$$

and its solution is $y = a\frac{e^{ct}+e^{-ct}}{2}$, or $y = a\cosh(ct)$, in another representation. If we replace a and c with $\widetilde{A}$ and $\widetilde{C}$, respectively, in the last representation, we get

$$\widetilde{Y} = \widetilde{A}\, \cosh(t\, \widetilde{C}),$$

which coincides with the solution according to Zadeh's extension principle (see Figure 2).

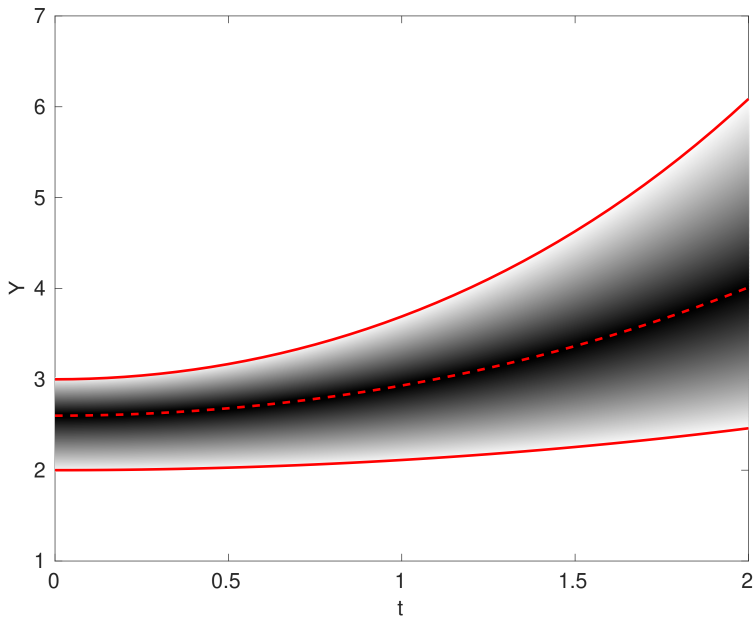

Fig. 2. The fuzzy solution, obtained by Zadeh's extension principle, for Example 4.3. Our proposed approach gives the same solution.

Remark. Let us emphasize the following circumstance concerning the incorrect implementation of Zadeh's extension principle. In the two examples given above (Examples 4.2 and 4.3), changing real parameters to the related fuzzy values gave the solution by the extension principle. However, this procedure is not valid in general. To be sure, let us consider Example 4.3 and represent the crisp solution as $y = \frac{1}{2}a(e^{ct} + e^{-ct})$. If we replace the real parameters (a and c) with the corresponding fuzzy values ($\widetilde{A}$ and $\widetilde{C}$), we obtain

$$\widetilde{Y} = \frac{1}{2}\widetilde{A}\left(e^{t\widetilde{C}} + e^{-t\widetilde{C}}\right) = \frac{1}{2}(2, 2.6, 3)\left(e^{\left(\frac{1}{3}t, \frac{1}{2}t, \frac{2}{3}t\right)} + e^{\left(-\frac{2}{3}t, -\frac{1}{2}t, -\frac{1}{3}t\right)}\right), \tag{10}$$

for which

$$\underline{y}(t) = e^{\frac{1}{3}t} + e^{-\frac{2}{3}t} \quad \text{and} \quad \overline{y}(t) = \frac{3}{2}\left(e^{\frac{2}{3}t} + e^{-\frac{1}{3}t}\right).$$

However, our approach gives a fuzzy solution (see Figure 2) with

$$\underline{y}(t) = e^{\frac{1}{3}t} + e^{-\frac{1}{3}t} \quad \text{and} \quad \overline{y}(t) = \frac{3}{2}\left(e^{\frac{2}{3}t} + e^{-\frac{2}{3}t}\right).$$

It is easy to see that the lower bound obtained by the extension principle is less by $\Delta\underline{y}(t) = e^{-\frac{1}{3}t} - e^{-\frac{2}{3}t}$, while the upper bound is larger by $\Delta\overline{y}(t) = \frac{3}{2}\left(e^{-\frac{1}{3}t} - e^{-\frac{2}{3}t}\right)$. Therefore, the fuzziness of the solution obtained by the improperly implemented extension principle is rougher than that of our approach.

Considering the steps taken in this example, we can state two significant difficulties encountered in the implementation of the extension principle to fuzzy differential equations. The first and main difficulty is that the solution to the associated real problem may not be expressed analytically. The second difficulty arises in the case when a fuzzy-valued function is involved as a parameter. In this case, when constructing the associated real problem, it is unclear what kind of real function to replace the fuzzy-valued function with. (Note that using the concept of a "fuzzy bunch of functions" helps overcome this difficulty.) In addition, it is usually very difficult to solve differential equations analytically when there is a function $f(t)$ among the parameters.

The extension principle is a basic principle of fuzzy set theory. The new approaches being proposed should give results compatible with it. Therefore, the extension principle can be used as a tool for verifying new approaches, and if a newly proposed approach does not give compatible results, it should be questioned.

4.3. Advantage of our proposed approach over Zadeh's extension principle

We provide two examples to explain the advantages of the proposed approach more clearly.

Example 4.4. Consider the FBVP

$$\begin{cases} \widetilde{Y}'' = \widetilde{C}^2\, \widetilde{Y} \\ \widetilde{Y}(0) = \widetilde{A} \\ \widetilde{Y}(l) = \widetilde{B} \end{cases} \tag{11}$$

where $\widetilde{C} = \left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$, $\widetilde{A} = (2, 2.6, 3)$, $\widetilde{B} = (-0.2, 0, 0.3)$ and $l = 3$.

The associated real BVP is

$$\begin{cases} y'' = c^2\, y \\ y(0) = a \\ y(l) = b \end{cases} \tag{12}$$

and its solution is

$$y = y(t, a, b, c) = \frac{1}{\sinh cl}\left(a \sinh c(l-t) + b \sinh ct\right) \tag{13}$$

The solution of FBVP (11) by the Extension principle is

$$\left[\widetilde{Y}\left(t, \widetilde{A}, \widetilde{B}, \widetilde{C}\right)\right]_\alpha = y\left(t, A_\alpha, B_\alpha, C_\alpha\right),$$

where $y = y(t, a, b, c)$ is defined by (13).

