KYBERNETIKA — VOLUME *60* (2024), NUMBER 6, PAGES 723–739

CHARACTERIZATION OF THE ORDER INDUCED BY UNINORM WITH THE UNDERLYING DRASTIC PRODUCT OR DRASTIC SUM

Zhi-qiang Liu

In this article, we investigate the algebraic structures of the partial orders induced by uninorms on a bounded lattice. For a class of uninorms with the underlying drastic product or drastic sum, we first present some conditions making a bounded lattice also a lattice with respect to the order induced by such uninorms. And then we completely characterize the distributivity of the lattices obtained.

Keywords: uninorm, triangular norm, divisibility, partial order, distributive lattice

Classification: 03B20, 06B05, 94D05

1. INTRODUCTION

Yager and Rybalov[31] introduced uninorms on the unit interval $[0,1]$, which are a special kind of aggregation functions that generalize the notions of triangular norms (t-norms, for short) and triangular conorms (t-conorms, for short). Uninorms are important from a theoretical viewpoint and their potential applications like fuzzy logic, expert systems and fuzzy system modeling [5, 8, 32], and the interest in uninorms had considerably grown in the last decade [6, 14, 21, 22, 26].

The partial orders induced by logical operators are a very active research area in recent years[7, 10, 12, 17, 18, 23, 24, 28, 29]. For instance, a T-partial order induced by a t-norm on a bounded lattice was defined, and some conditions for the new partial order to be a bounded lattice were obtained by Karaçal and Kesicioğlu [13]. Aşıcı and Karaçal[1], Kesicioğlu et al. [16] studied the T-partial order and its properties. An order induced by uninorms on bounded lattices was given and was discussed by Ertuğrul et al. [9]. The orders induced by uninorms and its properties have been widely investigated in[17]. An order induced by a nullnorm on a bounded lattice was defined by Aşıcı [2]. Further, Aşıcı [3] discussed some properties of such partial orders obtained from nullnorms. Kesicioğlu[19] studied the relationships between the orders induced by uninorms and nullnorms. Moreover, some properties concerning the order induced by uninorms and nullnorms are obtained[15], Recently, Mesiarová-Zemánková[27] studied natural partial order induced by a commutative, associative and idempotent function.

DOI: 10.14736/kyb-2024-6-0723

Gupta and Jayaram[11] discussed order based on associative operations, they raised an open problem: how is the structure of the poset obtained from the order induced by uninorms related in terms of the posets obtained from the order induced by the underlying t-norm and t-conorm? Therefore, the characterization of the orders induced by uninorms related to algebraic structures on bounded lattices has recently attracted much attention. However, to the best of our knowledge, the lattice structures for the partial orders induced by the uninorms on a bounded lattice are not known yet in the current literatures. As a theoretical continuation and development of the partial orders induced by uninorms, the main aim is to characterize the lattice structures of a partial order induced by a uninorm on bounded lattices. From the mathematical point of view, it is interesting to examine the change the underlying operators of uninorm on the lattice changes. We aim to investigate the lattice structures for the partial orders induced by uninorms U with divisible (resp. non-divisible) the underlying t-norms $\mathcal{T}_{\mathcal{U}}$ and t-conorms $\mathcal{S}_{\mathcal{U}}$, more precisely, (i) $\mathcal{T}_{\mathcal{U}}$ and $\mathcal{S}_{\mathcal{U}}$ are divisible, (ii) $\mathcal{T}_{\mathcal{U}}$ is the drastic product and $\mathcal{S}_{\mathcal{U}}$ is divisible, (iii) $\mathcal{T}_{\mathcal{U}}$ is divisible and $\mathcal{S}_{\mathcal{U}}$ is the drastic sum, (iv) $\mathcal{T}_{\mathcal{U}}$ is the drastic product and $\mathcal{S}_{\mathcal{U}}$ is the drastic sum.

The rest of this article is organized as follows. In Section 2 we provide the necessary background material. In Section 3, we show some conditions for the order induced by uninorm on bounded lattices to be a lattice. Also, we deal with the distributivity of the lattices obtained by the order derived from uninorms. A conclusion is drawn in Section 4.

2. PREVIOUS RESULTS

This section contains a short overview of the basic notions that are essential for the presented research. For more detailed expositions about posets and lattices we recommend [4].

A partially ordered set (*poset*) is a structure $(P, \leqslant)$ where P is a nonempty set and $\leqslant$ is an ordering (reflexive, antisymmetric and transitive) relation on P. Let $(P, \leqslant)$ be a poset and $p, q \in P$. If $p < q$ and there is no element $e \in P$ such that $p < e < q$, then we say that p is *covered* by q (or q *covers* p), and we write $p \lhd q$ (or $q \rhd p$). A *lattice* is a poset $(L, \leqslant)$ in which every two elements subset $\{x, y\}$ has the greatest lower bound, meet, denoted by $x \wedge y$, and the least upper bound, join, denoted by $x \vee y$. Let L be a lattice. $x \parallel y$ denotes that x is *incomparable* with y, i. e., $x \ngeqslant y$ and $x \nleqslant y$, and $x \nparallel y$ denotes that $x \geqslant y$ or $x \leqslant y$. We denote the set of elements which are incomparable with e by $\mathcal{I}_e$, i. e. $\mathcal{I}_e = \{x \in L \mid x \parallel e\}$.

Lemma 2.1. (Birkhoff [4]) A lattice L is distributive if and only if none of its sublattices is isomorphic to M_3 or N_5.

Definition 2.2. (Klement et al. [20], Saminger [30]) Let $(L, \leqslant, 0, 1)$ be a bounded lattice. A *t-norm* $\mathcal{T}$ (resp. *t-conorm* $\mathcal{S}$) is a binary operation on L which is commutative, associative, non-decreasing in each variable, and has a neutral element 1 (resp. 0).

Definition 2.3. (Mayor and Torrens [25]) A t-norm $\mathcal{T}$ (resp. t-conorm $\mathcal{S}$) on a bounded lattice L is *divisible* if the following condition holds:

For any $x, y \in L$ with $x \leqslant y$ there is a $z \in L$ such that $x = \mathcal{T}(y, z)$ (resp. $y = \mathcal{S}(x, z)$).

Example 2.4. Let $(L, \leqslant, 0, 1)$ be a bounded lattice. The greatest t-norm $\mathcal{T}_M$ (resp. the smallest t-conorm $\mathcal{S}_M$) and the smallest t-norm $\mathcal{T}_D$ (resp. the greatest t-conorm $\mathcal{S}_D$) are given as, respectively. For all $x, y \in L$,

$$\mathcal{T}_M(x,y) = x \wedge y, \quad \mathcal{T}_D(x,y) = \begin{cases} x & y = 1, \\ y & x = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{S}_M(x,y) = x \vee y, \quad \mathcal{S}_D(x,y) = \begin{cases} x & y = 0, \\ y & x = 0, \\ 1 & \text{otherwise.} \end{cases}$$

$\mathcal{T}_D$ and $\mathcal{S}_D$ are the *drastic product* and the *drastic sum*, respectively. Clearly, both $\mathcal{T}_M$ and $\mathcal{S}_M$ are divisible, neither $\mathcal{T}_D$ nor $\mathcal{S}_D$ is divisible.

