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PROPERTIES OF QUANTUM LOGIC MAPS AS FUZZY RELATIONS ON A SET OF ALL SYMMETRIC AND IDEMPOTENT BINARY MATRICES

Reinis Isaks

A quantum logic is one of possible mathematical models for non-compatible random events. In this work we solve a problem proposed at the conference FSTA 2006. Namely, it is proved that s-maps are symmetric fuzzy relations on a set of all symmetric and idempotent binary matrices. Consequently s-maps are not antisymmetric fuzzy relations. This paper also explores other properties of s-maps, j-maps and d-maps. Specifically, it is proved that s-maps are neither reflexive, nor irreflexive, and nor transitive, j-maps have the same properties as s-maps and d-maps are reflexive and not transitive.

Keywords: quantum logic, s-map, fuzzy relations

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1. INTRODUCTION

In 2006, at the 8th Conference on Fuzzy Set Theory and Applications, a number of problems related to the classification of strict t-norms, Lipschitz t-norms, interval semigroups, copulas, semicopulas and quasicopulas, fuzzy implications, mean values, fuzzy relations, MV-algebra, and effect algebras were raised [3]. This work offers a solution for the problem of s-maps as fuzzy relations, which was introduced in this conference. Namely, it is proved that s-maps are symmetric fuzzy relations on a set of all symmetric and idempotent binary matrices. This paper also explores properties of s-maps, j-maps and d-maps.

The paper is structured in the following way: in the Section 2 the initial problem is given; we study the properties of s-maps in Section 3; we propose the solution for the initial problem in Section 4; in Section 5 and 6, we examine j-maps and d-maps on a set of all symmetric and idempotent binary matrices.

2. THE GIVEN PROBLEM

Let $\mathcal{A}$ be a set of all symmetric matrices, such that each element of a matrix is either 0 or 1 and for each $A \in \mathcal{A}$ we have $A \cdot A = A$. Let I be the identity matrix and let Θ be the zero matrix.

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Definition 2.1. (Nánásiová and Pulmannová [5]) A fuzzy relation $s:\ \mathcal{A}\times\mathcal{A}\ \to [0,\ 1]$ is called an s-map, if:

1. $s\,(I,\ I) = 1$;
2. if $A,\ B\in\mathcal{A}$ such that $A\cdot B = B\cdot A = \Theta$, then $s\,(A,\ B) = 0$;
3. if $A,\ B\in\mathcal{A}$ such that $A\cdot B = B\cdot A = \Theta$, then for all $C\in\mathcal{A}$:
   - $s\,(A+B,\ C) = \ s\,(A,C) + s\,(B,C)$,
   - $s\,(C,\ A+B) = \ s\,(C,A) + s\,(C,B)$.

Let us formulate the main theorem that is ought to be proven:

Theorem 2.2. (Mesiar and Klement [3]) If the fuzzy relation $s:\ \mathcal{A}\times\mathcal{A}\ \to [0,\ 1]$ is an s-map, then it is symmetric, i. e., $s\,(A,\ B) = s\,(B,A)$ for all $A,B\in\mathcal{A}$.

3. PROPERTIES OF S-MAPS

From the given conditions, we will formulate the properties of the s-map, which will help us to solve the symmetry problem and understand the given fuzzy relation.

Let us prove that the s-map of any matrix from set $\mathcal{A}$ and the identity matrix is symmetric:

Lemma 3.1. (Al-Adilee and Nánásiová [1]) If the fuzzy relation s is an s-map, then for all $A\in\mathcal{A}: s\,(A,\ I) = s\,(I,A)$.

Proof. For a given matrix A, put $B = I - A$. Then $A+B = I$, $A\cdot B = B\cdot A = \Theta$ and using of 2. of Definition 2.1, we compute $s(A,I) = s(A,A) = s(I,A)$. □

Definition 3.2. A set of matrices $\mathcal{B}\subset\mathcal{A}$ is called a null set if for all $A,B\in\mathcal{B}$ and $A\neq B: s\,(A,\ B) = s\,(B,A) = 0$, where s is an s-map.

Let us prove that s-maps have the distributive property over addition:

Lemma 3.3. If the fuzzy relation s is an s-map, $H_1,H_2,\ldots,\ H_i\in\mathcal{B}$ $(i\in\mathbb{N}\setminus\{1\})$, then for all $A\in\mathcal{A}$:

$$s\,(H_1+H_2+\ldots+\ H_i,A) = s\,(H_1,A)+\ s\,(H_2,A)+\ldots+s\,(H_i,A)\,,$$

$$s\,(A,H_1+H_2+\ldots+\ H_i) = s\,(A,H_1)+\ s\,(A,H_2)+\ldots+s\,(A,H_i)\,.$$

Proof. The proof follows from 2. of Definition 2.1 by induction. □

Next we will look at the properties of s-maps to examine them more closely. We will start by proving that s-maps are not reflexive or irreflexive fuzzy relations.

