KYBERNETIKA — VOLUME *61* (2025), NUMBER 1, PAGES 133–140

SOLVABILITY OF (MAX,+) AND (MIN,+)–EQUATION SYSTEMS

Karel Zimmermann

Properties of (max,+)-linear and (min,+)-linear equation systems are used to study solvability of the systems. Solvability conditions of the systems are investigated. Both one-sided and two-sided systems are studied. Solvability of one class of (max,+)-nonlinear problems will be investigated. Small numerical examples illustrate the theoretical results.

Keywords: max-algebraic and min-algebraic linear equation systems, solvability conditions, two-sided max-/min- algebraic linear equation systems

Classification: 90C31,15A99

1. INTRODUCTION

Systems of max-linear equation systems and related optimization problems were studied since the sixties of the last century under various names in the literature e. g. in [1, 2, 3, 4, 6]. Later some authors studied also such systems with interval coefficients in the right-hand sides and/or in matrices of the systems (see e. g. [5] and references therein). Variables appeared originally in the equation systems only on one side of the equations (so called one-sided systems in [1] − [4])). Later appeared papers studying so called two-sided equation systems with variables on both sides of the equations see e. g. [1, 6]. Approximate solution methods of the two-sided equations were proposed in [6, 7, 8]. Some authors use as a basic algebraic structure other structures than (max,+)- or (min,+)-algebra, which are used in this paper. For instance in [4] the author studies one sided systems on (min-product)-algebra study. The present paper studies conditions of solvability of (max,+)-linear and (min,+)-linear equation systems and one class of nonlinear equation systems. Small numerical examples illustrate the theoretical results.

2. THEORETICAL BACKGROUND

In this section we will consider systems of so called one-sided (max,+)-linear and (min,+)-linear equations. We will recall some known results which were proved in the cited literature (e. g. [1, 2]), which form a theoretical background for further investigation.

DOI: 10.14736/kyb-2025-1-0133

Systems of one-sided (max,+)-linear/(min,+)-linear equations have the following form:

$$\max_{i\in J}(a_{ij}+x_j)=b_i,\ i\in I, \tag{1}$$

$$\min_{i\in J}(a_{ij}+x_j)=b_i,\ i\in I, \tag{2}$$

where $I=\{1,\dots,m\}$, $J=\{1,\dots,n\}$, a_{ij}, $b_i\in R$, $i\in I$, $j\in J$. The set of all solutions of system (1), (2) will be denoted $M(A,b)$, $M'(A,b)$ respectively. Systems (1),(2) will be called one-sided systems, because they contain variables only on one side of the equations. We will summarize in this section several known properties of the systems (the proofs see e. g. [1, 2]).

We can introduce also a more compact matrix-vector notation to describe the systems above. If we set

$$(A\ o\ x)_i=\max_{j\in J}(a_{ij}+x_j),\ i\in I,\ (A\ o'\ x)_i=\min_{j\in J}(a_{ij}+x_j),\ i\in I,$$

$$A\ o\ x=((A\ o\ x)_1,\dots(A\ o\ x)_m)^T,\ A\ o'\ x=((A\ o'\ x)_1,\dots(Ao'x)_m)^T,$$

then systems (1) can be written in the matrix-vector form as $Aox=b$, $Ao'x=b$.

Definition 2.1.

An element $x^{\max}$ ($x^{\min}$) of any set $M\subset R^n$ with the property $x\le x^{\max}$ ($x\ge x^{\min}$) for all $x\in M$ is called maximum (minimum) element of M.

Lemma 2.1.

(a) Let $M^{\le}(A,b)=\{x\in R^n \mid A\ o\ x\le b\}$. Then the set $M^{\le}(A,b)$ has the maximum element $\overline{x}(A,b)$ with the following components:

$$\overline{x}_j(A,b)=\min_{i\in I}(b_i-a_{ij}),\ j\in J. \tag{3}$$

If set $M(A,b)=\{x\in R^n \mid A\ o\ x=b\}$ is non-empty, then it has the unique maximum element and it is defined by relation (3).