Definition 2.5. (Karaçal and Mesiar [14]) Let $(L, \leqslant, 0, 1)$ be a bounded lattice. An operation $\mathcal{U} : L^2 \to L$ is called a *uninorm* on L, if it is commutative, associative, non-decreasing in each variable and has a neutral element $e \in L$.

For convenience, the symbol $\mathfrak{U}(e)$ will be used for the set of all uninorms on L with neutral element $e \in L$.

Definition 2.6. (Ertuğrul et al. [9]) Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$. Define the following relation, for $x, y \in L$, as

$$x \sqsubseteq_{\mathcal{U}} y :\Leftrightarrow \begin{cases} \text{if } x, y \in [0,e] \text{ and there exists } k \in [0,e] \text{ such that } \mathcal{U}(k,y) = x \text{ or,} \\ \text{if } x, y \in [e,1] \text{ and there exists } \ell \in [e,1] \text{ such that } \mathcal{U}(x,\ell) = y \text{ or,} \\ \text{if } (x,y) \in L^* \text{ and } x \leqslant y, \end{cases} \tag{1}$$

where $\mathcal{I}_e = \{x \in L \mid x \parallel e\}$ and $L^* = [0,e] \times [e,1] \cup [0,e] \times \mathcal{I}_e \cup [e,1] \times \mathcal{I}_e \cup [e,1] \times [0,e] \cup \mathcal{I}_e \times [0,e] \cup \mathcal{I}_e \times [e,1] \cup \mathcal{I}_e \times \mathcal{I}_e$.

Ertuğrul et al.[9] verified that the relation $\sqsubseteq_{\mathcal{U}}$ defined in (1) is a partial order on L. We denote by $A(e) = [0,e] \times [e,1] \cup [e,1] \times [0,e]$ for $e \in L \setminus \{0,1\}$.

For $A \subseteq L$, we denote by $\underline{A}_{\mathcal{U}} = \{x \in L : x \sqsubseteq_{\mathcal{U}} y \text{ for all } y \in A\}$ and $\overline{A}_{\mathcal{U}} = \{x \in L : y \sqsubseteq_{\mathcal{U}} x \text{ for all } y \in A\}$, respectively. If there exist the greatest element of $\underline{A}_{\mathcal{U}}$ and the least element of $\overline{A}_{\mathcal{U}}$ with respect to $\sqsubseteq_{\mathcal{U}}$, we will denote by $\inf_{\mathcal{U}} A$ and $\sup_{\mathcal{U}} A$, respectively.

Lemma 2.7. (Ertuğrul et al. [9]) Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$. If $x \sqsubseteq_{\mathcal{U}} y$ for any $x, y \in L$, then $x \leqslant y$.

Lemma 2.8. (Ertuğrul et al. [9]) Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$. Then $(L, \sqsubseteq_{\mathcal{U}})$ is a bounded partially ordered set.

Lemma 2.9. (Karaçal and Mesiar [14]) Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $e \in L \setminus \{0,1\}$. Then

(i) $\mathcal{T}_{\mathcal{U}} : \mathcal{U} \mid [0,e]^2 : [0,e]^2 \to [0,e]$ is a t-norm on $[0,e]$.

(ii) $\mathcal{S}_{\mathcal{U}} : \mathcal{U} \mid [e,1]^2 : [e,1]^2 \to [e,1]$ is a t-conorm on $[e,1]$.

$\mathcal{T}_{\mathcal{U}}$ and $\mathcal{S}_{\mathcal{U}}$ given in Lemma 2.9 are called the underlying t-norm and t-conorm of a uninorm $\mathcal{U}$ on a bounded lattice L with the neutral element e, respectively.

Lemma 2.10. (Ertuğrul et al. [9]) Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $e \in L \setminus \{0,1\}$. Then the underlying t-norm and t-conorm of $\mathcal{U}$ are divisible if and only if $\sqsubseteq_{\mathcal{U}} = \leqslant$.

Example 2.11. (Karaçal and Mesiar [14]) Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $e \in L \setminus \{0,1\}$. Then the following uninorms $\mathcal{U}_{\mathcal{S}_\vee} : L^2 \to L$ and $\mathcal{U}_{\mathcal{T}_\wedge} : L^2 \to L$, respectively, are the smallest and the greatest uninorm on L[14].

$$\mathcal{U}_{\mathcal{S}_\vee}(x,y) = \begin{cases} x \vee y & (x,y) \in [e,1]^2, \\ x \wedge y & (x,y) \in [0,e) \times [e,1] \\ & \cup [e,1] \times (0,e], \\ y & x \in [e,1], y \in \mathcal{I}_e, \\ x & y \in [e,1], x \in \mathcal{I}_e, \\ 0 & \text{otherwise}, \end{cases} \qquad \mathcal{U}_{\mathcal{T}_\wedge}(x,y) = \begin{cases} x \wedge y & (x,y) \in [0,e]^2, \\ x \vee y & (x,y) \in [0,e] \times (e,1] \\ & \cup (e,1] \times [0,e], \\ y & x \in [0,e], y \in \mathcal{I}_e, \\ x & y \in [0,e], x \in \mathcal{I}_e, \\ 1 & \text{otherwise}. \end{cases}$$

3. THE LATTICE STRUCTURES OF THE POSET $(L, \sqsubseteq_{\mathcal{U}})$

In this section, we deal with the lattice structures of the partial orders induced by uninorms $\mathcal{U}$.

3.1. $\mathcal{T}_{\mathcal{U}}$ and $\mathcal{S}_{\mathcal{U}}$ are divisible

Proposition 3.1. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$. If $\mathcal{T}_{\mathcal{U}}$ and $\mathcal{S}_{\mathcal{U}}$ are divisible, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice.

Proof. As an immediate consequence of Lemma 2.10 one gets that $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice. □

In particular, we have the following corollary when $\mathcal{T}_{\mathcal{U}}$ and $\mathcal{S}_{\mathcal{U}}$ are divisible.

Corollary 3.2. Let $(L, \leqslant, 0, 1)$ be a bounded distributive lattice and $\mathcal{U} \in \mathfrak{U}(e)$. Then so is $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$.

Example 3.3. Let $(L, \leqslant, 0, 1)$ be a bounded distributive lattice and $\mathcal{U}$ an idempotent uninorm on L. It is straightforward that $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded distributive lattice from Lemma 2.10 and Proposition 3.1.

3.2. $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$ and $\mathcal{S}_{\mathcal{U}}$ is divisible

In this subsection, assume that $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$ and $\mathcal{S}_{\mathcal{U}}$ is divisible, such the uninorm $\mathcal{U}$ is common. In Example 2.11, Karaçal and Mesiar presented the smallest uninorm $\mathcal{U}_{\mathcal{S}_\vee}$ (resp. the greatest uninorm $\mathcal{U}_{\mathcal{T}_\wedge}$) on L which require that $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$ and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_M$ (resp. $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_M$ and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$).

Proposition 3.4. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{S}_{\mathcal{U}}$ is divisible and $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$.

(i) If $(x, y) \in (0, e)^2$ with $x \neq y$, then $x \parallel y$ with respect to $\sqsubseteq_{\mathcal{U}}$.

(ii) If $(x, y) \in (0, e)^2$ with $x \neq y$, then $\sup_{\mathcal{U}}\{x, y\} = e$ and $\inf_{\mathcal{U}}\{x, y\} = 0$.

