KYBERNETIKA — VOLUME *61* (2025), NUMBER 2, PAGES 221–237

OPTIMALITY CONDITIONS FOR AN INTERVAL-VALUED VECTOR PROBLEM

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The present article considers a nonsmooth interval-valued vector optimization problem with inequality constraints. We first figure out Fritz John and Karush–Kuhn–Tucker type necessary optimality conditions for the interval-valued problem designed in the paper under quasidifferentiable $\mathfrak{F}$-convexity in connection with compact convex sets. Subsequently, sufficient optimality conditions are extrapolated under aforesaid quasidifferentiability supported by a suitable numerical example.

Keywords: interval-valued vector optimization problem, quasidifferentiable $\mathfrak{F}$-convexity, LU-Pareto optimality,

Classification: 49J52, 90C26, 90C30, 90C29

1. INTRODUCTION

The problems where we simultaneously optimize two or more objective functions are categorized as vector optimization problems. In vector optimization, the concept of optimality has a significant impact on statistical decision theory, game theory, economics, and all noncomparable criteria in optimal decision problems. Our intention in such problems is to extract the best leading solutions, or, more precisely, nondominated solutions, from the set of all feasible solutions.

There are several tools to handle uncertainty arising in mathematical programming problems. Various approaches like stochastic processes, fuzzy numbers, etc. emerged as mathematical devices to handle uncertainty in recent years. Uncertainty in real-world scenarios can also be sculptured by means of interval-valued programming problems. In linear and nonlinear interval-valued optimization problems, either at least one of the constraints or the objective function or both the objective function and constraints are considered interval-valued. The present paper intends to study interval-valued optimization programming problems assuming components incurred in objective values vary over some fixed intervals. Several authors had put their efforts into preparing the groundwork to develop sufficiency results along with duality criteria for interval-valued problems. In 2007, Wu [33] succeeded in deriving the KKT optimality conditions for the interval-valued optimization problem. Later on, Zhou and Wang [36] formulated

DOI: 10.14736/kyb-2025-2-0221

sufficient optimality conditions and derived duality theorems for interval-valued problems under convexity, whereas Jayswal *et al.* [19] studied duality results and sufficiency conditions for interval-valued problems under generalized convexity. Bhurjee and Panda [4] proposed an innovative approach to determine whether or not an efficient solution to the interval-valued optimization problem exists. Zhang [35] formulated the KKT conditions of optimality in a class of nonconvex interval-valued problems. Optimality and duality issues related to nondifferentiable interval-valued programming problems were addressed by Sun and Wang [29]. The nonsmooth vector optimization problem having each component locally Lipschitz produces outstanding results in optimality as discussed in Clarke [9], Craven [11], and Luc [23].

Abdouni and Thibault [14] intensively studied Lagrange multipliers for multiobjective nonsmooth problems. Brandao et al. [6] set up conditions of optimality for nonsmooth and nonconvex problems, whereas Coladas et al. [10] studied the same for nonsmooth multiobjective minimization problems. The work is further extended by Minami [26], who generalized it for nondifferentiable multiobjective problems. Jeyakumar and Yang [20] studied a class of problems where both the objective function and constraints are designed by taking functions that are not only convex but also locally Lipschitz and Gâteaux differentiable. A number of authors like Kanniappan [21], Wang [31], Bolintin*é*anu [5], Miettinen [25], Chinchuluun and Pardalos [8], Huang et al. [18], and Bhatia and Jain [3] worked on nonsmooth optimization problems and derived optimality conditions and other results.

The concept of quasidifferentiability was introduced by Demyanov and Rubinov [12] in the 1980's. Demyanov and Rubinov [13] worked on some approaches in order to deal with the nonsmooth optimization problem. Optimality and duality results for quasidifferentiable optimization problems can be found in many works (e. g., Luderer and Rösiger [24], Eppler and Luderer [15], Gao([16, 17]), Uderzo[30], Kuntz and Scholtes [22], Ward [32], Polyakova [27], Xia et al. [34], Shapiro [28]).

The objective of the present article is to establish optimality conditions for interval-valued vector optimization problems under quasidifferentiability. In this article, we introduced the Fritz John and KKT-type necessary optimality criteria for nonsmooth multiobjective interval-valued vector optimization problems with the help of $\mathfrak{F}$-convexity in connection with the compact convex set. Moreover, sufficient optimality criteria have been derived for the designed nonsmooth multiobjective interval-valued problem assuming the functions quasidifferentiable $\mathfrak{F}$-convex that satisfy compactness and convexity in the required domain.

This article is shaped up as follows: Section 2 recalls notations and definitions to set up the results derived in the continuation of the article. In Section 3, we established the Fritz John and the KKT-type necessary optimality results under quasidifferentiability. Finally, Section 4 deals with sufficiency optimality criteria for the (weak) LU-Pareto solution. An appropriate example of a non-convex quasidifferentiable interval-valued vector problem is engineered to get a better insight into the work.

2. PRELIMINARIES

In this section, we begin with the following convention for inequalities and equalities, which is utilized in the later part of this paper. For any $p = (p_1, p_2, \ldots, p_n)$, $q =$

$(q_1, q_2, \ldots, q_n)$ in $\Re^n$, we have

(i) $p = q \Leftrightarrow p_i = q_i, \forall i = 1, \ldots, n;$

(ii) $p > q \Leftrightarrow p_i > q_i, \forall i = 1, \ldots, n;$

(iii) $p \geqq q \Leftrightarrow p_i \geqq q_i, \forall i = 1, \ldots, n;$

(iv) $p \geq q \Leftrightarrow p \geqq q, p \neq q.$

In the sequel of the paper, $\mathcal{I}$ denotes the set of bounded and closed intervals of $\Re$. For any two intervals $A_1 = [a_1^L, a_1^U],\ A_2 = [a_2^L, a_2^U]\ \in \mathcal{I}$, the usual operations are defined as follows:

(i) $A_1 + A_2 = \{a_1 + a_2 : a_1 \in A_1 \text{ and } a_2 \in A_2\} = [a_1^L + a_2^L, a_1^U + a_2^U],$

(ii) $-A_1 = \{-a_1 : a_1 \in A_1\} = [-a_1^U, -a_1^L],$

(iii) $A_1 - A_2 = \{A_1 + (-A_2)\} = [a_1^L - a_2^U, a_1^U - a_2^L],$

(iv) $\alpha + A_1 = \{\alpha + a_1 : a_1 \in A_1\} = [\alpha + a_1^L, \alpha + a_1^U],$

(v) $\alpha A_1 = \{\alpha a_1 : a_1 \in A_1\} \ = \begin{cases} [\alpha a_1^L, \alpha a_1^U],\ \alpha > 0, \\ [\alpha a_1^U, \alpha a_1^L],\ \alpha \leqq 0, \end{cases}$

where α is any real number. If we take $a_1^L = a_1^U = a_1$, then the interval A_1 reduces to a real number.

Let $\emptyset \neq \mathrm{X} \subseteq \Re^n$, where $\Re^n$ symbolizes Euclidean space of n-dimension. If the function ϕ is interval-valued, then it can be represented more appropriately by $\phi(\pi) = [\phi^L(\pi), \phi^U(\pi)]$, where $\phi^L(\pi) : \Re^n \to \Re$ and $\phi^U(\pi) : \Re^n \to \Re$ have the components satisfying conditions $\phi^L(\pi) \leqq \phi^U(\pi),\ \forall\ \pi \in \Re^n$. Shortly, in place of $[\phi(\pi)]^L$ and $[\phi(\pi)]^U$ we write $\phi^L(\pi)$ and $\phi^U(\pi)$, respectively.

Symbolically, we write $A_1 \leqq_{LU} A_2$ to denote $a_1^L \leqq a_2^L$ and $a_1^U \leqq a_2^U$. For notational convenience, we use $\leqq_{LU}$ as partial ordering defined on $\mathcal{I}$. Moreover, $A_1 <_{LU} A_2 \Leftrightarrow A_1 \leqq_{LU} A_2,\ A_1 \neq A_2$.
Identically, $A_1 <_{LU} A_2$ means

$$a_1^L < a_2^L,\ a_1^U < a_2^U,$$

or,

$$a_1^L \leqq a_2^L,\ a_1^U < a_2^U,$$

or,

$$a_1^L < a_2^L,\ a_1^U \leqq a_2^U.$$

Definition 2.1. (Antczak [1]) A mapping $f : \Re^n \to \Re$ is termed as directionally differentiable at a point $\sigma \in \Re^n$ along the direction $d \in \Re^n$ provided the limit

$$f'(\sigma; d) := \lim_{\gamma \downarrow 0} \frac{f(\sigma + \gamma d) - f(\sigma)}{\gamma}$$

exists and is finite.