Remark 3.4. If a fuzzy relation s is an s-map, then it is neither reflexive nor irreflexive.

Namely, $s(\Theta, \Theta) = 0$ and $s(I, I) = 1$ which contradicts the definitions of reflexivity and irreflexivity [2], respectively.

Now let us prove that s-maps are not transitive.

Theorem 3.5. If a fuzzy relation s is an s-map, then it is not transitive.

Proof. Suppose we are given an s-map of s matrices of order n. We find two matrices $A, B \in \mathcal{A}$ such that $A \neq \Theta$ and $B \neq \Theta$, and $A \cdot B = B \cdot A = \Theta$. We assume that the s-map is transitive. Then by the definition of transitivity [2] it follows that: $s\,(A, B) \wedge s\,(B, C) \leq s\,(A, C)$ for all $A, B, C \in \mathcal{A}$. Therefore:

$$s\,(A, I) \wedge s\,(I, B) \leq s\,(A, B)\,.$$

But since $s\,(A, B) = s\,(B, A) = 0$, it follows that:

$$s\,(A, I) \wedge s\,(I, B) \leq 0$$

Hence, one of the values $s\,(A, I)$ or $s\,(I, B)$ is equal to 0.

Assume that $s\,(A, I) = 0$ (An analogous proof if one chooses that $s\,(I, B) = 0$). But then from Lemma 3.1 it follows that:

$$s\,(A, I) = s\,(A, A) = 0.$$

But in that case, the only matrix for which $s\,(A, A) = 0$ is satisfied is the zero matrix, i. e. $A = \Theta$. But that contradicts our initial assumption. □

4. SYMMETRY OF S-MAPS

It is easy to see that the only matrices which belong to set $\mathcal{A}$ are diagonal. To make it more convenient to write the matrices, let's denote them respectively by tuples of order n, which consist of the elements of the main diagonal of matrices from the set $\mathcal{A}$. In general:

$$\begin{pmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix} \leftrightharpoons (a_{11}, a_{22}, \ldots, a_{nn}) \leftrightharpoons (a_1, a_2, \ldots, a_n)\,.$$

Now we will define the terms base matrices, basis set and explore their properties.

Definition 4.1. A matrix $C \in \mathcal{A}$ is called a base matrix if only one of its elements is 1.

The set of all possible basis matrices $\Delta \subset \mathcal{A}$ will be called a basis set:

$$\Delta = \{C_1 = (1, 0, \ldots, 0)\,, C_2 = (0, 1, \ldots, 0)\,, \ldots, C_n = (0, 0, \ldots, 1)\}.$$

It is easy to see that every matrix A from set $\mathcal{A}$ can be expressed as the sum of base matrices. In fact, if $A = (a_1, a_2, \ldots, a_n)$, then $A = a_1 \cdot C_1 + a_2 \cdot C_2 + \ldots + a_n \cdot C_n$.

Lemma 4.2. The basis set $\Delta \subset \mathcal{A}$ is a null set.

Proof. We choose arbitrary matrices $A, B \in \Delta$, $A \neq B$. Then, for an arbitrary index i, if $a_i = 1$ then $b_i = 0$ and vice versa, $b_i = 1$ implies $a_i = 0$. Therefore $A \cdot B = B \cdot A = \Theta$ and this gives Δ is a null set. □

Now we have defined and proved everything necessary for the solution of the problem. Let us prove that from the given conditions, it follows that the s-map is always symmetric on a set of symmetric binary matrices.

Theorem 4.3. If the fuzzy relation s is an s-map, then $s(A, B) = s(B, A)$ for all $A, B \in \mathcal{A}$.

Proof. We consider the s-map $s : \mathcal{A} \times \mathcal{A} \to [0, 1]$ and choose matrices $A, B \in \mathcal{A}$, which are of order n. We divide the matrix A into the sum of the basis matrices:

$$A = a_1 \cdot C_1 + a_2 \cdot C_2 + \ldots + a_n \cdot C_n = H_1 + H_2 + \ldots + H_j = \sum_{t=1}^{j} H_t \ (1 \leq j \leq n),$$

where H_t are the basis matrices for which $a_i = 1$ $(i = 1, 2, \ldots, n)$.

$$s(A, B) = s\left(\sum_{t=1}^{j} H_t, B\right).$$

We use Lemma 3.1:

$$s\left(\sum_{t=1}^{j} H_t, B\right) = \sum_{t=1}^{j} s(H_t, B).$$

Next, we divide matrix B into the sum of base matrices. As with the matrix A, let's denote by G_k $(1 \leq k \leq n)$ those base matrices for which $b_i = 1$ $(i = 1, 2, \ldots, n)$.