(iii) If $(x, y) \in [e, 1]^2 \cup L^*$, then $x \sqsubseteq_{\mathcal{U}} y$ if and only if $x \leqslant y$.

P r o o f. Inasmuch as $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$ and $\mathcal{S}_{\mathcal{U}}$ is divisible.

(i) If $(x, y) \in (0, e)^2$ with $x \neq y$, assume that $x \nparallel y$ for some $x, y \in (0, e)$ with respect to $\sqsubseteq_{\mathcal{U}}$. Without loss of generality, let $x \sqsubseteq_{\mathcal{U}} y$. Then there exists an element $\ell \in [0, e]$ such that $x = \mathcal{U}(y, \ell)$. By $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$, if $\ell = e$, then $x = y$, a contradiction; If $\ell \in (0, e)$, then $x = 0$, contrary to the fact that $x \in (0, e)$. Therefore, for $x \neq y$, $x \parallel y$ with respect to $\sqsubseteq_{\mathcal{U}}$.

(ii) For any $(x, y) \in (0, e)^2$ with $x \neq y$, it follows from (i) that $x \parallel y$ with respect to $\sqsubseteq_{\mathcal{U}}$ for any $x, y \in (0, e)$. Furthermore, it is clear that $0 \sqsubseteq_{\mathcal{U}} x \sqsubseteq_{\mathcal{U}} e$ for any $x \in (0, e)$. Consequently, $\sup_{\mathcal{U}}\{x, y\} = e$ and $\inf_{\mathcal{U}}\{x, y\} = 0$ for any $(x, y) \in (0, e)^2$ with $x \neq y$.

(iii) If $(x, y) \in [e, 1]^2$, inasmuch as $\mathcal{S}_{\mathcal{U}}$ is divisible, then $x \sqsubseteq_{\mathcal{U}} y$ if and only if $x \leqslant y$ by Lemmas 2.9 and 2.10. Moreover, if $(x, y) \in L^*$, then it follows from Definition 2.6 that $x \sqsubseteq_{\mathcal{U}} y$ if and only if $x \leqslant y$. Therefore, for any $(x, y) \in [e, 1]^2 \cup L^*$, $x \sqsubseteq_{\mathcal{U}} y$ if and only if $x \leqslant y$.

□

In what follows, we shall prove that $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice and discuss their distributivity when $\mathcal{I}_e = \emptyset$.

Theorem 3.5. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{S}_{\mathcal{U}}$ is divisible and $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$. If $\mathcal{I}_e = \emptyset$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice.

P r o o f. It follows from Lemma 2.8 that $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded partially ordered set. Next, we only prove that both $\sup_{\mathcal{U}}\{x, y\}$ and $\inf_{\mathcal{U}}\{x, y\}$ always exist for any $x, y \in L$. Due to $\mathcal{I}_e = \emptyset$, we need consider the following three cases.

(i) If $x, y \in [0, e]$, then $0 \sqsubseteq_{\mathcal{U}} x \sqsubseteq_{\mathcal{U}} e$ for any $x \in L \setminus \{0, e\}$ and $\sup_{\mathcal{U}}\{x, y\} = e$ and $\inf_{\mathcal{U}}\{x, y\} = 0$ for any $x, y \in (0, e)$ from Proposition 3.4 (ii).

(ii) If $x, y \in [e, 1]$, then $\sup_{\mathcal{U}}\{x, y\} = x \vee y$ and $\inf_{\mathcal{U}}\{x, y\} = x \wedge y$ from Proposition 3.4 (iii), where $\wedge, \vee$ are obtained from $\leqslant$ of the lattice $(L, \leqslant, 0, 1)$. Consequently, $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice.

(iii) If $(x, y) \in L^*$, due to $\mathcal{I}_e = \emptyset$, i. e., $(x, y) \in A(e)$, then it follows that $\sup_{\mathcal{U}}\{x, y\} = x \vee y$ and $\inf_{\mathcal{U}}\{x, y\} = x \wedge y$ from Proposition 3.4 (iii).

Therefore, $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice. □

For the sake of convenience, we denote the cardinality of the subset $A \subseteq L$ by $|A|$. In particular, if $(L, \sqsubseteq_{\mathcal{U}})$ is distributive and $\mathcal{I}_e = \emptyset$, we have the following result.

Theorem 3.6. Let $(L, \leqslant, 0, 1)$ be a bounded distributive lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{S}_{\mathcal{U}}$ is divisible and $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$, $\mathcal{I}_e = \emptyset$.

(i) If $|(0, e)| \leqslant 2$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is distributive.

(ii) If $|(0, e)| \geqslant 3$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is nondistributive.

Proof. For any $x, y \in L$, because of $x \sqsubseteq_{\mathcal{U}} y$ if and only if $x \leqslant y$ for any $(x, y) \in [e, 1]^2 \cup A(e)$ from Proposition 3.4 (iii). Thus, we only need to check the distributivity of $(L, \sqsubseteq_{\mathcal{U}})$ when $(x, y) \in [0, e]^2$.

(i) If $|(0, e)| \leqslant 2$, then we distinguish three cases as follows.

(*a*) If $|(0, e)| = 0$, then $[0, e] = \{0, e\}$. It is trivial to check that the order $\sqsubseteq_{\mathcal{U}}$ coincides with $\leqslant$. Consequently, $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is distributive since $(L, \leqslant, 0, 1)$ is distributive.

(*b*) If $|(0, e)| = 1$, then there exists a unique element $\ell \in (0, e)$ such that $0 \lhd \ell \lhd e$ with respect to $\leqslant$, which means that $0 \sqsubseteq_{\mathcal{U}} \ell \sqsubseteq_{\mathcal{U}} e$. It is not difficult to check that the order $\sqsubseteq_{\mathcal{U}}$ coincides with $\leqslant$. Consequently, $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is distributive since $(L, \leqslant, 0, 1)$ is distributive.

(*c*) If $|(0, e)| = 2$, then there exist two elements $\ell, \xi \in (0, e)$, we have either $\ell \nparallel \xi$ or $\ell \parallel \xi$ with respect to $\leqslant$. Next, we consider the following two subcases.

- If $\ell \nparallel \xi$ with respect to $\leqslant$, then $0 \lhd \ell \lhd \xi \lhd e$ or $0 \lhd \xi \lhd \ell \lhd e$. By Proposition 3.4 (i), we have that $\ell \parallel \xi$ with respect to $\sqsubseteq_{\mathcal{U}}$, i. e., $0 \lhd \ell \lhd e$ and $0 \lhd \xi \lhd e$ with respect to $\sqsubseteq_{\mathcal{U}}$. Hence, the lattice L has no sublattice isomorphic to M_3 or N_5 with respect to $\sqsubseteq_{\mathcal{U}}$ since $(L, \leqslant, 0, 1)$ is distributive. Therefore, $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is distributive.
- If $\ell \parallel \xi$ with respect to $\leqslant$, then $0 \lhd \ell \lhd e$ and $0 \lhd \xi \lhd e$ with respect to $\leqslant$. We obtain that the order $\sqsubseteq_{\mathcal{U}}$ coincides with the order $\leqslant$. Thus, it follows from Definition 2.6 that $x \sqsubseteq_{\mathcal{U}} y$ if and only if $x \leqslant y$. Therefore, $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is distributive.