The fuzzy solution is depicted in Figure 3. Our proposed approach gives the same solution except for the difference in interpretation.

Below we consider a new example that is obtained from FBVP (11) by replacing the fuzzy number $\widetilde{C}$ with a fuzzy function $\widetilde{C}(t)$ and adding a new fuzzy function $\widetilde{F}(t)$.

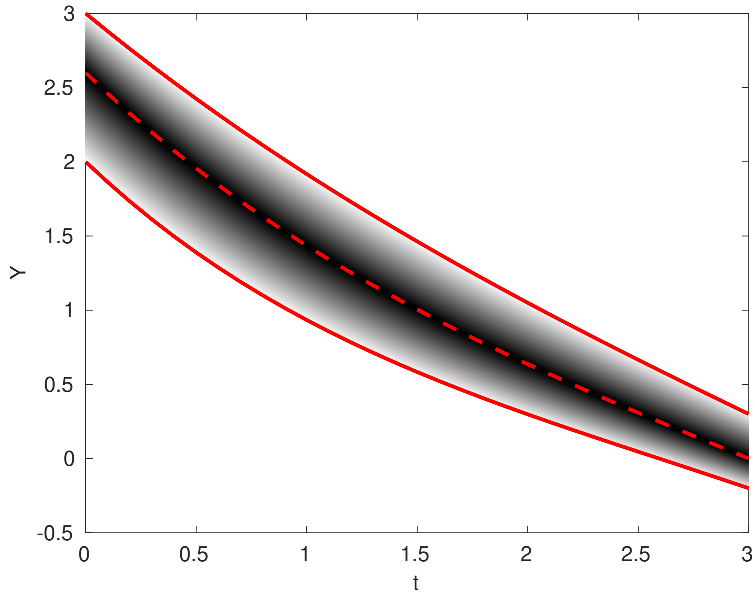

Fig. 3. The fuzzy solution, obtained by Zadeh's extension principle, for Example 4.4.

Example 4.5. Consider the FBVP

$$\begin{cases} \widetilde{Y}'' = \widetilde{C}^2(t)\,\widetilde{Y} + \widetilde{F}(t) \\ \widetilde{Y}(0) = \widetilde{A} \\ \widetilde{Y}(3) = \widetilde{B} \end{cases} \tag{14}$$

where $\widetilde{C}(t) = \left(\frac{1}{3} - \frac{1}{6}e^{-(t-1)^2},\, \frac{1}{2},\, \frac{2}{3} + \frac{1}{6}e^{-(t-1)^2}\right)$, $\widetilde{F}(t) = (-1 + \sin t, 0, 1 - \cos t)$, $\widetilde{A} = (2,\, 2.6,\, 3)$ and $\widetilde{B} = (-0.2,\, 0,\, 0.3)$.

In contrast to the previous example, it is difficult to construct an associated real BVP for the considered FBVP (14). The main reason is that it is unclear what real functions can represent the fuzzy functions $\widetilde{C}$ and $\widetilde{F}$. (Note that even if we could describe the fuzzy functions by real functions c and f, respectively, it would be difficult to find an analytical solution to the differential equation $y'' = c^2(t)y + f(t)$.) Therefore, no associated real solution can be obtained, which would then be transformed into a fuzzy solution. Thus, the extension principle does not work for the considered example. However, if we interpret the fuzzy functions $\widetilde{C}$ and $\widetilde{F}$ as triangular fuzzy functions (namely, if we represent $\widetilde{C} = \left\langle \frac{1}{3} - \frac{1}{6}e^{-(t-1)^2},\, \frac{1}{2},\, \frac{2}{3} + \frac{1}{6}e^{-(t-1)^2} \right\rangle$ and $\widetilde{F} = \langle -1 + \sin t, 0, 1 - \cos t \rangle$), we can reformulate FBVP (14) as a set of real BVPs. Based on the solutions of these real BVPs, we can construct a fuzzy solution to FBVP (14). We depict the fuzzy solution in Figure 4. The following points regarding the solution can be noted. When moving from

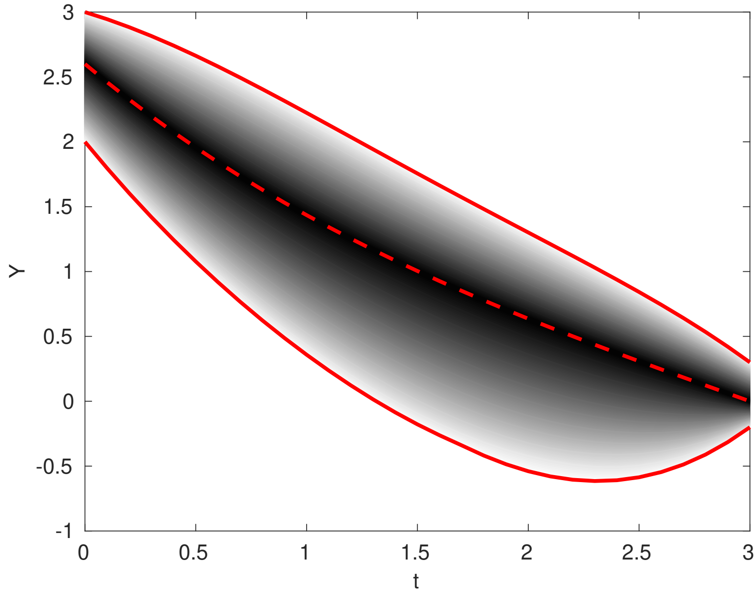

Fig. 4. The fuzzy solution, obtained by the proposed approach, for Example 4.5.

FBVP (11) to FBVP (14), the boundary values have not changed, but the fuzziness of $\widetilde{C}$ has increased. In addition, a new fuzziness, $\widetilde{F}$, has been added to the equation. Therefore, in the graphical representation of the solution, the left and right boundaries should remain the same, but the increase in the fuzziness of the input parameters should lead to an increase in the fuzziness of the solution in the intermediate part. It can be confirmed from the presented figures that this circumstance indeed takes place. The considered example shows that our proposed approach works even in cases where the extension principle becomes insufficient. The above-mentioned cases can be specified as cases where it is difficult to construct an associated real problem and where it is hard to represent a real solution by an analytical formula.