$$B = b_1 \cdot C_1 + b_2 \cdot C_2 + \ldots + b_n \cdot C_n = G_1 + G_2 + \ldots + G_k = \sum_{m=1}^{k} G_m.$$

Again using Lemma 3.1 we get:

$$\sum_{t=1}^{j} s(H_t, B) = \sum_{t=1}^{j} s\left(H_t, \sum_{m=1}^{k} G_m\right) = \sum_{t=1}^{j} \sum_{m=1}^{k} s(H_t, G_m).$$

Consider an arbitrary s-map $s(H_i, G_r)$, where $1 \leq i \leq j$ and $1 \leq r \leq k$.

- if $H_i = G_r$, then $s(H_i, G_r) = s(G_r, H_i) = s(H_i, H_i) = s(G_r, G_r)$;
- if $H_i \neq G_r$, then $s(H_i, G_r) = s(G_r, H_i) = 0$.

Hence:

$$s(A,B)=\sum_{t=1}^{j}\sum_{m=1}^{k}s(H_t,G_m)=\sum_{t=1}^{j}\sum_{m=1}^{k}s(G_m,H_t)=$$

$$\sum_{m=1}^{k}\sum_{t=1}^{j}s(G_m,H_t)=\sum_{m=1}^{k}s(G_m,\sum_{t=1}^{j}H_t)=s\left(\sum_{m=1}^{k}G_m,\sum_{t=1}^{j}H_t\right)=s(B,A).$$

□

Consequences: If a fuzzy relation s is an s-map, then it is not antisymmetric.

Example 4.4. For matrices of order n, the fuzzy relation $s(A,B)=\frac{tr(A\cdot B)}{n}$ is an s-map (see Table 1). Note that in this and the following examples $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ and $B=\begin{pmatrix}0&0\\0&1\end{pmatrix}$.

$s(A,B)$	Θ	A	B	I
Θ	0	0	0	0
A	0	$\frac{1}{2}$	0	$\frac{1}{2}$
B	0	0	$\frac{1}{2}$	$\frac{1}{2}$
I	0	$\frac{1}{2}$	$\frac{1}{2}$	1

Tab. 1. S-map values for 2×2 matrices.

5. PROPERTIES OF J-MAPS

Let us start with the definition of a j-map.

Definition 5.1. (Al-Adilee and Nánásiová [1]) A fuzzy relation $q:\ \mathcal{A}\times\mathcal{A}\ \to[0,\ 1]$ is called a j-map, if:

1. $q(\Theta,\Theta)=0$ and $q(I,I)=1$;
2. if $A,B\in\mathcal{A}$ such that $A\cdot B=B\cdot A=\Theta$, then $q(A,B)=q(A,A)+q(B,B)$;
3. if $A,B\in\mathcal{A}$ such that $A\cdot B=B\cdot A=\Theta$, then for all $C\in\mathcal{A}$:
   - $q(A+B,C)=\ q(A,C)+q(B,C)-q(C,C)$,
   - $q(C,A+B)=\ q(C,A)+q(C,B)-q(C,C)$.

It is easy to see from the second property of j-maps, that j-maps are symmetric for all matrices in the null set.

Lemma 5.2. (Al-Adilee and Nánásiová [1]) If $A, B \in \mathcal{A}$ such that $A \cdot B = B \cdot A = \Theta$, then $q(A+B, A+B) = q(A, B)$.

Proof. The proof follows from 3. of Definition 5.1. □

It is easy to see, that j-maps are neither reflexive nor irreflexive nor transitive. The proofs are very similar to the ones in chapter "Properties of s-maps". Now we are going to prove that j-maps are symmetric for all $A, B \in \mathcal{A}$.

Theorem 5.3. If the fuzzy relation q is a j-map, then $q(A,\ B) = q(B, A)$ for all $A, B \in \mathcal{A}$.

Proof. We consider the j-map $q\ :\ \mathcal{A} \times \mathcal{A}\ \to [0,\ 1]$. We can then write matrices A and B which are of order n as sum of their respective base matrices:

$$A = \sum_{t=1}^{j} H_t \ (1 \le j \le n) \text{ and } B = \sum_{k=1}^{m} G_k \ (1 \le m \le n).$$

Hence from 3. of Definition 5.1:

$$q(A, B) = q(\sum_{t=1}^{j} H_t\ , \sum_{k=1}^{m} G_k) = \sum_{t=1}^{j} q(H_t, \sum_{k=1}^{m} G_k)) - (j-1) \cdot q(\sum_{k=1}^{m} G_k, \sum_{k=1}^{m} G_k)$$

$$= \sum_{t=1}^{j} \sum_{k=1}^{m} q(H_t, G_k) - (m-1) \cdot \sum_{t=1}^{j} q(H_t, H_t) - (j-1) \cdot q(\sum_{k=1}^{m} G_k, \sum_{k=1}^{m} G_k).$$

