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AN LMI-BASED CONVEX FAULT TOLERANT CONTROL OF NONLINEAR DESCRIPTOR SYSTEMS VIA UNKNOWN INPUT OBSERVERS

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This paper proposes a fault tolerant control scheme for nonlinear systems in descriptor form. The approach is based on the design of an unknown input observer in order to estimate the missing state variables as well as actuator faults, such design is carried out once a proper estimation error system is obtained via a recent factorization method; then, the estimated signals are employed in the control law in order to drive the states asymptotically to the origin despite actuator faults. The designing conditions are given in terms of linear matrix inequalities. Numerical as well as physical systems are used to illustrate the advantages of the proposal.

Keywords: Takagi–Sugeno model, descriptor system, fault tolerant control, linear matrix inequality, Lyapunov method, unknown input observer

Classification: 93B53, 93B50,93C10,93C15,93D05

1. INTRODUCTION

The behavior of a closed-loop system is affected by uncertainties, disturbances, and faults; such misbehavior may lead to physical deterioration of the system. In the case of faults, we can identify the ones occurring at sensors, actuators or in the process [2, 59]. A fault can be defined as an abnormal behavior or deviation of at least one characteristic property or parameter that consequently modifies the average performance of the system [24]. Many approaches have been developed to increase the safety operation of systems as to achieve control tasks such as asymptotic stability, trajectory tracking, among others. For example, fault diagnosis and isolation [13, 42, 44, 53], fault estimation [23, 49], fault tolerant controllers [5, 15]. Many of these approaches are based on observers, in particular unknown input (UI) ones [20, 36, 37, 41, 46, 56]; such observer allows estimating both the system states as well as the actuator faults; in this case, the better the observer scheme is the more efficient the task of diagnosis or the performance of the fault tolerant scheme is. Particularly, convex (LPV systems, Takagi–Sugeno (TS) models) approaches are highly appreciated because the designing conditions are given in terms of linear matrix inequalities (LMIs) [6] as a result of using the direct Lyapunov method together with convex models (constituted by a blending of linear vertex models

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and scalar convex functions) [51]. Nevertheless, for observer schemes, an important problem related to the scalar convex functions must be faced, that is, when the functions depend on signals (scheduling variables) that are not available/measured [30]; such issue has been overcome by means of uncertain/perturbed setups [34], Lipschitz constraints [57, 58], the differential mean value theorem [18]. Recently, in [43] a methodology based on factorization allows overcoming this persistent problem, thus a greater family of nonlinear systems can be treated.

On the other hand, the presence of a fault does not always mean the stoppage of the system, therefore the fault tolerant control (FTC) must allow the system to continue operating with a behavior as close as possible to the nominal, maintain stability, and provide a desirable performance [35]. The FTC scheme hereby proposed belongs to the active type, that is, it permits reconfiguration of the control parameters [33, 60]. Observer-based FTC schemes have been design via several techniques such as virtual actuators/sensors [4], adaptive setups [32, 54], convex approaches [1, 21, 45, 40]. In particular, in [21], two FTC approaches are presented, but these algorithms only consider available scheduling variables and they do not contemplate descriptors models; although [45] considers descriptor systems, conditions cannot be used because the scheduling vector must be known.

Contribution: An extension of the factorization methodology presented in [43] to descriptor systems is developed; this allows constructing an UI observer to estimate the missing states and the actuator faults. Thus, in the context of convex approaches, the proposal overcomes traditional shortcomings from [1, 21, 45, 40], especially with respect to unmeasurable premise variables; thus, the family of systems under consideration is larger. In contrast with sliding mode and high-gain observers, the proposal provides a systematic way of computing the observer and controller gains, with the possibility of adding performance requirements (speed convergence, bounds on the input/output) within a single framework. As a result, an active observer-based fault tolerant control scheme capable of dealing with descriptor systems is designed by means of LMIs.

Organization: The rest of the document is organized as follows: in Section 2, the problem to be studied is stated while a brief on the factorization method is given and illustrated, convex modeling is also introduced in this section. In Section 3, the main results are given, they are divided in three parts: (1) the design of an UI observer, (2) the design of nonlinear controller (3) the integration of the the complete FTC scheme. In Section 4 the performance of the proposal is illustrated via two examples, the first one being a numerical nonlinear system while the second one is the well-known pendulum on a cart. Finally, Section 5 presents the conclusions and final remarks.

