NEW RESULTS ON ADDITIVE GENERATOR PAIRS OF OVERLAP AND GROUPING FUNCTIONS

Li-zhi Liang and Xue-ping Wang

In this article, we deeply reveal the relationship between functions θ and ϑ in an overlap function additively generated by an additive generator pair (θ, ϑ), which is used to characterize the conditions for an overlap function additively generated by the pair being a triangular norm by terms of functions θ and ϑ. We also establish the conditions that an overlap function additively generated by the additive generator pair can be obtained by a distortion of a triangular norm and a (pseudo) automorphism. Finally, we dually give the related results concerned grouping functions.

Keywords: overlap function, grouping function, triangular norm, additive generator pair, distortion

Classification: 03B52, 03E72

1. INTRODUCTION

Overlap and grouping functions introduced by Bustince et al. [5, 6] arise from some problems in image processing [4], classification and decision making [24], for which triangular norms (t-norms for short) and triangular conorms (t-conorms for short) are widely considered [13] but the associativity is not strongly required in the real world. As two special cases of aggregation functions [3, 18], overlap and grouping functions are given by increasing continuous commutative bivariate functions defined over the unit square and with appropriate boundary conditions.

In recent years, overlap and grouping functions have a very rapid development both in applications and theory. In applications, overlap and grouping functions play an important role in image processing [5, 16, 17], data stream clustering [26], convolutional neural networks [22], classification [1, 25, 29] and decision making problems [23, 28] when we need to assign a given object into one of two different classes which are not evidently distinct. In theory, several explorations have also been made. For example, Bedregal et al. [2] discussed some significant properties related to overlap and grouping functions, such as migrativity, homogeneity, idempotency and the existence of generators. Particularly, they obtained important results related to the action of automorphisms on overlap and grouping functions, and investigated the preservation of those properties and also the Lipschitz condition. More theoretic achievements are referred in [7, 8, 9, 12, 14].

As we know, there exists a close relationship between overlap and grouping functions and some particular class of t-norms and t-conorms. Because a very significant way for building t-norms or t-conorms is the use of additive and multiplicative generators [15, 20, 27], it seems natural for us to investigate whether such a building method are suitable for the overlap and grouping functions or not. Moreover, with an eye kept on application, the use of additive and multiplicative generators may simplify the choice of an appropriate overlap or grouping function for a given problem since we only need to consider one-variable functions instead of a bivariate one, reducing the computational complexity in this way [19]. Based on this idea, Dimuro et al. [11] supplied the notion of an additive generator pair for grouping functions. Later, they also suggested the notion of an additive generator pair for overlap functions and studied the overlap function obtained by the distortion of a positive continuous t-norm and a pseudo automorphism. Specific speaking, they gave the definition of an additive generator pair for overlap functions as follows.

Definition 1.1. Let $\theta : [0,1] \to [0,\infty]$ and $\vartheta : [0,\infty] \to [0,1]$ be continuous and decreasing functions such that

(1) $\theta(x) = \infty$ if and only if $x = 0$;

(2) $\theta(x) = 0$ if and only if $x = 1$;

(3) $\vartheta(x) = 1$ if and only if $x = 0$;

(4) $\vartheta(x) = 0$ if and only if $x = \infty$.

Then the function $O_{\theta,\vartheta} : [0,1]^2 \to [0,1]$, defined by $O_{\theta,\vartheta}(x,y) = \vartheta(\theta(x) + \theta(y))$, is an overlap function. (θ, ϑ) is called an additive generator pair of the overlap function $O_{\theta,\vartheta}$, and $O_{\theta,\vartheta}$ is said to be additively generated by the pair (θ, ϑ).

In 2018, Junsheng Qiao [21] gave another way for building overlap and grouping functions by a multiplicative generator pair. The definition of a multiplicative generator pair is as follows.

Definition 1.2. Let $g, h : [0,1] \to [0,1]$ be two continuous and increasing functions such that

(1) $h(x) = 0$ if and only if $x = 0$;

(2) $h(x) = 1$ if and only if $x = 1$;

(3) $g(x) = 0$ if and only if $x = 0$;

(4) $g(x) = 1$ if and only if $x = 1$.

Then, the function $O_{g,h} : [0,1]^2 \to [0,1]$, defined by $O_{g,h}(x,y) = g(h(x)h(y))$, is an overlap function. (g, h) is called a multiplicative generator pair of the overlap function $O_{g,h}$, and $O_{g,h}$ is said to be multiplicatively generated by the pair (g, h).

In particular, Feng-qing Zhu [30] supplied a positive answer to an open problem raised by Dimuro et al. [10] by constructing an example and presented the conditions under which an overlap function can be obtained by a $(\mathcal{F}, T)$-distortion for a positive and continuous t-norm T and a (pseudo) automorphism $\mathcal{F}$.

Notice that the condition of an additive generator pair proposed in [10] (resp. the condition of a multiplicative generator pair presented in [21]) is sufficient but not necessary for the function $O_{\theta,\vartheta}$ (resp. $O_{g,h}$) being an overlap function. For instance, when h and g do not satisfy (2) and (4) of Definition 1.2, respectively, $O_{g,h} : [0,1]^2 \rightarrow [0,1]$ can also be an overlap function (see Example 3.1 in [21]). Thus some conditions of the additive (resp. multiplicative) generator pair can be omitted or replaced. Therefore, an interesting question naturally arise: do we characterize the functions θ and ϑ (resp. g and h) and explore whether we can construct overlap and grouping functions by an additive generator pair (θ, ϑ) (resp. (g, h)) with weaker conditions or not? This article will positively answer this problem.

The rest of this article is organized as follows: In Section 2, we present some preliminary concepts and results. In Section 3, we first introduce the concept of an additive generator pair (θ, ϑ) of an overlap function, and then investigate the relationship between functions θ and ϑ. In Section 4, we shall give the conditions for an overlap function additively generated by the pair(θ, ϑ) being a t-norm. In Section 5, we explore the conditions that an overlap function additively generated by the pair(θ, ϑ) can be obtained by a distortion of a t-norm and a (pseudo) automorphism. Section 6 is devoted to introducing the concept of an additive generator pair of grouping functions and some related results. A conclusion is drawn in Section 7.

2. PRELIMINARIES

In this section, we recall some fundamental concepts and results that will be used in the sequel.

Definition 2.1. (Klement et al. [15]) A t-norm is a binary operator $T : [0,1]^2 \rightarrow [0,1]$ such that for all $x, y, z \in [0,1]$, the following conditions are satisfied:

(T1) $T(x, y) = T(y, x)$;

(T2) $T(T(x, y), z) = T(x, T(y, z))$;

(T3) $T(x, y) \leq T(x, z)$ whenever $y \leq z$;

(T4) $T(x, 1) = x$.

A t-norm T is called positive if $T(x, y) = 0$ then either $x = 0$ or $y = 0$. A binary operator $T : [0,1]^2 \rightarrow [0,1]$ is called t-subnorm if it satisfies $(T1), (T2), (T3)$ and $T(x, y) \leq \min\{x, y\}$ for all $x, y \in [0,1]$.

Let T be a t-norm. Define for each n-tuple $(x_1, x_2, \ldots, x_n) \in [0,1]^n$

$$\mathop{\mathsf{T}}_{i=1}^{n} x_i = T(\mathop{\mathsf{T}}_{i=1}^{n-1} x_i, x_n) = T(x_1, x_2, \ldots, x_n).$$

If, in particular, we have $x_1 = x_2 = \cdots = x_n = x$, we shall briefly write

$$x_T^{(n)} = T(x, x, \ldots, x).$$

Definition 2.2. (Klement et al. [15]) Let $p, q, s, t \in [-\infty, \infty]$ with $p < q, s < t$ and $f : [p, q] \to [s, t]$ be an increasing (resp. a decreasing) function. Then the function $f^{(-1)} : [s, t] \to [p, q]$ defined by

$$f^{(-1)}(y) = \sup\{x \in [p, q] \mid f(x) < y\} \ (\text{resp. } f^{(-1)}(y) = \sup\{x \in [p, q] \mid f(x) > y\})$$

is called a pseudo-inverse of the increasing (resp. decreasing) function f.

Definition 2.3. (Bustince et al. [5]) A bivariate function $O : [0, 1]^2 \to [0, 1]$ is said to be an overlap function if it satisfies the following conditions:

(O1) O is commutative;

(O2) $O(x, y) = 0$ if and only if $xy = 0$;

(O3) $O(x, y) = 1$ if and only if $xy = 1$;

(O4) O is increasing;

(O5) O is continuous.

3. ADDITIVE GENERATOR PAIRS OF OVERLAP FUNCTIONS

This section first introduces the concept of an additive generator pair (θ, ϑ) of an overlap function, and then it deeply reveals the relationship between functions θ and ϑ in the overlap function additively generated by the pair.

Similar to Definition 3.1 of [21], we may give the following one.

Definition 3.1. Let $\theta : [0, 1] \to [0, \infty]$ and $\vartheta : [0, \infty] \to [0, 1]$ be continuous and decreasing functions, respectively. If the bivariate function $O_{\theta, \vartheta} : [0, 1]^2 \to [0, 1]$, given by

$$O_{\theta, \vartheta}(x, y) = \vartheta(\theta(x) + \theta(y)),$$

is an overlap function, then (θ, ϑ) is called an additive generator pair of the overlap function $O_{\theta, \vartheta}$ and $O_{\theta, \vartheta}$ is said to be additively generated by the pair (θ, ϑ).

Let $\theta : [0, 1] \to [0, \infty]$ be a function. For any $a \in [0, \infty)$, define functions $2\theta : [0, 1] \to [0, \infty]$ and $\theta + \frac{a}{2} : [0, 1] \to [0, \infty]$, respectively, by

$$(2\theta)(x) = 2\theta(x)$$

for any $x \in [0, 1]$ and

$$(\theta + \frac{a}{2})(x) = \theta(x) + \frac{a}{2}$$

for any $x \in [0, 1]$. Then the following theorem characterizes the functions $\theta : [0, 1] \to [0, \infty]$ and $\vartheta : [0, \infty] \to [0, 1]$, respectively, whenever $O_{\theta, \vartheta}$ is an overlap function additively generated by the pair (θ, ϑ).

Theorem 3.2. Let $\theta : [0, 1] \to [0, \infty]$ and $\vartheta : [0, \infty] \to [0, 1]$ be continuous and decreasing functions, respectively, and $O_{\theta, \vartheta} : [0, 1]^2 \to [0, 1]$ be an overlap function additively generated by the pair (θ, ϑ). Then the following two statements hold:

(i) $\theta(x) = \infty$ if and only if $x = 0$;

(ii) $\vartheta(x) = 0$ if and only if $x = \infty$.