(ii) If $|(0, e)| \geqslant 3$, then it follows from Proposition 3.4 (i) that $x \parallel y$ with respect to $\sqsubseteq_{\mathcal{U}}$ for any $x, y \in (0, e)$ with $x \neq y$. Due to $0 \lhd x \lhd e$ with respect to $\sqsubseteq_{\mathcal{U}}$ for any $x \in (0, e)$. Consequently, the lattice $(L, \sqsubseteq_{\mathcal{U}})$ has a sublattice isomorphic to M_3 (see Figure 1 for a sublattice $(L|_{[0,e]}, \leqslant)$ with $|(0,e)| = 5$ and Figure 2 for the corresponding sublattice $(L|_{[0,e]}, \sqsubseteq_{\mathcal{U}})$. Therefore, $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is nondistributive. □

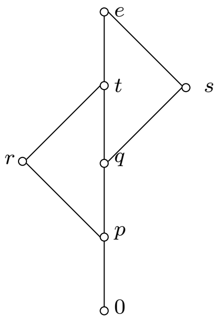

Fig. 1. The order $\leqslant$ on $[0,e]$.

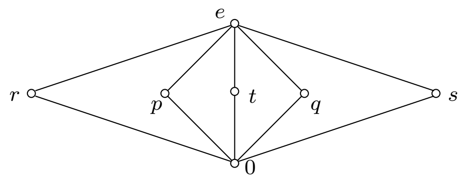

Fig. 2. The order $\sqsubseteq_{\mathcal{U}}$ on $[0,e]$.

In the following, we shall prove that $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice under some additional assumptions, and then discuss their distributivity when $\mathcal{I}_e \neq \emptyset$.

Notice that, for a bounded lattice $(L, \leqslant, 0, 1)$, $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ may not be a lattice when $\mathcal{I}_e \neq \emptyset$ in general.

Example 3.7. Consider the lattice $(L = \{0, b, c, d, e, f, g, h, k, 1\}, \leqslant, 0, 1)$, whose lattice diagram is presented in Figure 3. It is clear that $\mathcal{I}_e = \{c, g\}$. Let us consider $\mathcal{U}_t$ on L (see Theorem 1 in [6]) given as:

$$\mathcal{U}_t(x,y) = \begin{cases} \mathcal{T}_{\mathcal{U}}(x,y) & (x,y) \in [0,e]^2, \\ y & x \in [0,e], y \parallel e, \\ x & y \in [0,e], x \parallel e, \\ x \vee y & \text{otherwise}. \end{cases} \tag{2}$$

$\mathcal{U}_e^t$	0	b	d	e	c	f	g	h	k	1
0	0	0	0	0	c	f	g	h	k	1
b	0	0	0	b	c	f	g	h	k	1
d	0	0	0	d	c	f	g	h	k	1
e	0	b	d	e	c	f	g	h	k	1
c	c	c	c	c	c	f	h	h	f	1
f	f	f	f	f	f	f	h	h	f	1
g	g	g	g	g	h	h	g	h	h	1
h	h	h	h	h	h	h	h	h	h	1
k	k	k	k	k	f	g	h	h	k	1
1	1	1	1	1	1	1	1	1	1	1

Tab. 1. The uninorm $\mathcal{U}$ on $L = \{0, b, c, d, e, f, g, h, k, 1\}$.

Obviously, $\mathcal{S}_\mathcal{U}$ is divisible. If $\mathcal{T}_\mathcal{U} = \mathcal{T}_D$, then $\mathcal{U}_t$ on L can be seen in detail in Table 1. It is immediately that $b \leqslant d$ but $b \not\sqsubseteq_{\mathcal{U}_t} d$ from Proposition 3.4 (i). Furthermore, one gets that $[b \lhd e$ and $b \lhd g]$ and $[d \lhd e$ and $d \lhd g]$ with respect to $\sqsubseteq_{\mathcal{U}_t}$, which implies that $\sup_{\mathcal{U}_t}\{b, d\}$ does not exist. On the other hand, one has that $[g \rhd b$ and $g \rhd c]$ and $[e \rhd b$ and $e \rhd c]$ with respect to $\sqsubseteq_{\mathcal{U}_t}$, which implies that $\inf_{\mathcal{U}_t}\{e, g\}$ does not exist. Therefore, $(L, \sqsubseteq_{\mathcal{U}_t}, 0, 1)$ is not lattice as given in Figure 4.

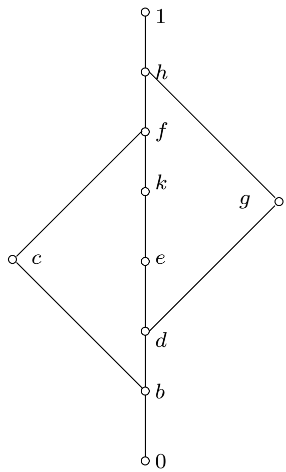

Fig. 3. The order $\leqslant$ on L.

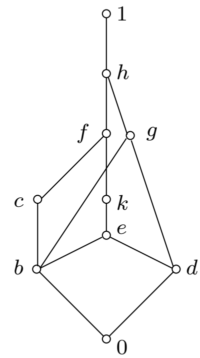

Fig. 4. The order $\sqsubseteq_{\mathcal{U}_t}$ on L.

Therefore, an interesting problem is to discuss the conditions for $(L, \sqsubseteq_\mathcal{U})$ to be a lattice when $\mathcal{I}_e \neq \emptyset$. We first have the following proposition.

Proposition 3.8. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{S}_\mathcal{U}$ is divisible and $\mathcal{T}_\mathcal{U} = \mathcal{T}_D$. If $0 \lhd x \wedge e \lhd e$ for any $x \in \mathcal{I}_e$ with respect to $\leqslant$, then the order $\sqsubseteq_\mathcal{U}$ coincides with the order $\leqslant$.

Proof. It follows from Proposition 3.4 (iii) that $x \sqsubseteq_\mathcal{U} y$ if and only if $x \leqslant y$ for $(x, y) \in [e, 1]^2 \cup L^*$. Thus, it only remains to prove that $x \sqsubseteq_\mathcal{U} y$ if and only if $x \leqslant y$ for any $x, y \in [0, e]$. Since $0 \lhd x \wedge e \lhd e$ for any $x \in \mathcal{I}_e$ with respect to $\leqslant$, then there exists a unique element $\ell = x \wedge e$ in $(0, e)$ such that $0 \lhd \ell \lhd e$ with respect to $\leqslant$. Consequently, it follows from Definition 2.6 that $0 \sqsubseteq_\mathcal{U} \ell \sqsubseteq_\mathcal{U} e$ if and only if $0 \leqslant \ell \leqslant e$. Therefore, the order $\sqsubseteq_\mathcal{U}$ coincides with the order $\leqslant$. □

The following example illustrates Proposition 3.8.

Example 3.9. Consider the lattice $(L = \{0, b, e, c, a, d, f, 1\}, \leqslant, 0, 1)$ as shown in Figure 5. It is obvious that $\mathcal{I}_e = \{c\}$. One can verify that $0 \lhd c \wedge e = b \lhd e$ with respect to $\leqslant$. Define $\mathcal{U}_t$ (see Example 3.7) on L with $\mathcal{T}_\mathcal{U} = \mathcal{T}_D$ and its lattice diagram is as in Figure 6. It follows from Proposition 3.8 that $\sqsubseteq_{\mathcal{U}_t} = \leqslant$ (see Figure 6).