At the end of this subsection, we would like to address the issue of the applicability of our proposed approach. The proposed approach can be applied to any fuzzy differential equation, regardless of whether it is linear or not. However, in the linear case, the approach works effectively from the point of view of computational complexity. Namely, to build the fuzzy solution, in this case, it is sufficient to solve a much smaller number of real problems. For example, assume that we solve a FIVP for a second-order crisp differential equation with fuzzy initial values. If the differential equation is linear, then to construct the fuzzy solution, it is sufficient to solve only 4 classical IVPs (for edge values of the initial values), i. e., the computational complexity of the approach is $O(1)$ in terms of real problems to be solved. But, if the differential equation is non-linear, then the complexity can increase up to $O(n^2)$, where n is the number of grid points used to approximate each fuzzy initial value. It should be noted that, in the case where the solution of the associated real problem monotonically depends on the input values, the evaluation can be improved.

4.4. A comparison based on boundary value problem

We first present a real BVP, which we will then use to solve the subsequent FBVP.

Example 4.6. Consider the second-order boundary value problem

$$\begin{cases} y'' + 5y = 2y', \\ y(0) = a, \\ y\left(\frac{3\pi}{4}\right) = b. \end{cases} \tag{15}$$

The solution is

$$y = a\ e^t \cos 2t - b\ e^{t-\frac{3\pi}{4}} \sin 2t. \tag{16}$$

Example 4.7. Now let us examine the second-order fuzzy boundary value problem

$$\begin{cases} \widetilde{Y}'' + 5\widetilde{Y} = 2\widetilde{Y}', \\ \widetilde{Y}(0) = \widetilde{A}, \\ \widetilde{Y}\left(\frac{3\pi}{4}\right) = \widetilde{B}, \end{cases} \tag{17}$$

where $\widetilde{A} = (-0.5,\ 0,\ 0.5)$ and $\widetilde{B} = (1,\ 2,\ 3)$.

To obtain the solution by Zadeh's extension principle, we replace the real values in (16) with the corresponding fuzzy values:

$$\widetilde{Y} = \left(e^t \cos 2t\right) \widetilde{A} + \left(-e^{t-\frac{3\pi}{4}} \sin 2t\right) \widetilde{B}. \tag{18}$$

Thus, the solution by Zadeh's extension principle is (See Figure 5)

$$\widetilde{Y} = \left(e^t \cos 2t\right) (-0.5,\ 0,\ 0.5) + \left(-e^{t-\frac{3\pi}{4}} \sin 2t\right) (1,\ 2,\ 3). \tag{19}$$

For the solution by our approach, we have:

$$\overline{y}(t) = \begin{cases} 0.5\ e^t \cos 2t - e^{t-\frac{3\pi}{4}} \sin 2t, & 0 \le t < \frac{\pi}{4} \\ -0.5\ e^t \cos 2t - e^{t-\frac{3\pi}{4}} \sin 2t, & \frac{\pi}{4} \le t \le \frac{\pi}{2} \\ -0.5\ e^t \cos 2t - 3\ e^{t-\frac{3\pi}{4}} \sin 2t, & \frac{\pi}{2} \le t \le \frac{3\pi}{4} \end{cases} \tag{20}$$

$$y_{cr}(t) = -2\ e^{t-\frac{3\pi}{4}} \sin 2t \tag{21}$$

$$\underline{y}(t) = \begin{cases} -0.5\ e^t \cos 2t - 3\ e^{t-\frac{3\pi}{4}} \sin 2t, & 0 \le t < \frac{\pi}{4} \\ 0.5\ e^t \cos 2t - 3\ e^{t-\frac{3\pi}{4}} \sin 2t, & \frac{\pi}{4} \le t \le \frac{\pi}{2} \\ 0.5\ e^t \cos 2t - e^{t-\frac{3\pi}{4}} \sin 2t, & \frac{\pi}{2} \le t \le \frac{3\pi}{4}. \end{cases} \tag{22}$$

Graphically, the solution by our approach is the same as the one obtained by Zadeh's extension principle (See Figure 5).

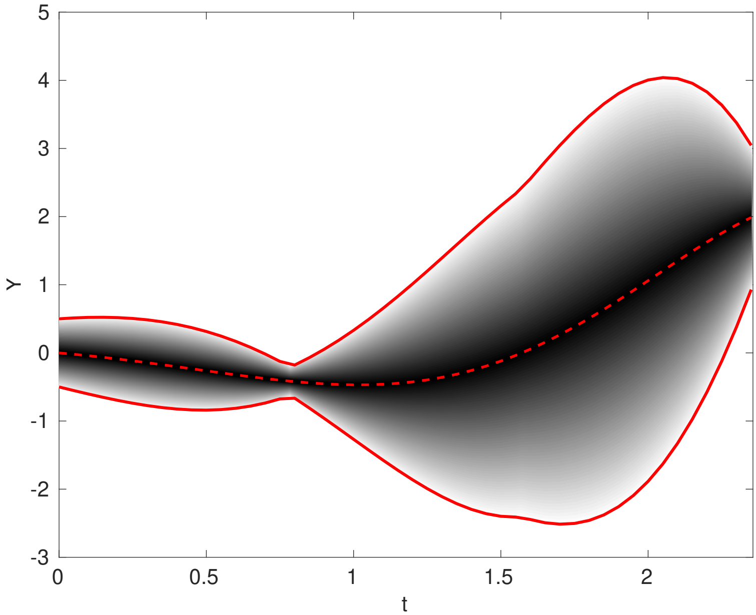

Fig. 5. The fuzzy solution, obtained by Zadeh's extension principle, for Example 4.7. This solution coincides with the solution obtained by the proposed approach.

