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A NOVEL KERNEL FUNCTION BRIDGING ITERATION BOUNDS IN INTERIOR-POINT ALGORITHMS FOR LINEAR PROGRAMMING

Imene Touil and Sajad Fathi-Hafshejani

Kernel functions play an important role in designing and analyzing interior-point methods. They are not only used for determining search directions but also for measuring the distance between the given iterate and the μ-center in the algorithms. Currently, interior-point methods based on kernel functions are among the most effective methods for solving different types of optimization problems and are very active research area in mathematical programming. Therefore, in this work, we introduce a novel kernel function that bridges the gap between the iteration bounds for large-update and small-update methods. To the best of our knowledge, we are the first to achieve these results.

Keywords: linear programming, primal-dual interior point methods, kernel functions, complexity analysis, large- and small-update methods

Classification: 90C05, 90C51

1. INTRODUCTION

Consider the following primal linear programming (LP) problem in standard form:

$$(P) \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \end{array}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $c, x \in \mathbb{R}^n$.

Its dual problem is written as follows:

$$(D) \quad \begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y + s = c, \\ & s \geq 0, \end{array}$$

where $s \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.

The sets of strictly feasible solutions of (P) and (D) are:

$$\mathcal{F}^0(P) = \{x \in \mathbb{R}^n_{++} : Ax = b\},$$
$$\mathcal{F}^0(D) = \{(y, s) \in \mathbb{R}^m \times \mathbb{R}^n_{++} : A^T y + s = c\},$$

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respectively.

Without any loss of generality, we assume throughout this paper that:

- (H_1) The matrix A satisfies $\text{rank}(A) = m \leq n$.
- (H_2) $\mathcal{F}^0(P) \times \mathcal{F}^0(D) \neq \emptyset$ (called the Interior Point Condition (IPC)).

Over the years, various methods have been developed for solving LP problems. Among these, Interior Point Methods (IPMs) have proven to be particularly effective for large-scale optimization problems. Initially introduced by N. Karmarkar in 1984 [13], IPMs represent a class of polynomial-time algorithms that have become fundamental for solving not only LPs [4, 10, 11, 13, 15, 17, 21] but also convex quadratic programming (CQP) [1, 6, 7, 12] and semidefinite programming (SDP) [2, 8, 15, 18, 19, 20, 23]. Their theoretical foundations, combined with practical innovations and wide applicability, make them indispensable in fields ranging from operations research and finance to engineering and machine learning.

Kernel functions (KFs) have played a crucial role in the development of IPMs for optimization problems. N. Karmarkar [13] introduced the polynomial IPM for LP in 1984, laying a foundational step despite not explicitly utilizing KFs. In 1994, R. E. Nesterov and A.Ṡ. Nemirovskii [14] introduced self-concordant barriers, which essentially function as KFs for convex optimization problems. This framework unified existing approaches and established a robust theoretical foundation. Y.Q. Bai and C. Roos [3] subsequently made significant contributions to the understanding of KFs, particularly in long- and short-step methods, and explored the impact of various barrier functions on the efficiency of interior-point algorithms (IPAs).

In the past two decades, many new KFs have been developed to improve the complexity analysis and efficiency of IPMs for LP problems. KFs were first introduced by Peng et al. [15], where it was shown that some of them almost closed the gap between the iteration bounds for large-update and small-update methods. In 2004, Bai et al. [4] introduced the first new KF with a trigonometric barrier term. The IPM with this function was further analyzed by El Ghami et al. [9] in 2012. They established the iteration complexity as $\mathcal{O}(n^{\frac{3}{4}} \log \frac{n}{\epsilon})$ for large-update methods. A year later, Wang et al. [23] introduced the first new KF with a simple algebraic expression. They proved that the IPM based on this KF had iteration bounds of $\mathcal{O}(qn \log \frac{n}{\epsilon})$ and $\mathcal{O}(q^2 \sqrt{n} \log \frac{n}{\epsilon})$, $q > 1$ for large-update and small-update IPMs, respectively. Since then, several KFs using similar barrier terms have been proposed and analyzed. In 2020, Touil et al. [20, 22] introduced the first KF with a hyperbolic barrier term, establishing iteration complexities of $\mathcal{O}(n^{\frac{2}{3}} \log \frac{n}{\epsilon})$ and $\mathcal{O}(n^{\frac{3}{4}} \log \frac{n}{\epsilon})$ for large-update methods. Subsequently, researchers have sought to introduce KFs that could yield improved iteration bounds for small-update methods, as large-update methods are generally more efficient in practice. However, none of the KFs introduced to date have surpassed the iteration bounds established by [5, 16]. At best, their iteration bounds are comparable to those reported in these earlier works, indicating that the pursuit of an optimal KF remains unresolved.