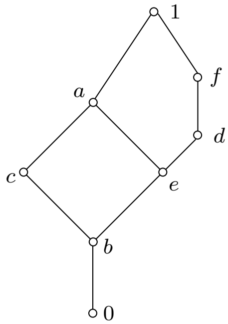

Fig. 5. The order $\leqslant$ on L.

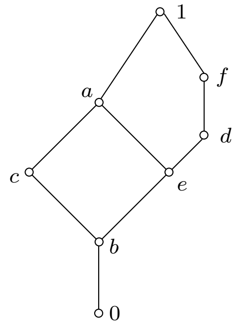

Fig. 6. The order $\sqsubseteq_{\mathcal{U}_t}$ on L.

Fortunately, we have the following two propositions.

Proposition 3.10. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{S}_{\mathcal{U}}$ is divisible and $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$. If $x \wedge e = 0$ for any $x \in \mathcal{I}_e \neq \emptyset$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice.

Proof. If $(x, y) \in [e, 1]^2 \cup L^*$, then it follows from Proposition 3.4 (iii) that $x \sqsubseteq_{\mathcal{U}} y$ if and only if $x \leqslant y$. Thus, $\sup_{\mathcal{U}}\{x, y\} = x \vee y$ and $\inf_{\mathcal{U}}\{x, y\} = x \wedge y$. If $x \wedge e = 0$ for any $x \in \mathcal{I}_e \neq \emptyset$, it only remains to prove that both $\sup_{\mathcal{U}}\{x, y\}$ and $\inf_{\mathcal{U}}\{x, y\}$ exist for any $(x, y) \in [0, e]^2$. As a matter of fact, it follows from Proposition 3.4 (ii) that $\sup_{\mathcal{U}}\{x, y\} = e$ and $\inf_{\mathcal{U}}\{x, y\} = 0$ for any $x, y \in (0, e)$ with $x \neq y$. Therefore, both $\sup_{\mathcal{U}}\{x, y\}$ and $\inf_{\mathcal{U}}\{x, y\}$ exist for any $x, y \in L$. □

Proposition 3.11. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{S}_{\mathcal{U}}$ is divisible and $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$. If $0 \lhd x \wedge e$ for any $x \in \mathcal{I}_e \neq \emptyset$ with respect to $\leqslant$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice.

Proof. Because of $0 \lhd x \wedge e$ for any $x \in \mathcal{I}_e$, it follows from Proposition 3.4 (ii) that $\sup_{\mathcal{U}}\{x, y\} = e$ and $\inf_{\mathcal{U}}\{x, y\} = 0$ for any $x, y \in (0, e)$ with $x \neq y$. Moreover, since $x \sqsubseteq_{\mathcal{U}} y$ if and only if $x \leqslant y$ for any $(x, y) \in [e, 1]^2 \cup L^*$ from Proposition 3.4 (ii). Consequently, both $\sup_{\mathcal{U}}\{x, y\}$ and $\inf_{\mathcal{U}}\{x, y\}$ exist for any $x, y \in L$. □

From Propositions 3.10 and 3.11, we immediately have the following theorem.

Theorem 3.12. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{S}_{\mathcal{U}}$ is divisible and $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$. Then $x \wedge e = 0$ or $0 \lhd x \wedge e$ for any $x \in \mathcal{I}_e \neq \emptyset$ with respect to $\leqslant$ if and only if $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice.

Proof. It follows from Propositions 3.10 and 3.11 that $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice since $x \wedge e = 0$ or $0 \lhd x \wedge e$ for any $x \in \mathcal{I}_e$.

Reciprocally, assume that there exist $\ell \in (0, e)$ and $\lambda \in \mathcal{I}_e$ such that $0 \lhd \ell \lhd \lambda \wedge e \lhd e$ and $0 \lhd \ell \lhd \lambda \wedge e \lhd \lambda$ with respect to $\leqslant$. Due to $\ell, \lambda \wedge e \in (0, e)$, thus, it follows from Proposition 3.4 (iii) that $\ell \parallel \lambda \wedge e$ with respect to $\sqsubseteq_{\mathcal{U}}$. Furthermore, we have $0 \sqsubseteq_{\mathcal{U}} \ell \sqsubseteq_{\mathcal{U}} e$ and $0 \sqsubseteq_{\mathcal{U}} \lambda \wedge e \sqsubseteq_{\mathcal{U}} e$ with respect to $\sqsubseteq_{\mathcal{U}}$, and $0 \sqsubseteq_{\mathcal{U}} \ell \sqsubseteq_{\mathcal{U}} \lambda$ and $0 \sqsubseteq_{\mathcal{U}} \lambda \wedge e \sqsubseteq_{\mathcal{U}} \lambda$ with respect to $\sqsubseteq_{\mathcal{U}}$. Consequently, both $\sup_{\mathcal{U}}\{\ell, \lambda \wedge e\}$ and $\inf_{\mathcal{U}}\{e, \lambda\}$ do not exist, contrary to the fact that $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice. Therefore, one concludes that $x \wedge e = 0$ or $0 \lhd x \wedge e$ with respect to $\leqslant$ for any $x \in \mathcal{I}_e$. □

Theorem 3.13. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{I}_e \neq \emptyset$ such that $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a lattice.

(i) If $|(0, e)| \leqslant 1$, then $(L, \leqslant, 0, 1)$ is distributive if and only if so is $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$.

(ii) If $|(0, e)| = 2$, then there are two situations as follows.

(a) If $(L, \leqslant, 0, 1)$ is distributive, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is distributive.

(b) If $(L, \leqslant, 0, 1)$ is nondistributive, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is nondistributive except $(L, \leqslant)$ contains exactly one nondistributive sublattice as presented in Figure 7 (a).

(iii) If $|(0, e)| \geqslant 3$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ has a sublattice isomorphic to M_3, i. e., $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is nondistributive.

Fig. 7. The order $\leqslant$ on L and the corresponding order $\sqsubseteq_{\mathcal{U}}$ on L.

Proof.

(i) If $|(0, e)| \leqslant 1$, then we claim that $\sqsubseteq_{\mathcal{U}} = \leqslant$. Indeed, for any $(x, y) \in [e, 1]^2 \cup L^*$, it follows from Proposition 3.4 (ii) that $x \sqsubseteq_{\mathcal{U}} y$ if and only if $x \leqslant y$. Therefore, it is enough to prove that $x \sqsubseteq_{\mathcal{U}} y$ if and only if $x \leqslant y$ when $x, y \in [0, e]$. There are two cases as follows.

(a) If $|(0, e)| = 0$, then $[0, e] = \{0, e\}$, and it is easy to see that $0 \sqsubseteq_{\mathcal{U}} e$ if and only if $0 \leqslant e$.

(b) If $|(0, e)| = 1$, then $0 \lhd x \wedge e \lhd e$ for any $x \in \mathcal{I}_e$, thus, $x \sqsubseteq_{\mathcal{U}} y$ if and only if $x \leqslant y$ for any $x, y \in [0, e]$ by Proposition 3.8.

Therefore, $\sqsubseteq_{\mathcal{U}} = \leqslant$ when $|(0, e)| \leqslant 1$.