P r o o f. (i) ($\Leftarrow$) If $x = 0$, then we affirm that $\theta(x) = \infty$. Indeed, if $\theta(0) < \infty$ then by Definition 2.3, $\vartheta(u) \leq \vartheta(2\theta(0)) = \vartheta(\theta(0) + \theta(0)) = O_{\theta,\vartheta}(0,0) = 0$, i. e.,

$$\vartheta(u) = 0 \tag{1}$$

for any $u \in [2\theta(0), \infty]$. Note that $2\theta : [0,1] \to [0,\infty]$ is continuous since $\theta : [0,1] \to [0,\infty]$ is continuous. In the following, we prove that $\vartheta(y) \neq 0$ for any $y \in [0, 2\theta(0))$ by distinguishing two cases.

Case 1. If $y \in (2\theta(1), 2\theta(0))$, then there exists a $z \in (0,1)$ such that $y = 2\theta(z)$ by the continuity of 2θ. Assume that $\vartheta(y) = 0$. Then one has that

$$\begin{aligned}
O_{\theta,\vartheta}(z,z) =& \vartheta(\theta(z) + \theta(z)) \\
=& \vartheta(2\theta(z)) \\
=& \vartheta(y) \\
=& 0,
\end{aligned}$$

which contradicts (O2) of Definition2.3. Therefore,

$$\vartheta(y) \neq 0 \tag{2}$$

for any $y \in (2\theta(1), 2\theta(0))$.

Case 2. If $y \in [0, 2\theta(1)]$ and $\vartheta(y) = 0$, then it is obvious that $\vartheta(u) = 0$ for any $u \in (2\theta(1), 2\theta(0))$ since ϑ is a decreasing function, contrary to (2). Therefore, $\vartheta(y) \neq 0$ for any $y \in [0, 2\theta(1)]$.

Cases 1 and 2 imply that $\vartheta(y) \neq 0$ for any $y \in [0, 2\theta(0))$, which together with (1) implies that

$$\vartheta(u) = 0 \text{ if and only if } u \in [2\theta(0), \infty]. \tag{3}$$

On the other hand, since $O_{\theta,\vartheta}$ is an overlap function, by (O2) of Definition 2.3 we have that

$$O_{\theta,\vartheta}(0,y) = \vartheta(\theta(0) + \theta(y)) = 0$$

for any $y \in [0,1]$. Thus from (3) we get $\theta(0) + \theta(y) \geq 2\theta(0)$, i.e., $\theta(y) \geq \theta(0)$ for any $y \in [0,1]$. In particular, $\theta(0) \leq \theta(1)$. Thus $\theta(1) = \theta(0)$ since $\theta(1) \leq \theta(0)$. Furthermore, we conclude that

$$\begin{aligned}
O_{\theta,\vartheta}(1,1) =& \vartheta(\theta(1) + \theta(1)) \\
=& \vartheta(\theta(0) + \theta(0)) \\
=& O_{\theta,\vartheta}(0,0) \\
=& 0 \quad \text{by (O2) of Definition 2.3}
\end{aligned}$$

which contradicts (O3) of Definition 2.3. Therefore, $\theta(0) = \infty$.

($\Rightarrow$) If $\theta(x) = \infty$ then we assert that $x = 0$. In fact, if there exists a $u \in (0, 1]$ such that $\theta(u) = \infty$, then

$$O_{\theta,\vartheta}(u, u) = \vartheta(\theta(u) + \theta(u))$$
$$= \vartheta(\infty),$$

Since $\theta(0) = \infty$, by (O2) of Definition 2.3, we further have

$$O_{\theta,\vartheta}(u, u) = \vartheta(\infty) = \vartheta(\theta(0) + \theta(0))) = O_{\theta,\vartheta}(0, 0) = 0, \tag{4}$$

contrary to (O2) of Definition 2.3.

(ii) ($\Leftarrow$) If $x = \infty$, then by (4), we know that $\vartheta(x) = 0$.
($\Rightarrow$) From (i), we know that $\theta(x) = \infty$ if and only if $x = 0$. Thus if $\vartheta(x) = 0$ then we assert that $x = \infty$. Otherwise, there are two cases as follows.

Case a. If there exists a $u \in (2\theta(1), 2\theta(0))$ such that $\vartheta(u) = 0$, then there exists a $z \in (0, 1)$ such that $u = 2\theta(z)$ by the continuity of 2θ. Thus

$$O_{\theta,\vartheta}(z, z) = \vartheta(\theta(z) + \theta(z))$$
$$= \vartheta(u)$$
$$= 0,$$

which is contrary to (O2) of Definition 2.3. Consequently,

$$\vartheta(y) \neq 0 \tag{5}$$

for any $y \in (2\theta(1), 2\theta(0))$.

Case b. If there exists a $u \in [0, 2\theta(1)]$ such that $\vartheta(u) = 0$, then $\vartheta(y) \leq \vartheta(u) = 0$ for any $y \in (2\theta(1), \infty)$, contrary to (5).

Therefore, from Cases a and b, $\vartheta(u) \neq 0$ for any $u \in [0, \infty)$, i. e., $x = \infty$. $\qquad \square$

The following theorem shows the conditions that a pair (θ, ϑ) can additively generate overlap functions.

Theorem 3.3. For a given $a \in [0, \infty)$, let $\theta : [0, 1] \to [0, \infty]$ and $\vartheta : [0, \infty] \to [0, 1]$ be continuous and decreasing functions such that

(i) $\theta(x) = \infty$ if and only if $x = 0$;

(ii) $\theta(x) = \frac{a}{2}$ if and only if $x = 1$;

(iii) $\vartheta(x) = 1$ if and only if $x \in [0, a]$;

(iv) $\vartheta(x) = 0$ if and only if $x = \infty$.

Then, the function $O_{\theta,\vartheta} : [0, 1]^2 \to [0, 1]$, defined by

$$O_{\theta,\vartheta}(x, y) = \vartheta(\theta(x) + \theta(y)),$$

is an overlap function.

Proof. We only need to show that Definition 2.3 holds. Indeed, the conditions (O1) and (O5) are obvious. So that we just verify the other conditions as follows.

(O2) First note that from (iv), $O_{\theta,\vartheta}(x,y) = 0$ if and only if $\vartheta(\theta(x) + \theta(y)) = 0$ if and only if $\theta(x) + \theta(y) = \infty$.

Now, suppose that $O_{\theta,\vartheta}(x,y) = 0$. We claim that $xy = 0$. In fact, if $xy \neq 0$. Then $x \neq 0$ and $y \neq 0$. Thus from (i), we have $\theta(x) + \theta(y) \neq \infty$, a contradiction.

Conversely, if $xy = 0$, then $x = 0$ or $y = 0$, say $x = 0$, thus from (i) we have $\theta(x) = \infty$. This follows that $\theta(x) + \theta(y) = \infty$, which implies that $O_{\theta,\vartheta}(x,y) = 0$.

(O3) First note that from (iii), $O_{\theta,\vartheta}(x,y) = 1$ if and only if $\vartheta(\theta(x) + \theta(y)) = 1$ if and only if $\theta(x) + \theta(y) \in [0, a]$.

Now, suppose that $O_{\theta,\vartheta}(x,y) = 1$. Then we assert that $xy = 1$. In fact, if $xy \neq 1$. Then $x \neq 1$ or $y \neq 1$, say $x \neq 1$, thus from (ii) we have $\theta(x) > \theta(1) = \frac{a}{2}$. Then $\theta(x) + \theta(y) > \frac{a}{2} + \theta(y) \geq \frac{a}{2} + \frac{a}{2} = a$, which is in conflict with $\theta(x) + \theta(y) \in [0, a]$.

Conversely, if $xy = 1$, then $x = 1$ and $y = 1$. Thus from (ii) we have that $\theta(x) + \theta(y) = \theta(1) + \theta(1) = \frac{a}{2} + \frac{a}{2} = a$, i.e., $\theta(x) + \theta(y) \in [0, a]$. Therefore, $O_{\theta,\vartheta}(x,y) = 1$.

(O4) For any $y, z \in [0, 1]$ with $y \leq z$, $\theta(y) \geq \theta(z)$ since θ is decreasing, which together with ϑ being decreasing means that

$$O_{\theta,\vartheta}(x,y) = \vartheta(\theta(x) + \theta(y)) \leq \vartheta(\theta(x) + \theta(z)) = O_{\theta,\vartheta}(x,z),$$

i.e., (O4) holds. □

The following two examples illustrate Theorem 3.3.

Example 3.4. For a given $a \in [0, \infty)$, consider the functions $\theta : [0, 1] \to [0, \infty]$ and $\vartheta : [0, \infty] \to [0, 1]$ defined by

$$\theta(x) = \begin{cases} \frac{a}{2} - \ln x & \text{if } x \neq 0, \\ \infty & \text{if } x = 0 \end{cases}$$

and

$$\vartheta(x) = \begin{cases} 1 & \text{if } x \in [0, a), \\ e^{-x+a} & \text{if } x \in [a, \infty], \end{cases}$$

respectively. Then it is easy to see that both θ and ϑ are continuous and decreasing functions satisfying the conditions of Theorem 3.3. Therefore, $O_{\theta,\vartheta}(x,y) = xy$ is an overlap function. In particular, $O_{\theta,\vartheta} = T_p$.

Example 3.5. For a given $a \in [0, \infty)$, consider the functions $\theta : [0, 1] \to [0, \infty]$ and $\vartheta : [0, \infty] \to [0, 1]$, defined, respectively, by

$$\theta(x) = \begin{cases} \frac{a}{2} - \ln x & \text{if } x \neq 0, \\ \infty & \text{if } x = 0 \end{cases}$$

and

$$\vartheta(x) = \begin{cases} 1 & \text{if } x \in [0, a), \\ \frac{a}{x} & \text{if } x \in [a, \infty]. \end{cases}$$

Then we can see that both θ and ϑ are continuous and decreasing functions, satisfying the conditions of Theorem 3.3.

In a simple computation, we have

$$O_{\theta,\vartheta}(x,y) = \begin{cases} 0 & \text{otherwise,} \\ \dfrac{a}{a - \ln xy} & \text{if } xy \neq 0. \end{cases}$$

Therefore, by Theorem 3.3, $O_{\theta,\vartheta}$ is an overlap function that is not associative, and 1 is not its neutral element.

The following proposition reveals that the conditions (ii) and (iii) of Theorem 3.3 are closely associated whenever $O_{\theta,\vartheta} : [0,1]^2 \to [0,1]$ is an overlap function additively generated by the pair (θ, ϑ).

Proposition 3.6. For a given $a \in [0, \infty)$, let $\theta : [0,1] \to [0, \infty]$ and $\vartheta : [0, \infty] \to [0,1]$ be two continuous and decreasing functions such that a binary function $O_{\theta,\vartheta} : [0,1]^2 \to [0,1]$ with $O_{\theta,\vartheta}(x,y) = \vartheta(\theta(x) + \theta(y))$ is an overlap function. Then the following statements are equivalent:

(i) $x \in [0, a]$ if and only if $\vartheta(x) = 1$;

(ii) $\theta(x) = \frac{a}{2}$ if and only if $x = 1$.