A function $F = (f_1, \dots, f_k) : \Re^n \to \Re^k$ where each component $f_i,\ i = \{1, \dots, k\}$, is directionally differentiable at a point σ is called directionally differentiable at $\sigma \in \Re^n$ in the specified direction $d \in \Re^n$.

Definition 2.2. (Demyanov and Rubinov [12]) A directionally differentiable function $f : \Re^n \to \Re$ is known as quasidifferentiable at a point $\sigma \in \Re^n$ if there exists a pair of ordered compact convex sets $D_f(\sigma) = [\underline{\partial} f(\sigma), \overline{\partial} f(\sigma)]$ corresponding to the function f satisfying

$$f'(\sigma; d) := \max_{\lambda \in \underline{\partial} f(\sigma)} \langle \lambda, d \rangle + \min_{\varrho \in \overline{\partial} f(\sigma)} \langle \varrho, d \rangle.$$

Here, $\underline{\partial} f(\sigma)$ is subdifferential, and $\overline{\partial} f(\sigma)$ is superdifferential of f at a point σ. Moreover, the pair of ordered sets $D_f(\sigma) = [\underline{\partial} f(\sigma), \overline{\partial} f(\sigma)]$ is quasidifferential of f at a point σ.

Example 2.3. Let us consider a nonsmooth function $f : \Re^2 \to \Re$ defined by $f(\pi) = \pi_1^2 + |\pi_1| + |\pi_2|$, and $\sigma = (0, 0)$. Our aim is to find the quasidifferential of f at σ. Using the definition of directional differentiable function at σ along $d \in \Re^2$, we obtain $f'(\sigma; d) = |d_1| + |d_2|$. Hence,

$$f'(\sigma; d) := \max_{\lambda \in \operatorname{conv}\{(1,0),(-1,0)\}} \langle \lambda, d \rangle + \min_{\varrho \in \operatorname{conv}\{(0,1),(0,-1)\}} \langle \varrho, d \rangle.$$

Therefore, by Definition 2.2, f is quasidifferentiable at σ, the subdifferential being $\underline{\partial} f(\sigma)$= $\operatorname{conv}\{(1, 0), (-1, 0)\}$ and superdifferential $\overline{\partial} f(\sigma) = \operatorname{conv}\{(0, 1), (0, -1)\}$. Moreover, $D_f(\sigma) = [\operatorname{conv}\{(1, 0), (-1, 0)\}, \operatorname{conv}\{(0, 1), (0, -1)\}]$ is a pair of quasidifferential ordered sets of the function f at a point σ.

Note: The uniqueness of a quasidifferential function f may fail at some point $\sigma \in \Re^n$. Consequently, if $D_f(\sigma) = [\underline{\partial} f(\sigma), \overline{\partial} f(\sigma)]$ is a quasidifferential of f at a point $\sigma \in \Re^n$, then the ordered sets $[\underline{\partial} f(\sigma) + V, \overline{\partial} f(\sigma) - V]$ is also its quasidifferential for any compact set V.

Definition 2.4. (Antczak [1]) A vector-valued function $F = (f_1, \dots, f_k) : \Re^n \to \Re^k$ where each of its component f_i is quasidifferentiable at a point σ having quasidifferential $D_{f_i}(\sigma) = [\underline{\partial} f_i(\sigma), \overline{\partial} f_i(\sigma)]$ is termed as quasidifferentiable at a point $\sigma \in \Re^n$.

3. NECESSARY OPTIMALITY CONDITIONS

Let us examine the following nonsmooth interval-valued vector optimization problem:

$$\text{(IVOP)} \qquad \text{minimize} \qquad \aleph(\pi) = \Big([\aleph_1^L(\pi), \aleph_1^U(\pi)], \dots, [\aleph_k^L(\pi), \aleph_k^U(\pi)] \Big)$$
$$\text{subject to} \quad \psi_j(\pi) \leqq 0;\ j \in \mathfrak{J} = \{1, \dots, m\};\ \pi \in \Re^n.$$

where, $\aleph_i : \Re^n \to \mathcal{I},\ i = \{1, \dots, k\}$ are interval-valued functions, whereas on the other hand, $\aleph_i^L(\pi), \aleph_i^U(\pi)$ and $\psi_j(\pi) : \Re^n \to \Re,\ j \in \mathfrak{J} = \{1, \dots, m\}$, are quasidifferentiable functions on $\Re^n$.

For convenience, we will use $\aleph = (\aleph_1, \dots, \aleph_k) : \Re^n \to \mathcal{I}$, where $\aleph_i = [\aleph_i^L, \aleph_i^U]$ and $\aleph^L, \aleph^U : \Re^n \to \Re^k$ and $\psi = (\psi_1, \dots, \psi_m) : \Re^n \to \Re^m$. Let $\Omega = \{\pi \in \Re^n : \psi_j(\pi) \leqq$

$0;\ j \in \mathfrak{J}\}$ represent the set of all feasible solutions to the problem (IVOP). Moreover, $\mathfrak{J}(\bar{\pi}) := \{j \in \mathfrak{J} : \psi_j(\bar{\pi}) = 0\}$ denotes the set of active constraints at a point $\bar{\pi}$.

Note: If we confine the objective function to be vector-valued instead of interval-valued, then the above problem reduces to the problem considered in Antczak [1].

Definition 3.1. (Antczak [2]) A feasible point $\bar{\pi} \in \Omega$ is known as an *LU*-Pareto (*LU*-efficient) solution to (IVOP) if there does not exist any point $\pi \in \Omega$ satisfying

$$\aleph_i(\pi) \leqq_{LU} \aleph_i(\bar{\pi}), \text{ for each } i \in \{1, \dots, k\}$$

and

$$\aleph_i(\pi) <_{LU} \aleph_i(\bar{\pi}), \text{ for at least one } i \in \{1, \dots, k\}.$$

Definition 3.2. (Antczak [2]) A feasible point $\bar{\pi} \in \Omega$ is known as a weak *LU*-Pareto (weak *LU*-efficient) solution to (IVOP) if there does not exist any point $\pi \in \Omega$ satisfying

$$\aleph_i(\pi) <_{LU} \aleph_i(\bar{\pi}), \text{ for each } i \in \{1, \dots, k\}.$$

Definition 3.3. (Antczak [1]) A function $\mathfrak{F} : \mathrm{X} \times \mathrm{X} \times \Re^n \to \Re$ is sublinear with respect to the third component, if for all $\pi, \sigma \in \mathrm{X} \subseteq \Re^n$, we have

(i) $\mathfrak{F}(\pi, \sigma; \rho_1 + \rho_2) \leqq \mathfrak{F}(\pi, \sigma; \rho_1) + \mathfrak{F}(\pi, \sigma; \rho_2),\ \forall\ \rho_1, \rho_2 \in \Re^n$,

(ii) $\mathfrak{F}(\pi, \sigma; \alpha\rho) = \alpha\mathfrak{F}(\pi, \sigma; \rho),\ \forall\ \alpha \in \Re_+,\ \forall\ \rho \in \Re^n$.

Taking $\alpha = 0$ in (ii), we get,

$$\mathfrak{F}(\pi, \sigma; 0) = 0. \tag{1}$$

Now, we recall the definition of $\mathfrak{F}$-convex function. Let σ be chosen from $\Re^n$ arbitrarily, and $S_f(\sigma)$ be a nonempty, compact, and convex subset of $\Re^n$.

Definition 3.4. (Antczak [1]) The function $f : \Re^n \to \Re$ is known as $\mathfrak{F}$-convex at a point σ on $\Re^n$ in connection with the compact convex set $S_f(\sigma)$, if there exists a sublinear function $\mathfrak{F}$ satisfying

$$f(\pi) - f(\sigma) \ \geqq\ \mathfrak{F}(\pi, \sigma; \bar{\varrho}), \text{ for all } \bar{\varrho} \in S_f(\sigma),\ \pi \in \Re^n. \tag{2}$$

Remark 3.5. If the function $f : \Re^n \to \Re$ is locally Lipschitz at every point σ of $\Re^n$ and $S_f(\sigma)$ is the same as that of Clarke subdifferential [9] of the function f at a point σ, then we arrive at the definition of locally Lipschitz $\mathfrak{F}$-convex function defined on $\Re^n$ as reflected in [3]. If the function f is differentiable at each point $\sigma \in \Re^n$ and $S_f(\sigma) = \{\nabla f(\sigma)\}$, then the Definition 3.4 reduces to the definition of differentiable $\mathfrak{F}$-convex function (see [7]).