Now, let us consider whether the FBVP (17) has a solution under fuzzy calculus (under Bede–Gal derivative). Since $\widetilde{B}$ (the right-end value) is wider than $\widetilde{A}$ (the left-end value), the solution can only be with expanding fuzziness, i. e., it can be either (1,1) or (1,2)-differentiable. First, let us search for the (1,1)-differentiable solution. For the fuzzy solution we use the α-cut representation: $Y_\alpha(t) = [\underline{y_\alpha}(t), \overline{y_\alpha}(t)]$. For globally (1,1)-solution, $Y'_\alpha = [\underline{y_\alpha}', \overline{y_\alpha}']$ and $Y''_\alpha = [\underline{y_\alpha}'', \overline{y_\alpha}'']$. If to put in (17), we have:

$$\left\{ \begin{array}{ll} \overline{y_\alpha}'' + 5\overline{y_\alpha} = 2\overline{y_\alpha}', & \underline{y_\alpha}'' + 5\underline{y_\alpha} = 2\underline{y_\alpha}', \\ \overline{y_\alpha}(0) = 0.5(1-\alpha), & \underline{y_\alpha}(0) = -0.5(1-\alpha), \\ \overline{y_\alpha}\left(\frac{3\pi}{4}\right) = 3-\alpha, & \underline{y_\alpha}\left(\frac{3\pi}{4}\right) = 1+\alpha. \end{array} \right. \tag{23}$$

We can see that the equations and boundary conditions for $\overline{y_\alpha}$ and $\underline{y_\alpha}$ are independent. Therefore, they can be determined separately. For $\alpha = 0$, we have:

$$\overline{y_0}(t) = 0.5\ e^t \cos 2t - 3\ e^{t-\frac{3\pi}{4}} \sin 2t,$$
$$\underline{y_0}(t) = -0.5\ e^t \cos 2t - e^{t-\frac{3\pi}{4}} \sin 2t.$$

These functions are depicted in Figure 6. It can be seen that the upper bound passes lower on some intervals. Therefore, a proper fuzzy solution is not determined. Thus, Example 4.7 does not have a globally (1,1)-differentiable solution.

Similar (but more cumbersome) calculations show that a global (1,2)-solution also does not exist. To sum up, FBVP (17) has solutions according to our approach and

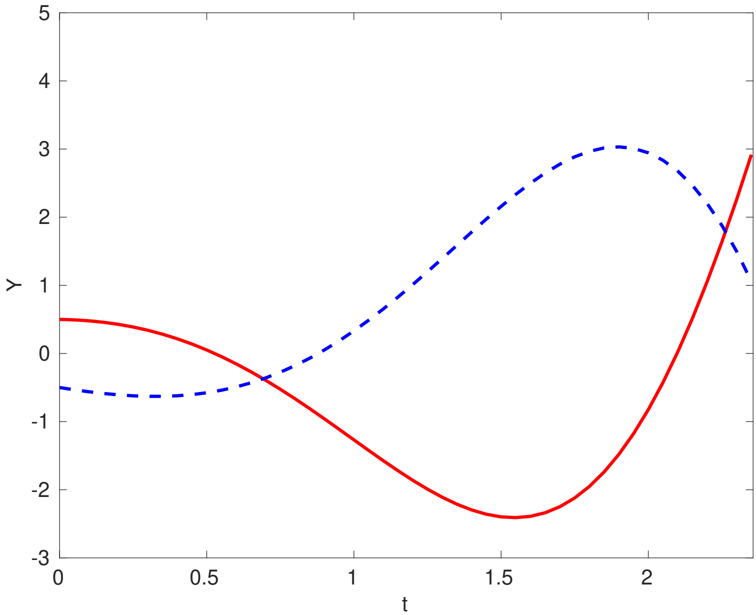

Fig. 6. The upper and lower solutions to the (1,1)-problem, created for Example 4.7. The upper solution (continuous line) passes lower on some intervals. Therefore, Example 4.7 does not have a globally (1,1)-differentiable solution.

by Zadeh's extension principle, and these solutions are in accordance (see Figure 5). However, the problem has no solution under the Bede–Gal differentiability concept. This fact once again indicates that the fuzzy calculus approach is not sufficient for solving fuzzy differential equations.

5. COMPARISON WITH THE APPROACH USING INTERACTIVE FUZZY NUMBERS

One of the important concepts discussed by some researchers recently is the concept of "interactive (or correlated) fuzzy numbers". Let us illustrate this concept through the following example.

Assume that during a production process, it is necessary to heat a substance to a temperature of $a = 50°C$. Since the production conditions and the equipment used are not perfect, this value cannot be achieved with certainty. Let us assume that the provided temperature is modeled by the fuzzy number $\widetilde{A} = (45, 50, 55)°C$. If the above-mentioned values are expressed in Fahrenheit, then they become $b = f(a) = \frac{9}{5}a + 32 = 122°F$ and $\widetilde{B} = (113, 122, 131)°F$, respectively. Now let us consider the fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$. In the example under consideration, each time, when, for example, the value $a = 52.5$ is taken from $\widetilde{A}$, the value $b = 126.5$ will be realized from $\widetilde{B}$. In other words, there is a relationship between the elements of $\widetilde{A}$ and $\widetilde{B}$. In such a case, $\widetilde{A}$ and $\widetilde{B}$ are called correlated (or interactive, in the general sense of the concept) fuzzy numbers. Since the function $b = f(a) = \frac{9}{5}a + 32$, which determines the relationship between the elements of sets $\widetilde{A}$ and $\widetilde{B}$, is linear, then $\widetilde{A}$ and $\widetilde{B}$ are called linearly correlated fuzzy numbers.

More generally, if there is a relationship between elements of sets $\widetilde{A}$ and $\widetilde{B}$ via a function (linear or non-linear) $b = f(a)$ (or, in other words, if $\widetilde{B} = f(\widetilde{A})$), then $\widetilde{A}$ and $\widetilde{B}$ are simply called correlated fuzzy numbers.

Interactive fuzzy numbers are a generalization of correlated fuzzy numbers. In detail, this statement can be explained as follows. For given independent fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$, the Cartesian product $\widetilde{A} \times \widetilde{B}$ consists of real pairs (a, b) with membership degrees, defined as $\mu_{\widetilde{A}\times\widetilde{B}}(a, b) = \min\left\{\mu_{\widetilde{A}}(a),\ \mu_{\widetilde{B}}(b)\right\}$. Thus, if $\widetilde{A}$ and $\widetilde{B}$ are independent, geometrically, the set $\widetilde{A} \times \widetilde{B}$ forms a fuzzy rectangle in the xy-coordinate plane. But, if $\widetilde{B} = f(\widetilde{A})$, i. e., if they are correlated, then the pairs $(a, b) = (a, f(a))$ constitute a fuzzy curve $\widetilde{C}$ (i. e., a subset of $\widetilde{A} \times \widetilde{B}$), lying in the above rectangle. In the general case, if in the problem under consideration, $(a, b) \in \widetilde{S}$ for all possible pairs, where $\widetilde{S} \subset \widetilde{A} \times \widetilde{B}$, then the fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ are called interactive fuzzy numbers.