P r o o f . (i)$\Rightarrow$(ii): If $\theta(x) = \frac{a}{2}$, then from (i) we have $\vartheta(\theta(x) + \theta(x)) = \vartheta(a) = 1$. Hence $O_{\theta,\vartheta}(x,x) = 1$. Since $O_{\theta,\vartheta}$ is an overlap function, by (O3) of Definition 2.3, we have that $x = 1$.

In the converse implication, if $x = 1$, then $O_{\theta,\vartheta}(1,1) = 1$ since $O_{\theta,\vartheta}$ is an overlap function, i.e., $\vartheta(\theta(1) + \theta(1)) = 1$. Hence, it follows from (i) that $\theta(1) \in [0, \frac{a}{2}]$. We assert that $\theta(1) = \frac{a}{2}$. In fact, if $\theta(1) < \frac{a}{2}$, then there exists an $x_0 \in [0,1)$ such that $\theta(x_0) = \frac{a}{2}$ since θ is continuous and $\theta(0) = \infty$, thus $\vartheta(\theta(x_0) + \theta(x_0)) = \vartheta(a) = 1$, which violates (O3) of Definition 2.3. Therefore, $\theta(1) = \frac{a}{2}$.

(ii)$\Rightarrow$(i): Necessity. If $x \in [0, a]$, then $\vartheta(x) \geq \vartheta(a)$ since ϑ is decreasing. Meanwhile, from (ii) we have $\vartheta(a) = \vartheta(\theta(1)+\theta(1)) = O_{\theta,\vartheta}(1,1) = 1$ since $O_{\theta,\vartheta}$ is an overlap function. Thus $1 \geq \vartheta(x) \geq \vartheta(a) = 1$ for any $x \in [0, a]$, i.e., $\vartheta(x) = 1$ for any $x \in [0, a]$.

Sufficiency. If $\vartheta(x) = 1$, Then we claim $x \in [0, a]$. Indeed, if there exists an $x_0 \in (a, \infty]$ such that $\vartheta(x_0) = 1$, then there exist two elements $y, z \in [0,1]$ such that $\theta(y) + \theta(z) = x_0$ since θ is continuous. Thus $yz < 1$ and $\vartheta(\theta(y) + \theta(z)) = \vartheta(x_0) = 1$, contrary to (O3) of Definition 2.3 since $O_{\theta,\vartheta}$ is an overlap function. $\square$

Remark 3.7. Compared with the additive generator pair of an overlap function in [10] (see Corollary 4.1 in [10]), the additive generator pair of an overlap function appeared in Theorem 3.3 gives a way to construct overlap functions with more relaxed boundary conditions. Indeed, if $a = 0$, then Theorem 3.3 coincides with Corollary 4.1 in [10].

4. CONDITIONS THAT AN OVERLAP FUNCTION IS A T-NORM

From Examples 3.4 and 3.5, one easily sees that not every overlap function additively generated by the pair (θ, ϑ) is a t-norm. Then a natural question arises: what are the

conditions for an overlap function additively generated by the pair (θ,ϑ) being a t-norm? This section will answer this problem.

First, we have the following proposition.

Proposition 4.1. Let $\theta : [0,1] \to [0,\infty]$ and $\vartheta : [0,\infty] \to [0,1]$ be two continuous and decreasing functions, and let $O_{\theta,\vartheta} : [0,1]^2 \to [0,1]$ be an overlap function additively generated by the pair (θ, ϑ) with $\theta(1) = \frac{a}{2}$ where $a \in [0,\infty)$. Then the following two statements are equivalent:

(1) $\vartheta \circ (\theta + \frac{a}{2}) = id_{[0,1]}$;

(2) 1 is a neutral element of $O_{\theta,\vartheta}$.

P r o o f. $(1) \Rightarrow (2)$ If $\vartheta \circ (\theta + \frac{a}{2}) = \mathrm{id}_{[0,1]}$, then

$$O_{\theta,\vartheta}(x,1) = \vartheta(\theta(x) + \theta(1))$$
$$= \vartheta(\theta(x) + \frac{a}{2})$$
$$= x$$

for each $x \in [0,1]$ since $\theta(1) = \frac{a}{2}$. Therefore, 1 is a neutral element of $O_{\theta,\vartheta}$.
$(2) \Rightarrow (1)$ If 1 is a neutral element of $O_{\theta,\vartheta}$, then

$$\vartheta((\theta(x) + \frac{a}{2})) = \vartheta((\theta(x) + \theta(1)))$$
$$= O_{\theta,\vartheta}(x,1)$$
$$= x$$

for any $x \in [0,1]$ since $\theta(1) = \frac{a}{2}$. Therefore, $\vartheta \circ (\theta + \frac{a}{2}) = \mathrm{id}_{[0,1]}$. $\square$

Before presenting results that are associating a t-norm with an overlap function additively generated by the pair (θ,ϑ), we need the following lemma.

Lemma 4.2. (Bustince et al. [5]) $O : [0,1]^2 \to [0,1]$ is an associative overlap function if and only if O is a continuous and positive t-norm.

Proposition 4.3. For a given $a \in [0,\infty)$, let $\theta : [0,1] \to [0,\infty]$ be a continuous and strictly decreasing function such that:

(1) $\theta(x) = \infty$ if and only if $x = 0$;

(2) $\theta(x) = \frac{a}{2}$ if and only if $x = 1$.

If a function $\vartheta : [0,\infty] \to [0,1]$ satisfies $\vartheta(x) = (\theta + \frac{a}{2})^{(-1)}(x)$ for any $x \in [0,1]$, then a binary function $O_{\theta,\vartheta} : [0,1]^2 \to [0,1]$ with $O_{\theta,\vartheta}(x,y) = \vartheta(\theta(x) + \theta(y))$ is a positive t-norm.

P r o o f. Since θ is strictly decreasing, $\theta + \frac{a}{2}$ is also strictly decreasing. Then $\vartheta = (\theta + \frac{a}{2})^{(-1)}$ is continuous and decreasing. In the following, we verify that the $O_{\theta,\vartheta}$: $[0,1]^2 \to [0,1]$ is an overlap function according to Theorem 3.3. From the hypothesis it is clearly that both (i) and (ii) of Theorem 3.3 hold. Next, we prove that (iii) and (iv) of Theorem 3.3 are true, respectively.

(3) From Definition 2.2, $\vartheta(x) = 1$ if and only if $(\theta + \frac{a}{2})^{(-1)}(x) = 1$ if and only if $\sup\{z \in [0,1] \mid \theta(z) + \frac{a}{2} > x\} = 1$.

Suppose that $\vartheta(x) = 1$. If $x \in (a, \infty]$, then there exists a $u \in [0,1)$ such that $x = \theta(u) + \frac{a}{2}$ since θ is continuous and strictly decreasing. Thus $\vartheta(x) = (\theta + \frac{a}{2})^{(-1)}(x) = \sup\{z \in [0,1] \mid \theta(z) + \frac{a}{2} > \theta(u) + \frac{a}{2}\} = u \neq 1$, which is contrary to $\vartheta(x) = 1$. Therefore, $x \in [0, a]$.

Conversely, if $x = a$, then we get

$$\begin{aligned}
\vartheta(a) &= (\theta + \frac{a}{2})^{(-1)}(a) \\
&= \sup\{z \in [0,1] \mid \theta(z) + \frac{a}{2} > a\} \\
&= \sup\{z \in [0,1] \mid \theta(z) > \frac{a}{2}\} \\
&= 1.
\end{aligned}$$

Thus $1 \geq \vartheta(x) \geq \vartheta(a) = 1$, i. e., $\vartheta(x) = 1$ for any $x \in [0, a]$.

(4) From Definition 2.2, $\vartheta(x) = 0$ if and only if $(\theta + \frac{a}{2})^{(-1)}(x) = 0$ if and only if $\sup\{z \in [0,1] \mid \theta(z) + \frac{a}{2} > x\} = 0$.

($\Rightarrow$) Supposing that $\vartheta(x) = 0$, we assert that $x = \infty$. Otherwise, there exists an $x_0 \in [0, \infty)$ such that $\vartheta(x_0) = 0$. Then, because of the continuity of θ, we have

$$\begin{aligned}
\vartheta(x_0) &= (\theta + \frac{a}{2})^{(-1)}(x_0) \\
&= \sup\{z \in [0,1] \mid \theta(z) + \frac{a}{2} > x_0\} \\
&\neq 0
\end{aligned}$$

since $\theta(x) = \infty$ if and only if $x = 0$, which violates $\vartheta(x_0) = 0$.

($\Leftarrow$) If $x = \infty$, then $\vartheta(x) = \sup\{z \in [0,1] \mid \theta(z) + \frac{a}{2} > \infty\} = 0$.

Therefore, from Theorem 3.3, $O_{\theta,\vartheta}$ is an overlap function.

Now, we prove the associativity of $O_{\theta,\vartheta}$.

For each $x, y, z \in [0,1]$, there exists an $w \in [0,1]$ such that $\theta(x) + \theta(y) = \theta(w) + \frac{a}{2}$ since θ is continuous and $\theta(x) + \theta(y) \geq \theta(x) + \frac{a}{2}$. On the other hand, $\theta + \frac{a}{2}$ is strictly

decreasing since θ is strictly decreasing, which means $(\theta + \frac{a}{2})^{(-1)}(\theta(w) + \frac{a}{2}) = w$. Thus

$$
\begin{aligned}
O_{\theta,\vartheta}(O_{\theta,\vartheta}(x,y),z) =& O_{\theta,\vartheta}((\theta + \frac{a}{2})^{(-1)}(\theta(x) + \theta(y)), z)\\
=& (\theta + \frac{a}{2})^{(-1)}(\theta((\theta + \frac{a}{2})^{(-1)}(\theta(x) + \theta(y))) + \theta(z))\\
=& (\theta + \frac{a}{2})^{(-1)}(\theta((\theta + \frac{a}{2})^{(-1)}(\theta(w) + \frac{a}{2})) + \theta(z))\\
=& (\theta + \frac{a}{2})^{(-1)}(\theta(w) + \theta(z))\\
=& (\theta + \frac{a}{2})^{(-1)}(\theta(x) + \theta(y) + \theta(z) - \frac{a}{2}).
\end{aligned}
$$

Analogously, $O_{\theta,\vartheta}(x, O_{\theta,\vartheta}(y,z)) = (\theta + \frac{a}{2})^{(-1)}(\theta(x) + \theta(y) + \theta(z) - \frac{a}{2})$. Therefore, $O_{\theta,\vartheta}(x, O_{\theta,\vartheta}(y,z)) = O_{\theta,\vartheta}(O_{\theta,\vartheta}(x,y),z)$.

Finally, by Lemma 4.2, $O_{\theta,\vartheta}$ is a continuous and positive t-norm. $\qquad \square$

Notice that we can directly prove Proposition 4.3 by the definition of the t-norm. Meanwhile, the condition that a function $\vartheta : [0, \infty] \to [0, 1]$ satisfies $\vartheta(x) = (\theta + \frac{a}{2})^{(-1)}(x)$ for any $x \in [0, 1]$ cannot be deleted generally.