Definition 3.6. (Antczak [1]) Let $F = (f_1, \dots, f_k) : \Re^n \to \Re^k$ be a vector valued function, and each $S_{f_i}(\sigma),\ i = 1, \dots, k$ be a nonempty, compact, as well as convex subset of $\Re^n$. The function f_i is known as $\mathfrak{F}$-convex at a point σ of $\Re^n$ in connection with $S_{f_i}(\sigma)$ if each component f_i of F satisfies (2). Furthermore, the function F is known as $\mathfrak{F}$-convex at a point σ of $\Re^n$ in connection with $S_F(\sigma) = S_{f_1}(\sigma) \times \cdots \times S_{f_k}(\sigma)$.

Definition 3.7. The interval-valued function $\aleph : \Re^n \to \mathcal{I}$ is known as the $\mathfrak{F}$-convex function at a point σ on $\Re^n$ in connection with the compact convex set $S_\aleph(\sigma)$, if all components of $\aleph^L = (\aleph_1^L, \dots, \aleph_k^L)$ and $\aleph^U = (\aleph_1^U, \dots, \aleph_k^U)$ are $\mathfrak{F}$-convex at a point σ on $\Re^n$ in connection with compact convex sets $S_{\aleph_i^L}(\sigma)$ and $S_{\aleph_i^U}(\sigma)$, respectively. That is,

$$\aleph_i^L(\pi) - \aleph_i^L(\sigma) \geqq \mathfrak{F}(\pi, \sigma; \bar{\varrho}_i^L), \text{ for all } \bar{\varrho}_i^L \in S_{\aleph_i^L}(\sigma), \tag{3}$$

$$\aleph_i^U(\pi) - \aleph_i^U(\sigma) \geqq \mathfrak{F}(\pi, \sigma; \bar{\varrho}_i^U), \text{ for all } \bar{\varrho}_i^U \in S_{\aleph_i^U}(\sigma), \ \pi \in \Re^n. \tag{4}$$

Note: Every convex and quasidifferentiable function is a $\mathfrak{F}$-convex quasidifferentiable function in connection with the convex compact set, but the converse is not true.

Now, in order to substantiate the necessary Fritz John-type optimality results for constructed quasidifferentiable interval-valued vector optimization problem (IVOP), we apply the ε-constraint technique in which one of the objectives is optimized (let it be rth component), and the remaining are shielded by setting an upper bound. Therefore, the coupled scalar initial value problem assumes the following form:

$$\begin{aligned}
(\mathrm{P}_r)_\varepsilon \quad & \text{minimize} && \aleph_r(\pi) = [\aleph_r^L(\pi), \aleph_r^U(\pi)] \\
& \text{subject to} && [\aleph_i^L(\pi), \aleph_i^U(\pi)] \leqq_{LU} [\varepsilon_i^L, \varepsilon_i^U], \ i = \{1, \dots, k\}, \ i \neq r, \\
& && \psi_j(\pi) \leqq 0, \ (j \in \mathfrak{J}), \ \pi \in \Re^n.
\end{aligned}$$

Theorem 3.8. The feasible point $\bar{\pi} \in \Omega$ becomes an LU-Pareto solution to (IVOP) iff it is a minimal solution to the ε-constraint problem $(P_r)_\varepsilon$, where r can take any value from 1 to k, $\varepsilon_i^L = \aleph_i^L(\bar{\pi})$, and $\varepsilon_i^U = \aleph_i^U(\bar{\pi})$, $\forall\ i = \{1, \dots, k\}, \ i \neq r$.

In the light of the above theorem, we can rewrite our problem as

$$\begin{aligned}
(\mathrm{P}_r(\bar{\pi})) \quad & \text{minimize} && \aleph_r(\pi) = [\aleph_r^L(\pi), \aleph_r^U(\pi)] \\
& \text{subject to} && [\aleph_i^L(\pi), \aleph_i^U(\pi)] \leqq_{LU} [\aleph_i^L(\bar{\pi}), \aleph_i^U(\bar{\pi})], \ i = \{1, \dots, k\}, \ i \neq r, \\
& && \psi_j(\pi) \leqq 0, \ (j \in \mathfrak{J}), \ \pi \in \Re^n.
\end{aligned}$$

Theorem 3.9. (Necessary criteria of Fritz John-type) Let the feasible point $\bar{\pi} \in \Omega$ be a weak LU-Pareto solution to (IVOP). Further, assume that each $\aleph_i^L, \aleph_i^U$, $i = \{1, \dots, k\}$, is quasidifferentiable at a point $\bar{\pi}$ together with the quasidifferential $D_{\aleph_i^L}(\bar{\pi}) = [\underline{\partial}\aleph_i^L(\bar{\pi}), \overline{\partial}\aleph_i^L(\bar{\pi})]$ and $D_{\aleph_i^U}(\bar{\pi}) = [\underline{\partial}\aleph_i^U(\bar{\pi}), \overline{\partial}\aleph_i^U(\bar{\pi})]$ respectively. Moreover, ψ_j $(j \in \mathfrak{J})$, are quasidifferentiable at a point $\bar{\pi}$ together with the quasidifferential $D_{\psi_j}(\bar{\pi}) = [\underline{\partial}\psi_j(\bar{\pi}), \overline{\partial}\psi_j(\bar{\pi})]$. Then, for any sets of $\varrho_i^L \in \overline{\partial}\aleph_i^L(\bar{\pi})$, $\varrho_i^U \in \overline{\partial}\aleph_i^U(\bar{\pi})$ $(i = \{1, \dots, k\})$, and $\vartheta_j \in \overline{\partial}\psi_j(\bar{\pi})$ $(j \in \mathfrak{J})$, there exist Lagrange multipliers $\bar{\mu}^L(\eth) \in \Re^k$, $\bar{\mu}^U(\eth) \in \Re^k$ and $\bar{\delta}(\eth) \in \Re^m$ so that

$$0 \in \sum_{i=1}^{k} \left[\bar{\mu}_i^L(\eth)(\underline{\partial}\aleph_i^L(\bar{\pi}) + \varrho_i^L) + \bar{\mu}_i^U(\eth)(\underline{\partial}\aleph_i^U(\bar{\pi}) + \varrho_i^U) \right] + \sum_{j=1}^{m} \bar{\delta}_j(\eth)(\underline{\partial}\psi_j(\bar{\pi}) + \vartheta_j), \tag{5}$$

$$\bar{\delta}_j(\eth)\psi_j(\bar{\pi}) = 0 \ (j \in \mathfrak{J}), \tag{6}$$

$$\big(\bar{\mu}^L(\eth), \bar{\mu}^U(\eth), \bar{\delta}(\eth)\big) \geq 0, \tag{7}$$

where as, Lagrange multipliers $\bar{\mu}^L(\eth) = \big(\bar{\mu}^L_1(\eth), \ldots, \bar{\mu}^L_k(\eth)\big)$, $\bar{\mu}^U(\eth) = \big(\bar{\mu}^U_1(\eth), \ldots, \bar{\mu}^U_k(\eth)\big)$ and $\bar{\delta}(\eth) = \big(\bar{\delta}_1(\eth), \ldots, \bar{\delta}_m(\eth)\big)$ rely on the particularly selected $\eth = (\varrho^L, \varrho^U, \vartheta) = (\varrho^L_1, \ldots, \varrho^L_k, \varrho^U_1, \ldots, \varrho^U_k, \vartheta_1, \ldots, \vartheta_m)$.

Proof. It is given that the feasible point $\bar{\pi} \in \Omega$ is a weak LU-Pareto solution to (IVOP). In view of Theorem 3.8, $\bar{\pi}$ is a minimal solution to $(P_r(\bar{\pi}))$, which due to Proposition (2.1) (Gao [17]) guarantees the existence of $\bar{\mu}^L_i(\eth) \geqq 0$, $\bar{\mu}^U_i(\eth) \geqq 0$ $(i = \{1, \ldots, k\})$, and $\bar{\delta}_j \geqq 0$ $(j \in \mathfrak{J})$, all not being zero simultaneously for all $\varrho^L_i \in \overline{\partial}\aleph^L_i(\bar{\pi})$, $\varrho^U_i \in \underline{\partial}\aleph^U_i(\bar{\pi})$, and $\vartheta_j \in \overline{\partial}\psi_j(\bar{\pi})$ $(j \in \mathfrak{J})$, satisfying

$$0 \in \ \bar{\mu}^L_r(\eth)(\underline{\partial}\aleph^L_r(\bar{\pi}) + \varrho^L_r) + \bar{\mu}^U_r(\eth)(\underline{\partial}\aleph^U_r(\bar{\pi}) + \varrho^U_r) + \sum_{i=1, i\neq r}^{k} \Big[\bar{\mu}^L_i(\eth)(\underline{\partial}\aleph^L_i(\bar{\pi}) + \varrho^L_i) + \bar{\mu}^U_i(\eth)(\underline{\partial}\aleph^U_i(\bar{\pi}) + \varrho^U_i)\Big] + \sum_{j=1}^{m} \bar{\delta}_j(\eth)(\underline{\partial}\psi_j(\bar{\pi}) + \vartheta_j), \tag{8}$$