1.1. Contribution

This paper presents the following contributions:

- We propose a novel IPA for solving LP problems, utilizing a new logarithmic KF.
- We demonstrate that the IPM derived from our algorithm achieves an iteration complexity of $O(\sqrt{n}\log\frac{n}{\epsilon})$ for the large-update method, thereby improving upon the previously established result of $O(\sqrt{n}\log n\log\frac{n}{\epsilon})$. The obtained result is on par with small-update methods, which were previously considered more efficient in practice.
- The new KF represents an advancement in KF theory by bridging the gap between large- and small-update methods. This innovation addresses a long-standing issue in IPM research.

The paper is organized as follows: Section 2 reviews fundamental concepts related to IPMs, presents the new KF, and establishes its properties. Section 3 analyzes the complexity of the algorithm. Finally, Section 4 concludes with remarks on the findings.

In this paper, we adopt the following notational conventions. The Euclidean norm of a vector is denoted by $\|\cdot\|$. The non-negative and positive orthants are represented by $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$, respectively. The expressions xs and $\frac{x}{s}$ denote component-wise operations on the vectors x and s, resulting in components x_is_i and $\frac{x_i}{s_i}$, respectively. Additionally, we use diagonal matrices X and S, whose diagonal entries are the components of vectors x and s, respectively. The function $\lceil . \rceil$ represents the ceiling function, which rounds any number up to the next nearest integer. We say that $f(t)=\Theta\left(g(t)\right)$, if there exist positive constants ω_1 and ω_2 such that $\omega_1 g(t)\le f(t)\le \omega_2 g(t)$ for all $t\in\mathbb{R}_{++}$. Similarly, we say that $f(t)=O\left(g(t)\right)$, if there exists a positive constant ω such that $f(t)\le \omega g(t)$ for all $t\in\mathbb{R}_{++}$.

2. SEARCH DIRECTION AND NEW KERNEL FUNCTION

For the current iterate $(x,y,s)\in\mathcal{F}^0(P)\times\mathcal{F}^0(D)$, finding the optimal solutions of both problems (P) and its dual (D) is equivalent to solving the following system:

$$\begin{cases} Ax=b,\ x\ge 0,\\ A^Ty+s=c,\ s\ge 0,\\ XS\mathbf{e}=0, \end{cases}$$

where $\mathbf{e}=(1,1,\ldots,1)^T$. In IPMs, the last equation of this system is generally replaced by the parameterized equation $XS\mathbf{e}=\mu\mathbf{e}$, where $\mu>0$. This leads to the following perturbed system:

$$\begin{cases} Ax=b,\ x>0,\\ A^Ty+s=c,\ s>0,\\ XS\mathbf{e}=\mu\mathbf{e},\ \mu>0, \end{cases}$$

where μ is a positive parameter that decreases over iterations, driving the solution toward optimality.

It is well known that under the IPC assumption, the perturbed system has a unique solution, denoted (x_μ,y_μ,s_μ), for each $\mu>0$. Note that x_μ is called the μ-center of (P), and (y_μ,s_μ) is called the μ-center of (D). The set $\{(x_\mu,y_\mu,s_\mu);\ \mu>0\}$ is called the central path of (P) and (D) (or the set of μ-centers). As μ approaches 0, the limit of the central path exists and yields optimal solutions for both problems (P) and (D) [4].

By applying the Newton method, we linearize the perturbed system as follows:

$$\begin{cases} A\Delta x = 0, \\ A^T\Delta y + \Delta s = 0, \\ S\Delta x + X\Delta s = \mu\mathbf{e} - XS\mathbf{e}, \end{cases} \tag{1}$$

where $(\Delta x, \Delta y, \Delta s)$ represents the unique search direction obtained by solving this system.

To follow the central path more efficiently, for any strictly feasible primal and dual (x, s), where $x > 0$ and $s > 0$, and for any positive parameter μ, we define the scaled vectors v and v^{-1} as:

$$v := \sqrt{\frac{xs}{\mu}}, \qquad v^{-1} = \sqrt{\frac{\mu}{xs}},$$

where $\sqrt{\frac{xs}{\mu}}$ denotes the component-wise square root of the vector $\frac{xs}{\mu}$. The new search directions d_x and d_s are then defined as follows:

$$d_x = \frac{v\Delta x}{x}, \qquad d_s = \frac{v\Delta s}{s}. \tag{2}$$

With this, system (1) is reduced to

$$\begin{cases} \overline{A}d_x = 0, \\ \overline{A}^T\Delta y + d_s = 0, \\ d_x + d_s = v^{-1} - v = -\nabla\Psi_c(v), \end{cases} \tag{3}$$

where $\overline{A} = \frac{1}{\mu}AV^{-1}X, \;\; V = \mathrm{diag}(v)$.