It is worth highlighting that the interactivity is not a mathematical property: the numbers $\widetilde{A} = (45, 50, 55)$ and $\widetilde{B} = (113, 122, 131)$, which are seen as interactive fuzzy numbers above, may not be interactive in another context (for example, if $\widetilde{A}$ is a temperature value and $\widetilde{B}$ is a location value). The fact that some parameters in the problem are correlated can be interpreted as additional information on the problem, and allows one to express the fuzziness of the solution more accurately (precisely). As an example, let us consider the FBVP, which was examined in [110] and [266]:

$$\begin{cases} \widetilde{X}''(t) + 16\widetilde{X}(t) = 0, \\ \widetilde{X}(0) = \widetilde{A} = (-1,\ 0,\ 1), \\ \widetilde{X}(2) = \widetilde{B} = (-0.5,\ 0,\ 0.5). \end{cases} \tag{24}$$

If $\widetilde{A}$ and $\widetilde{B}$ are considered to be non-interactive (or independent), the solution will be $\widetilde{X}(t) = \frac{\sin(8-4t)}{\sin 8}(-1,\ 0,\ 1) + \frac{\sin 4t}{\sin 8}(-0.5,\ 0,\ 0.5)$ or, in other words,

$$\begin{aligned} \widetilde{X}(t) &= \tfrac{1}{\sin 8}\left[2\sin(8-4t)\ (-0.5,\ 0,\ 0.5) + \sin 4t\ (-0.5,\ 0,\ 0.5)\right] \\ &= \tfrac{1}{\sin 8}\left[|2\sin(8-4t)| + |\sin 4t|\right]\ (-0.5,\ 0,\ 0.5). \end{aligned}$$

But if they are considered to be interactive through $\widetilde{A} = 2\widetilde{B}$, the solution will be

$$\begin{aligned} \widetilde{X}_{interactive}(t) &= \tfrac{1}{\sin 8}\left(2\sin(8-4t) + \sin 4t\right)\ (-0.5,\ 0,\ 0.5) \\ &= \tfrac{1}{\sin 8}\left|2\sin(8-4t) + \sin 4t\right|\ (-0.5,\ 0,\ 0.5). \end{aligned}$$

The fuzziness of the second solution is less. For example, at $t = 0.5$,

$\widetilde{X}(0.5) = \frac{1}{\sin 8}\left[2\sin 6\ (-0.5,\ 0,\ 0.5) + \sin 2\ (-0.5,\ 0,\ 0.5)\right] \approx (-0.6007,\ 0,\ 0.6007)$, but

$\widetilde{X}_{interactive}(0.5) = \frac{1}{\sin 8}\left(2\sin 6 + \sin 2\right)\ (-0.5,\ 0,\ 0.5) \approx (-0.3183,\ 0,\ 0.3183)$.

In summary, the concept of interactive fuzzy numbers is undoubtedly a useful tool that allows us to determine the uncertainty of the solution more carefully. However, we believe that constructing a calculus based on this concept should be done with the utmost care.

First of all, as we noted before, a priori, two fuzzy numbers are neither interactive nor non-interactive. Based on the problem under consideration, we accept them to be interactive or independent. Therefore, interactivity is not a general property of fuzzy numbers but represents an additional condition on them.

On the other hand, the concept can be easily used in the framework of the extension principle without any special means. We argue this statement as follows. Suppose we have a fuzzy problem with two fuzzy inputs, say $\widetilde{A}$ and $\widetilde{B}$, which are fuzzy numbers, and let them be interactive through a function $b = f(a)$. Also, let $x(t, a, b)$ be the solution of the associated real problem. Then, by the extension principle, the solution of the fuzzy problem is $\left[\widetilde{X}(t, \widetilde{A}, \widetilde{B})\right]_\alpha = \left\{ x(t, a, f(a)) \mid a \in \left[\widetilde{A}\right]_\alpha \right\}$.

Our proposed approach is also easily adopted in the case of interactive fuzzy numbers. In this case, we interpret the fuzzy problem as a set of real problems that are built by taking a real number a from $\widetilde{A}$ and simultaneously $b = f(a)$ from $\widetilde{B}$. Therefore, we have a set of real problems with a single parameter a. Based on their solutions, we can constitute the fuzzy solution.

Finally, hastily created arithmetic and calculus are difficult to implement in practice. Often, even their authors themselves get confused when working with them. Below, we provide some examples to support our claim.

First, we think that there is a difficulty with the interactive differentiability ($\mathcal{F}$-differentiability), proposed in [241]. Below, we give our arguments. Let us consider the function $\widetilde{F}(t) = (-t^2,\ 0,\ t^2)$, or $\widetilde{F}(t) = t^2(-1,\ 0,\ 1)$, in another representation. It is natural to expect its derivative be $\widetilde{F}'(t) = 2t(-1,\ 0,\ 1)$. Note that the considered function is represented by its α-cuts as follows: $\left[\widetilde{F}(t)\right]_\alpha = \left[\underline{f_\alpha}(t),\ \overline{f_\alpha}(t)\right]$, where $\underline{f_\alpha}(t) = t^2(\alpha - 1)$ and $\overline{f_\alpha}(t) = t^2(1 - \alpha)$.

Below, we examine what is the interactive derivative of $\widetilde{F}(t)$ according to Theorem 2, provided by the authors of [241]. Since $\widetilde{F}(t) = t^2(-1,\ 0,\ 1)$ and $\widetilde{F}(t+h) = (t+h)^2(-1,\ 0,\ 1)$, we can see that $\widetilde{F}(t+h) = \frac{(t+h)^2}{t^2}\widetilde{F}(t) = \left(1 + \frac{h}{t}\right)^2 \widetilde{F}(t)$. Thus, $\widetilde{F}(t+h) = \mathcal{F}_{t,h}\left(\widetilde{F}(t)\right)$, where $\mathcal{F}_{t,h}(z) = \left(1 + \frac{h}{t}\right)^2 z$. Then, $\mathcal{F}'_{t,h}(z) = \left(1 + \frac{h}{t}\right)^2$. For clarity, let $t \in [1,\ 2]$. It can be seen that $\mathcal{F}'_{t,h}(z) > 1$, for positive h, while $0 < \mathcal{F}'_{t,h}(z) < 1$, for negative h. Since h can be both positive and negative in each of the three items of Theorem 2, the condition of no item is satisfied. Thus, even for a simple fuzzy function, Theorem 2 does not provide a derivative.