$M(A,b)\ne\emptyset$ if and only if $\overline{x}(A,b)\in M(A,b)$.

(b) Let $M^{\ge}(A,b)=\{x\in R^n \mid A\ o'\ x\ge b\}$. Then the set $M^{\ge}(A,b)$ has the minimum element $\underline{x}(A,b)$ with the following components:

$$\underline{x}_j(A,b)=\max_{i\in I}(b_i-a_{ij}),\ j\in J. \tag{4}$$

If set $M'(A,b)=\{x\in R^n \mid A\ o'x=b\}$ is non-empty, then it has the unique minimum element and it is defined by relation (4).

$M'(A,b)\ne\emptyset$ *if and only if* $\underline{x}(A,b)\in M'(A,b)$.

Proof. See [1, 2].

We will bring a motivation based on examples of machine time scheduling problems, which have appeared in literature (see [1, 2]). □

Example 2.1.

We have to carry out n operations at each of m places (worksops). Each operation $j \in J$ is carried out without interruption. Operation $j \in J = \{1, \dots, n\}$ at place $i \in I = \{1, \dots, m\}$ needs processing time a_{ij} units of time, i. e. if x_j is the time at which the operation j begins, then the operation is finished at place i at time $x_j + a_{ij}$. All n operations are therefore finished at place i at time equal to $\max_{j \in J}(a_{ij} + x_j)$. Let $b_i,\ i \in I$ denotes a required time at which all operations at place (workshop) i should be finished. It means that feasible starting times of operations x_j must satisfy the following system of m inequalities:

$$\max_{j \in J}(a_{ij} + x_j) \leq b_i,\ i \in I.$$

Our aim is to find out whether there is possible to find such starting times, which avoid unnecessary waiting times for all $i \in I$ i. e. whether it is possible to solve the following equation system with respect to variables $x_j,\ j \in J$:

$$\max_{j \in J}(a_{ij} + x_j) = b_i,\ i \in I.$$

3. SOLVABILITY OF (MAX,+)- AND (MIN,+)-LINEAR EQUATION SYSTEMS

In general only the inequalities $Ao\overline{x}(A,b) \leq b,\ A\ o'\underline{x}(A,b) \geq b$ hold for $\overline{x}(A,b),\ \underline{x}(A,b)$. The next Theorem establishes necessary and sufficient conditions under which $Ao\overline{x}(A,b) = b,\ A\ o'\underline{x}(A,b) = b$.

Theorem 3.1.

(a) Let $b \in R^n$. Then $\overline{x}(A,b) \in M(A,b)$ if and only if

$$(\forall i \in I)(\exists j(i) \in J) \text{ such that } \overline{x}_{j(i)}(A,b) = \min_{k \in I}(b_k - a_{kj(i)}) = b_i - a_{ij(i)}. \tag{5}$$

(b) Let $b \in R^n$. Then $\underline{x}(A,b) \in M'(A,b)$ if and only if

$$(\forall i \in I)(\exists j(i) \in J) \text{ such that } \underline{x}_{j(i)}(A,b) = \max_{k \in I}(b_k - a_{kj(i)}) = b_i - a_{ij(i)}. \tag{6}$$

Proof. We will prove only part (a). Part (b) can be proved by analogy.

Let condition (5) be fulfilled. It follows from (2) that we have for each $i \in I$:

$$a_{ij(i)} + \overline{x}_{j(i)}(A,b) = a_{ij(i)} + \min_{k \in I}(b_k - a_{kj(i)}) = a_{ij(i)} + b_i - a_{ij(i)} = b_i. \tag{7}$$

Since $\overline{x}(A,b) \in M^{\leq}(A,b)$, it holds for all $j \neq j(i)$ $(a_{ij} + \overline{x}_j(A,b)) \leq b_i$ (see Lemma 5.1)(a)). Therefore

$$\max_{j \in J}(a_{ij} + \overline{x}_j(A,b)) = a_{ij(i)} + \overline{x}_{j(i)}(A,b) = b_i. \tag{8}$$