Here $\Psi_c(v) = \sum_{i=1}^n \psi_c(v_i)$, is the logarithmic barrier (or proximity) function of the classical KF $\psi_c(t)$, where

$$\psi_c(t) = \left(\frac{t^2-1}{2} - \log t\right).$$

The basic idea in IPMs for approximating the central path is to replace $\psi_c(t)$ by any strictly convex function $\psi(t) : \,]0, +\infty[\to [0, +\infty[$ which is minimal at $t = 1$ with $\psi(1) = 0$.

By replacing the proximity function $\Psi_c(v)$ with a new proximity function $\Psi(v) = \sum_{i=1}^n \psi(v_i)$, where $\psi(t)$ is our new KF defined in (6), we obtain:

$$\begin{cases} \overline{A}d_x = 0, \\ \overline{A}^T\Delta y + d_s = 0, \\ d_x + d_s = -\nabla\Psi(v). \end{cases} \tag{4}$$

Thanks to (H_1), system (4) possesses a unique solution denoted as $(d_x, \Delta y, d_s)$. Having d_x and d_s, the directions Δx and Δs can be readily computed according to (2). By taking a suitable step size, denoted as α, along the search direction, the new iterates x_+, y_+ and s_+ are updated as:

$$x_+ := x + \alpha \Delta x, \quad y_+ := y + \alpha \Delta y \text{ and } s_+ := s + \alpha \Delta s, \tag{5}$$

where $\alpha \in (0,1)$ is chosen to be the largest possible value such that $x_+ > 0$ and $s_+ > 0$. The above generic method can be outlined in the following primal-dual IPA:

Algorithm 1 : Primal-dual algorithm for LP

Input
A proximity function $\Psi(v)$;
a threshold parameter $\tau \geq 1$;
an accuracy parameter $\varepsilon > 0$;
a barrier update parameter θ, $0 < \theta < 1$;
$(x^0, y^0, s^0) \in \mathcal{F}^0(P) \times \mathcal{F}^0(D)$, $\mu^0 = 1$ and $v^0 = \sqrt{\frac{x^0 s^0}{\mu^0}}$ such that $\Phi\left(x^0, s^0; \mu^0\right) := \Psi(v^0) \leq \tau$;
begin
$x := x^0; y := y^0; s := s^0; \mu := \mu^0;\ v := v^0;$
while $n\mu \geq \epsilon$ **do**
  begin (outer iteration)
  $\mu: = (1-\theta)\mu;$
  while $\Psi(v) > \tau$ **do**
    begin (inner iteration)
    Solve system (4) and use (2) to obtain $(\Delta x, \Delta y, \Delta s)$;
    $x := x + \alpha \Delta x;$
    $s := s + \alpha \Delta s;$
    $y := y + \alpha \Delta y;$
    $v := \sqrt{\frac{xs}{\mu}};$
  end while (inner iteration)
end while (outer iteration)

2.1. New kernel function and its properties

The new KF is defined as follows:

$$\psi(t) = \frac{t^2 - 1}{2} + 2\log\left(\frac{1+t}{2t}\right), \forall t > 0. \tag{6}$$

Subsequently, for all $t > 0$ the first three derivatives of $\psi(t)$ are:

$$\psi'(t) = t - \frac{2}{t^2 + t}, \quad \forall t > 0, \tag{7}$$

$$\psi''(t) = 1 + \frac{4t + 2}{(t^2 + t)^2} > 1, \quad \forall t > 0, \tag{8}$$

$$\psi'''(t) \quad = \quad -4\frac{3t^2+3t+1}{(t^2+t)^3} < 0, \quad \forall t > 0. \tag{9}$$

Lemma 2.1. Let $\psi(t)$ be the KF defined in (6), then we have

(i) $t\psi''(t) + \psi'(t) > 0,\ \forall t > 0$, (called as the *e*-convex property of $\psi(t)$), i. e.,

$$\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}\left(\psi(t_1) + \psi(t_2)\right),\ \forall t_1, t_2 > 0.$$

(ii) $t\psi''(t) - \psi'(t) > 0,\ \forall t > 0.$

(iii) $\psi''(t)$ is monotonically decreasing on $]0, +\infty[$.