(ii) (a) If $|(0, e)| = 2$, say $p, q \in (0, e)$, and $(L, \leqslant, 0, 1)$ is distributive, then we claim that $p \parallel q$ with respect to $\leqslant$ and $\sqsubseteq_{\mathcal{U}} = \leqslant$. Assume that $p \nparallel q$, say $p \lhd q$ with respect to $\leqslant$. Then $0 \lhd p \lhd q \lhd e$ with respect to $\leqslant$. Thus we have either $0 = x \wedge e$ or $0 \lhd p = x \wedge e$ for any $x \in \mathcal{I}_e$ from Theorem 3.12, that is, $(L, \leqslant, 0, 1)$ has a sublattice isomorphic to N_5, a contradiction. Now, we prove $\sqsubseteq_{\mathcal{U}} = \leqslant$. As a matter of fact, we have that $0 \lhd p = x \wedge e \lhd e$ or $0 \lhd q = x \wedge e \lhd e$ for any $x \in \mathcal{I}_e$. As an immediate consequence of Definition 2.6 we get that $\sqsubseteq_{\mathcal{U}} = \leqslant$. Therefore, $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is distributive.

(b) If $|(0, e)| = 2$ and $(L, \leqslant, 0, 1)$ is nondistributive, then we distinguish two cases as follows.

- If $(L, \leqslant)$ contains exactly one nondistributive sublattice as presented in Figure 7, then from Definition 2.6 it is easy to verify that $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ has no sublattice isomorphic to M_3 or N_5 with respect to $\sqsubseteq_{\mathcal{U}}$. Consequently, $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is distributive.
- Otherwise, from Definition 2.6 and Proposition 3.4 (i), we always obtain that $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ has a sublattice isomorphic to M_3 or N_5 with respect to $\sqsubseteq_{\mathcal{U}}$. Therefore, $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is nondistributive.

(iii) It is straightforward to verify that $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is nondistributive.

□

Example 3.14. Consider the lattice $(L = \{0, b, d, e, c, f, a, 1\}, \leqslant, 0, 1)$ whose lattice diagram is displayed in Figure 8, it is obvious that $\mathcal{I}_e = \{c, f\} \neq \emptyset$. It is easy to check that $(L = \{0, b, d, e, c, f, a, 1\}, \leqslant, 0, 1)$ is nondistributive since L has a sublattice isomorphic to N_5. Define $\mathcal{U}_t$ (see Example 3.7) on L with $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$ and its lattice diagram is as in Figure 9, one concludes that $(L = \{0, a, b, c, d, e, f, 1\}, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is nondistributive with respect to $\sqsubseteq_{\mathcal{U}}$ since $(L = \{0, b, d, e, c, f, a, 1\}, \sqsubseteq_{\mathcal{U}})$ has a sublattice isomorphic to N_5.

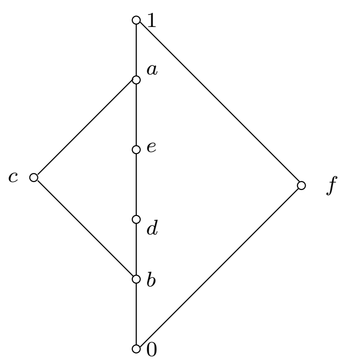

Fig. 8. The order $\leqslant$ on L.

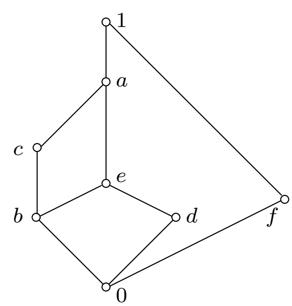

Fig. 9. The order $\sqsubseteq_{\mathcal{U}}$ on L.

3.3. $\mathcal{T}_{\mathcal{U}}$ is divisible and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$

In this subsection, by the same token, the results will be listed without proof.

Proposition 3.15. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{T}_{\mathcal{U}}$ is divisible and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$.

(i) If $(x, y) \in (e, 1)^2$ with $x \neq y$, then $x \parallel y$ with respect to $\sqsubseteq_{\mathcal{U}}$.

(ii) If $(x, y) \in (e, 1)^2$ with $x \neq y$, then $\sup_{\mathcal{U}}\{x, y\} = 1$ and $\inf_{\mathcal{U}}\{x, y\} = e$.

(iii) If $(x, y) \in [0, e]^2 \cup L^*$, then $x \sqsubseteq_{\mathcal{U}} y$ if and only if $x \leqslant y$.

Theorem 3.16. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{T}_{\mathcal{U}}$ is divisible and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$. If $\mathcal{I}_e = \emptyset$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice.

Theorem 3.17. Let $(L, \leqslant, 0, 1)$ be a bounded distributive lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{T}_{\mathcal{U}}$ is divisible and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$, $\mathcal{I}_e = \emptyset$.

(i) If $|(e, 1)| \leqslant 2$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is distributive.

(ii) If $|(e, 1)| \geqslant 3$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is nondistributive.

Proposition 3.18. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{T}_{\mathcal{U}}$ is divisible and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$. If $e \lhd x \vee e \lhd 1$ for any $x \in \mathcal{I}_e$ with respect to $\leqslant$, then the order $\sqsubseteq_{\mathcal{U}}$ coincides with the order $\leqslant$.

Proposition 3.19. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{T}_{\mathcal{U}}$ is divisible and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$. If $\mathcal{I}_e \neq \emptyset$ and $x \vee e = 1$ for any $x \in \mathcal{I}_e$ with respect to $\leqslant$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice.

Proposition 3.20. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{T}_{\mathcal{U}}$ is divisible and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$. If $\mathcal{I}_e \neq \emptyset$ and $x \vee e \lhd 1$ for any $x \in \mathcal{I}_e$ with respect to $\leqslant$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice.

Theorem 3.21. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{T}_{\mathcal{U}}$ is divisible and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$. Then $x \vee e = 1$ or $x \vee e \lhd 1$ for any $x \in \mathcal{I}_e \neq \emptyset$ with respect to $\leqslant$ if and only if $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice.

Theorem 3.22. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a lattice and $\mathcal{T}_{\mathcal{U}}$ is divisible, $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$, and $\mathcal{I}_e \neq \emptyset$.

(i) If $|(e, 1)| \leqslant 1$, then $(L, \leqslant, 0, 1)$ is distributive if and only if so is $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$.

(ii) If $|(e, 1)| = 2$, then there are two situations as follows.

(a) If $(L, \leqslant, 0, 1)$ is distributive, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is distributive.

(b) If $(L, \leqslant, 0, 1)$ is nondistributive, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is nondistributive except $(L, \leqslant)$ contains exactly one nondistributive sublattice as presented in Figure 10 (a).

(iii) If $|(e, 1)| \geqslant 3$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ has a sublattice isomorphic to M_3, i. e., $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is nondistributive.

(a) (b)

Fig. 10. The order $\leqslant$ on L and the corresponding order $\sqsubseteq_{\mathcal{U}}$ on L.

3.4. $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$ and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$

In this subsection, the proofs of the following results are immediate from both Subsections 3.2 and 3.3 for $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$ and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$.

Proposition 3.23. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$ and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$.

(i) If $x, y \in (0, e)$ with $x \neq y$, then $x \parallel y$ with respect to $\sqsubseteq_{\mathcal{U}}$.