Let us assume that condition (5) is not fulfilled for an index $i \in I$. Then $\overline{x}_j(A,b) = \min_{k \in I}(b_k - a_{kj}) < b_i - a_{ij}$ for all $j \in J$, i. e. $(Ao\overline{x}(A,b))_i < b_i$ and $\overline{x}(A,b) \notin M(A,b)$. □

The following Theorem establishes structural conditions of matrix A, which ensure the solvability of systems $Aox = b,\ Ao'x = b$.

Theorem 3.2.

(a) Let matrix A satisfy the following conditions:

$$(\forall i \in I)(\exists j(i) \in J) \textit{ such that } \max_{j \in J} a_{ij} = a_{ij(i)} = \max_{r \in I} a_{rj(i)}. \tag{9}$$

Let $\overline{x}(A, b)$ be defined as in (3). Then $Ao\overline{x}(A, b) = b$ for any $b \in R^m$.

(b) Let matrix A satisfy the following conditions:

$$(\forall i \in I)(\exists j(i) \in J) \textit{ such that } \min_{j \in J} a_{ij} = a_{ij(i)} = \min_{r \in I} a_{rj(i)}. \tag{10}$$

Let $\underline{x}(A, b)$ be defined as in (4). Then $Ao'\underline{x}(A, b) = b$ for any $b \in R^m$.

Proof. We will prove only part (a). Part (b) can be proved by analogy.

If conditions (9) are satisfied, we have $\overline{x}_{j(i)}(A, 0) = \min_{r \in I} -a_{rj(i)} = -a_{ij(i)}$ for all $i \in I$. Let $i \in I$ be arbitraily chosen. It holds:

$$\max_{j \in J}(a_{ij} + \overline{x}_j(A, 0)) = a_{ij(i)} - a_{ij(i)} = 0.$$

Since $a_{ij} + \overline{x}_j(A, 0) \leq 0$ for all $j \in J$, we obtain $\max_{j \in J}(a_{ij} + \overline{x}(A, 0)) = a_{ij(i)} + \overline{x}_{j(i)}(A, 0) = 0$. We add b_i to both sides of this equality and have b_i on the right-hand side. On the left-hand side we obtain:

$$\max_{j \in J}(a_{ij} + \overline{x}(A, 0)) + b_i = a_{ij(i)} + \overline{x}_{j(i)}(A, 0) + b_i = a_{ij(i)} - a_{ij(i)} + b_i = b_i,$$

Since $i \in I$ was arbitrarily chosen, it holds $Ao\overline{x}(A, b) = b$. □

4. SOLVABILITY OF TWO-SIDED SYSTEMS

Let us consider equation two-sided equation systems

$$\max_{j \in J}(a_{ij} + x_j) = \max_{k \in J}(b_{ik} + y_k),\ i \in I, \tag{11}$$

$$\min_{j \in J}(a_{ij} + x_j) = \min_{k \in J}(b_{ik} + y_k),\ i \in I, \tag{12}$$

$$\max_{j \in J}(a_{ij} + x_j) = \min_{k \in J}(b_{ik} + y_k),\ i \in I, \tag{13}$$

where $a_{ij}, b_{ik} \in R$. Using a more compact matrix-vector notation the systems can be written as $Aox = Boy,\ A\ o'x = B\ o'y,\ Aox = B\ o'y$.

Assume that matrices $A,\ B$ satisfy for each $i \in I$ relations (9) i. e. it holds $\max_{j \in J} a_{ij} = a_{ij(i)} = \max_{r \in I} a_{rj(i)}$ and $\max_{k \in J} b_{ik} = b_{ik(i)} = \max_{r \in I} b_{rk(i)}$. Then according to Theorem 3.2(a) $Ao\overline{x}(A, b) = Bo\overline{y}(B, b) = b$. Therefore the pair $(\overline{x}(A, b)), \overline{y}(B, b))$ is a solution of the two-sided system (11).