(iv) $\psi''(t)\,\psi'(\beta t) - \beta\psi'(t)\,\psi''(\beta t) > 0, \forall t > 1, \beta > 1.$

Proof. For (i) and (ii), using (7) and (8), we have

$$t\psi''(t) + \psi'(t) = 2t + \frac{2}{t^2+t}\left(1 - \frac{1}{t+1}\right) > 0,\ \forall t > 0.$$

$$t\psi''(t) - \psi'(t) = \frac{4t^2+t}{t(t^2+t)^2} + \frac{2}{t^2+t} > 0,\ \forall t > 0.$$

To show that $\psi''(t)$ is monotonically decreasing on the interval $]0, +\infty[$, we need to demonstrate that its derivative, $\psi'''(t)$, is negative for all $t > 0$.

It is clear that:

$$\psi'''(t) = -4\frac{3t^2+3t+1}{(t^2+t)^3} < 0, \qquad \forall\ t > 0.$$

Thus, it shows that $\psi''(t)$ is monotonically decreasing for all $t > 0$.

For the last item, and based on Lemma 2.4 in [4], if $\psi(t)$ satisfies (ii) and (iii) of this Lemma, then $\psi(t)$ satisfies:

$$\psi''(t)\,\psi'(\beta t) - \beta\psi'(t)\,\psi''(\beta t) > 0, \quad \forall t > 1, \quad \beta > 1.$$

For $t > 1$, let us define the function $f(\beta)$ as follows:

$$f(\beta) := \psi''(t)\psi'(\beta t) - \beta\psi'(t)\psi''(\beta t), \qquad \beta \geq 1.$$

Given that $f(1) = 0$, we can establish (iv) by showing that $f'(\beta) > 0$ for $\beta > 1$. This is demonstrated as follows:

$$f'(\beta) = t\psi''(t)\psi''(\beta t) - \psi'(t)\psi''(\beta t) - \beta t\psi'(t)\psi'''(\beta t).$$

This simplifies to:

$$f'(\beta) = \psi''(\beta t)[t\psi''(t) - \psi'(t)] - \beta t\psi'(t)\psi'''(\beta t) > 0.$$

The inequality holds because $\psi''(\beta t) \geq 0$ (from (8)), $t\psi''(t) - \psi'(t) \geq 0$ (as stated in item (ii) of this Lemma), and $-\beta t\psi'(t)\psi'''(\beta t) > 0$. Given that $t > 1$, it follows that $\psi'(t) > 0$ (from (7)) and $\psi'''(\beta t) < 0$ (as shown in item (iii) of this Lemma).

Hence, the proof is completed. □

Lemma 2.2. For $\psi(t)$, we have

(i) $(t-1)^2 \leq \psi(t) \leq \frac{1}{2}(\psi'(t))^2, \quad \forall t > 0.$

(ii) $\psi(t) \leq \frac{5}{4}(t-1)^2, \ \forall t \geq 1.$

Proof. For (i), since $\psi''(t) > 1$ from (8), we have

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)\,\mathrm{d}\zeta\mathrm{d}\xi \geq \int_1^t \int_1^\xi \mathrm{d}\zeta\mathrm{d}\xi = (t-1)^2,$$

and

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)\,\mathrm{d}\zeta\mathrm{d}\xi \leq \int_1^t \int_1^\xi \psi''(\xi)\psi''(\zeta)\,\mathrm{d}\zeta\mathrm{d}\xi = \frac{1}{2}\left(\psi'(t)\right)^2.$$

To prove (ii), we apply the Taylor theorem, we get:

$$\psi(t) = \psi(1) + \psi'(1)(t-1) + \frac{1}{2}\psi''(1)(t-1)^2 + \frac{1}{3!}\psi'''(\zeta)(\zeta-1)^3,$$

where $1 < \zeta < t$. Since $\psi(1) = \psi'(1) = 0$, $\psi''(1) = \frac{5}{2}$ and $\psi'''(\zeta) < 0$, we can conclude that:

$$\psi(t) \leq \frac{5}{4}(t-1)^2.$$

This completes the proof. □

Lemma 2.3. Let $\varrho : [0,+\infty) \longrightarrow [1,+\infty)$ be the inverse function of $\psi(t)$ for $t \geq 1$, and let $\rho : [0,+\infty) \longrightarrow (0,1]$ be the inverse function of $-\frac{1}{2}\psi'(t)$ for $0 < t \leq 1$. Then, we have:

(i) $1 + 2\sqrt{\frac{z}{5}} \leq \varrho(z) \leq 1 + \sqrt{z}, \quad \forall z \in [0,+\infty).$

(ii) $\rho(z) \geq \sqrt{\frac{2}{\lambda(2z+1)}}$, where $z = -\frac{1}{2}\psi'(t) \geq 0$ and λ is an upper bound for $\left(1 + \frac{1}{t}\right)$ over $t \in (0,1]$.