$$\bar{\delta}_j(\eth)\psi_j(\bar{\pi}) = 0, \ j \in \mathfrak{J}, \tag{9}$$

$$\big(\bar{\mu}^L(\varrho), \bar{\mu}^U(\varrho), \bar{\delta}(\varrho)\big) \geq 0, \tag{10}$$

where as, Lagrange multipliers $\bar{\mu}^L(\eth) = \big(\bar{\mu}^L_1(\eth), \ldots, \bar{\mu}^L_k(\eth)\big)$, $\bar{\mu}^U(\eth) = \big(\bar{\mu}^U_1(\eth), \ldots, \bar{\mu}^U_k(\eth)\big)$, $\bar{\delta}(\eth) = \big(\bar{\delta}_1(\eth), \ldots, \bar{\delta}_m(\eth)\big)$ rely on the particular choice of $\eth = (\varrho^L, \varrho^U, \vartheta) = (\varrho^L_1, \ldots, \varrho^L_k, \varrho^U_1, \ldots, \varrho^U_k, \vartheta_1, \ldots, \vartheta_m)$. Using (5), we can easily establish (8). Moreover, equation (9) is same as that of equation (6), whereas inequality (10) is same as that of inequality (7). □

Now, in order to prove the KKT-type necessary optimality criteria for the formulated interval-valued problem, we impose appropriate constraint qualifications. The constraint qualifications are satisfied for the studied quasidifferentiable problem (IVOP) at a point $\bar{\pi}$, if there exists $d \in \Re^n$ so that

$$\max_{\sigma_j \in \underline{\partial}\psi_j(\bar{\pi})} \langle \sigma_j, d\rangle + \max_{\vartheta_j \in \overline{\partial}\psi_j(\bar{\pi})} \langle \vartheta_j, d\rangle \ < \ 0, \qquad j \in \mathfrak{J}(\bar{\pi}). \tag{11}$$

Hence, constraint qualifications are satisfied at $\bar{\pi}$ $(\forall \ j \in \mathfrak{J}(\bar{\pi}))$, if there exists a quasidifferential $D_{\psi_j}(\bar{\pi}) = [\underline{\partial}\psi_j(\bar{\pi}), \overline{\partial}\psi_j(\bar{\pi})]$ so that

$$0 \notin \ \text{conv} \bigcup_{j \in \mathfrak{J}(\bar{\pi})} (\underline{\partial}\psi_j(\bar{\pi}), \overline{\partial}\psi_j(\bar{\pi})). \tag{12}$$

Theorem 3.10. (Necessary criteria of KKT-type) Let the feasible point $\bar{\pi} \in \Omega$ be a weak LU-Pareto solution to the problem (IVOP). Suppose the functions $\aleph^L_i$ and $\aleph^U_i$ are quasidifferentiable at a point $\bar{\pi}$ having quasidifferential $D_{\aleph^L_i}(\bar{\pi}) = [\underline{\partial}\aleph^L_i(\bar{\pi}), \overline{\partial}\aleph^L_i(\bar{\pi})]$ and $D_{\aleph^U_i}(\bar{\pi}) = [\underline{\partial}\aleph^U_i(\bar{\pi}), \overline{\partial}\aleph^U_i(\bar{\pi})]$ respectively. Moreover, ψ_j $(j \in \mathfrak{J})$ are quasidifferentiable at $\bar{\pi}$, having quasidifferential $D_{\psi_j}(\bar{\pi}) = [\underline{\partial}\psi_j(\bar{\pi}), \overline{\partial}\psi_j(\bar{\pi})]$ and constraint qualifications fulfilled at a point $\bar{\pi}$ to (IVOP). Then for the sets $\varrho_i^L \in \overline{\partial}\aleph_i^L(\bar{\pi})$, $\varrho_i^U \in \overline{\partial}\aleph_i^U(\bar{\pi})$, and $\vartheta_j \in \overline{\partial}\psi_j(\bar{\pi})$ $(j \in \mathfrak{J})$, there exist vectors $\bar{\mu}^L(\eth), \bar{\mu}^U(\eth) \in \Re^k$ and $\bar{\delta}(\eth) \in \Re^m$ so that

$$0 \in \sum_{i=1}^{k} \Big[\bar{\mu}_i^L(\eth)(\underline{\partial}\aleph_i^L(\bar{\pi}) + \varrho_i^L) + \bar{\mu}_i^U(\eth)(\underline{\partial}\aleph_i^U(\bar{\pi}) + \varrho_i^U)\Big] + \sum_{j=1}^{m} \bar{\delta}_j(\eth)(\underline{\partial}\psi_j(\bar{\pi}) + \vartheta_j), \quad (13)$$

$$\bar{\delta}_j(\eth)\psi_j(\bar{\pi}) = 0,\ j \in \mathfrak{J} \quad (14)$$

$$\bar{\mu}^L(\eth) \geq 0,\ \bar{\mu}^U(\eth) \geq 0,\ \bar{\delta}(\eth) \geqq 0, \quad (15)$$

where as, Lagrange multipliers $\bar{\mu}^L(\eth) = \big(\bar{\mu}_1^L(\eth), \dots, \bar{\mu}_k^L(\eth)\big)$, $\bar{\mu}^U(\eth) = \big(\bar{\mu}_1^U(\eth), \dots, \bar{\mu}_k^U(\eth)\big)$ and $\bar{\delta}(\eth) = \big(\bar{\delta}_1(\eth), \dots, \bar{\delta}_m(\eth)\big)$ rely on the particular selection of $\eth = (\varrho^L, \varrho^U, \vartheta) = (\varrho_1^L, \dots, \varrho_k^L,\ \varrho_1^U, \dots, \varrho_k^U, \vartheta_1, \dots, \vartheta_m)$.

P r o o f. Let the feasible point $\bar{\pi} \in \Omega$ be a weak LU-Pareto solution to (IVOP), and the necessary conditions (5) – (7) of optimality are satisfied at $\bar{\pi}$. It is sufficient to show that $\bar{\mu}^L(\eth), \bar{\mu}^U(\eth) \neq 0$ for all $\eth$. Let us assume that there exists $\eth^*$ for which $\bar{\mu}^L(\eth^*) = 0$, and $\bar{\mu}^U(\eth^*) = 0$. Then, it is clear from (5) that there exists $\vartheta_j^* \in \bar{\partial}\psi_j(\bar{\pi})$ $(j \in \mathfrak{J})$ such that

$$0 \in \sum_{j=1}^{m} \bar{\delta}_j(\eth^*)(\underline{\partial}\psi_j(\bar{\pi}) + \vartheta_j^*). \quad (16)$$

With the help of Fritz John-type necessary criteria (7), one can get $\bar{\delta}(\eth^*) \geq 0$, and hence, summing up from $j = \{1, \dots, m\}$, we get

$$\sum_{j=1}^{m} \bar{\delta}_j(\eth^*)\ >\ 0. \quad (17)$$

On dividing (16) by $\sum_{t=1}^{m} \bar{\delta}_t(\eth^*)$, we obtain

$$0 \in \sum_{j=1}^{m} \frac{\bar{\delta}_j(\eth^*)}{\sum_{t=1}^{m} \bar{\delta}_t(\eth^*)}(\underline{\partial}\psi_j(\bar{\pi}) + \vartheta_j^*). \quad (18)$$

Next, we define

$$\alpha_j(\eth^*) = \frac{\bar{\delta}_j(\eth^*)}{\sum_{t=1}^{m} \bar{\delta}_t(\eth^*)}, \qquad j \in \mathfrak{J}(\bar{\pi}), \quad (19)$$

which reveals that $0 \leqq \alpha_j(\eth^*) \leqq 1$, and $\sum_{j \in \mathfrak{J}(\bar{\pi})} \alpha_j(\eth^*) = 1$.
On combining (18) and (19), we get

$$0 \in \sum_{j=1}^{m} \alpha_j(\eth^*)(\underline{\partial}\psi_j(\bar{\pi}) + \vartheta_j^*). \quad (20)$$

The relation (19) together with (20) implies that

$$0 \in \text{ conv } \bigcup_{j \in \mathfrak{J}(\bar{\pi})} (\underline{\partial}\psi_j(\bar{\pi}) + \vartheta_j^*). \tag{21}$$

Using (12) and the fact that $\bar{\pi}$ satisfies the constraint qualifications, it reveals that

$$0 \notin \text{ conv } \bigcup_{j \in \mathfrak{J}(\bar{\pi})} (\underline{\partial}\psi_j(\bar{\pi}) + \vartheta_j), \quad \forall\, \vartheta_j \in \overline{\partial}\psi_j(\bar{\pi}),\ j \in \mathfrak{J}. \tag{22}$$

In particular, $\vartheta_j = \vartheta_j^* \in \overline{\partial}\psi_j(\bar{\pi})$ $(j \in \mathfrak{J})$. Therefore, using (22) it can be seen that

$$0 \notin \text{ conv } \bigcup_{j \in \mathfrak{J}(\bar{\pi})} (\underline{\partial}\psi_j(\bar{\pi}) + \vartheta_j^*)$$

which contradicts (21). This means that $\bar{\mu}^L(\eth), \bar{\mu}^U(\eth) \neq 0$ for all choices of $\eth$. This completes the proof. □

4. SUFFICIENCY RESULTS

In the present section, we formulate sufficient optimality criteria for a weak *LU*-Pareto solution and a *LU*-Pareto solution.