Example 4.4. For a given $a \in [0, \infty)$, consider the function $\theta : [0, 1] \to [0, \infty]$ defined by

$$
\theta(x) = \begin{cases} \frac{a}{2} - \ln x & \text{if } x \neq 0, \\ \infty & \text{if } x = 0. \end{cases}
$$

For each $b \in [0, \infty)$ with $a \neq b$, let a function $\vartheta : [0, \infty] \to [0, 1]$ satisfy $\vartheta(x) = (\theta + \frac{b}{2})^{(-1)}(x)$ for any $x \in [0, 1]$. Then by a simple calculation, we have

$$
O_{\theta,\vartheta}(x, y) = e^{\frac{b-a}{2}} xy,
$$

which is not a t-norm.

Proposition 4.5. Let $\theta : [0, 1] \to [0, \infty]$ and $\vartheta : [0, \infty] \to [0, 1]$ be two continuous and decreasing functions, and $O_{\theta,\vartheta} : [0, 1]^2 \to [0, 1]$ be an overlap function additively generated by the pair (θ, ϑ) with 1 as neutral element. If for a given $a \in [0, \infty)$, θ satisfies the condition (ii) of Theorem 3.3, then $O_{\theta,\vartheta}$ is associative.

P r o o f. From the condition (ii) of Theorem 3.3 we get $\theta(x) = \frac{a}{2}$ if and only if $x = 1$. Then we have

$$
O_{\theta,\vartheta}(x, 1) = \vartheta(\theta(x) + \theta(1)) = \vartheta(\theta(x) + \frac{a}{2}) = x \tag{6}
$$

for any $x \in [0, 1]$ since 1 is the neutral element of $O_{\theta,\vartheta}$. Moreover, because θ is decreasing, $\theta(x) + \theta(y) \geq \theta(x) + \frac{a}{2}$ for any $x, y \in [0, 1]$. Thus $\theta(x) + \theta(y) \in \mathrm{Ran}(\theta + \frac{a}{2})$ since θ is continuous, i. e., there exists an $w \in [0, 1]$ such that $\theta(x) + \theta(y) = \theta(w) + \frac{a}{2}$. It follows

that for any $x, y, z \in [0, 1]$,

$$
\begin{aligned}
O_{\theta,\vartheta}(O_{\theta,\vartheta}(x,y), z) &= \vartheta(\theta(O_{\theta,\vartheta}(x,y)) + \theta(z)) \\
&= \vartheta(\theta(\vartheta(\theta(x) + \theta(y))) + \theta(z)) \\
&= \vartheta(\theta(\vartheta(\theta(w) + \frac{a}{2})) + \theta(z)) \\
&= \vartheta(\theta(w) + \theta(z)) \quad \text{by (6)} \\
&= \vartheta(\theta(x) + \theta(y) + \theta(z) - \frac{a}{2}).
\end{aligned}
$$

Similarly, we have $O_{\theta,\vartheta}(x, O_{\theta,\vartheta}(y, z)) = \vartheta(\theta(x) + \theta(y) + \theta(z) - \frac{a}{2})$ for any $x, y, z \in [0, 1]$. Therefore, $O_{\theta,\vartheta}(O_{\theta,\vartheta}(x,y), z) = O_{\theta,\vartheta}(x, O_{\theta,\vartheta}(y, z))$ for any $x, y, z \in [0, 1]$. $\qquad \square$

Lemma 4.2, Propositions 4.1 and 4.5 imply the following theorem.

Theorem 4.6. Let $\theta : [0, 1] \to [0, \infty]$ and $\vartheta : [0, \infty] \to [0, 1]$ be two continuous and decreasing functions and, $O_{\theta,\vartheta} : [0, 1]^2 \to [0, 1]$ be an overlap function additively generated by the pair (θ, ϑ). If for a given $a \in [0, \infty)$, θ satisfies the condition (ii) of Theorem 3.3, then the following three statements are equivalent:

(1) $O_{\theta,\vartheta}$ is a t-norm;

(2) 1 is a neutral element of $O_{\theta,\vartheta}$;

(3) $\vartheta \circ (\theta + \frac{a}{2}) = id_{[0,1]}$.

Remark 4.7. The differences between the pair (θ, ϑ) that generates t-norms in Theorem 4.6 and the classical additive generator t of continuous t-norms in [15] are as follows.

(1) The classical additive generator t must be strictly decreasing and the pair (θ, ϑ) only need be decreasing.

(2) The classical additive generator t satisfies that $t(x) = 0$ if and only if $x = 1$ and, the pair (θ, ϑ) satisfies that for a given $a \in [0, \infty)$, $\theta(x) = \frac{a}{2}$ if and only if $x = 1$, and $\vartheta \circ (\theta + \frac{a}{2}) = id_{[0,1]}$.

5. CONDITIONS THAT OVERLAP FUNCTIONS ADDITIVELY GENERATED BY THE PAIR (θ,ϑ) CAN BE OBTAINED BY A DISTORTION

Dimuro et al. [10] introduced the definition of a (pseudo) automorphism to discuss the conditions on which an overlap function can be obtained by a distortion of a t-norm and a (pseudo) automorphism. This section investigates the conditions that an overlap function additively generated by the pair (θ,ϑ) can be obtained by a distortion of a t-norm and a (pseudo) automorphism.

We first need the following definitions.

Definition 5.1. (Dimuro et al. [10]) A function $\mathcal{F} : [0, 1] \to [0, 1]$ is said to be a pseudo automorphism if the following conditions hold:

($\mathcal{F}$1) $\mathcal{F}$ is increasing;

($\mathcal{F}2$) $\mathcal{F}$ is continuous;

($\mathcal{F}3$) $\mathcal{F}(x) = 1$ if and only if $x = 1$;

($\mathcal{F}4$) $\mathcal{F}(x) = 0$ if and only if $x = 0$.

An automorphism $\mathcal{F} : [0,1] \to [0,1]$ is a strictly increasing pseudo automorphism.

Proposition 5.2. (Dimuro et al. [10]) Let $\mathcal{F} : [0,1] \to [0,1]$ be a pseudo automorphism. Then for every positive and continuous t-norm $T : [0,1]^2 \to [0,1]$, the function $O_{\mathcal{F},T} : [0,1]^2 \to [0,1]$, given by

$$O_{\mathcal{F},T}(x,y) = \mathcal{F}(T(x,y)),$$

is an overlap function.

The function $O_{\mathcal{F},T}$ in Proposition 5.2 is called an overlap function obtained by the distortion of the t-norm T by the pseudo automorphism $\mathcal{F}$ or an overlap function obtained by the $(\mathcal{F},T)$-distortion.

Proposition 5.3. Let $\theta : [0,1] \to [0,\infty]$ and $\vartheta : [0,\infty] \to [0,1]$ be two continuous and decreasing functions and, $O_{\theta,\vartheta} : [0,1]^2 \to [0,1]$ be an overlap function additively generated by the pair (θ,ϑ) that satisfies one of the following two conditions: for a given $a \in [0,\infty)$,

  (1) $\theta(x) = \frac{a}{2}$ if and only if $x = 1$;

  (2) $\vartheta(x) = 1$ if and only if $x \in [0,a]$.

Then the overlap function $O_{\theta,\vartheta}$ can be obtained by a $(\mathcal{F},T_{sub})$-distortion for a (pseudo) automorphism $\mathcal{F}$ and a t-subnorm T_{sub}.

P r o o f. Define functions $T_{sub} : [0,1]^2 \to [0,1]$ and $\mathcal{F} : [0,1] \to [0,1]$, respectively, by

$$T_{sub}(x,y) = (\theta + \frac{a}{2})^{(-1)}(\theta(x) + \theta(y)) \tag{7}$$

for any $x,y \in [0,1]$ and

$$\mathcal{F}(x) = \vartheta(\theta(x) + \frac{a}{2}) \tag{8}$$

for any $x \in [0,1]$. The rest of the proof is completed by three steps as below.
Step 1. We prove that T_{sub} is a t-subnorm.

  (i) The commutativity of T_{sub} is obvious.

  (ii) The associativity of T_{sub}. Because θ is decreasing, from the hypothesis and Proposition 3.6 $\theta(x) + \theta(y) \geq \theta(x) + \frac{a}{2}$ for each $x,y \in [0,1]$ since $O_{\theta,\vartheta}$ is an overlap function. Then $\theta(x) + \theta(y) \in \text{Ran}(\theta + \frac{a}{2})$ since θ is continuous, i.e., there exists an $w \in [0,1]$ such that

$$\theta(x) + \theta(y) = \theta(w) + \frac{a}{2}. \tag{9}$$

Let

$$w_1 = (\theta + \frac{a}{2})^{(-1)}(\theta(w) + \frac{a}{2}). \tag{10}$$

Thus, from the continuity of θ we have

$$\theta(w) = \theta(w_1). \tag{11}$$

Therefore, for each $x, y, z \in [0,1]$,

$$
\begin{aligned}
T_{sub}(T_{sub}(x,y),z) &= T_{sub}((\theta + \frac{a}{2})^{(-1)}(\theta(x) + \theta(y)), z) \\
&= (\theta + \frac{a}{2})^{(-1)}(\theta((\theta + \frac{a}{2})^{(-1)}(\theta(x) + \theta(y))) + \theta(z)) \\
&= (\theta + \frac{a}{2})^{(-1)}(\theta(w_1) + \theta(z)) \quad \text{by (9) and (10)} \\
&= (\theta + \frac{a}{2})^{(-1)}(\theta(w) + \theta(z)) \quad \text{by (11)} \\
&= (\theta + \frac{a}{2})^{(-1)}(\theta(x) + \theta(y) + \theta(z) - \frac{a}{2}).
\end{aligned}
$$

Analogously, $T_{sub}(x, T_{sub}(y,z)) = (\theta + \frac{a}{2})^{(-1)}(\theta(x) + \theta(y) + \theta(z) - \frac{a}{2})$ for each $x, y, z \in [0,1]$. Consequently, $T_{sub}(x, T_{sub}(y,z)) = T_{sub}(T_{sub}(x,y), z)$ for each $x, y, z \in [0,1]$.

(iii) The monotonicity of T_{sub}. For $y, z \in [0,1]$ with $y \le z$, $\theta(y) \ge \theta(z)$ since θ is decreasing. This follows that $T_{sub}(x,y) = (\theta + \frac{a}{2})^{(-1)}(\theta(x) + \theta(y)) \le (\theta + \frac{a}{2})^{(-1)}(\theta(x) + \theta(z)) = T_{sub}(x,z)$.