Notation: An asterisk $(*)$ will be used in-line expressions to denote the transpose of the terms on its left side, i. e., $A + B + A^T + B^T + C = A + B + (*) + C$; in matrix expressions it denotes the transpose of the symmetric element. Additionally, $P > 0\,(< 0)$ means that $P \in \mathbb{R}^{n\times n}$ is positive (negative) definite. Arguments will be omitted when their meaning can be inferred from the context.

2. PROBLEM STATEMENT

Consider a nonlinear descriptor system under actuator faults

$$E(y)\dot{x}(t)=f(x)+g(x)\left(u(t)+f_a(t)\right),\quad y(t)=h(x), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^o$ is the output vector, $f_a \in \mathbb{R}^m$ is the actuator fault vector, the nonlinear mappings $f(x) : \mathbb{R}^n \mapsto \mathbb{R}^n$, $g(x) : \mathbb{R}^n \mapsto \mathbb{R}^{n\times m}$, and $h(x) : \mathbb{R}^n \mapsto \mathbb{R}^o$ are assumed to be sufficiently smooth, $E(y) : \mathbb{R}^o \mapsto \mathbb{R}^{n\times n}$ is assumed to be invertible for all x in a region around the origin [17]. It is also assumed that the origin $x = 0$ is an equilibrium point of the system. The task is to design a nonlinear control law of the form

$$u(t) = K(\tilde{x})\hat{x}(t) - \hat{f}_a(t), \tag{2}$$

$K(\tilde{x}) \in \mathbb{R}^{m\times n}$, $\tilde{x} \in \mathbb{R}^n$ is employed to indicate that the controller gain only depends on available signals $\hat{x}$ and y, such that the closed-loop system holds $\lim_{t\to\infty} x(t) = 0$, i. e., the system trajectories go asymptotically to zero despite actuator faults. The design is carried out under the assumption[1] $\dot{f}_a^{(p)}(t) \approx 0$, p is the derivative order, the estimated signals $\hat{x}(t)$ and $\hat{f}_a(t)$ are calculated from an UI observer such that the estimation errors $e_x(t) = x(t) - \hat{x}(t)$ and $e_{f_a}(t) = f_a(t) - \hat{f}_a(t)$ satisfy $\lim_{t\to\infty} e_x(t)=0$ and $\lim_{t\to\infty} e_{f_a}(t)=0$. Once the signals $\hat{x}$ and $\hat{f}_a$ are recovered, they can be employed in the fault tolerant control scheme (2). In [9], an UI observer for descriptor systems has been presented; however, this observer cannot deal with unmeasurable scheduling variables, thus reducing its applicability to real systems. In order to overcome this issue, section 3.1 presents an UI observer based on the factorization proposed by [43].

2.1. Factorization of error signals

The observation problem for nonlinear systems obliges the designer to deal with expressions of the form $f(x) - f(\hat{x})$ in order to factorize the observation error $e = x - \hat{x}$ as to write the time derivative of the Lyapunov function in the form $e^T Q(\cdot)e$, so LMI conditions can be derived. Most of the existing works, especially those based on convex structures, consider $f(x) = A(y)x$ and thus $A(y)x - A(y)\hat{x} = A(y)(x - \hat{x}) = A(y)e$ [17, 29]; nevertheless, in general this is not always the case [18, 21]. In [43], a methodology to factorize at the left-hand side the error signal is provided as long as the hypotheses of the differential mean value theorem (DMVT) hold. Thus, such approach allows obtaining $f(x) - f(\hat{x}) = F(x, \hat{x})(x - \hat{x})$ where $f(x)$ could be a function with multivariate polynomial or non-polynomial expressions; the former can be directly arranged for factorization while the later ones are approximated with any degree of accuracy by Taylor series, i. e.,

$$f(x) - f(\hat{x}) \approx \sum_{i=0}^{v} \frac{f^{(i)}(0)}{i!} x^i - \sum_{i=0}^{v} \frac{f^{(i)}(0)}{i!} \hat{x}^i.$$

Expanding up to a v-order (including the most significant terms) suffices to factorize the error signal and perform the required manipulations to achieve the corresponding

[1] This assumption is customary in the context of proportional multi-integral (PMI) observers [26, 28, 22, 11]. In practice, even piecewise continuously differentiable but bounded signals can be estimated.

analysis and design, since from the numerical point of view the high order terms make no difference.