Theorem 4.1. (Sufficiency optimality criteria for weak *LU*-Pareto solution) A feasible point $\bar{\pi}$ becomes a weak *LU*-Pareto solution to (IVOP) if it satisfies the following sufficient optimality conditions:

(i) $\bar{\pi}$ satisfies KKT-type necessary conditions given by (13)-(15) with the quasidifferentials $D_{\aleph_i^L}(\bar{\pi}) = [\underline{\partial}\aleph_i^L(\bar{\pi}), \overline{\partial}\aleph_i^L(\bar{\pi})]$, $D_{\aleph_i^U}(\bar{\pi}) = [\underline{\partial}\aleph_i^U(\bar{\pi}), \overline{\partial}\aleph_i^U(\bar{\pi})]$ $(i = \{1, \dots, k\})$, and $D_{\psi_j}(\bar{\pi}) = [\underline{\partial}\psi_j(\bar{\pi}), \overline{\partial}\psi_j(\bar{\pi})]$ $(j \in \mathfrak{J})$, respectively;

(ii) functions $\aleph_i^L, \aleph_i^U$ $(i = \{1, \dots, k\})$, are $\mathfrak{F}$-convex quasidifferentiable at a point $\bar{\pi} \in \Omega$ in connection with $S_{\aleph_i^L}(\bar{\pi}) = \underline{\partial}\aleph_i^L(\bar{\pi}) + \overline{\partial}\aleph_i^L(\bar{\pi})$ and $S_{\aleph_i^U}(\bar{\pi}) = \underline{\partial}\aleph_i^U(\bar{\pi}) + \overline{\partial}\aleph_i^U(\bar{\pi})$ respectively;

(iii) ψ_j $(j \in \mathfrak{J}(\bar{\pi}))$ are $\mathfrak{F}$-convex quasidifferentiable at any point $\bar{\pi}$ in Ω in connection with $S_{\psi_j}(\bar{\pi}) = \underline{\partial}\psi_j(\bar{\pi}) + \overline{\partial}\psi_j(\bar{\pi})$.

Proof. Since the feasible solution $\bar{\pi} \in \Omega$ satisfies conditions (i) − (ii), then there exist $\bar{\mu}^L(\eth) \in \Re^k, \bar{\mu}^U(\eth) \in \Re^k$ and $\bar{\delta}(\eth) \in \Re^m$ satisfying the conditions (13)-(15). On the other hand, if the point $\bar{\pi}$ is not a weak *LU*-Pareto solution in (IVOP), then there exist $\tilde{\pi} \in \Omega$ satisfying

$$\aleph_i(\tilde{\pi}) <_{LU} \aleph_i(\bar{\pi}) \quad (i = \{1, \dots, k\}), \tag{23}$$

that is,

$$\begin{matrix} \aleph_i^L(\tilde{\pi}) < \aleph_i^L(\bar{\pi}) & \text{or} & \aleph_i^L(\tilde{\pi}) \leqq \aleph_i^L(\bar{\pi}) & \text{or} & \aleph_i^L(\tilde{\pi}) < \aleph_i^L(\bar{\pi}) \\ \aleph_i^U(\tilde{\pi}) < \aleph_i^U(\bar{\pi}) & & \aleph_i^U(\tilde{\pi}) < \aleph_i^L(\bar{\pi}) & & \aleph_i^L(\tilde{\pi}) \leqq \aleph_i^L(\bar{\pi}). \end{matrix}$$

By assumption, each function $\aleph_i^L, \aleph_i^U$ $(i = \{1, \ldots, k\})$ is $\mathfrak{F}$-convex quasidifferentiable at a point $\bar{\pi}$ in Ω in connection with $S_{\aleph_i^L}(\bar{\pi}) = \underline{\partial}\aleph_i^L(\bar{\pi}) + \overline{\partial}\aleph_i^L(\bar{\pi})$ and $S_{\aleph_i^U}(\bar{\pi}) = \underline{\partial}\aleph_i^U(\bar{\pi}) + \overline{\partial}\aleph_i^U(\bar{\pi})$ respectively; each ψ_j $(j \in \mathfrak{J}(\bar{\pi}))$ is a $\mathfrak{F}$-convex quasidifferentiable function at a point $\bar{\pi}$ on Ω in connection with $S_{\psi_j}(\bar{\pi}) = \underline{\partial}\psi_j(\bar{\pi}) + \overline{\partial}\psi_j(\bar{\pi})$. With the help of $\mathfrak{F}$-convexity, we can show that

$$\aleph_i^L(\pi) - \aleph_i^L(\bar{\pi}) \geqq \mathfrak{F}(\pi, \bar{\pi}, \bar{\varrho}_i^L), \ \forall \bar{\varrho}_i^L \in S_{\aleph_i^L}(\bar{\pi}), \ i = \{1, \ldots, k\}, \tag{24}$$

$$\aleph_i^U(\pi) - \aleph_i^U(\bar{\pi}) \geqq \mathfrak{F}(\pi, \bar{\pi}, \bar{\varrho}_i^U), \ \forall \bar{\varrho}_i^U \in S_{\aleph_i^U}(\bar{\pi}), \ i = \{1, \ldots, k\}, \tag{25}$$

$$\psi_j(\pi) - \psi_j(\bar{\pi}) \geqq \mathfrak{F}(\pi, \bar{\pi}; \bar{\vartheta}_j), \ \forall \bar{\vartheta}_j \in S_{\psi_j}(\bar{\pi}), \ j \in \mathfrak{J}(\bar{\pi}) \tag{26}$$

are satisfied for all points π belonging to Ω and, in particular, for $\pi = \tilde{\pi}$. Therefore, the inequalities (24), (25), and (26) yield

$$\aleph_i^L(\tilde{\pi}) - \aleph_i^L(\bar{\pi}) \geqq \mathfrak{F}(\tilde{\pi}, \bar{\pi}, \bar{\varrho}_i^L), \ \forall \bar{\varrho}_i^L \in S_{\aleph_i^L}(\bar{\pi}), \ i = \{1, \ldots, k\}, \tag{27}$$

$$\aleph_i^U(\tilde{\pi}) - \aleph_i^U(\bar{\pi}) \geqq \mathfrak{F}(\tilde{\pi}, \bar{\pi}, \bar{\varrho}_i^U), \ \forall \bar{\varrho}_i^U \in S_{\aleph_i^U}(\bar{\pi}), \ i = \{1, \ldots, k\}, \tag{28}$$

$$\psi_j(\tilde{\pi}) - \psi_j(\bar{\pi}) \geqq \mathfrak{F}(\tilde{\pi}, \bar{\pi}; \bar{\vartheta}_j), \ \forall \bar{\vartheta}_j \in S_{\psi_j}(\bar{\pi}), \ j \in \mathfrak{J}(\bar{\pi}). \tag{29}$$

On combining (23), (27), and (28), we get

$$\left.\begin{array}{c} \mathfrak{F}(\tilde{\pi}, \bar{\pi}; \bar{\varrho}_i^L) < 0, \ \forall \bar{\varrho}_i^L \in S_{\aleph_i^L}(\bar{\pi}), \\ \mathfrak{F}(\tilde{\pi}, \bar{\pi}; \bar{\varrho}_i^U) < 0, \ \forall \bar{\varrho}_i^U \in S_{\aleph_i^U}(\bar{\pi}). \\ \textbf{or} \\ \mathfrak{F}(\tilde{\pi}, \bar{\pi}; \bar{\varrho}_i^L) \leqq 0, \ \forall \bar{\varrho}_i^U \in S_{\aleph_i^U}(\bar{\pi}), \\ \mathfrak{F}(\tilde{\pi}, \bar{\pi}; \bar{\varrho}_i^U) < 0, \ \forall \bar{\varrho}_i^U \in S_{\aleph_i^U}(\bar{\pi}). \\ \textbf{or} \\ \mathfrak{F}(\tilde{\pi}, \bar{\pi}; \bar{\varrho}_i^L) < 0, \ \forall \bar{\varrho}_i^L \in S_{\aleph_i^L}(\bar{\pi}), \\ \mathfrak{F}(\tilde{\pi}, \bar{\pi}; \bar{\varrho}_i^U) \leqq 0, \ \forall \bar{\varrho}_i^U \in S_{\aleph_i^U}(\bar{\pi}). \end{array}\right\} \tag{30}$$