(iv) For each $x, y \in [0,1]$, we have that $T_{sub}(x,y) = (\theta + \frac{a}{2})^{(-1)}(\theta(x) + \theta(y)) \le (\theta + \frac{a}{2})^{(-1)}(\theta(x) + \frac{a}{2}) \le x$ since both θ and $\theta + \frac{a}{2}$ are decreasing. Similarly, $T_{sub}(x,y) \le y$. Therefore, $T_{sub}(x,y) \le \min\{x,y\}$. (i), (ii), (iii) and (iv) mean that T_{sub} is a t-subnorm.

Step 2. We prove that $\mathcal{F}$ is a (pseudo) automorphism.

In fact, it is easy to see that $\mathcal{F}$ is continuous and increasing since both θ and ϑ are continuous and decreasing. From $\mathcal{F}(x) = \vartheta(\theta(x) + \frac{a}{2})$, we have that $\mathcal{F}(x) = 0$ if and only if $\vartheta(\theta(x) + \frac{a}{2}) = 0$. Moreover, by Theorem 3.2, $\vartheta(\theta(x) + \frac{a}{2}) = 0$ if and only if $\theta(x) = \infty$ if and only if $x = 0$ since $O_{\theta,\vartheta}$ is an overlap function. Then $\mathcal{F}(x) = 0$ if and only if $x = 0$. Again from $\mathcal{F}(x) = \vartheta(\theta(x) + \frac{a}{2})$, we have that $\mathcal{F}(x) = 1$ if and only if $\vartheta(\theta(x) + \frac{a}{2}) = 1$. Moreover, from the hypothesis, it follows from Proposition 3.6 that we have $\vartheta(\theta(x) + \frac{a}{2}) = 1$ if and only if $\theta(x) + \frac{a}{2} \in [0, a]$ if and only if $\theta(x) = \frac{a}{2}$ if and only if $x = 1$ since $O_{\theta,\vartheta}$ is an overlap function. Therefore, $\mathcal{F}(x) = 1$ if and only if $x = 1$. Consequently, by Definition 5.1, $\mathcal{F}$ is a (pseudo) automorphism.

Step 3. We prove that the overlap function $O_{\theta,\vartheta}$ can be obtained by a $(\mathcal{F}, T_{sub})$-distortion for a (pseudo) automorphism $\mathcal{F}$ and a t-subnorm T_{sub}.

For any $x, y \in [0, 1]$,

$$
\begin{aligned}
\mathcal{F}(T_{sub}(x, y)) &= \vartheta(\theta(\theta + \tfrac{a}{2})^{(-1)}(\theta(x) + \theta(y)) + \tfrac{a}{2}) \quad \text{by (7) and (8)} \\
&= \vartheta(\theta(\theta + \tfrac{a}{2})^{(-1)}(\theta(w) + \tfrac{a}{2}) + \tfrac{a}{2}) \quad \text{by (9)} \\
&= \vartheta(\theta(w_1) + \tfrac{a}{2}) \quad \text{by (10)} \\
&= \vartheta(\theta(w) + \tfrac{a}{2}) \quad \text{by (11)} \\
&= \vartheta(\theta(x) + \theta(y)) \quad \text{by (9)} \\
&= O_{\theta, \vartheta}(x, y),
\end{aligned}
$$

i.e., $O_{\theta, \vartheta}(x, y) = \mathcal{F}(T_{sub}(x, y))$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following theorem gives two descriptions that an overlap function $O_{\theta, \vartheta}$ additively generated by the pair (θ, ϑ) can be obtained by a $(\mathcal{F}, T)$-distortion for a (pseudo) automorphism $\mathcal{F}$ and a t-norm T.

Theorem 5.4. For a function $O : [0, 1]^2 \to [0, 1]$ the following are equivalent:

(1) There exist a continuous and strictly decreasing function $\theta : [0, 1] \to [0, \infty]$ and a continuous and decreasing function $\vartheta : [0, \infty] \to [0, 1]$ such that the function O is an overlap function additively generated by the pair (θ, ϑ) satisfying one of the following two conditions: for a given $a \in [0, \infty)$,

   (a) $\theta(x) = \tfrac{a}{2}$ if and only if $x = 1$;
   (b) $\vartheta(x) = 1$ if and only if $x \in [0, a]$.

(2) The function O is an overlap function obtained by a $(\mathcal{F}, T)$-distortion for a (pseudo) automorphism $\mathcal{F}$ and a strict t-norm T;

(3) There exist a strictly increasing bijection $\varphi : [0, 1] \to [0, 1]$ and a (pseudo) automorphism $\mathcal{H}$ such that $O(x, y) = \mathcal{H}(\varphi(x)\varphi(y))$ for all $(x, y) \in [0, 1]^2$.

P r o o f. $(1) \Rightarrow (2)$: By Proposition 5.3 and its proof, the overlap function O can be obtained by a $(\mathcal{F}, T_{sub})$-distortion for a (pseudo) automorphism $\mathcal{F}$ and a t-subnorm T_{sub} where $T_{sub}(x, y) = (\theta + \tfrac{a}{2})^{(-1)}(\theta(x) + \theta(y))$ for any $x, y \in [0, 1]$. From the hypothesis, by Proposition 3.6 we always get that the condition (a) holds since O is the overlap function. It is obvious that T_{sub} is a continuous t-subnorm since θ is a continuous and strictly decreasing function, moreover,

$$
\begin{aligned}
T_{sub}(x, 1) &= (\theta + \tfrac{a}{2})^{(-1)}(\theta(x) + \theta(1)) \\
&= (\theta + \tfrac{a}{2})^{(-1)}(\theta(x) + \tfrac{a}{2}) \\
&= x.
\end{aligned}
$$

Then T_{sub} is a continuous t-norm. The rest of the proof are split into two parts as below.

Part A. T_{sub} is Archimedean.

By Definition 2.2 we have

$$x_{T_{sub}}^{(n)} = (\theta + \frac{a}{2})^{(-1)}(n\theta(x) - (n-2)\frac{a}{2})$$

$$= \sup\{z \in [0,1] \mid \theta(z) + \frac{a}{2} > n\theta(x) - (n-2)\frac{a}{2}\} \tag{12}$$

for any $x \in (0,1)$. Since $n\theta(x) - (n-2)\frac{a}{2} = n(\theta(x) - \frac{a}{2}) + a$ and $\theta(x) > \frac{a}{2}$ for any $x \in (0,1)$, we get that $\lim_{n \to \infty} n(\theta(x) - \frac{a}{2}) + a = \infty$. This together with (12) yields $\lim_{n \to \infty} x_{T_{sub}}^{(n)} = 0$. Thus by Theorem 2.12 of [15], T_{sub} is Archimedean.

Part B. T_{sub} is a t-norm with no zero divisors.

Indeed, by Theorem 3.2 the condition (i) of Theorem 3.3 holds since O is an overlap function. Thus, we get $xy = 0$ whenever $\sup\{z \in [0,1] \mid \theta(z) + \frac{a}{2} > \theta(x) + \theta(y)\} = 0$. Then it follows from

$$T_{sub}(x,y) = (\theta + \frac{a}{2})^{(-1)}(\theta(x) + \theta(y))$$

$$= \sup\{z \in [0,1] \mid \theta(z) + \frac{a}{2} > \theta(x) + \theta(y)\}$$

that, $T_{sub}(x,y) = 0$ if and only $xy = 0$. Thus T_{sub} is a continuous t-norm with no zero divisors.

Therefore, T_{sub} is a continuous Archimedean t-norm with no zero divisors, i.e., T_{sub} is strict by Theorem 2.18 of [15].

$(2) \Rightarrow (1)$: Suppose that $O(x,y) = \mathcal{F}(T(x,y))$ for a (pseudo) automorphism $\mathcal{F}$ and a strict t-norm T. Thus, from Theorem 5.1 of [15], we know that T has a continuous additive generator $t : [0,1] \to [0,\infty]$ where t is a continuous and strictly decreasing function satisfying $t(1) = 0$. Define $\theta : [0,1] \to [0,\infty]$ by

$$\theta(x) = t(x) + \frac{a}{2} \tag{13}$$

for any $x \in [0,1]$. Then from Corollary 3.30 of [15], θ is continuous and strictly decreasing and satisfies the following two conditions:

(i) $\theta(x) = \frac{a}{2}$ if and only if $x = 1$;

(ii) $\theta(x) = \infty$ if and only if $x = 0$.

Define $\vartheta : [0,\infty] \to [0,1]$ by

$$\vartheta(x) = \mathcal{F}((\theta + \frac{a}{2})^{(-1)}(x)). \tag{14}$$

for any $x \in [0,\infty]$. Then, ϑ is a continuous and decreasing function since $\mathcal{F}$ is increasing and continuous, and $(\theta + \frac{a}{2})^{(-1)}$ is decreasing and continuous.

By Definition 5.1, $\mathcal{F}((\theta + \frac{a}{2})^{(-1)}(x)) = 0$ if and only if $(\theta + \frac{a}{2})^{(-1)}(x) = 0$. Then from (14), $\vartheta(x) = 0$ if and only if $(\theta + \frac{a}{2})^{(-1)}(x) = 0$. On the other hand, from (ii), we have

$\sup\{z \in [0,1] \mid \theta(z) + \frac{a}{2} > x\} = 0$ if and only if $x = \infty$. Thus $\vartheta(x) = 0$ if and only if $x = \infty$.

From (14) and Definition 5.1, it is obvious that $\vartheta(x) = 1$ if and only if $(\theta + \frac{a}{2})^{(-1)}(x) = 1$. From (i), we have $\sup\{z \in [0,1] \mid \theta(z) + \frac{a}{2} > x\} = 1$ if and only if $x \in [0, a]$. Thus $\vartheta(x) = 1$ if and only if $x \in [0, a]$.