(ii) If $x, y \in (0, e)$ with $x \neq y$, then $\sup_{\mathcal{U}}\{x, y\} = e$ and $\inf_{\mathcal{U}}\{x, y\} = 0$.

(iii) If $x, y \in (e, 1)$ with $x \neq y$, then $x \parallel y$ with respect to $\sqsubseteq_{\mathcal{U}}$.

(iv) If $x, y \in (e, 1)$ with $x \neq y$, then $\sup_{\mathcal{U}}\{x, y\} = 1$ and $\inf_{\mathcal{U}}\{x, y\} = e$.

Theorem 3.24. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$ and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$. If $\mathcal{I}_e = \emptyset$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice.

Observe that Theorem 3.24 generalizes Corollary 14 in [9].

Theorem 3.25. Let $(L, \leqslant, 0, 1)$ be a bounded distributive lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$ and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$, $\mathcal{I}_e = \emptyset$.

(i) If $|(0, e)| \leqslant 2$ and $|(e, 1)| \leqslant 2$ on $(L, \leqslant, 0, 1)$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is distributive.

(ii) If $|(0, e)| \geqslant 3$ or $|(e, 1)| \geqslant 3$ on $(L, \leqslant, 0, 1)$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is nondistributive.

Theorem 3.26. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$ such that $\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$ and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$. Then $[x \wedge e = 0$ or $0 \lhd x \wedge e]$ and $[x \vee e \lhd 1$ or $x \vee e \lhd 1]$ with respect to $\leqslant$ for any $x \in \mathcal{I}_e \neq \emptyset$ if and only if $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice.

Theorem 3.27. Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$, $\mathcal{I}_e \neq \emptyset$ such that $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a lattice.

(i) If $|(0, e)| \leqslant 1$ and $|(e, 1)| \leqslant 1$, then $(L, \leqslant, 0, 1)$ is distributive if and only if so is $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$.

(ii) If $|(0,e)| = 2$ and $|(e,1)| = 2$, then there are two situations as follows.

  (a) If $(L, \leqslant, 0, 1)$ is distributive, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is distributive.

  (b) If $(L, \leqslant, 0, 1)$ is nondistributive, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is nondistributive except $(L, \leqslant)$ contains exactly one nondistributive sublattice as presented in Figure 11.

(iii) If $|(0,e)| \geqslant 3$ or $|(e,1)| \geqslant 3$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ has a sublattice isomorphic to M_3. Therefore, $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is nondistributive.

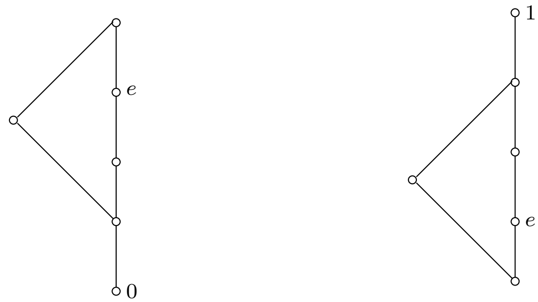

Fig. 11. The order $\leqslant$ on L.

To better observe the relationship between the underlying operators of $\mathcal{U}$ and the structure of $(L, \sqsubseteq_{\mathcal{U}})$, a summary is given in Table 2.

$(L, \sqsubseteq_{\mathcal{U}})$ / $\mathcal{T}_{\mathcal{U}}$ and $\mathcal{S}_{\mathcal{U}}$	The lattice structure of $(L, \sqsubseteq_{\mathcal{U}})$	The distributivity of $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$
$\mathcal{T}_{\mathcal{U}}$ and $\mathcal{S}_{\mathcal{U}}$ are divisible	Proposition 3.1	Corollary 3.2
$\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$ and $\mathcal{S}_{\mathcal{U}}$ is divisible	Theorem 3.5 ($\mathcal{I}_e = \emptyset$) Theorem 3.12 ($\mathcal{I}_e \neq \emptyset$)	Theorem 3.6 ($\mathcal{I}_e = \emptyset$) Theorem 3.13 ($\mathcal{I}_e \neq \emptyset$)
$\mathcal{T}_{\mathcal{U}}$ is divisible and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$	Theorem 3.16 ($\mathcal{I}_e = \emptyset$) Theorem 3.21 ($\mathcal{I}_e \neq \emptyset$)	Theorem 3.17 ($\mathcal{I}_e = \emptyset$) Theorem 3.22 ($\mathcal{I}_e \neq \emptyset$)
$\mathcal{T}_{\mathcal{U}} = \mathcal{T}_D$ and $\mathcal{S}_{\mathcal{U}} = \mathcal{S}_D$	Theorem 3.24 ($\mathcal{I}_e = \emptyset$) Theorem 3.26 ($\mathcal{I}_e \neq \emptyset$)	Theorem 3.25 ($\mathcal{I}_e = \emptyset$) Theorem 3.27 ($\mathcal{I}_e \neq \emptyset$)

Tab. 2. The lattice structures for the poset $(L, \sqsubseteq_{\mathcal{U}})$.

4. CONCLUSIONS

In [9], Ertuğrul et al. verified that $(L, \sqsubseteq_{\mathcal{U}})$ is a poset. In this work, we investigated the lattice structures of $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$. In particular, we provided some sufficient and necessary conditions for the poset $(L, \sqsubseteq_{\mathcal{U}}\ 0, 1)$ to be a lattice when the underlying t-norm of $\mathcal{U}$ is either divisible or $\mathcal{T}_D$, and t-conorm of $\mathcal{U}$ is either divisible or $\mathcal{S}_D$, and characterized the distributivity of lattices $(L, \sqsubseteq_{\mathcal{U}}\ 0, 1)$. More precisely:

Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $\mathcal{U} \in \mathfrak{U}(e)$.

(i) If $\mathcal{T}_{\mathcal{U}}$ and $\mathcal{S}_{\mathcal{U}}$ are divisible, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice.

(ii) If $\mathcal{T}_{\mathcal{U}}$ is the drastic product $\mathcal{T}_D$ and $\mathcal{S}_{\mathcal{U}}$ is divisible, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice if and only if $x \wedge e = 0$ or $0 \lhd x \wedge e$ for any $x \in \mathcal{I}_e \neq \emptyset$ with respect to $\leqslant$. In addition, the distributivity of $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ depends on the cardinality of $(0, e)$.

(iii) If $\mathcal{T}_{\mathcal{U}}$ is divisible and $\mathcal{S}_{\mathcal{U}}$ is the drastic sum $\mathcal{S}_D$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice if and only if $x \vee e = 1$ or $x \vee e \lhd 1$ for any $x \in \mathcal{I}_e \neq \emptyset$ with respect to $\leqslant$. In addition, the distributivity of $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ depends on the cardinality of $(e, 1)$.

(iv) If $\mathcal{T}_{\mathcal{U}}$ is the drastic product $\mathcal{T}_D$ and $\mathcal{S}_{\mathcal{U}}$ is the drastic sum $\mathcal{S}_D$, then $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ is a bounded lattice if and only if $[x \wedge e = 0$ or $0 \lhd x \wedge e]$ and $[x \vee e \lhd 1$ or $x \vee e \lhd 1]$ with respect to $\leqslant$ for any $x \in \mathcal{I}_e \neq \emptyset$.