Now, let us focus on another circumstance. Since $\widetilde{F}(t) = t^2(-1,\ 0,\ 1)$ and $\widetilde{F}(t+h) = (t+h)^2(-1,\ 0,\ 1)$, we can also interpret that $\widetilde{F}(t+h) = -\left(1 + \frac{h}{t}\right)^2 \widetilde{F}(t)$. Then, $\widetilde{F}(t+h) = \mathcal{F}_{t,h}\left(\widetilde{F}(t)\right)$ with $\mathcal{F}_{t,h}(z) = -\left(1 + \frac{h}{t}\right)^2 z$. Since $\mathcal{F}'_{t,h}(z) = -\left(1 + \frac{h}{t}\right)^2$, we have $\mathcal{F}'_{t,h}(z) < 0$, for sufficiently small h. Then, by Theorem 2, item *iii*, we get: $\left[\widetilde{F}'_{\mathcal{F}}(t)\right]_\alpha = \left\{ \left(\underline{f_\alpha}\right)'(t) \right\} = \{2t(\alpha - 1)\}$, i. e., $\left[\widetilde{F}'_{\mathcal{F}}(t)\right]_\alpha = \{-2t(1 - \alpha)\}$. The obtained derivative is not a proper fuzzy value. To be sure, take $t = 1.5$, for example. We have $\left[\widetilde{F}'_{\mathcal{F}}(1.5)\right]_1 = \{0\}$, $\left[\widetilde{F}'_{\mathcal{F}}(1.5)\right]_0 = \{-3\}$ and, therefore, $\left[\widetilde{F}'_{\mathcal{F}}(1.5)\right]_1 \not\subseteq \left[\widetilde{F}'_{\mathcal{F}}(1.5)\right]_0$, i. e., the nesting condition of α-cuts is not satisfied.

Thus, the main theorem about the $\mathcal{F}$-differentiability does not work even for simple functions. The derivatives in items *i* and *ii* are 1st and 2nd Bede–Gal derivatives. Finally, there is confusion, as indicated above, with the item *iii*.

In the next two subsections, we provide solutions to two fuzzy application problems and explain some misunderstandings.

5.1. The population growth problem

Let us consider the following fuzzy model of population dynamics, namely Verhulst logistic growth model [241]:

$$\begin{cases} \widetilde{X}'(t) = a\ \widetilde{X}(t)\left(k - \widetilde{X}(t)\right), \\ \widetilde{X}(0) = \widetilde{X}_0, \end{cases} \tag{25}$$

where a and k are positive real constants, $\widetilde{X}_0$ is a "positive" fuzzy number. Below, we obtain the solution of the FIVP by the extension principle.

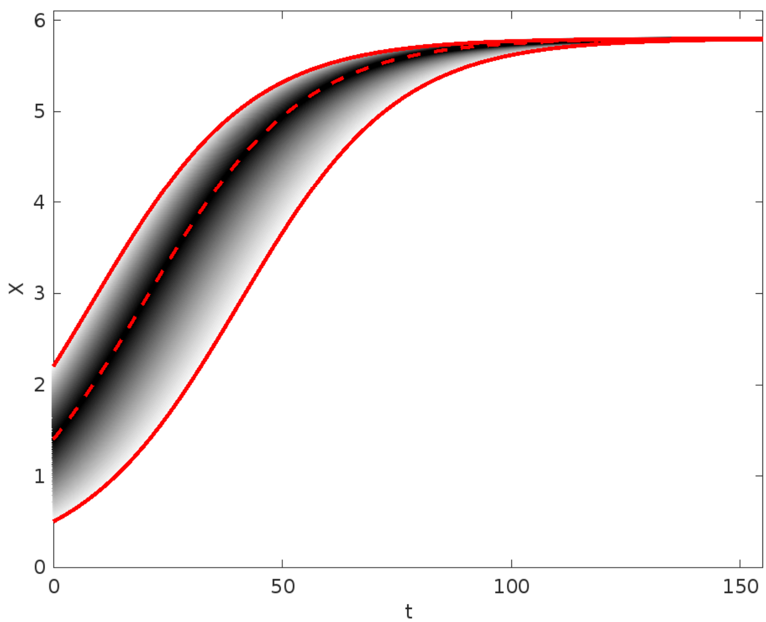

Fig. 7. The solution of FIVP (25), obtained by Zadeh's extension principle. Our proposed approach gives the same solution.

The associated real problem is as follows:

$$\begin{cases} x'(t) = a\ x(t)\,(k - x(t)), \\ x(0) = x_0. \end{cases} \tag{26}$$

This first-order differential equation, also called the logistic equation, can be solved by separating variables or by treating it as a Bernoulli equation. The obtained solution of IVP (26) can be represented as

$$x(t,x_0)=\frac{k}{1+\left(\frac{k}{x_0}-1\right)e^{-akt}}. \tag{27}$$

It can be seen that the solution increases with x_0. Then, the solution to the fuzzy problem (25) is easily obtained by the extension principle:

$$X_\alpha(t)=\left[\underline{X_\alpha}(t),\ \overline{X_\alpha}(t)\right]=\left[x\left(t,\underline{X_{0,\alpha}}\right),\ x\left(t,\overline{X_{0,\alpha}}\right)\right], \tag{28}$$

where the function $x(t,x_0)$ is given by (27).

Figure 7 shows the solution of (25) with $a=0.01$, $k=5.8$ and $\widetilde{X}_0=(0.5,\ 1.4,\ 2.2)$. Let us compare Figure 7 with the solution obtained using the interactivity concept by the authors of [241]. It can be seen that their solution differs from the solution by the extension principle. This circumstance indicates that the arithmetic and calculus proposed by the authors have some shortcomings.