Therefore,

$$
\begin{aligned}
\vartheta(\theta(x) + \theta(y)) &= \mathcal{F}((\theta + \frac{a}{2})^{(-1)}(\theta(x) + \theta(y))) \quad \text{by (14)} \\
&= \mathcal{F}(\sup\{z \in [0,1] \mid \theta(z) + \frac{a}{2} > \theta(x) + \theta(y)\}) \quad \text{by Definition 2.2} \\
&= \mathcal{F}(\sup\{z \in [0,1] \mid t(z) + a > t(x) + t(y) + a\}) \quad \text{by (13)} \\
&= \mathcal{F}(\sup\{z \in [0,1] \mid t(z) > t(x) + t(y)\}) \\
&= \mathcal{F}(t^{(-1)}(t(x) + t(y))) \quad \text{by Definition 2.2} \\
&= \mathcal{F}(T(x, y)) \\
&= O(x, y).
\end{aligned}
$$

$(2) \Leftrightarrow (3)$ is immediately from [30]. $\qquad\square$

Example 5.5. For a given $a \in [0, \infty)$, let $\theta : [0,1] \to [0, \infty]$ and $\vartheta : [0, \infty] \to [0,1]$ be defined, respectively, by

$$
\theta(x) = \begin{cases} \frac{a}{2} + \ln\frac{2-x}{x} & \text{if } x \neq 0, \\ \infty & \text{if } x = 0 \end{cases}
$$

for any $x \in [0,1]$ and

$$
\vartheta(x) = \begin{cases} 1 & \text{if } x \in [0, a), \\ \frac{4}{e^{2(x-a)} + 2e^{x-a} + 1} & \text{if } x \in [a, \infty] \end{cases}
$$

for any $x \in [0, \infty]$. Then, it is obvious that θ is a continuous and strictly decreasing function and, ϑ is a continuous and decreasing function. By a simple computation, we get that

$$
O_{\theta, \vartheta}(x, y) = \vartheta(\theta(x) + \theta(y)) = \left(\frac{xy}{2 - x - y + xy}\right)^2.
$$

Hence, from Theorem 5.4, the function $O_{\theta, \vartheta}$ is an overlap function obtained by a $(\mathcal{F}, T)$-distortion for a (pseudo) automorphism $\mathcal{F}$ and a strict t-norm T. In fact, the Hamacher t-norm $T(x, y) = \frac{xy}{2-x-y+xy}$ is a strict t-norm and let $\mathcal{F}(x) = x^2$. Then

$$
O_{\theta, \vartheta}(x, y) = \left(\frac{xy}{2 - x - y + xy}\right)^2 = \mathcal{F}(T(x, y)).
$$

Meanwhile, from Theorem 5.4, there exist a strictly increasing bijection $\varphi : [0,1] \to [0,1]$ and a (pseudo) automorphism $\mathcal{H}$ such that $O(x, y) = \mathcal{H}(\varphi(x)\varphi(y))$ for each $(x, y) \in [0,1]^2$. Indeed, let $\varphi(x) = \frac{x}{2-x}$ and $\mathcal{H}(x) = \left(\frac{2x}{1+x}\right)^2$. Obviously, φ is a strictly increasing bijection and $\mathcal{H}$ is an automorphism. Moreover,

$$
\mathcal{H}(\varphi(x)\varphi(y)) = \left(\frac{xy}{2 - x - y + xy}\right)^2
$$

for all $(x, y) \in [0,1]^2$.

As a conclusion of this section, we investigate when an overlap function additively generated by the pair (θ, ϑ) cannot be obtained by a $(\mathcal{F}, T)$-distortion for a positive and continuous t-norm T and a (pseudo) automorphism $\mathcal{F}$. First, we introduce two necessary lemmas.

Lemma 5.6. (Feng-qing Zhu and Xue-ping Wang [30]) Let T be a positive continuous t-norm. Then, one of the following three cases is valid for T:

(1) $T(x, y) = min\{x, y\}$;

(2) $T(x, y)$ is a strict t-norm;

(3) There exists a countable family $\{[a_\alpha, e_\alpha], T_\alpha\}$ such that T is an ordinal sum of this family and each T_α is a continuous Archimedean t-norm, and T_α is a strict t-norm when $a_\alpha = 0$.

Lemma 5.7. (Feng-qing Zhu and Xue-ping Wang [30]) Let $O : [0, 1]^2 \to [0, 1]$ be an overlap function and $\mathcal{F} : [0, 1] \to [0, 1]$ be a (pseudo) automorphism with $\mathcal{F}(x) = O(x, 1)$. Then the following are equivalent:

(1) The overlap function O can be determined by a $(\mathcal{F}, T)$-distortion for the (pseudo) automorphism $\mathcal{F}$ and a positive continuous t-norm T.

(2) One of the following three cases is valid for the overlap function O:

   (i) $O(x, y) = \min\{\mathcal{F}(x), \mathcal{F}(y)\}$;

   (ii) There exists a continuous and strictly decreasing function $t : [0, 1] \to [0, \infty]$ with $t(1) = 0$ and $t(0) = \infty$ such that

$$O(x, y) = \mathcal{F}(t^{(-1)}(t(x) + t(y)))$$

   for all $(x, y) \in [0, 1]^2$;

   (iii) There exists a countable family $[a_\alpha, e_\alpha]$ of non-overlapping, closed, non-trivial, proper subintervals of $[0, 1]$ such that, for all $(x, y) \in [0, 1]^2$,

$$O(x, y) = \begin{cases} \mathcal{F}(h_\alpha^{-1}(h_\alpha(x) + h_\alpha(y))) & if \ (x, y) \in [a_\alpha, e_\alpha]^2, \\ \min\{\mathcal{F}(x), \mathcal{F}(y)\} & \text{otherwise,} \end{cases}$$

   where $h_\alpha : [a_\alpha, e_\alpha] \to [0, \infty]$ is a continuous and strictly decreasing function which satisfies $h_\alpha(e_\alpha) = 0$ for each α and $h_\alpha(a_\alpha) = \infty$ when $a_\alpha = 0$.

Then we have the following proposition.

Proposition 5.8. Let $\theta : [0, 1] \to [0, \infty]$ be a continuous and decreasing but not strict function and $\vartheta : [0, \infty] \to [0, 1]$ be a continuous and decreasing function. Let $O_{\theta,\vartheta} : [0, 1]^2 \to [0, 1]$ be an overlap function additively generated by the pair (θ, ϑ) that satisfies one of the following two conditions: for a given $a \in [0, \infty)$,

(1) $\theta(x) = \frac{a}{2}$ if and only if $x = 1$;

(2) $\vartheta(x) = 1$ if and only if $x \in [0, a]$.

Then the overlap function $O_{\theta,\vartheta}$ cannot be obtained by a $(\mathcal{F}, T)$-distortion for a positive and continuous t-norm T and a (pseudo) automorphism $\mathcal{F}$.

P r o o f.   Suppose that the overlap function $O_{\theta,\vartheta}$ can be obtained by a $(\mathcal{F}, T)$-distortion for a positive and continuous t-norm T and a (pseudo) automorphism $\mathcal{F}$. Then by Lemma 5.6, there are three cases as follows.

Case 1. If $T(x, y) = \min\{x, y\}$, then $O_{\theta,\vartheta}(x, y) = \min\{\mathcal{F}(x), \mathcal{F}(y)\}$. In particular,

$$O_{\theta,\vartheta}(x, x) = \vartheta(\theta(x) + \theta(x)) = \mathcal{F}(x) \tag{15}$$

for any $x \in (0, 1)$. From the hypothesis, it follows from Proposition 3.6 that we have $\theta(x) = \frac{a}{2}$ if and only if $x = 1$ since $O_{\theta,\vartheta}$ is an overlap function. Thus

$$O_{\theta,\vartheta}(x, 1) = \vartheta(\theta(x) + \frac{a}{2}) = \mathcal{F}(x) \tag{16}$$

and, $\theta(x) > \frac{a}{2}$ for any $x \in (0, 1)$ since θ is decreasing. The inequality means that $2\theta(x) > \theta(x) + \frac{a}{2}$. Hence there exists a y with $0 < y < x$ such that $\theta(x) + \frac{a}{2} < \theta(y) + \frac{a}{2} < 2\theta(x)$. This together with (15) and (16) results in $\mathcal{F}(x) = \vartheta(2\theta(x)) = \vartheta(\theta(y) + \frac{a}{2}) = \vartheta(2\theta(y)) = \mathcal{F}(y)$. By induction, we can get that $\mathcal{F}(u) = \mathcal{F}(x)$ for any $u \in (0, x)$. Then $\mathcal{F}$ is a constant over $(0, x)$ and $\mathcal{F}(0) = 0$, which violate the continuity of $\mathcal{F}$.

Case 2. If $T(x, y)$ is a strict t-norm, then by Theorem 5.4, θ is a strict function, a contradiction.

Case 3. If there exists a countable family $\{[a_\alpha, e_\alpha], T_\alpha\}$ such that T is an ordinal sum of this family and each T_α is a continuous Archimedean t-norm, and T_α is a strict t-norm when $a_\alpha = 0$, then from Lemma 5.7, there exists a countable family $[a_\alpha, e_\alpha]$ of non-overlapping, closed, non-trivial, proper subintervals of $[0, 1]$ such that, for all $(x, y) \in [0, 1]^2$,

$$O(x, y) = \begin{cases} \mathcal{F}(h_\alpha^{(-1)}(h_\alpha(x) + h_\alpha(y))) & \text{if } (x, y) \in [a_\alpha, e_\alpha]^2 \\ \min\{\mathcal{F}(x), \mathcal{F}(y)\} & \text{otherwise,} \end{cases} \tag{17}$$

where $h_\alpha : [a_\alpha, e_\alpha] \to [0, \infty]$ is a continuous and strictly decreasing function which satisfies $h_\alpha(e_\alpha) = 0$ for each α and $h_\alpha(a_\alpha) = \infty$ when $a_\alpha = 0$.

It is easy to see that the interval $[0, 1]$ can be represented as a union of a countable family of pairwise disjoint intervals $(J_\alpha)_{\alpha \in A}$ where, for each $\alpha \in A$, either $J_\alpha = [u_\alpha, v_\alpha]$ or $J_\alpha = (u_\alpha, v_\alpha)$ for suitable $u_\alpha, v_\alpha \in [0, 1]$. From Theorem 3.2, we have $\theta(x) = \infty$ if and only if $x = 0$ since $O_{\theta,\vartheta}$ is an overlap function and, from hypothesis and Proposition 3.6 we get $\theta(x) = \frac{a}{2}$ if and only if $x = 1$. Then there exist suitable u_α, v_α such that $u_\alpha \neq 0$ and $\theta(u_\alpha) > \theta(v_\alpha)$ and $[u_\alpha, v_\alpha]$ or $(u_\alpha, v_\alpha) \in (J_\alpha)_{\alpha \in A}$. Hence, for some fixed x with $0 < x < u_\alpha$, we have $\vartheta(\theta(x) + \theta(v_\alpha)) = \vartheta(\theta(x) + \theta(u_\alpha)) = \mathcal{F}(x)$ by (17). Because θ is decreasing and continuous, there exists an x_0 with $0 < x_0 < x$ such that $\theta(x) + \theta(v_\alpha) < \theta(x_0) + \theta(v_\alpha) < \theta(x) + \theta(u_\alpha)$ and, $\mathcal{F}(x_0) = \vartheta(\theta(x_0) + \theta(v_\alpha))$ from (17). Then, $\mathcal{F}(x_0) = \vartheta(\theta(x_0) + \theta(v_\alpha)) = \vartheta(\theta(x + \theta(v_\alpha))) = \mathcal{F}(x)$. Therefore, by induction,

$\vartheta(\theta(u) + \theta(v_\alpha)) = \mathcal{F}(u) = \mathcal{F}(x) \neq 0$ for any $u \in (0, x)$, which violates the continuity of $\mathcal{F}$ since $\mathcal{F}(x) = 0$ if and only if $x = 0$.

This completes the proof. $\square$

The following example explains Proposition 5.8.