Similarly if matrices $A,\ B$ satisfy for each $i \in I$ relations (10) i. e. it holds $\min_{j \in J} a_{ij} = a_{ij(i)} = \min_{r \in I} a_{rj(i)}$ and $\min_{k \in J} b_{ik} = b_{ik(i)} = \min_{r \in I} b_{rk(i)}$, then according to Theorem 3.2(b) $Ao'\underline{x}(A, b) = Bo'\underline{y}(B, b) = b$. Therefore the pair $(\underline{x}(A, b)), \underline{y}(B, b))$ is a solution of the two sided system (12).

Finally if relations (9) hold for matrix A and relations (10) for matrix B, then $Ao\overline{x}(A, b) = Bo'\underline{y}(B, b) = b$ and therefore the pair $(\overline{x}(A, b), \underline{y}(B, b))$ is a solution of system (13).

5. SOLVABILITY OF ONE CLASS OF (MAX,+)-NONLINEAR EQUATION SYSTEMS

Let us consider the following equation systems:

$$\max_{j\in J}(a_{ij}+p_{ij}(x_j))=b_i,\ i\in I, \tag{14}$$

$$\min_{j\in J}(a_{ij}+p_{ij}(x_j))=b_i,\ i\in I, \tag{15}$$

where $p_{ij}:R\to R$ is a strictly increasing continuous function and $p_{ij}(0)=0$ for all $i\in I, j\in J$.

Let P be the matrix with elements $p_{ij}(.), i\in I, j\in J$, $M(A,P,b)$ denote the set of solutions of system (14) and $M'(A,P,b)$ denote the set of solutions of system (15)

Lemma 5.1.

(a) Let $\overline{x}_j(A,P,b)=\min_{r\in I}p_{rj}^{-1}(b_r-a_{rj}), j\in J$. If $\overline{x}(A,P,b)\in M(A,P,b)$, then $\overline{x}(A,P,b)$ is the maximum element of the set $M(A,P,b)$.

(b) Let $\underline{x}_j(A,P,b)=\max_{r\in I}p_{rj}^{-1}(b_r-a_{rj}), j\in J$. If $\underline{x}(A,P,b)\in M'(A,P,b)$, then $\underline{x}(A,P,b)$ is the minimum element of the set $M'(A,P,b)$.

P r o o f. We will prove only part (a) of the theorem, part (b) can be proved by analogy.

Let x be any element of $M(A,P,b)$ and $j\in J$ arbitrarily chosen. Then $x_j\le p_{rj}^{-1}(b_r-a_{rj})\ \forall r\in I$. It follows that $x_j\le\overline{x}_j(A,P,b)$. □

Theorem 5.1.

(a) Assume the system (14) satisfies the following relations:

$$(\forall i\in I)\max_{j\in J}a_{ij}=a_{ij(i)};\qquad (\forall i\in I)\min_{r\in I}p_{rj(i)}^{-1}(-a_{rj(i)})=p_{ij(i)}^{-1}(-a_{ij(i)}). \tag{16}$$

Then equation system (14) is solvable for any $b\in R^m$.

(b) Assume the system (15) satisfies the following relatons:

$$(\forall i\in I)\min_{j\in J}a_{ij}=a_{ij(i)};\qquad (\forall i\in I)\max_{r\in I}p_{rj(i)}^{-1}(-a_{rj(i)})=p_{ij(i)}^{-1}(-a_{ij(i)}). \tag{17}$$

Then equation system (15) is solvable for any $b\in R^m$.

P r o o f. We will prove only part (a), part (b) can be proved by analogy.