Then we can apply Lemma 5.2 and 3. of Definition 5.1 to get:

$$q(\sum_{k=1}^{m} G_k, \sum_{k=1}^{m} G_k) = \sum_{k=1}^{m} q(G_k, G_k).$$

We can acknowledge that $q(H_t, G_k) = q(G_k, H_t)$ for all base matrices, where $t = 1, 2, \ldots, j$ and $k = 1, 2, \ldots, m$. Hence we can conclude that:

$$q(A, B) = \sum_{t=1}^{j} \sum_{k=1}^{m} q(H_t, G_k) - (m-1) \cdot \sum_{t=1}^{j} q(H_t, H_t) - (j-1) \cdot \sum_{k=1}^{m} q(G_k, G_k)$$

$$= \sum_{t=1}^{j} \sum_{k=1}^{m} q(G_k, H_t) - (m-1) \cdot \sum_{t=1}^{j} q(H_t, H_t) - (j-1) \cdot \sum_{k=1}^{m} q(G_k, G_k) = q(B, A).$$

□

$q(A,B)$	Θ	A	B	I
Θ	0	$\frac{1}{2}$	$\frac{1}{2}$	1
A	$\frac{1}{2}$	$\frac{1}{2}$	1	1
B	$\frac{1}{2}$	1	$\frac{1}{2}$	1
I	1	1	1	1

Tab. 2. J-map values for 2×2 matrices.

Example 5.4. For matrices of order n, the fuzzy relation $q(A,B) = \frac{\sum_{i=1}^{n} max(a_i,b_i)}{n}$ is a j-map (see Table 2).

6. PROPERTIES OF D-MAPS

Definition 6.1. (Al-Adilee and Nánásiová [1]) A fuzzy relation $d : \mathcal{A} \times \mathcal{A} \to [0,\ 1]$ is called a d-map, if:

1. $d(\Theta, I) = d(I, \Theta) = 1$;
2. $d(A,A) = 0$ for all $A \in \mathcal{A}$
3. if $A, B \in \mathcal{A}$ such that $A \cdot B = B \cdot A = \Theta$, then $d(A,B) = d(A,\Theta) + d(\Theta,B)$;
4. if $A, B \in \mathcal{A}$ such that $A \cdot B = B \cdot A = \Theta$, then for all $C \in \mathcal{A}$:
    - $d(A+B, C) = d(A,C) + d(B,C) - d(\Theta, C)$,
    - $d(C, A+B) = d(C,A) + d(C,B) - d(C,\Theta)$.

From 2. of Definition 6.1, we get that d-maps are reflexive. As with all other quantum logic maps, d-maps are not transitive.

From the following examples, we can notice that the symmetry of d-maps does not follow from the given properties.

Example 6.2.

$d(A,B)$	Θ	A	B	I
Θ	0	$\frac{1}{3}$	$\frac{2}{3}$	1
A	$\frac{2}{3}$	0	1	$\frac{1}{3}$
B	$\frac{1}{3}$	1	0	$\frac{2}{3}$
I	1	$\frac{2}{3}$	$\frac{1}{3}$	0

Tab. 3. D-map values for 2×2 matrices.

Example 6.3. (Nánásiová and Valášková [4])

$$d(A, B) = s(A, A) + s(B, B) - 2s(A, B),$$

where $s\colon \mathcal{A} \times \mathcal{A} \to [0,\ 1]$ is an s-map.

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REFERENCES

[1] A. M. Al-Adilee and O. Nánásiová: Copula and s-map on a quantum logic. Inform. Sci. *24* (2009), 4199–4207. DOI:10.1016/j.ins.2009.08.011

[2] A. Šostak: L-sets and L-valued structures. Nr. 2. University of Latvia, 2003.

[3] R. Mesiar and E. P. Klement: Open problems posed at the eighth international conference on fuzzy set theory and applications. Kybernetika *42* (2006), 2, 225–235.

[4] O. Nánásiová and L. Valášková: Maps on a quantum logic. Soft Computing *14* (2010), 1047–1052. DOI:10.1007/s00500-009-0483-4

[5] O. Nánásiová and S. Pulmannová: S-map and tracial states. Inform. Sci. *5* (2009), 515–520. DOI:10.1016/j.ins.2008.07.032

Reinis Isaks, Department of Mathematics, University of Latvia, Jelgavas street 3, LV-1004 Riga. Latvia.
e-mail: reinis.isaks@inbox.lv