Example 5.9. For a given $a \in [0, \infty)$, consider the functions $\theta : [0, 1] \to [0, \infty]$ and $\vartheta : [0, \infty] \to [0, 1]$ defined by

$$
\theta(x) = \begin{cases}
\infty & \text{if } x = 0, \\
\frac{a}{2} - \ln x & \text{if } 0 < x \leq \frac{1}{3}, \\
\frac{a}{2} + \ln 3 & \text{if } \frac{1}{3} < x \leq \frac{2}{3}, \\
\frac{a}{2} + \ln \frac{4-3x}{x} & \text{if } \frac{2}{3} < x \leq 1
\end{cases}
$$

and

$$
\vartheta(x) = \begin{cases}
1 & \text{if } x \in [0, a), \\
e^{-x+a} & \text{if } x \in [a, \infty],
\end{cases}
$$

respectively. Then it is easy to see that both θ and ϑ are continuous and decreasing functions satisfying the conditions of Theorem 3.3. By a simple calculation, the overlap function $O_{\theta,\vartheta}$ generated by θ and ϑ is

$$
O_{\theta,\vartheta}(x,y) = \begin{cases}
xy & \text{if } (x,y) \in [0, \frac{1}{3})^2, \\
\min\{\frac{x}{3}, \frac{y}{3}\} & \text{if } (x,y) \in [0, \frac{1}{3}) \times [\frac{1}{3}, \frac{2}{3}] \cup [\frac{1}{3}, \frac{2}{3}] \times [0, \frac{1}{3}), \\
\frac{1}{9} & \text{if } (x,y) \in [\frac{1}{3}, \frac{2}{3}]^2, \\
\frac{\min\{x,y\}\max\{x,y\}}{4 - 3\max\{x,y\}} & \text{if } (x,y) \in [0, \frac{1}{3}) \times (\frac{2}{3}, 1] \cup (\frac{2}{3}, 1] \times [0, \frac{1}{3}), \\
\frac{\max\{x,y\}}{12 - 9\max\{x,y\}} & \text{if } (x,y) \in [\frac{1}{3}, \frac{2}{3}] \times (\frac{2}{3}, 1] \cup (\frac{2}{3}, 1] \times [\frac{1}{3}, \frac{2}{3}], \\
\frac{xy}{(4-3x)(4-3y)} & \text{if } (x,y) \in (\frac{2}{3}, 1]^2.
\end{cases}
$$

Therefore, by Proposition 5.8, $O_{\theta,\vartheta}$ cannot be obtained by $(\mathcal{F},T)$-distortion. In fact, if $O_{\theta,\vartheta}$ can be obtained by $(\mathcal{F},T)$-distortion, then from Lemma 5.1 of [10], we have that

$$
\mathcal{F}(x) = O_{\theta,\vartheta}(x, 1) = \begin{cases}
x & \text{if } x \in [0, \frac{1}{3}), \\
\frac{1}{3} & \text{if } x \in [\frac{1}{3}, \frac{2}{3}], \\
\frac{x}{4-3x} & \text{if } x \in (\frac{2}{3}, 1].
\end{cases}
$$

Thus $O_{\theta,\vartheta}(x,y) = \mathcal{F}(T(x,y)) = \frac{1}{3}$ whenever $T(x,y) \in [\frac{1}{3}, \frac{2}{3}]$. On the other hand, by a simple calculation, we get $O_{\theta,\vartheta}(x,y) = \frac{1}{3}$ when $x = 1, y \in [\frac{1}{3}, \frac{2}{3}]$ or $y = 1, x \in [\frac{1}{3}, \frac{2}{3}]$ or $x, y \in (0, 1)$ with $6(x + y) - 3xy = 8$. Therefore, $T(x,y) < \frac{1}{3}$ when $x, y \in (0, 1)$ with $6(x + y) - 3xy < 8$, $T(x,y) > \frac{2}{3}$ when $x, y \in (0, 1)$ with $6(x + y) - 3xy > 8$, contrary to the fact that T is continuous.

6. ADDITIVE GENERATOR PAIRS OF GROUPING FUNCTIONS

In this section, we introduce the concept of an additive generator pair of grouping functions and some related results whose proofs are omitted since grouping functions are given as dual operations of overlap functions.

First, we give the definitions of both a grouping function and its an additive generator pair.

Definition 6.1. (Bustince et al. [6]) A bivariate function $G : [0,1]^2 \to [0,1]$ is said to be a grouping function if it satisfies the following conditions:

(G1) G is commutative;

(G2) $G(x,y) = 0$ if and only if $x = y = 0$;

(G3) $G(x,y) = 1$ if and only if $x = 1$ or $y = 1$;

(G4) G is increasing;

(G5) G is continuous.

Definition 6.2. Let $t : [0,1] \to [0,\infty]$ and $s : [0,\infty] \to [0,1]$ be continuous and increasing functions, respectively. If the bivariate function $G_{t,s} : [0,1]^2 \to [0,1]$, given by

$$G_{t,s}(x,y) = s(t(x) + t(y)),$$

is a grouping function, then (t,s) is called an additive generator pair of the grouping function $G_{t,s}$ and $G_{t,s}$ is said to be additively generated by the pair (t,s).

The following theorem characterizes the functions $t : [0,1] \to [0,\infty]$ and $s : [0,\infty] \to [0,1]$, respectively, whenever $G_{t,s}$ is a grouping function additively generated by the pair (t,s).

Theorem 6.3. Let $t : [0,1] \to [0,\infty]$ and $s : [0,\infty] \to [0,1]$ be continuous and increasing functions, respectively. Let $G_{t,s} : [0,1]^2 \to [0,1]$ be a grouping function additively generated by the pair (t,s). Then the following statements hold:

(1) $t(x) = \infty$ if and only if $x = 1$;

(2) $s(x) = 1$ if and only if $x = \infty$.

The following theorem shows the conditions that a pair (t,s) can additively generate grouping functions.

Theorem 6.4. For a given $a \in [0,\infty)$, let $t : [0,1] \to [0,\infty]$ and $s : [0,\infty] \to [0,1]$ be continuous and increasing functions such that

a. $t(x) = \infty$ if and only if $x = 1$;

b. $t(x) = \frac{a}{2}$ if and only if $x = 0$;

c. $s(x) = 0$ if and only if $x \in [0,a]$;

   d. $s(x) = 1$ if and only if $x = \infty$.

Then, the function $G_{t,s} : [0,1]^2 \to [0,1]$, defined by

$$G_{t,s}(x,y) = s(t(x) + t(y)),$$

is a grouping function.

   The following proposition reveals that the conditions (b) and (c) of Theorem 6.4 are closely associated whenever $G_{t,s} : [0,1]^2 \to [0,1]$ is a grouping function additively generated by the pair (t,s).

Proposition 6.5. For a given $a \in [0,\infty)$, let $t : [0,1] \to [0,\infty]$ and $s : [0,\infty] \to [0,1]$ be two continuous and increasing functions such that a binary function $G_{t,s} : [0,1]^2 \to [0,1]$ with $G_{t,s}(x,y) = s(t(x) + t(y))$ is a grouping function. Then the following statements are equivalent:

  (i) $x \in [0,a]$ if and only if $s(x) = 0$;

  (ii) $t(x) = \frac{a}{2}$ if and only if $x = 0$.

   In what follows, we shall investigate the conditions under which a grouping function additively generated by the pair (t,s) is a t-conorm.

Proposition 6.6. Let $t : [0,1] \to [0,\infty]$ and $s : [0,\infty] \to [0,1]$ be continuous and increasing functions, respectively. Let $G_{t,s} : [0,1]^2 \to [0,1]$ be a grouping function additively generated by the pair (t,s) with $t(0) = \frac{a}{2}$ for some $a \in [0,\infty)$. Then the following statements are equivalent:

  (1) $s \circ (t + \frac{a}{2}) = id_{[0,1]}$;

  (2) 0 is a neutral element of $G_{t,s}$.

Proposition 6.7. For a given $a \in [0,\infty)$, let $t : [0,1] \to [0,\infty]$ be a continuous and strictly increasing function such that:

  (1) $t(x) = \infty$ if and only if $x = 1$;

  (2) $t(x) = \frac{a}{2}$ if and only if $x = 0$.

If a function $s : [0,\infty] \to [0,1]$ satisfies $s(x) = (t + \frac{a}{2})^{(-1)}(x)$ for any $x \in [0,1]$, then a binary function $G_{t,s} : [0,1]^2 \to [0,1]$ with $G_{t,s}(x,y) = s(t(x) + t(y))$ is a positive t-conorm.

Proposition 6.8. Let $t : [0,1] \to [0,\infty]$ and $s : [0,\infty] \to [0,1]$ be two continuous and increasing functions, and $G_{t,s} : [0,1]^2 \to [0,1]$ be a grouping function additively generated by the pair (t,s) with 0 as neutral element. If for a given $a \in [0,\infty)$, t satisfies the condition (b) of Theorem 6.4, then $G_{t,s}$ is associative.

Theorem 6.9. Let $t : [0,1] \to [0,\infty]$ and $s : [0,\infty] \to [0,1]$ be two continuous and increasing functions and, $G_{t,s} : [0,1]^2 \to [0,1]$ be a grouping function additively generated by the pair (t,s). If for a given $a \in [0,\infty)$, t satisfies the condition (b) of Theorem 6.4, then the following three statements are equivalent:

(1) $G_{t,s}$ is a t-conorm;

(2) 0 is a neutral element of $G_{t,s}$;

(3) $s \circ (t + \frac{a}{2}) = id_{[0,1]}$.

In the following, we give the conditions that a grouping function additively generated by the pair (t,s) can be obtained by a distortion of a t-conorm and a (pseudo) automorphism.

Proposition 6.10. Let $\mathcal{F} : [0,1] \to [0,1]$ be a pseudo automorphism. Then for every positive and continuous t-conorm $S : [0,1]^2 \to [0,1]$, the function $G_{\mathcal{F},S} : [0,1]^2 \to [0,1]$, given by

$$G_{\mathcal{F},S}(x,y) = \mathcal{F}(S(x,y)),$$

is a grouping function.

Where $G_{\mathcal{F},S}$ is called a grouping function obtained by the distortion of the t-conorm S by the pseudo automorphism $\mathcal{F}$ or a grouping function obtained by a $(\mathcal{F}, S)$-distortion.

Proposition 6.11. Let $t : [0,1] \to [0,\infty]$ and $s : [0,\infty] \to [0,1]$ be two continuous and increasing functions and, $G_{t,s} : [0,1]^2 \to [0,1]$ be a grouping function additively generated by the pair (t,s) that satisfies one of the following two conditions: for a given $a \in [0,\infty)$,

(1) $t(x) = \frac{a}{2}$ if and only if $x = 0$;

(2) $s(x) = 0$ if and only if $x \in [0,a]$.

Then the grouping function $G_{t,s}$ can be obtained by a $(\mathcal{F}, T_{super})$-distortion for a (pseudo) automorphism $\mathcal{F}$ and a t-superconorm T_{super}.

The next theorem gives two descriptions that a grouping function $G_{t,s}$ additively generated by the pair (t,s) can be obtained by a $(\mathcal{F}, S)$-distortion for a (pseudo) automorphism $\mathcal{F}$ and a t-conorm S.