Let relations (16) hold. Then it is $\overline{x}_j(A,p,b)=p_{ij}^{-1}(b_i-a_{ij})\ i\in I, j\in J$.
It holds for $b=0$ and an arbitrarily chosen $i\in I$:

$$\max_{j\in J}(a_{ij}+p_{ij}(\overline{x}_j(A,P,0)))=a_{ij(i)}+p_{ij(i)}(\overline{x}_{j(i)}(A,p,0)) \tag{18}$$

and further

$$a_{ij(i)}+p_{ij(i)}(\min_{r\in I}p_{rj(i)}^{-1}(-a_{rj(i)})). \tag{19}$$

and according to the assumption (16) we have:

$$a_{ij(i)}+p_{ij(i)}(\min_{r\in I} p^{-1}_{rj(i)}(-a_{rj(i)})) = a_{ij(i)}+p_{ij(i)}(p^{-1}_{ij(i)}(-a_{ij(i)})) = a_{ij(i)}-a_{ij(i)} = 0. \quad (20)$$

It holds further $p_{ij(i)}(\overline{x}_{j(i)}(A,P,b)) = p_{ij}(p^{-1}_{ij(i)}(b_i - a_{ij(i)})) = b_i - a_{ij(i)}$ so that

$$a_{ij(i)} + p_{ij(i)}(\overline{x}_{j(i)}(A,p,b)) = a_{ij(i)} + p_{ij(i)}(p^{-1}_{ij(i)}(b_i - a_{ij(i)})), \quad (21)$$

and further

$$a_{ij(i)} + p_{ij(i)}(p^{-1}_{ij(i)}(b_i - a_{ij(i)}) = a_{ij(i)} + b_i - a_{ij(i)} = b_i. \quad (22)$$

It follows from relations (18) – (22) that

$$\max_{j\in J}(a_{ij} + p_{ij}(\overline{x}_j(A,P,b))) = a_{ij(i)} + p_{ij(i)}(\overline{x}_{j(i)}(A,P,b)) = b_i \quad (23)$$

Since $i \in I$ was arbitrarily chosen, it follows that $\overline{x}(A,P,b) \in M(A,P,b)$, which implies the solvability of system (15). □

Remark 5.1.
Taking into account the application to machine-time scheduling in Example 2.1, Theorem 3.1 may be interpreted as conditions of synchronization of two machine-time scheduling problems. Part (a) gives conditions of synchronization of termination times of all operations at place i, part (b) gives conditions of synchronization of release times.

Remark 5.2.
Let us note that, according to author's knowledge, a universal polynomial algorithm for solving two-sided systems of the form (11) – (13) has not been published until now. The general systems without any preliminary assumptions were solved only approximately (see e. g. [7, 8]). The results following from Theorem 3.2 make possible to point out a special class of two-sided systems, for which solutions can be found without any problems with the complexity of the caculations.

Remark 5.3.
The results of this paper concerning solvability of the systems can be extended to other max- or min-algebras considered in the literature, e. g. to max- or min-product algebra defined on positive numbers (see [4]). The role, which plays number 0 as the neutral element of the plus operation, which replaces in the (max,+)-algebra the multiplication will play in (max-product)-algebra number 1 as neutral element of the usual multiplication (see [2, 4]).

We will illustrate the results on small numerical examples.

Example 5.1.

(a) Let $m = n = 3$ so that $I = J = \{1, 2, 3\}$.

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 3 \\ 1 & 5 & 1 \end{pmatrix}.$$

Matrix A satifies conditions (9), indexes $i, j(i)$ and elements $a_{ij(i)}$ for $i \in I$ are the following:

$i = 1,\ j(1) = 3,\ a_{1j(1)} = a_{13} = 4,\ i = 2,\ j(2) = 1,\ a_{2j(2)} = a_{21},\ i = 3,\ j(3) = 2,\ a_{32} = 5.$

If $b = (0, 0, 0)^T$, then

$\overline{x}(A, 0) = (\overline{x}_1(A, 0),\ \overline{x}_2(A, 0),\ \overline{x}_3(A, 0))^T = (\overline{x}_{j(2)}(A, 0), \overline{x}_{j(3)}(A, 0),\ \overline{x}_{j(1)}(A, 0))^T = (-3, -5, -4)^T$

If $b = (4, 8, 5)^T$, then

$\overline{x}(A, b) = \overline{x}(A, 0) + b = (4, 8, 5)^T$ and $Ao\overline{x}(A, b) = b = (4, 8, 5)^T$. It is further $\overline{x}(A.b) = (-3, -5, -4)^T + (4, 8, 5)^T = (1, 3, 1)^T$.