For example, let us consider $x = [x_1 \ x_2]^T$ and $\hat{x} = [\hat{x}_1 \ \hat{x}_2]^T$, functions $f(x) = [x_1^2 \ \sin(x_2) + x_1]^T$ and $f(\hat{x}) = [\hat{x}_1^2 \ \sin(\hat{x}_2) + \hat{x}_1]^T$. Thus, we have the following

$$f(x)-f(\hat{x}) = \begin{bmatrix} (x_1 - \hat{x}_1)(x_1 + \hat{x}_1) \\ (x_1 - \hat{x}_1) + \sin x_2 - \sin \hat{x}_2 \end{bmatrix} \approx \begin{bmatrix} x_1 + \hat{x}_1 & 0 \\ 1 & 1 - \frac{1}{6}(x_2^2 + x_2\hat{x}_2 + \hat{x}_2^2) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

where $e_1 = x_1 - \hat{x}_1$, $e_2 = x_2 - \hat{x}_2$ and a third order approximation for $\sin x_2 - \sin \hat{x}_2$ have been used.

2.2. Convex structures

The sector nonlinearity [39] is employed to express bounded non-constant terms $z(\cdot) \in [z^0, z^1]$ as a convex sums of its bounds, that is, $z(\cdot) = w_0(z)z^0 + w_1(z)z^1$, where z^0 and z^1 are the minimum and maximum of $z(\cdot)$ in a region of interest; functions $w_0 = (z^1 - z(\cdot))/(z^1 - z^0)$ and $w_1 = 1 - w_0$ hold the convex sum property in the same region, i. e., $w_0(z) + w_1(z) = 1$ and $w_0, w_1 \in [0, 1]$. In our case, in order to obtain an useful convex rewriting, the non-constant terms are split into those depending exclusively on available signals ($\hat{x}$ and y), gathered in $z(\hat{x}, y) \in \mathbb{R}^s$, while the rest of terms are gathered in $\zeta(x, \hat{x}, y) \in \mathbb{R}^\sigma$. Due to designing specifications, each entry of $z(\hat{x}, y)$ and $\zeta(x, \hat{x}, y)$ is bounded in $x \in \Omega_x \subset \mathbb{R}^n$, $\hat{x} \in \Omega_{\hat{x}} \subset \mathbb{R}^n$, i. e., $z_i(\hat{x}, y) \in [z_i^0, \ z_i^1]$, $i \in \{1, 2, \ldots, s\}$ and $\zeta_j(x, \hat{x}, y) \in [\zeta_j^0, \ \zeta_j^1]$, $j \in \{1, 2, \ldots, \sigma\}$. Then, each $z_i(\hat{x}, y)$, $i \in \{1, 2, \ldots, s\}$ and $\zeta_j(x, \hat{x}, y)$, $j \in \{1, 2, \ldots, \sigma\}$ can be exactly rewritten as

$$\begin{aligned} z_i(\hat{x}, y) &= w_0^i(\hat{x}, y)z_i^0 + w_1^i(\hat{x}, y)z_i^1 \\ \zeta_j(x, \hat{x}, y) &= \omega_0^j(x, \hat{x}, y)\zeta_j^0 + \omega_1^j(x, \hat{x}, y)\zeta_j^1, \end{aligned}$$

where $w_0^i(\hat{x}, y) = (z_i^1 - z_i(\hat{x}, y))/(z_i^1 - z_i^0)$, $w_1^i(\hat{x}, y) = 1 - w_0^i(\hat{x}, y)$, $\omega_0^j(x, \hat{x}, y) = (\zeta_j^1 - \zeta_j(x, \hat{x}, y))/(\zeta_j^1 - \zeta_j^0)$, and $\omega_1^j(x, \hat{x}, y) = 1 - \omega_0^j(x, \hat{x}, y)$ are scalar functions holding the convex sum property in $\Omega_x \times \Omega_{\hat{x}}$, i. e., $w_0^i(\hat{x}, y) + w_1^i(\hat{x}, y) = 1$, $w_0^i(\hat{x}, y), w_1^i(\hat{x}, y) \in [0, \ 1]$, $\omega_0^j(x, \hat{x}, y) + \omega_1^j(x, \hat{x}, y) = 1$, $\omega_0^j(x, \hat{x}, y)$, $\omega_1^j(x, \hat{x}, y) \in [0, \ 1]$. Hence, the so-called scheduling functions are defined as:

$$\begin{aligned} \mathbf{w}_i(z) &= w_{i_1}^1(z_1)w_{i_2}^2(z_2)\cdots w_{i_s}^s(z_s), \\ \boldsymbol{\omega}_j(\zeta) &= \omega_{j_1}^1(\zeta_1)\omega_{j_2}^2(\zeta_2)\cdots \omega_{j_\sigma}^\sigma(\zeta_\sigma), \end{aligned}$$