Theorem 6.12. For a function $G : [0,1]^2 \to [0,1]$ the following are equivalent:

(1) There exist a continuous and strictly increasing function $t : [0,1] \to [0,\infty]$ and a continuous and increasing function $s : [0,\infty] \to [0,1]$ such that the function G is a grouping function additively generated by the pair (t,s) satisfying one of the following two conditions: for a given $a \in [0,\infty)$,

    (a) $t(x) = \frac{a}{2}$ if and only if $x = 0$;

    (b) $s(x) = 0$ if and only if $x \in [0,a]$.

(2) The function G is a grouping function obtained by a $(\mathcal{F},S)$-distortion for a (pseudo) automorphism $\mathcal{F}$ and a strict t-conorm S;

(3) There exist a strictly increasing bijection $\varphi : [0,1] \to [0,1]$ and a (pseudo) automorphism $\mathcal{H}$ such that $G(x,y) = \mathcal{H}(\varphi(x) + \varphi(y) - \varphi(x)\varphi(y))$ for all $(x,y) \in [0,1]^2$.

Finally, we investigate when a grouping function additively generated by the pair (t, s) cannot be obtained by a $(\mathcal{F}, S)$-distortion for a positive and continuous t-conorm S and a (pseudo) automorphism $\mathcal{F}$.

Proposition 6.13. Let $t : [0, 1] \to [0, \infty]$ be a continuous and increasing but not strict function and $s : [0, \infty] \to [0, 1]$ be a continuous and increasing function. Let $G_{t,s} : [0, 1]^2 \to [0, 1]$ be a grouping function additively generated by the pair (t, s) that satisfies one of the following two conditions: for a given $a \in [0, \infty)$,

(1) $t(x) = \frac{a}{2}$ if and only if $x = 0$;

(2) $s(x) = 0$ if and only if $x \in [0, a]$.

Then the grouping function $G_{t,s}$ cannot be obtained by a $(\mathcal{F}, S)$-distortion for a positive and continuous t-conorm S and a (pseudo) automorphism $\mathcal{F}$.

7. CONCLUSIONS

The article mainly explored the relationship between functions θ and ϑ in an overlap function additively generated by a relaxing boundary additive generator pair (θ, ϑ) (Proposition 3.6 and Theorem 4.6) and characterized the overlap function additively generated by the additive generator pair (θ, ϑ) (Theorem 5.4). Compared with the known definition, see [10], of an additive generator pair for overlap functions, our functions θ and ϑ in the additive generator pair of an overlap function are more general with a relaxed boundary conditions while the property of overlap functions can be preserved (Theorem 3.3). Clearly, multiplicative generator pairs of overlap and grouping functions can be discussed analogously.

R E F E R E N C E S

[1] T. D. C. Asmus, J. A. Sanz, G. P. Dimuro, B. Bedregal, J. Fernández, and H. Bustince: N-Dimensional admissibly ordered interval-valued overlap functions and its influence in interval-valued fuzzy-rule-based classification systems. IEEE Trans. Fuzzy Syst. *30* (2021), 1060–1072. DOI:10.1109/TFUZZ.2021.3052342

[2] B. Bedregal, G. P. Dimuro, H. Bustince, and E. Barrenechea: New results on overlap and grouping functions. Inf. Sci. *249* (2013), 148–170. DOI:10.1016/j.ins.2013.05.004

[3] G. Beliakov, A. Pradera, and T. Calvo: Aggregation Functions: A Guide for Practitioners. Springer, Berlin 2007.

[4] H. Bustince, E. Barrenechea, and M. Pagola: Image thresholding using restricted equivalence functions and maximizing the measures of similarity. Fuzzy Sets Syst. *158* (2007), 496–516. DOI:10.1016/j.fss.2006.09.012

[5] H. Bustince, J. Fernández, R. Mesiar, J. Montero, and R. Orduna: Overlap functions. Nonlinear Anal. *72* (2010), 1488–1499. DOI:10.1016/j.na.2009.08.033

[6] H. Bustince, M. Pagola, R. Mesiar, E. Hüllermeier, and F. Herrera: Grouping, overlaps, and generalized bientropic functions for fuzzy modeling of pairwise comparisons. IEEE Trans. Fuzzy Syst. *20* (2012), 405–415. DOI:10.1109/TFUZZ.2011.2173581

[7] G. P. Dimuro and B. Bedregal: Archimedean overlap functions: the ordinal sum and the cancellation, idempotency and limiting properties. Fuzzy Sets Syst. *252* (2014), 39–54. DOI:10.1016/j.fss.2014.04.008

[8] G. P. Dimuro and B. Bedregal: On residual implications derived from overlap functions. Inf. Sci. *312* (2015), 78–88. DOI:10.1016/j.ins.2015.03.049

[9] G. P. Dimuro and B. Bedregal: On the laws of contraposition for residual implications derived from overlap functions. In: 2015 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), IEEE, Los Alamitos 2015, pp. 1–7. DOI:10.1109/fuzz-ieee.2015.7337867

[10] G. P. Dimuro, B. Bedregal, H. Bustince, M. J. Asiáin, and R. Mesiar: On additive generators of overlap functions. Fuzzy Sets Syst. *287* (2016), 76–96. DOI:10.1016/j.fss.2015.02.008

[11] G. P. Dimuro, B. Bedregal, H. Bustince, R. Mesiar, and M. J. Asiáin: On additive generators of grouping functions. In: Information Processing and Management of Uncertainty in Knowledge-Based Systems (A. Laurent, O. Strauss, B. Bouchon-Meunier, R. R. Yager, eds.), Commun. Comput. Inf. Sci., vol. 444, Springer International Publishing 2014, pp. 252–261.

[12] G. P. Dimuro, B. Bedregal, and R. H. N. Santiago: On (G,N)-implications derived from grouping functions. Inf. Sci. *279* (2014), 1–17. DOI:10.1016/j.ins.2014.04.021

[13] J. Fodor and M. Roubens: Fuzzy preference modelling and multicriteria decision support. In: Theory and Decision Library, Kluwer Academic Publishers 1994.

[14] D. Gómez, J. T. Rodríguez, J. Montero, H. Bustince, and E. Barrenechea: n-dimensional overlap functions. Fuzzy Sets Syst. *287* (2016), 57–75. DOI:10.1016/j.fss.2014.11.023

[15] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. Kluwer Academic Publisher, Dordrecht 2000.

[16] C. Marco-Detchart, G. Lucca, C. Lopez-Molina, L. De Miguel, G. P. Dimuro, and H. Bustince: Neuro-inspired edge feature fusion using choquet integrals. Inf. Sci. *581* (2021), 740–754. DOI:10.1016/j.ins.2021.10.016

[17] C. Marco-Detchart, G. Lucca, M. A. Santos Silva, J. A. Rincon, V. Julian, and G. Dimuro: Sliding window based adaptative fuzzy measure for edge detection. Expert Systems *42* (2025), e13730. DOI:10.1111/exsy.13730

[18] G. Mayor and E. Trillas: On the representation of some aggregation functions. In: Proceedings of IEEE International Symposium on Multiple-Valued Logic, IEEE, Los Alamitos 1986, pp. 111–114.

[19] A. Mesiarová-Zemánková: Ranks of additive generators. Fuzzy Sets Syst. *160* (2009), 2032–2048. DOI:10.1016/j.fss.2008.12.015

[20] Yao Ouyang: On the construction of boundary weak triangular norms through additive generators. Nonlinear Anal. *66* (2007), 125–130. DOI:10.1016/j.na.2005.11.014

[21] Junsheng Qiao and Baoqing Hu: On multiplicative generators of overlap and grouping functions. Fuzzy Sets Syst. *332* (2018), 1–24. DOI:/10.1016/j.fss.2016.11.010

[22] I. Rodriguez-Martinez, T. da Cruz Asmus, G. P. Dimuro, F. Herrera, Z. Takáč, and H. Bustince: Generalizing max pooling via (a, b)-grouping functions for convolutional neural networks. Inform. Fusion *99* (2023), 101893. DOI:10.1016/j.compenvurbsys.2022.101893

[23] R. Santiago, B. Bedregal, G. P. Dimuro, J. Fernández, H. Bustince, and H. M. Fardoun: Abstract homogeneous functions and consistently influenced/disturbed multi-expert decision making. IEEE Trans. Fuzzy Syst. *30* (2021), 3447–3459. DOI:10.1109/TFUZZ.2021.3117438

[24] J. A. Sanz, A. Fernández, H. Bustince, and F. Herrera: Improving the performance of fuzzy rule-based classification systems with interval-valued fuzzy sets and genetic amplitude tuning. Inf. Sci. *180* (2010), 3674–3685. DOI:10.1016/j.ins.2010.06.018

[25] J. Sartori, G. Lucca, T. Asmus, H. Santos, E. Borges, B. Bedregal, H. Bustince, and G. P. Dimuro: d-cc-integrals: Generalizing cc-integrals by restricted dissimilarity functions with applications to fuzzy-rule based systems. In: Brazilian Conference on Intelligent Systems, Springer Nature Switzerland 2023, pp. 243–258.

[26] A. Urio-Larrea, H. Camargo, G. Lucca, T. Asmus, C. Marco-Detchart, L. Schick, C. Lopez-Molina, J. Andreu-Perez, H. Bustince, and G. P. Dimuro: Data stream clustering: introducing recursively extendable aggregation functions for incremental cluster fusion processes. IEEE Trans. Cybernetics *55* (2025), 1421–1435. DOI:10.1109/TCYB.2025.3527862

[27] P. Viceník: Additive generators of associative functions. Fuzzy Sets Syst. *153* (2005), 137–160. DOI:10.1016/j.fss.2004.11.016

[28] J. Wieczynski, G. Lucca, E. Borges, G. Dimuro, R. Lourenzutti, and H. Bustince: Application and comparison of cc-integrals in business group decision making. In: International Conference on Enterprise Information Systems, Springer International Publishing 2021, pp. 129–148.

[29] J. Wieczynski, G. Lucca, G. P. Dimuro, E. Borges, J. A. Sanz, T. da Cruz Asmus, J. Fernández, and H. Bustince: dC_F-integrals: generalizing C_F-integrals by means of restricted dissimilarity functions. IEEE Trans. Fuzzy Syst. *31* (2023), 160–173. DOI:10.1109/tfuzz.2022.3184054

[30] Feng-qing Zhu and Xue-ping Wang: A note on the conditions that an overlap function can be obtained by a distortion. Fuzzy Sets Syst. *448* (2022), 84–93. DOI:10.1016/j.fss.2022.01.005

Liang Li-zhi, School of Mathematical Sciences, Sichuan Normal University, Chengdu 610066, Sichuan. P. R. China.
    e-mail: 1902428258@qq.com

Wang Xue-ping, School of Mathematical Sciences, Sichuan Normal University, Chengdu 610066, Sichuan. P. R. China.
    e-mail: xpwang1@hotmail.com