(b) Let $m = n = 3$ so that $I = J = \{1, 2, 3\}$.

$$B = \begin{pmatrix} 4 & 3 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 5 \end{pmatrix}.$$

Matrix B satifies conditions (9), indexes $i, j(i)$ and elements $b_{ij(i)}$ for $i \in I$ are the following:

$b_{1j(1)} = b_{11} = 4,\ b_{2j(2)} = b_{22} = 5,\ b_{j(3)} = b_{33} = 5.$

Let $b = 0$. Then $\overline{y}(B, 0) = (-4, -5, -5)^T$ and $Bo\overline{y}(B, 0) = 0$.

Let $b = (4, 8, 5)^T$. Then it is:

$\overline{y}(B, b) = \overline{y}(B, 0) + b = (4, 8, 5)^T$ and $Bo\overline{y}(B, b) = b = (4, 8, 5)^T$. It is further $\overline{y}(B.b) = (-4, -5, -5)^T + (4, 8, 5)^T = (0, 3, 0)^T$.

(c) Let us consider two-sided system $Aox = Boy$. It is $Ao\overline{x}(A, 0) = Bo\overline{y}(B, 0) = 0$. It follows that the pair $(\overline{x}(A, 0),\ \overline{y}(B, 0)) = ((-3, -5, -4)^T, (-4, -5, -5)^T)$ is a solution of the two-sided system and both sides of the system are equal to $b = (0, 0, 0)^T$. If $b = (4, 8, 5)^T$, then the pair $(\overline{x}(A, b), \overline{y}(B, b)) = (1, 3, 1)^T, (0, 3, 0)^T$ solves the two-sided system and both sides of the system are equal to $(4, 8, 5)^T$.

ACKNOWLEDGEMENT

Under the support of GA ČR 22-11117S.

(Received October 7, 2024)

REFERENCES

[1] P. Butkovič: Max-linear Systems: Theory and Algorithms. Monographs in Mathematics, Springer Verlag 2010.

[2] R. A. Cuninghame-Green: Minimax Algebra. Lecture Notes in Economics and Mathematical Systems *166*, Springer Verlag, Berlin 1979.

[3] G. L. Litvinov, V. P. Maslov, and S. N. Sergeev (eds.): Idempotent and Tropical Mathematics and Problems of Mathematical Physics, vol. I. Independent University Moscow, 2007.

[4] N. N. Vorobjov: Extremal Algebra of Positive Matrices. (In Russian.) Datenverarbeitung und Kybernetik *3* (1967), 39–71.

[5] H. Myšková and J. Plávka: The robustness of interval matrices in max-plus algebra. Linear Algebra Appl. *445* (2014), 85–102. DOI:10.1016/j.laa.2013.12.008

[6] N. K. Krivulin: On the Solution of the two-sided vector equation. In: Tropical Algebra, Vestnik, St. Petersburg University, Mathematics *56* (2023) 2, 236–248. DOI:10.21638/spbu01.2023.205

[7] R. A. Cunninghame-Green and K. Zimmermann: Equation with residual functions. CMUC *42* (2001), 729–740. DOI:10.1046/j.1365-2958.2001.02638.x

[8] A. Aminu: On the solvability of homogeneous two-sided systems in max-algebra. Notes on Number Theory and Discrete Mathematics, ISSN 1310–5132 Volume 16, 2010, Number 2, pp. 5–15.

Karel Zimmermann, Faculty of Mathematics and Physics, Charles University, Praha. Czech Republic.

e-mail: karel.zimmermann@mff.cuni.cz