5.2. The problem of determining the shape of a suspension bridge cable

In the real case, the problem of finding the shape of a suspension bridge cable that is fastened at each end and carries a distributed load is modeled by the following BVP [266]:

$$\begin{cases} x''(t)=\beta\sqrt{1+\left(x'(t)\right)^2}, \\ x(0)=h_0, \\ x(T)=h_T, \end{cases} \tag{29}$$

where $\beta>0$ and $T>0$ are given parameters. The differential equation is non-linear, and its general solution is

$$x(t,h_0,h_T)=c_1+\frac{1}{\beta}\cosh\left(\beta\left(t+c_2\right)\right),$$

where the constants c_1 and c_2 depend on the boundary values h_0 and h_T. They can be represented as $c_1=c_1\left(h_0,h_T\right)$ and $c_2=c_2\left(h_0,h_T\right)$, and satisfy the following system:

$$\begin{cases} c_1+\frac{1}{\beta}\cosh\left(\beta c_2\right)=h_0, \\ c_1+\frac{1}{\beta}\cosh\left(\beta\left(T+c_2\right)\right)=h_T. \end{cases}$$

If to express c_1 through c_2 by using the first equation, the solution of the BVP can be rewritten as

$$x(t,h_0,h_T)=h_0+\frac{1}{\beta}\left[\cosh\left(\beta\left(t+c_2\right)\right)-\cosh\left(\beta c_2\right)\right], \tag{30}$$

where c_2 is the solution of the non-linear algebraic equation

$$\frac{1}{\beta}\left[\cosh\left(\beta\left(T+c_2\right)\right)-\cosh\left(\beta c_2\right)\right]=h_T-h_0. \tag{31}$$

The function on the left-hand side, $G(c_2) = \frac{1}{\beta}\left[\cosh\left(\beta\left(T+c_2\right)\right) - \cosh\left(\beta c_2\right)\right]$, is a strongly increasing function with range $(-\infty, \infty)$. Therefore, the solution of (31), $c_2 = c_2(h_0, h_T)$, is unique and increases if the right side $h_T - h_0$ increases. Consequently, $c_2(h_0, h_T)$ increases with respect to h_T and decreases with respect to h_0, that is, $\frac{\partial c_2}{\partial h_T} > 0$ and $\frac{\partial c_2}{\partial h_0} < 0$. Below we show that $\frac{\partial x}{\partial h_0} > 0$ and $\frac{\partial x}{\partial h_T} > 0$, i. e., the solution function $x(t, h_0, h_T)$ is increasing with respect to both h_0 and h_T.

From (30), we have

$$\frac{\partial x}{\partial h_T} = \left[\sinh\left(\beta\left(t+c_2\right)\right) - \sinh\left(\beta c_2\right)\right]\,\frac{\partial c_2}{\partial h_T}.$$

Since sinh is an increasing function and $\frac{\partial c_2}{\partial h_T} > 0$, we obtain $\frac{\partial x}{\partial h_T} > 0$ (on $t > 0$).

From (30) and (31), we also have

$$\frac{\partial x}{\partial h_0} = 1 + \left[\sinh\left(\beta\left(t+c_2\right)\right) - \sinh\left(\beta c_2\right)\right]\,\frac{\partial c_2}{\partial h_0}$$

and

$$\left[\sinh\left(\beta\left(T+c_2\right)\right) - \sinh\left(\beta c_2\right)\right]\,\frac{\partial c_2}{\partial h_0} = -1,$$

respectively. Then, on $t > 0$,

$$\frac{\partial x}{\partial h_0} = 1 - \frac{\sinh\left(\beta\left(t+c_2\right)\right) - \sinh\left(\beta c_2\right)}{\sinh\left(\beta\left(T+c_2\right)\right) - \sinh\left(\beta c_2\right)} > 1 - 1 = 0,$$

because sinh is an increasing function.

Now, let us consider the fuzzy version of the problem under investigation:

$$\begin{cases} \widetilde{X}''(t) = \beta\sqrt{1+\left(\widetilde{X}'(t)\right)^2}, \\ \widetilde{X}(0) = \widetilde{H}_0, \\ \widetilde{X}(T) = \widetilde{H}_T. \end{cases} \tag{32}$$

Since the solution of the associated classical BVP (29) is increasing with respect to both h_0 and h_T, the extension principle gives the solution

$$X_\alpha(t) = \left[\underline{X_\alpha}(t),\ \overline{X_\alpha}(t)\right], \tag{33}$$

where

$$\begin{aligned} \underline{X_\alpha}(t) &= x\left(t,\ \underline{H_{0,\alpha}},\ \underline{H_{T,\alpha}}\right), \\ \overline{X_\alpha}(t) &= x\left(t,\ \overline{H_{0,\alpha}},\ \overline{H_{T,\alpha}}\right), \end{aligned}$$

and the function $x(t, h_0, h_T)$ is given by (30) and (31).

In Figure 8, we show the solution of FBVP (32), where $\beta = 0.5$, $T = 1$, $\widetilde{H}_0 = (0.6,\ 1,\ 1.4)$ and $\widetilde{H}_T = (1.2,\ 1.5,\ 1.7)$. From this example, it can be seen that there is some confusion in the argumentation of the authors of [266]. In particular, their statement that "the fuzzy solution can be seen as the fuzzy number $\widetilde{H}_0$ moving along" the crisp function is not correct in general. It is correct if and only if the uncertainties of $\widetilde{H}_0$ and $\widetilde{H}_T$ are the same.

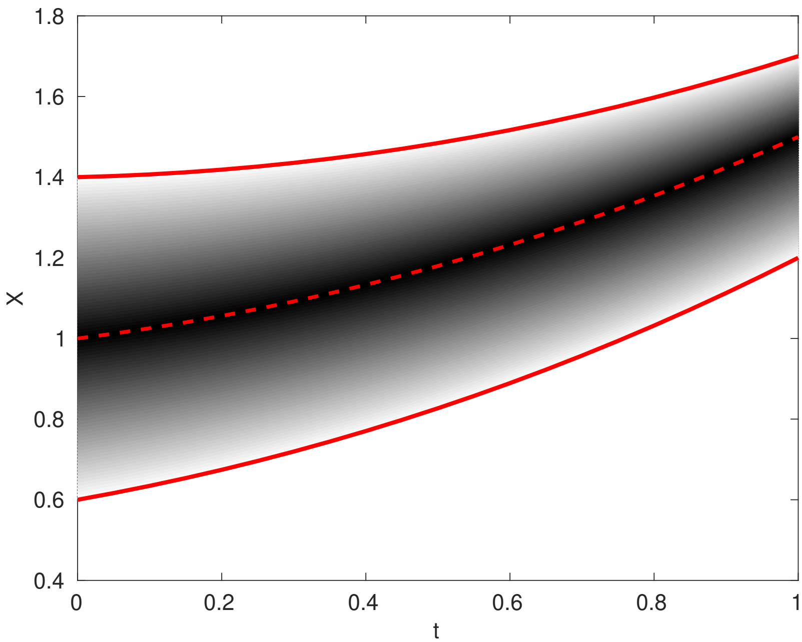

Fig. 8. The solution of FBVP (32), obtained by Zadeh's extension principle. Our proposed approach gives the same solution.