The main contribution of this work is the investigation of lattice structures derived from posets $(L, \sqsubseteq_{\mathcal{U}})$, and a study of the distributivity of lattices. Due to limitations on the underlying operations of uninorms, our future works include studying the algebraic structures of $(L, \sqsubseteq_{\mathcal{U}}, 0, 1)$ for uninorms with any underlying operations.

ACKNOWLEDGEMENT

The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the referees. This work is supported by National Natural Science Foundation of China (No. 12201443), and the Natural Science Foundation of Sichuan Province (No. 25QNJJ1456).

(Received February 29, 2024)

REFERENCES

[1] E. Aşıcı and F. Karaçal: On the T-partial order and properties. Inform. Sci. *267* (2014), 323–333. DOI:10.1016/j.ins.2014.01.032

[2] E. Aşıcı: An order induced by nullnorms and its properties. Fuzzy Sets Syst. *325* (2017), 35–46. DOI:10.1016/j.fss.2016.12.004

[3] E. Aşıcı: On the properties of the F-partial order and the equivalence of nullnorms. Fuzzy Sets Syst. *346* (2018), 72–84. DOI:10.1016/j.fss.2017.11.008

[4] G. Birkhoff: Lattice Theory, 3rd edition. Colloquium Publishers, American Mathematical Society, 1967.

[5] T. Calvo, B. De Baets, and J. Fodor: The functional equations of Frank and Alsina for uninorms and nullnorms. Fuzzy Sets Syst. *120* (2001), 385–394. DOI:10.1016/S0165-0114(99)00125-6

[6] G. D. Çaylı, F. Karaçal, and R. Mesiar: On a new class of uninorms on bounded lattices. Inform. Sci. *367–368* (2016), 221–231. DOI:10.1016/j.ins.2016.05.036

[7] Q. H. Duan and B. Zhao: Maximal chains on the interval $[0,1]$ with respect to t-norm-partial orders and uninorm-partial orders. Inform. Sci. *516* (2020), 419–428. DOI:10.1016/j.ins.2019.12.073

[8] D. Dubois and H. Prade: Fundamentals of Fuzzy Sets. Kluwer Academic Publisher, Boston 2000.

[9] Ü. Ertuğrul M N. Kesicioğlu, and F. Karaçal: Ordering based on uninorms. Inform. Sci. *330* (2016), 315–327. DOI:10.1016/j.ins.2015.10.019

[10] V. K. Gupta and B. Jayaram: Importation lattices. Fuzzy Sets Syst. *405* (2021), 1–17. DOI:10.1016/j.fss.2020.04.003

[11] V. K. Gupta and B. Jayaram: Order based on associative operations. Inform. Sci. *566* (2021), 326–346. DOI:10.1016/j.ins.2021.02.020

[12] V. K. Gupta and B. Jayaram: Clifford's order obtained from uninorms on bounded lattices. Fuzzy Sets Syst. *462* (2023), 108384. DOI:10.1016/j.fss.2022.08.016

[13] F. Karaçal and M. N. Kesicioğlu: A T-partial order obtained from t-norms. Kybernetika *47* (2011), 300–314. DOI:10.1159/000337863

[14] F. Karaçal and R. Mesiar: Uninorms on bounded lattices. Fuzzy Sets Syst. *261* (2015), 33–43. DOI:10.1016/j.fss.2014.05.001

[15] M. N. Kesicioğlu: Some notes on the partial orders induced by a uninorm and a nullnorm in a bounded lattice. Fuzzy Sets Syst. *346* (2018), 55–71. DOI:10.1016/j.fss.2017.08.010

[16] M. N. Kesicioğlu, F. Karaçal, and R. Mesiar: Order-equivalent triangular norms. Fuzzy Sets Syst. *268* (2015), 59–71. DOI:10.1016/j.fss.2014.10.006

[17] M. N. Kesicioğlu, Ü. Ertuğrul, and F. Karaçal: An equivalence relation based on the U-partial order. Inform. Sci. *411* (2017), 39–51. DOI:10.1016/j.ins.2017.05.020

[18] M. N. Kesicioğlu, Ü. Ertuğrul, and F. Karaçal: Some notes on U-partial order. Kybernetika *55* (2019), 518–530. DOI:10.14736/kyb-2019-3-0518

[19] M. N. Kesicioğlu: On the relationships between the orders induced by uninorms and nullnorms. Fuzzy Sets Syst. *378* (2020), 23–43. DOI:10.1016/j.fss.2018.12.020

[20] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. Springer Netherlands, 2000.

[21] G. Li, H. W. Liu, and Y. Su: On the conditional distributivity of nullnorms over uninorms. Inform. Sci. *317* (2015), 157–169. DOI:10.1016/j.ins.2015.04.040

[22] W. H. Li and F. Qin: Conditional distributivity equation for uninorms with continuous underlying operators. IEEE Trans. Fuzzy Syst. *28* (2020), 1664–1678. DOI:10.1109/TFUZZ.2019.2920809

[23] Z. Q. Liu: New R-implication generated by T-partial order. Comp. Appl. Math. *43* (2024), 425. DOI:10.1007/s40314-024-02939-5

[24] Z. Q. Liu: Clifford's order based on non-commutative operations. Iran. J. Fuzzy syst. *21(3)* (2024), 77–90. DOI:10.22111/IJFS.2024.47502.8363

[25] G. Mayor and J. Torrens: On a class of operators for expert systems. Int. J. Intell. Syst. *8* (1993), 771–778. DOI:10.1002/int.4550080703

[26] A. Mesiarová-Zemánková: Characterization of uninorms with continuous underlying t-norm and t-conorm by means of the ordinal sum construction. Int. J. Approx. Reason. *83* (2017), 176–192. DOI:10.1016/j.ijar.2017.01.007

[27] A. Mesiarová-Zemánková: Natural partial order induced by a commutative, associative and idempotent function. Inform. Sci. *545* (2021), 499–512. DOI:10.1016/j.ins.2020.09.028

[28] A. Mesiarová-Zemánková: Representation of non-commutative, idempotent, associative functions by pair-orders. Fuzzy Sets Syst. *475* (2023), 108759. DOI:10.1016/j.fss.2023.108759

[29] K. Nanavati and B. Jayaram: Order from non-associative operations. Fuzzy Sets Syst. *467* (2023), 108484. DOI:10.1016/j.fss.2023.02.005

[30] B. Saminger: On ordinal sums of triangular norms on bounded lattices. Fuzzy Sets Syst. *157* (2006), 1403–1416. DOI:10.1016/j.fss.2005.12.021

[31] R. R. Yager and A. Rybalov: Uninorm aggregation operators. Fuzzy Sets Syst. *80* (1996), 111–120. DOI:10.1016/0165-0114(95)00133-6

[32] R. R. Yager: Uninorms in fuzzy systems modeling. Fuzzy Sets Syst. *122* (2001), 167–175. DOI:10.1016/S0165-0114(00)00027-0

Zhi-qiang Liu, School of Mathematical Sciences, Sichuan Normal University, Chengdu, Sichuan 610066, P. R. China and V. C. & V.Ṙ. Key Lab of Sichuan Provence, Sichuan Normal University, Chengdu 610068, Sichuan. P. R. China.

e-mail: lzq220@outlook.com