with $i \in \{1, 2, \ldots, r\}$, $j \in \{1, 2, \ldots, \rho\}$ $i_j \in \{0, 1\}$, $r = 2^s$, $\rho = 2^\sigma$; the sets of indexes $[i_1 i_2 \cdots i_s]$ and $[j_1 j_2 \cdots j_\sigma]$ are a s-digit and σ-digit binary representation of $(i - 1)$ and $(j - 1)$; respectively. The scheduling functions also hold the convex sum property in $\Omega_x \times \Omega_{\hat{x}} \times \Omega_u$, i. e., $\sum_{i=1}^r \mathbf{w}_i(z) = 1$, $\mathbf{w}_i(z) \in [0, \ 1]$, $\sum_{j=1}^\rho \boldsymbol{\omega}_j(\zeta) = 1$, and $\boldsymbol{\omega}_j(\zeta) \in [0, \ 1]$.

The following lemma is useful in order to derive LMI conditions from convex inequalities:

Lemma 2.1. (Tuan et al. [52]) Let $\Upsilon_{ik}^j = \left(\Upsilon_{ik}^j\right)^T$, $(i, k) \in \{1, 2, \ldots, r\}^2$, $j \in \{1, 2, \ldots, \rho\}$, be matrices of adequate dimensions. Then $\sum_{i=1}^r \sum_{k=1}^r \sum_{j=1}^\rho \mathbf{w}_i(z)\mathbf{w}_k(z)\boldsymbol{\omega}_j(\zeta)\Upsilon_{ik}^j < 0$

holds if

$$\frac{2}{r-1}\Upsilon_{ii}^{j}+\Upsilon_{ik}^{j}+\Upsilon_{ki}^{j}<0, \tag{3}$$

for all $(i,k)\in\{1,2,\ldots,r\}^2$, $j\in\{1,2,\ldots,\rho\}$.

3. MAIN RESULT

As customary in descriptor approaches, system (1) is expressed as the following augmented system via the so-called *descriptor redundancy* approach [50]:

$$\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\ \ddot{x}\end{bmatrix}=\begin{bmatrix}\dot{x}\\ f(x)+g(x)(u+f_a)-E(y)\dot{x}\end{bmatrix},\quad \bar{x}=\begin{bmatrix}x\\ \dot{x}\end{bmatrix}. \tag{4}$$

Both the controller and UI observer gains are carried out for this augmented system.

3.1. The unknown input observer

Recall that the actuator faults (viewed as an unknown input) holds $f_a^{(p)}(t)\approx 0$, where p is the order of the derivative, this can be arranged as a set of first order differential equations by defining $d_1(t)=f_a(t)$, $d_2(t)=\dot{f}_a(t)$, $d_3(t)=\ddot{f}_a(t)$, …, $d_p(t)=f_a^{(p-1)}(t)$; thus

$$\dot{d}=Sd,\ \text{with}\ S=\begin{bmatrix}0 & I_m & 0 & \cdots & 0\\ 0 & 0 & I_m & \cdots & 0\\ 0 & 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix},\ d=\begin{bmatrix}d_1\\ d_2\\ d_3\\ \vdots\\ d_p\end{bmatrix}. \tag{5}$$

In general, as $f_a\in\mathbb{R}^m$ we have $d\in\mathbb{R}^{mp}$ and $S\in\mathbb{R}^{mp\times mp}$. Now, we are ready to define the following augmented system with $\chi=[x^T\ d^T]^T\in\mathbb{R}^{n+mp}$:

$$E_d(y)\dot{\chi}=f_d(\chi,u),\quad y=h_d(\chi), \tag{6}$$

where

$$E_d(y)=\begin{bmatrix}E(y) & 0\\ 0 & I_{mp}\end{bmatrix},\quad f_d(\chi,u)=\begin{bmatrix}f(x)+g(x)u+g(x)Wd\\ Sd\end{bmatrix},$$

$h_d(\chi)=h(x)$, and $W=[I_m\ 0_{m\times m(p-1)}]$. As before, using descriptor redundancy we have

$$\begin{bmatrix}I_{n+mp} & 0\\ 0 & 0_{n+mp}\end{bmatrix}\begin{bmatrix}\dot{\chi}\\ \ddot{\chi}\end{bmatrix}=\begin{bmatrix}\dot{\chi}\\ f_d(\chi,u)-E_d(y)\dot{\chi}\end{bmatrix}. \tag{7}$$