The above inequalities with the KKT condition (15) give

$$\sum_{i=1}^{k} \left[\bar{\mu}_i^L(\eth)\mathfrak{F}(\tilde{\pi}, \bar{\pi}; \bar{\varrho}_i^L) + \bar{\mu}_i^U(\eth)\mathfrak{F}(\tilde{\pi}, \bar{\pi}; \bar{\varrho}_i^U)\right] < 0, \tag{31}$$

for all $\bar{\varrho}_i^L \in \underline{\partial}\aleph_i^L(\bar{\pi}) + \varrho_i^L$, and $\bar{\varrho}_i^U \in \underline{\partial}\aleph_i^U(\bar{\pi}) + \varrho_i^U$.
Due to sublinearity of $\mathfrak{F}$, we get

$$\mathfrak{F}\left(\tilde{\pi}, \bar{\pi}; \sum_{i=1}^{k} \bar{\mu}_i^L(\eth)(\bar{\varrho}_i^L) + \sum_{i=1}^{k} \bar{\mu}_i^U(\eth)(\bar{\varrho}_i^U)\right) < 0, \tag{32}$$

for all $\bar{\varrho}_i^L \in \underline{\partial}\aleph_i^L(\bar{\pi}) + \varrho_i^L$, $\bar{\varrho}_i^U \in \underline{\partial}\aleph_i^U(\bar{\pi}) + \varrho_i^U$.
Since $\tilde{\pi}$ and $\bar{\pi} \in \Omega$ satisfy (14) and (15), therefore, we have

$$\bar{\delta}_j(\eth)\psi_j(\tilde{\pi}) \leqq \bar{\delta}_j(\eth)\psi_j(\bar{\pi}) = 0, \quad \forall j \in \mathfrak{J}(\bar{\pi}). \tag{33}$$

Using inequality (29) and condition (15), we get

$$\bar{\delta}_j(\eth)\psi_j(\tilde{\pi}) - \bar{\delta}_j(\eth)\psi_j(\bar{\pi}) \geqq \bar{\delta}_j(\eth)\mathfrak{F}(\tilde{\pi}, \bar{\pi}; \bar{\vartheta}_j), \quad \forall \bar{\vartheta}_j \in S_{\psi_j}(\bar{\pi}), \ \forall j \in \mathfrak{J}(\bar{\pi}). \tag{34}$$

In view of inequalities (33) and (34), we get

$$\bar{\delta}_j(\eth)\mathfrak{F}(\tilde{\pi},\bar{\pi};\bar{\vartheta}_j) \leqq 0, \quad \forall\bar{\vartheta}_j \in S_{\psi_j}(\bar{\pi}),\ \forall j \in \mathfrak{J}(\bar{\pi}). \tag{35}$$

By definition of $S_{\psi_j}(\bar{\pi}),\ \forall j \in \mathfrak{J}(\bar{\pi})$, and (35) yield

$$\sum_{j\in\mathfrak{J}(\bar{\pi})} \bar{\delta}_j(\eth)\mathfrak{F}\big(\tilde{\pi},\bar{\pi};\bar{\vartheta}_j\big) \leqq 0; \quad \forall\bar{\vartheta}_j \in \underline{\partial}\psi_j(\bar{\pi})+\vartheta_j, \tag{36}$$

which, due to sublinearity of $\mathfrak{F}$, produces

$$\mathfrak{F}\left(\tilde{\pi},\bar{\pi};\sum_{j\in\mathfrak{J}(\bar{\pi})} \bar{\delta}_j(\eth)\bar{\vartheta}_j\right) \leqq 0; \quad \forall\bar{\vartheta}_j \in \underline{\partial}\psi_j(\bar{\pi})+\vartheta_j. \tag{37}$$

With the help of (32) and (37), we obtain

$$\mathfrak{F}\left(\tilde{\pi},\bar{\pi};\sum_{i=1}^{k}\bar{\mu}_i^L(\eth)\bar{\varrho}_i^L+\sum_{i=1}^{k}\bar{\mu}_i^U(\eth)\bar{\varrho}_i^U+\sum_{j\in\mathfrak{J}(\bar{\pi})} \bar{\delta}_j(\eth)\bar{\vartheta}_j\right) < 0, \tag{38}$$

for each $\bar{\varrho}_i^L \in \underline{\partial}\aleph_i^L(\bar{\pi})+\varrho_i^L,\ \bar{\varrho}_i^U \in \underline{\partial}\aleph_i^U(\bar{\pi})+\varrho_i^U,\ (i=\{1,\ldots,k\})$, and $\bar{\vartheta}_j \in \underline{\partial}\psi_j(\bar{\pi})+\vartheta_j,\ j\in\mathfrak{J}(\bar{\pi})$. Eventually, it is clear from (38) that

$$0 \notin \sum_{i=1}^{k}\Big[\bar{\mu}_i^L(\eth)(\underline{\partial}\aleph_i^L(\bar{\pi})+\varrho_i^L)+\bar{\mu}_i^U(\eth)(\underline{\partial}\aleph_i^U(\bar{\pi})+\varrho_i^U)\Big]+\sum_{j\in\mathfrak{J}(\bar{\pi})}^{m}\bar{\delta}_j(\eth)(\underline{\partial}\psi_j(\bar{\pi})+\vartheta_j),$$

which opposes the KKT condition (13). Hence, the proof is complete. □

Theorem 4.2. (Sufficiency optimality criteria for LU-Pareto solution) A feasible point $\bar{\pi}$ becomes an LU-Pareto solution to (IVOP) if it satisfies the following sufficiency optimality conditions:

(i) $\bar{\pi}$ satisfies the KKT-type necessary conditions of optimality given by (13)-(15) with the quasidifferentials $D_{\aleph_i^L}(\bar{\pi})=[\underline{\partial}\aleph_i^L(\bar{\pi}),\overline{\partial}\aleph_i^L(\bar{\pi})]$, $D_{\aleph_i^U}(\bar{\pi})=[\underline{\partial}\aleph_i^U(\bar{\pi}),\overline{\partial}\aleph_i^U(\bar{\pi})]$; $i=\{1,\ldots,k\}$ and $D_{\psi_j}(\bar{\pi})=[\underline{\partial}\psi_j(\bar{\pi}),\overline{\partial}\psi_j(\bar{\pi})]$ $(j\in\mathfrak{J})$, respectively;

(ii) functions $\aleph_i^L,\aleph_i^U$ $(i=\{1,\ldots,k\})$, are strictly $\mathfrak{F}$-convex quasidifferentiable at a point $\bar{\pi}$ on Ω in connection with $S_{\aleph_i^L}(\bar{\pi})=\underline{\partial}\aleph_i^L(\bar{\pi})+\overline{\partial}\aleph_i^L(\bar{\pi})$ and $S_{\aleph_i^U}(\bar{\pi})=\underline{\partial}\aleph_i^U(\bar{\pi})+\overline{\partial}\aleph_i^U(\bar{\pi})$ respectively;

(iii) ψ_j $(j\in\mathfrak{J}(\bar{\pi}))$ are $\mathfrak{F}$-convex quasidifferentiable functions at a point $\bar{\pi}$ on Ω in connection with $S_{\psi_j}(\bar{\pi})=\underline{\partial}\psi_j(\bar{\pi})+\overline{\partial}\psi_j(\bar{\pi})$.