Proof. For (i), let $z = \psi(t) \geq 0$ for all $t \geq 1$. Thus, we have $\varrho(z) = t$. By applying parts (i) and (ii) of Lemma 2.2, we can derive the following:

$$z = \psi(t) \geq (t-1)^2,$$

which implies that

$$t \leq 1 + \sqrt{z}.$$

This establishes the right side of (i).

Next, from the inequality

$$z \leq \frac{5}{4}(t-1)^2,$$

we deduce that

$$t \geq 1 + 2\sqrt{\frac{z}{5}},$$

thereby confirming the left side of (i).

For (ii), consider $z \geq 0$ and $t \in (0, 1]$ such that $z = -\frac{1}{2}\psi'(t)$. Hence, we have $\rho(z) = t$. Using the relation from (7), we find:

$$z = -\frac{1}{2}\psi'(t) = -\frac{t}{2} + \frac{1}{t^2 + t}.$$

This leads us to the following equation:

$$\frac{2}{t^2 + t} = 2z + t \leq 2z + 1, \quad \forall t \in (0, 1]. \tag{10}$$

To manipulate the left-hand side, we write:

$$\frac{2}{t^2 + t} = \frac{2}{t^2(1 + \frac{1}{t})}.$$

Assuming $t \in (0, 1]$, we can define $t_0 > 0$ (however small) such that $t \in [t_0, 1]$. We denote:

$$\lambda = 1 + \frac{1}{t_0} \quad \text{then} \quad 1 + \frac{1}{t} \leq \lambda, \tag{11}$$

which implies:

$$\frac{2}{t^2(1 + \frac{1}{t})} \geq \frac{2}{\lambda t^2}.$$

From the previous inequality and (10), we can derive:

$$\frac{2}{\lambda t^2} \leq 2z + 1, \quad z \in [0, +\infty),$$

leading to:

$$t = \rho(z) \geq \sqrt{\frac{2}{\lambda(2z + 1)}},$$

thereby completing the proof of (ii). □

Now, we derive an estimate for the effect of updating the barrier parameter μ on the value of the proximity function during an iteration. We start with an important theorem which is valid for all KFs $\psi(t)$ that are strictly convex and satisfies (ii) and (iii) of Lemma 2.1.

Lemma 2.4. (Bai et al. [4]) For any $v > 0$ and $\beta > 1$, we have

$$\Psi(\beta v) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(v)}{n}\right)\right).$$

Corollary 2.5. Let θ be such that $0 < \theta < 1$. If $\Psi(v) \leq \tau$, then

$$\Psi(v_+) = \Psi(\beta v) \leq \frac{5}{4(1-\theta)}\left(\theta\sqrt{n} + \sqrt{\tau}\right)^2, \beta = \frac{1}{\sqrt{1-\theta}} > 1.$$

P r o o f. Using (ii) of Lemma 2.2 for $t \geq 1$, we have

$$\psi(t) \leq \frac{5}{4}(t-1)^2.$$

Therefore, based on the previous lemma, we obtain

$$\Psi(\beta v) \leq \frac{5n}{4}\left(\beta\varrho\left(\frac{\Psi(v)}{n}\right) - 1\right)^2.$$

Thus, according to property (i) of Lemma 2.3, we have

$$\Psi(\beta v) \leq \frac{5n}{4}\left(\beta\left(1 + \sqrt{\frac{\Psi(v)}{n}}\right) - 1\right)^2.$$

Given that $\beta = \frac{1}{\sqrt{1-\theta}}$, then

$$\begin{aligned}\Psi(\beta v) &\leq \frac{5n}{4(1-\theta)}\left(\sqrt{\frac{\Psi(v)}{n}} + (1 - \sqrt{1-\theta})\right)^2\\ &\leq \frac{5}{4(1-\theta)}\left(\theta\sqrt{n} + \sqrt{\Psi(v)}\right)^2,\end{aligned}$$

the last inequality is obtained from the fact that $1 - \sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$.
Under the assumption $\Psi(v) \leq \tau$ just before the $\mu-$update, we have

$$\Psi(v_+) \leq \frac{5}{4(1-\theta)}\left(\theta\sqrt{n} + \sqrt{\tau}\right)^2 := \Psi_0, \tag{12}$$

Consequently, Ψ_0 serves as an upper bound for $\Psi(v_+)$ throughout the execution of the algorithm. □

We define the proximity measure $\sigma(v)$ based on the norm as follows:

$$\sigma(v) = \frac{1}{2}\|\nabla\Psi(v)\| = \frac{1}{2}\|d_x + d_s\|, \quad \text{for any } v > 0. \tag{13}$$

The next Lemma provides a lower bound of $\sigma(v)$ in terms of the proximity function $\Psi(v)$.