6. SOME COMMENTS ON THE GRANULAR DIFFERENTIABILITY APPROACH

Let us consider another approach, which has also gained attention in recent years, namely, the granular differentiability approach. This approach can be briefly explained as follows. A fuzzy number $\widetilde{u}$ can be described by its μ-level sets (μ-cuts) $[\widetilde{u}]_\mu = \left[\underline{u_\mu},\ \overline{u_\mu}\right]$. Each μ-level set is an interval, which can be parameterized by its left-end point and width (diameter): $[\widetilde{u}]_\mu = \left\{\underline{u_\mu} + \left(\overline{u_\mu} - \underline{u_\mu}\right)\alpha_u \ \middle|\ \alpha_u \in [0,1]\right\}$. (To this end, note that the parametrization $[\widetilde{u}]_\mu = \left\{a \mid a \in \left[\underline{u_\mu},\ \overline{u_\mu}\right]\right\}$ is simpler). Based on this fact, it is proposed to match a fuzzy number $\widetilde{u}$ with its parametrization function $u_{gr}(\mu, \alpha_u) = \underline{u_\mu} + \left(\overline{u_\mu} - \underline{u_\mu}\right)\alpha_u$. (This function is named "horizontal membership function". The name is unfortunate because the value of the function is not a membership degree from $[0,1]$.) The followers of the approach hope that this matching can allow them to avoid the deficiencies of fuzzy arithmetic and calculus. Unfortunately, they get confused right at the start. For example, the authors of [209] state that $\widetilde{u} - \widetilde{u} = 0$ under granular arithmetic. This circumstance immediately leads to results that are hard to accept. To be sure, let us consider the function $\widetilde{f}(t) = \left(-(1+t),\ 0,\ 1+t^2\right) - (-1, 0, 1)$ on $[-0.5,\ 0.5]$. This function is expected to be continuous. Let us examine whether this function is continuous at $t = 0$ under granular arithmetic. At $t = 0$, we have

$\widetilde{f}(0) = (-1,0,1) - (-1,0,1) = 0$, since $\widetilde{u} - \widetilde{u} = 0$ by [209]. Let us denote the operands of $\widetilde{f}(t)$ as $\widetilde{v}$ and $\widetilde{u}$, respectively: $\widetilde{v} = \left(-(1+t),\ 0,\ 1+t^2\right)$ and $\widetilde{u} = (-1,0,1)$. Then, $v_{gr}(\mu,\alpha_v) = (1-\mu)\left[-(1+t) + \left(2+t+t^2\right)\alpha_v\right]$, $u_{gr}(\mu,\alpha_u) = (1-\mu)\left[-1+2\alpha_u\right]$ and $f_{gr}(t,\mu,\alpha_v,\alpha_u) = v_{gr}(\mu,\alpha_v) - u_{gr}(\mu,\alpha_u) = (1-\mu)\left[-t + \left(2+t+t^2\right)\alpha_v - 2\alpha_u\right]$. Then,

$$\underline{f_\mu}(t) = \inf_{\beta\geq\mu} \min_{\alpha_v,\ \alpha_u} f_{gr}(t,\beta,\alpha_v,\alpha_u) = (1-\mu)(-2-t) \quad \text{and}$$

$$\overline{f_\mu}(t) = \sup_{\beta\geq\mu} \max_{\alpha_v,\ \alpha_u} f_{gr}(t,\beta,\alpha_v,\alpha_u) = (1-\mu)\left(2+t^2\right).$$

Therefore, $\widetilde{f}(t) = \left(-(2+t),\ 0,\ 2+t^2\right)$. When t goes to 0, $\widetilde{f}(t)$ goes to $(-2,0,2)$, which is different from $\widetilde{f}(0) = 0$. Consequently, $\widetilde{f}(t)$ is not continuous at $t = 0$ under granular operations. The considered example also demonstrates that granular subtraction is not a continuous operation. This fact points to a significant drawback of granular arithmetic and, therefore, granular calculus.

We also would like to draw attention to the following circumstances related to studies on the granular approach. Usually, when a new arithmetic is proposed, the results for the base case are given first. However, followers of granular arithmetic do not explicitly indicate the sum, difference, product, and division of two triangular fuzzy numbers under granular arithmetic. These results would immediately highlight the difficulties with this arithmetic.

7. CONCLUSION

In this paper, we review more than 300 studies on fuzzy differential equations and compare different solution concepts. Most of these studies are devoted to developing a new fuzzy derivative and applying this derivative to solve differential equations. In this regard, we discuss the pros and cons of existing derivative concepts. Although a sufficient number of different derivative concepts have been proposed, they have not brought the expected progress in solving fuzzy differential equations. Our view on this issue is as follows. The derivative is defined through the subtraction operation. This operation is expected to be, first, the inverse of addition, and, second, to be defined for each pair of numbers. Everywhere in fuzzy arithmetic, Minkowski addition is used. If the subtraction operation is defined as the inverse of this addition, we have the Hukuhara difference. However, the Hukuhara difference is not defined for each pair of fuzzy numbers. And if any other subtraction is used, it will not be the inverse of addition. Thus, it is impossible to create an arithmetic, and consequently a calculus, that would have the same properties as in the case of real numbers. To summarize the above, we believe that unless a new addition operation is defined that can replace Minkowski's addition, the new derivative concepts that will be proposed will not be able to overcome the existing shortcomings of fuzzy calculus. Based on this, we believe that further developments in the theory of fuzzy differential equations will be associated with alternative approaches that do not use fuzzy derivatives.

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