Proof. The proof of the present theorem runs on the lines parallel to the proof of Theorem 4.1. Hence, the proof is omitted. □

Example 4.3. Consider the following nonsmooth interval-valued multiobjective programming problem:

$$
\begin{aligned}
(\mathrm{IVOP}_1)\quad \text{minimize}\quad \aleph(\pi) &= \Big([\aleph_1^L(\pi), \aleph_1^U(\pi)], [\aleph_2^L(\pi), \aleph_2^U(\pi)]\Big)\\
&= \Big([\pi_1^2+\pi_2^2+|\pi_1|-|\pi_2|,\ 5\pi_2^4+3\pi_1^2+|\pi_2+|\pi_1||],\\
&\qquad [\pi_1^4+|\pi_1|+|\pi_2|-\pi_1-\pi_2,\ 4\pi_1^2+\pi_2^2+|\pi_1|+2\pi_2+5]\Big),\\
\text{subject to } \psi_1(\pi) &= \pi_1^4+\pi_2^4+|\pi_1+\pi_2|+2\pi_1 \leqq 0,\ \pi\in\Re^2.
\end{aligned}
$$

Here, the set $\Omega = \{\pi = (\pi_1, \pi_2) \in \Re^2 : \pi_1^4 + \pi_2^4 + |\pi_1 + \pi_2| + 2\pi_1 \leqq 0\}$ and $\bar{\pi} = (0,0)$ represents the feasible solution to the interval-valued problem (IVOP_1). Moreover, it can be justified that the functions $\aleph_1^L(\pi)$, $\aleph_1^U(\pi)$, $\aleph_2^L(\pi)$, $\aleph_2^U(\pi)$ and $\psi_1(\pi)$ are quasidifferentiable at a point $\bar{\pi}$. By the definition of the directional derivative, we obtain ${\aleph_1^L}'((0,0);d) = |d_1| - |d_2|$, ${\aleph_1^U}'((0,0);d) = |d_2 + |d_1||$, ${\aleph_2^L}'((0,0);d) = |d_1| + |d_2| - d_1 - d_2$ and ${\aleph_2^U}'((0,0);d) = |d_1| + 2d_2$. Hence,

$$
{\aleph_1^L}'((0,0);d) = \max_{\lambda_1^L\in \mathrm{conv}\{(1,0),(-1,0)\}} (\lambda_1^L)^T d + \min_{\varrho_1^L\in \mathrm{conv}\{(0,1),(0,-1)\}} (\varrho_1^L)^T d,
$$

where,

$$
\underline{\partial}\aleph_1^L(0,0) = \mathrm{conv}\{(1,0),(-1,0)\},\ \overline{\partial}\aleph_1^L(0,0) = \mathrm{conv}\{(0,1),(0,-1)\}
$$

and

$$
{\aleph_1^U}'((0,0);d) = \max_{\lambda_1^U\in \mathrm{conv}\{(0,0),(-2,2),(2,2)\}} (\lambda_1^U)^T d + \min_{\varrho_1^U\in \{(-1,-1),(1,-1)\}} (\varrho_2^L)^T d.
$$

It is clear that $\underline{\partial}\aleph_1^U(0,0) = \mathrm{conv}\{(0,0),(-2,2),(2,2)\}$, $\overline{\partial}\aleph_1^U(0,0) = \{(-1,-1),\ (1,-1)\}$. Moreover

$$
{\aleph_2^L}'((0,0);d) = \max_{\lambda_2^L\in \mathrm{conv}\{(1,1),(-1,-1)\}} (\lambda_2^L)^T d + \min_{\varrho_2^L\in \{(-1,-1)\}} (\varrho_2^L)^T d,
$$

where $\underline{\partial}\aleph_2^L(0,0) = \mathrm{conv}\{(1,1),(-1,-1)\}$, $\overline{\partial}\aleph_2^L(0,0) = \{(-1,-1)\}$
and

$$
{\aleph_2^U}'((0,0);d) = \max_{\lambda_2^U\in \mathrm{conv}\{(1,0),(-1,0)\}} (\lambda_2^L)^T d + \min_{\varrho_2^U\in \mathrm{conv}\{(0,2)\}} (\varrho_2^U)^T d,
$$

where,

$$
\underline{\partial}\aleph_2^U(0,0) = \mathrm{conv}\{(1,0),(-1,0)\},\ \overline{\partial}\aleph_2^U(0,0) = \mathrm{conv}\{(0,2)\}.
$$

Therefore, by Definition 2.2, we can conclude that the functions $\aleph_1^L$, $\aleph_1^U$, $\aleph_2^L$, and $\aleph_2^U$ are quasidifferentiable at a point $\bar{\pi} = (0,0)$. Similarly, we have $\psi_1'((0,0),d) = |d_1 + d_2| + 2d_1$ and hence

$$
\psi_1'((0,0);d) = \max_{\omega_1\in \mathrm{conv}\{(1,1),(-1,-1)\}} \omega_1^T d + \min_{\vartheta_1\in\{(2,0)\}} {\vartheta_1}^T d,
$$

where, $\underline{\partial}\psi_1(0,0) = \mathrm{conv}\{(1,1),(-1,-1)\}$, $\overline{\partial}\psi_1(0,0) = \{(2,0)\}$.

Now, we will show that the necessary optimality criteria of KKT-type are satisfied at a point $\bar{\pi}$ in which the Lagrange multipliers are not constant. Evidently, it can be proved that, for any sets of $\varrho_1^L \in \overline{\partial}\aleph_1^L(\bar{\pi})$, $\varrho_1^U \in \overline{\partial}\aleph_1^U(\bar{\pi})$, $\varrho_2^L \in \overline{\partial}\aleph_2^L(\bar{\pi})$, $\varrho_2^U \in \overline{\partial}\aleph_2^U(\bar{\pi})$, and $\vartheta_1 \in \overline{\partial}\psi_1(\bar{\pi})$, there exist Lagrange multipliers $\bar{\mu}_i^L(\eth) \geq 0$, $\bar{\mu}_i^U(\eth) \geq 0$, $i = \{1,2\}$, and $\bar{\delta}_1(\eth) \geqq 0$ satisfying the KKT-type necessary optimality criteria. Let us consider the following example for the particular choice of $\eth = (\varrho_1^L, \varrho_1^U, \varrho_2^L, \varrho_2^U, \vartheta_1)$:

(a) if $\varrho_1^L = (0,1)$, $\varrho_1^U = (1,-1)$, $\varrho_2^L = (-1,-1)$, $\varrho_2^U = (0,2)$, $\vartheta_1 = (2,0)$, then we substitute $\bar{\mu}_1^L = 1$, $\bar{\mu}_1^U = 1$, $\bar{\mu}_2^L = 1$, $\bar{\mu}_2^U = 1$, and $\bar{\delta}_1 = 1$;

(b) if $\varrho_1^L = (0,-1)$, $\varrho_1^U = (1,-1)$, $\varrho_2^L = (-1,-1)$, $\varrho_2^U = (0,2)$, $\vartheta_1 = (2,0)$, then we substitute $\bar{\mu}_1^L = 1$, $\bar{\mu}_1^U = 1$, $\bar{\mu}_2^L = 1$, $\bar{\mu}_2^U = 1$, and $\bar{\delta}_1 = 1$.

We observe that necessary KKT-type optimality criteria are satisfied for both cases (a) and (b) for the particular selected Lagrange multipliers. Furthermore, if we consider different Lagrange multipliers, then the KKT-type necessary conditions may or may not be satisfied. Let the right side of the KKT-type necessary criteria of (13) be denoted by $W_\eth$. Therefore, we have

$$W_\eth = \bar{\mu}_1^L(\eth)(\underline{\partial}\aleph_1^L(\bar{\pi}) + \varrho_1^L) + \bar{\mu}_1^U(\eth)(\underline{\partial}\aleph_1^U(\bar{\pi}) + \varrho_1^U) + \bar{\mu}_2^L(\eth)(\underline{\partial}\aleph_2^L(\bar{\pi}) + \varrho_2^L)$$
$$+\bar{\mu}_2^U(\eth)(\underline{\partial}\aleph_2^U(\bar{\pi}) + \varrho_2^U) + \bar{\delta}_1(\eth)(\underline{\partial}\psi_1(\bar{\pi}) + \vartheta_1),$$

for the particular choice of $\eth = (\varrho_1^L, \varrho_1^U, \varrho_2^L, \varrho_2^U, \vartheta_1)$ and it depends on the Lagrange multipliers $\bar{\mu}_1^L$, $\bar{\mu}_1^U$, $\bar{\mu}_2^L$, $\bar{\mu}_2^U$ and $\bar{\delta}_1$.

(1) for $\eth' = (\varrho_1^L, \varrho_1^U, \varrho_2^L, \varrho_2^U, \vartheta_1) = ((0,1),(1,-1),(-1,-1),(0,2),(2,0))$ and $\bar{\mu}_1^L = 1$, $\bar{\mu}_1^U = 1$, $\bar{\mu}_2^L = 1$, $\bar{\mu}_2^U = 1$, $\bar{\delta}_1 = 1$
then $W_{\eth'} = \text{conv}\{(-4,1),(-2,-1),(-2,1),(-2,3),(0,-1),(0,1),(0,3),(0,5),$
$(2,-1),(2,1),(2,3),(2,5),(4,1),(4,3),(4,5),(6,3),(6,5),(8,5)\}$

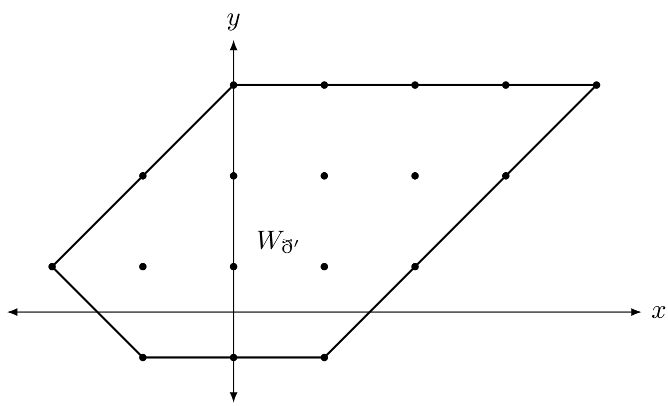

Fig. 1. Here $0 \in W_{\eth'}$, that is, the KKT-type necessary optimality criteria are satisfied.