Lemma 2.6. For any $v > 0$, we have

$$\sigma(v) \geq \sqrt{\frac{\Psi(v)}{2}}.$$

Proof. Utilizing (13) and from property (i) of Lemma 2.2, we get

$$\sigma^2(v) = \frac{1}{4}\|\nabla\Psi(v)\|^2 = \frac{1}{4}\sum_{i=1}^{n}(\psi'(v_i))^2 \geq \frac{1}{2}\sum_{i=1}^{n}\psi(v_i) = \frac{1}{2}\Psi(v).$$

This proves the Lemma. □

Remark 2.7. Throughout the entire paper, it is assumed that $\tau \geq 1$. By applying the aforementioned lemma and considering the assumption $\Psi(v) \geq \tau$, it follows that $\sigma(v) \geq \frac{1}{\sqrt{2}}$.

3. ANALYSIS OF THE ALGORITHM AND ITS COMPUTATIONAL COMPLEXITY

The objective of this section is to calculate a default step size α such that the tuple (x_+, y_+, s_+) defined in our algorithm remains feasible, and the proximity function decreases adequately.

It is worth noting that the parameter μ remains fixed during an inner iteration. Therefore, by substituting (2) into (5), we obtain:

$$x_+ = \frac{x}{v}(v + \alpha d_x), \quad s_+ = \frac{s}{v}(v + \alpha d_s).$$

Consequently, the updated vector v_+ is expressed as:

$$v_+ = \sqrt{\frac{x_+ s_+}{\mu}} = \sqrt{(v + \alpha d_x)(v + \alpha d_s)}.$$

Exploiting the e-convex property of $\psi(t)$, we can deduce that:

$$\Psi(v_+) \leq \frac{1}{2}\left(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)\right).$$

Let us define $f(\alpha) = \Psi(v_+) - \Psi(v)$.

For simplicity, in the subsequent part of this section, we denote σ as $\sigma(v)$.

Lemma 3.1. (Lemmas 4.4 and 4.7 in Touil and Chikouche [20]) The largest possible value of the step size α^* is:

$$\alpha^* = \frac{\rho(\sigma) - \rho(2\sigma)}{2\sigma}.$$

Furthermore

$$\alpha^* \geq \frac{1}{\psi''(\rho(2\sigma))},$$

and we have for all $\alpha \in [0, \alpha^*]$

$$f(\alpha) \leq -\alpha\sigma^2.$$

The next corollary presents the decrease of the proximity function in the inner iteration.

Corollary 3.2. Let us set $\bar{\alpha} = \frac{1}{\psi''(\rho(2\sigma))}$, as the default step size and let λ be defined in (11). Suppose that $\sigma \geq 1$, we have

$$f(\bar{\alpha}) \leq -\frac{\sigma^2}{\psi''(\rho(2\sigma))} \leq -\frac{\Psi(v)^{\frac{1}{2}}}{2(1+8\lambda)}. \tag{14}$$

Proof. From (8), for $t \in]0,1]$ or more precisely $t \in [t_0, 1]$, we have

$$\psi''(t) = 1 + \frac{4t+2}{(t^2+t)^2} < 1 + \frac{4t+4}{(t^2+t)^2} = 1 + \frac{4}{(1+t)t^2},$$

since $\frac{1}{1+t} < 1$ for $t > 0$, we obtain

$$\psi''(t) < 1 + \frac{4}{t^2}.$$

By setting $t = \rho(2\sigma)$ and using (ii) of Lemma 2.3 in this inequality, we obtain

$$\psi''(\rho(2\sigma)) < 1 + \frac{4}{\rho(2\sigma)^2} \leq 1 + 2\lambda(4\sigma+1).$$

Now, using the last inequality in Lemma 3.1, for $\bar{\alpha} = \frac{1}{\psi''(\rho(2\sigma))} \in [0, \alpha^*]$, we get

$$f(\bar{\alpha}) \leq -\sigma^2\bar{\alpha} \leq -\frac{\sigma^2}{1+2\lambda(4\sigma+1)} < -\frac{\sigma^2}{\sigma\left(\sqrt{2}+8\lambda+2\sqrt{2}\lambda\right)} = -\frac{\sigma}{\sqrt{2}+8\lambda+2\sqrt{2}\lambda}.$$

where the last inequality is due to Remark 2.7.

Using Lemma 2.6, we have

$$f(\bar{\alpha}) \leq -\frac{\sqrt{\frac{\Psi(v)}{2}}}{\sqrt{2}+8\lambda+2\sqrt{2}\lambda} = -\frac{\Psi(v)^{\frac{1}{2}}}{2+8\sqrt{2}\lambda+4\lambda} < -\frac{\Psi(v)^{\frac{1}{2}}}{2(1+8\lambda)}.$$

This completes the proof. □

We need to compute how many inner iterations are required to return to the situation where $\Psi(v) \leq \tau$ after $\mu-$update. Let us define the value of $\Psi(v)$ after $\mu-$update as Ψ_0, and the subsequent values in the same outer iteration as $\Psi_k, k = 1, \ldots, K$, where K stands for the total number of inner iterations in the outer iteration. By the definition of $f(\alpha)$ and according to (14), for $k = 1, \ldots, K-1$, we obtain

$$\Psi_{k+1} \leq \Psi_k - \frac{\Psi_k^{\frac{1}{2}}}{2(1+8\lambda)}.$$

Lemma 3.3. (Choi and Lee [8]) Suppose $t_0, t_1, \ldots, t_k$ be a sequence of positive numbers such that

$$t_{k+1} \leq t_k - \beta t_k^{1-\gamma}, k = 0, 1, \ldots, K-1,$$

where $\beta > 0$ and $0 < \gamma \leq 1$. Then $K \leq \left\lceil \frac{t_0^\gamma}{\beta\gamma} \right\rceil$, where $\lceil . \rceil$ is the ceiling function.

From this lemma and by taking $t_k = \Psi_k$, $\beta = \dfrac{1}{2(1+8\lambda)}$ and $\gamma = \dfrac{1}{2}$, we can get the following lemma.

Lemma 3.4. Let K be the total number of inner iterations in the outer iteration. Then, we have

$$K \leq \left\lceil \frac{\Psi_0^\gamma}{\beta\gamma} \right\rceil = 4(1+8\lambda)\Psi_0^{\frac{1}{2}}.$$

Now, we derive the complexity bounds for large- and small-update methods.

Theorem 3.5. The total number of iterations to obtain an approximate solution is bounded by:

$$\left(4(1+8\lambda)\Psi_0^{\frac{1}{2}}\right)\left(\frac{\log \frac{n}{\epsilon}}{\theta}\right). \tag{15}$$

P r o o f. We know that the number of outer iterations for the situation $n\mu \leq \epsilon$ is bounded by $\frac{1}{\theta}(\log \frac{n}{\epsilon})$.

Knowing that an upper bound for the total number of iterations is obtained by multiplying the number of inner and outer iterations, we obtain the result thanks to the above lemma. □

For large-update methods with $\tau = \mathcal{O}(n)$ and $\theta = \Theta(1)$, the complexity of the primal-dual IPMs based on our new KF is given by $\mathcal{O}\left(\sqrt{n}\log \frac{n}{\epsilon}\right)$. This complexity coincides with the currently best-known iteration bound for small-update methods with $\tau = \mathcal{O}(1)$ and $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$. In our Knowledge, these results are the first to reach this goal.

4. CONCLUSIONS AND REMARKS

In this work, we have introduced a new KF with a logarithmic barrier term. Firstly, we have presented various properties of this new KF, and through simple analytical tools, we have shown that the large-update IPM based on this KF has an $\mathcal{O}\left(\sqrt{n}\log \frac{n}{\epsilon}\right)$ iteration bound in the worst case. This bound coincides with the best-known complexity bound of IPMs for small-update methods. This result is significant as it is the first to close the gap between the large-update and small-update IPMs.

Some interesting topics for future work include the extension to semidefinite programming (SDP), second-order cone programming (SOCP), and convex quadratic semidefinite programming (CQSDP). Furthermore, numerical tests to investigate the behavior of the algorithm so as to be compared with other approaches.

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REFERENCES

[1] M. Achache: A new primal-dual path-following method for convex quadratic programming. Comput. Appl. Math. *25* (2006), 1, 97–110. DOI:10.1590/S0101-82052006000100005

[2] F. Alizadeh: Interior point methods in semidefinite programming with applications to combinatorial optimization. SIAM J. Optim. *5* (1995), 13–51. DOI:10.1137/0805002

[3] Y. Q. Bai and C. Roos: A primal-dual interior-point method based on a new kernel function with linear growth rate. In: Proc. Industrial Optimization Symposium and Optimization Day 2002.

[4] Y. Q. Bai, M. El Ghami, and C. Roos: A comparative study of kernel functions for primal-dual interior-point algorithms in linear optimization. SIAM J. Optim. *15* (2004), 101–128. DOI:10.1137/S1052623403423114

[5] M. Bouafia, D. Benterki, and A. Yassine: An efficient primal-dual interior point method for linear programming problems based on a new kernel function with a trigonometric barrier term. J. Optim. Theory. Appl. *170* (2016), 528–545. DOI:10.1007/s10957-016-0895-0

[6] N. Boudjellal, H. Roumili, and D. Benterki: A primal-dual interior point algorithm for convex quadratic programming based on a new parametric kernel function. Optimization *70* (2020) 8, 1–22. DOI:10.1080/02331934.2020.1751156

[7] Y. Bouhenache, W. Chikouche, I. Touil, and S. Fathi-Hafshejani: Complexity analysis of primal-dual interior-point methods for convex quadratic programming based on a new twice parameterized kernel function. J. Math. Model. *12* (2024), 2, 247–265.