(2) for $\eth'' = (\varrho_1^L, \varrho_1^U, \varrho_2^L, \varrho_2^U, \vartheta_1) = ((0,-1),(1,-1),(-1,-1),(0,2),(2,0))$ and $\bar{\mu}_1^L = 1$, $\bar{\mu}_1^U = 1$, $\bar{\mu}_2^L = 1$, $\bar{\mu}_2^U = 1$, $\bar{\delta}_1 = 1$
then $W_{\eth''} = \text{conv}\{(-4,-1),(-2,-3),(-2,-1),(-2,1),(0,-3),(0,-1),(0,1),$
$(0,3),(2,-3),(2,-1),(2,1),(2,3),(4,-1),(4,1),(4,3),(6,1),(6,3),(8,3)\}$

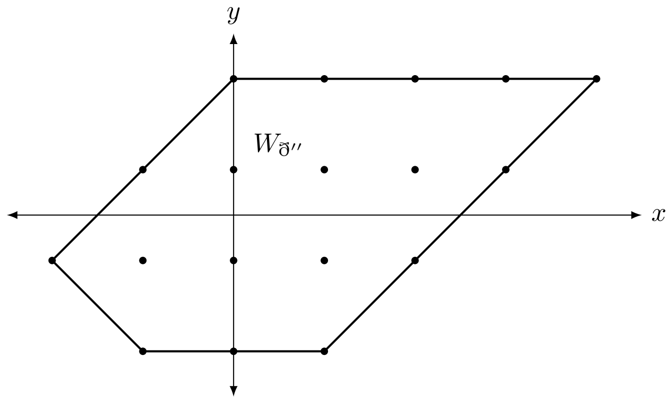

Fig. 2. Here $0 \in W_{\eth''}$, that is, the KKT-type necessary optimality criteria are satisfied.

(3) for $\eth' = (\varrho_1^L, \varrho_1^U, \varrho_2^L, \varrho_2^U, \vartheta_1) = ((0,1),(1,-1),(-1,-1),(0,2),(2,0))$ and $\bar{\mu}_1^L = 2$, $\bar{\mu}_1^U = 1$, $\bar{\mu}_2^L = 1$, $\bar{\mu}_2^U = 2$, $\bar{\delta}_1 = 1$
then $W_{\eth'} = \text{conv}\{(-6,4),(-4,6),(-4,2),(-2,8),(-2,4),(0,2),(0,6),(2,4),$
$(2,8),(4,2),(4,6),(6,4),(6,8),(8,6),(10,8)\}$

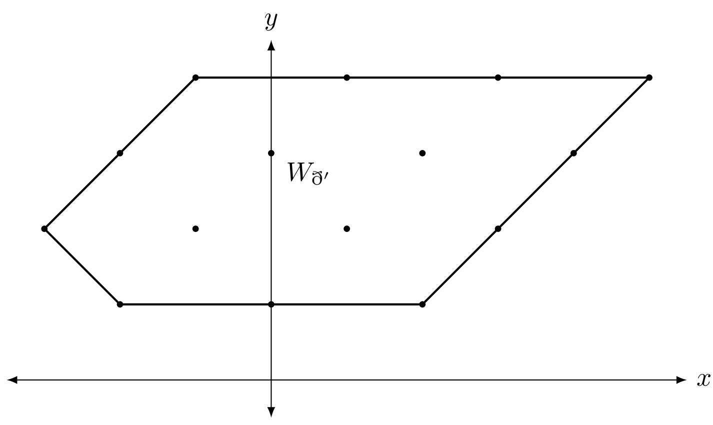

Fig. 3. Here $0 \notin W_{\eth'}$, that is, the KKT-type necessary optimality criteria are not satisfied.

Therefore, we can conclude that the Lagrange multipliers depend on the particular choice of $\eth$. Furthermore, in order to prove that KKT-type sufficiency criteria are applicable to the considered interval-valued problem (IVOP_1), it is sufficient to show that the functions $\aleph_1^L$, $\aleph_1^U$, $\aleph_2^L$ and $\aleph_2^U$ are $\mathfrak{F}$-convex quasidifferentiable at a point $\bar{\pi}$ on Ω in connection with η as well as in connection with convex compact set $S_{\aleph_1^L}(\bar{\pi}) = \underline{\partial}\aleph_1^L(\bar{\pi}) + \overline{\partial}\aleph_1^L(\bar{\pi})$, $S_{\aleph_1^U}(\bar{\pi}) = \underline{\partial}\aleph_1^U(\bar{\pi}) + \overline{\partial}\aleph_1^U(\bar{\pi})$, $S_{\aleph_2^L}(\bar{\pi}) = \underline{\partial}\aleph_2^L(\bar{\pi}) + \overline{\partial}\aleph_2^L(\bar{\pi})$ and $S_{\aleph_2^U}(\bar{\pi}) = \underline{\partial}\aleph_2^U(\bar{\pi}) + \overline{\partial}\aleph_2^U(\bar{\pi})$ respectively. Moreover, the function ψ_1 is $\mathfrak{F}$-convex quasidifferentiable at a point $\bar{\pi}$ on Ω in connection with η as well as in connection with convex compact set $S_{\psi_1}(\bar{\pi}) = \underline{\partial}\psi_1(\bar{\pi}) + \overline{\partial}\psi_1(\bar{\pi})$. Let us define $\mathfrak{F}$ as $\mathfrak{F}(\pi, \bar{\pi}; \rho) = (\rho_1+\rho_2)[(|\pi_1|+|\pi_2|)-(|\bar{\pi}_1|+|\bar{\pi}_2|)]$. Then, using the definition of $\mathfrak{F}$-convexity, we can conclude that the functions $\aleph_1^L$, $\aleph_1^U$, $\aleph_2^L$ and $\aleph_2^U$ are $\mathfrak{F}$-convex quasidifferentiable at a point $\bar{\pi}$ on Ω in connection with η as well as in connection with convex compact set $S_{\aleph_1^L}(\bar{\pi}) = \underline{\partial}\aleph_1^L(\bar{\pi}) + \overline{\partial}\aleph_1^L(\bar{\pi})$, $S_{\aleph_1^U}(\bar{\pi}) = \underline{\partial}\aleph_1^U(\bar{\pi}) + \overline{\partial}\aleph_1^U(\bar{\pi})$, $S_{\aleph_2^L}(\bar{\pi}) = \underline{\partial}\aleph_2^L(\bar{\pi}) + \overline{\partial}\aleph_2^L(\bar{\pi})$ and $S_{\aleph_2^U}(\bar{\pi}) = \underline{\partial}\aleph_2^U(\bar{\pi}) + \overline{\partial}\aleph_2^U(\bar{\pi})$ respectively. Furthermore, the function ψ_1 is $\mathfrak{F}$-convex quasidifferentiable at a point $\bar{\pi}$ on Ω in connection with η as well as in connection with the convex compact set $S_{\psi_1}(\bar{\pi}) = \underline{\partial}\psi_1(\bar{\pi}) + \overline{\partial}\psi_1(\bar{\pi})$.

As all the assumptions of Theorem 4.1 are satisfied at a point $\bar{\pi}$, therefore, we can conclude that the feasible point $\bar{\pi}$ becomes a weak LU-Pareto solution to the problem (IVOP_1) under the quasidifferentiable $\mathfrak{F}$-convex functions in connection with compact convex sets, which are equivalent to the Minkowski sum of their subdifferentials and superdifferentials at a point $\bar{\pi}$.

5. CONCLUSIONS

The present article envisaged the class of nonsmooth interval-valued programming problems. We have framed Fritz John and KKT-type necessary optimality criteria. Further, sufficiency results for feasible solutions are accustomed for nonsmooth multiobjective interval-valued programming problems, assuming functions to be $\mathfrak{F}$-convex quasidifferentiable in connection with a suitably defined set that is compact as well as convex. Finally, an example of a nonsmooth interval-valued programming problem was constructed consisting of $\mathfrak{F}$-convex quasidifferential function in connection with convex compact sets and extracted sufficient optimality conditions.

(Received October 26, 2024)

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