[8] B.K. Choi and G.M. Lee: On complexity analysis of the primal-dual interior-point method for semidefinite optimization problem based on a new proximity function. Nonlinear Anal. *71* (2009), 2628–2640. DOI:10.1016/j.na.2009.05.078

[9] M. El Ghami, Z.A. Guennoun, S. Bouali, and T. Steihaug: Interior point methods for linear optimization based on a kernel function with a trigonometric barrier term. J. Comput. Appl. Math. *236* (2012), 3613–3623. DOI:10.1016/j.cam.2011.05.036

[10] S. Guerdouh, W. Chikouche, and I. Touil: An efficient primal-dual interior point algorithm for linear optimization problems based on a novel parameterized kernel function with a hyperbolic barrier term. 2021. halshs-03228790. `https://shs.hal.science/halshs-03228790/`

[11] S. Fathi-Hafshejani, H. Mansouri, M.R. Peyghami, and S. Chen: Primal-dual interior-point method for linear optimization based on a kernel function with trigonometric growth term. Optimization (2018), 1029–4945. DOI:10.1080/02331934.2018.1482297

[12] S. Fathi-Hafshejani and Z. Moaberfard: An interior-point algorithm for linearly constrained convex optimization based on kernel function and application in non-negative matrix factorization. Optim. Engrg. *21* (2020), 1019–1051. DOI:10.1007/s11081-020-09514-x

[13] N. K. Karmarkar: A new polynomial-time algorithm for linear programming. In: Proc. 16th Annual ACM Symposium on Theory of Computing *4* (1984), 373–395.

[14] R. E. Nesterov and A. S. Nemirovskii: Interior-Point Ploynomial Methods in Convex Programming. Society for Industrial and Applied Mathematics 1994.

[15] J. Peng, C. Roos, and T. Terlaky: Self-regular functions and new search directions for linear and semidefinite optimization. Math. Program. *93* (2002), 129–171. DOI:10.1007/s101070200296

[16] M. R. Peyghami, S. Fathi-Hafshejani, and S. Chen: A primal-dual interior-point method for semidefinite optimization based on a class of trigonometric barrier functions. Oper. Res. Lett. *44* (2016), 319–323. DOI:10.1016/j.orl.2016.02.013

[17] C. Roos, T. Terlaky, and J. Ph. Vial: Theory and algorithms for linear optimization. In: An interior point Approach. John Wiley and Sons, Chichester 1997.

[18] I. Touil, D. Benterki, and A. Yassine: A feasible primal-dual interior point method for linear semidefinite programming. J. Comput. Appl. Math. *312* (2017), 216–230. DOI:10.1016/j.cam.2016.05.008

[19] I. Touil and D. Benterki: A primal-dual interior-point method for the semidefinite programming problem based on a new kernel function. J. Nonlinear Funct. Anal. (2019), Article ID 25.

[20] I. Touil and W. Chikouche: Primal-dual interior point methods for semidefinite programming based on a new type of kernel functions. Filomat. *34* (2020), 12, 3957–3969. DOI:10.2298/FIL2012957T

[21] I. Touil and W. Chikouche: Polynomial-time algorithm for linear programming based on a kernel function with hyperbolic-logarithmic barrier term. PJM *11* (2022), 127–135.

[22] I. Touil and W. Chikouche: Novel kernel function with a hyperbolic barrier term to primal-dual interior point algorithm for SDP problems. Acta Math. Sinica (Engl. Ser.) *38* (2022), 1, 44–67. DOI:10.1007/s10255-022-1061-0

[23] G. Q. Wang, Y. Q. Bai, and C. Roos: Primal-dual interior-point algorithms for semidefinite optimization based on a simple kernel function. J. Math. Model. Algorithms Oper. Res. *4* (2005), 4, 409–433. DOI:10.1007/s10852-005-3561-3

Imene Touil, LMPA, Department of Mathematics, Mohammed Seddik Ben Yahia University, P.O. Box 98, Ouled Aissa, 18000, Jijel. Algeria.
e-mail: touilimene@univ-jijel.dz

Sajad Fathi-Hafshejani, Faculty of Mathematics, Shiraz University of Technology, P.O. Box 71555.313, Shiraz. Iran.
e-mail: s.fathi@sutech.ac.ir