Inspired by developments in [17], consider $\beta(t)\in\mathbb{R}^{n+mp}$ and the following UI observer

$$\begin{bmatrix}I_{n+mp} & 0\\ 0 & 0_{n+mp}\end{bmatrix}\begin{bmatrix}\dot{\hat{\chi}}\\ \dot{\beta}\end{bmatrix}=\begin{bmatrix}\beta\\ f_d(\hat{\chi},u)-E_d(y)\beta\end{bmatrix}+\begin{bmatrix}N_{d1}(z)\\ N_{d2}(z)\end{bmatrix}(y-\hat{y}), \tag{8}$$

where $N_{d1}(z), N_{d2}(z) \in \mathbb{R}^{(n+mp)\times o}$ are the observer gains to be designed via LMIs; they only depend on available signals gathered in $z(\hat{x}, y)$. Then, the error system is

$$\bar{E}_d \dot{\bar{e}}_d = \left(\bar{A}_d(\chi, \hat{\chi}, u) - \bar{N}_d(z)\bar{C}_d(\chi, \hat{\chi})\right) \bar{e}_d, \tag{9}$$

with

$$\bar{e}_d = \begin{bmatrix} e_\chi \\ e_\beta \end{bmatrix} = \begin{bmatrix} \chi - \hat{\chi} \\ \dot{\chi} - \beta \end{bmatrix}, \bar{A}_d(\chi, \hat{\chi}, u) = \begin{bmatrix} 0 & I \\ A_d(\chi, \hat{\chi}, u) & -E_d(y) \end{bmatrix},$$
$$\bar{E}_d = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \bar{N}_d(z) = \begin{bmatrix} N_{d1}(z) \\ N_{d2}(z) \end{bmatrix}, \bar{C}_d(\chi, \hat{\chi}) = \begin{bmatrix} C_d(\chi, \hat{\chi}) & 0 \end{bmatrix},$$

where $A_d(\chi, \hat{\chi}, u)e_\chi = f_d(\chi, u) - f_d(\hat{\chi}, u)$ and $C_d(\chi, \hat{\chi})e_\chi = h_d(\chi) - h_d(\hat{\chi})$ have been obtained from the factorization method given in [43] (see section 2.1). The following result provides LMI conditions for the stabilization of (9):

Theorem 3.1. The origin of the error system (9) is asymptotically stable if there exist matrices $P_{d1} = P_{d1}^T$, P_{d3k}, $P_{d4k} \in \mathbb{R}^{(n+mp)\times(n+mp)}$, L_{d1k}, $L_{d2k} \in \mathbb{R}^{(n+mp)\times o}$, $k \in \{1, 2, \ldots, r\}$ such that $P_{d1} > 0$ and LMIs (3) hold with

$$\Upsilon_{ik}^j := \begin{bmatrix} P_{d3k}^T A_{dij} - L_{d1k}C_{dij} + (*) & (*) \\ P_{d4k}^T A_{dij} - L_{d2k}C_{dij} + P_{d1} - E_{di}^T P_{d3k} & -E_{di}^T P_{d4k} + (*) \end{bmatrix},$$

for $(i, k) \in \{1, 2, \ldots, r\}^2$, $j \in \{1, 2, \ldots, \rho\}$. If found feasible, the UI observer structure is

$$E_d(y)\dot{\hat{\chi}} = f_d(\hat{\chi}, u) + \begin{bmatrix} E_d(y) & I \end{bmatrix} \begin{bmatrix} N_{d1}(z) \\ N_{d2}(z) \end{bmatrix} (y - \hat{y}). \tag{10}$$

P r o o f. Consider the Lyapunov function candidate

$$V(\bar{e}_d) = \bar{e}_d^T \bar{E}_d^T \bar{P}_d(z)\bar{e}_d, \quad \bar{P}_d(z) = \begin{bmatrix} P_{d1} & 0 \\ P_{d3}(z) & P_{d4}(z) \end{bmatrix},$$

with $P_{d1}, P_{d3}(z), P_{d4}(z) \in \mathbb{R}^{(n+mp)\times(n+mp)}$, holding $\bar{E}_d^T \bar{P}_d(z) = \bar{P}_d^T(z)\bar{E}_d \geq 0$ and $P_{d1} > 0$; its time derivative is

$$\dot{V} = \dot{\bar{e}}_d^T \bar{E}_d^T \bar{P}_d(z)\bar{e}_d + \bar{e}_d^T \bar{P}_d^T(z)\bar{E}_d \dot{\bar{e}}_d.$$

Once system (9) is substituted; then, holds $\dot{V} < 0$ if

$$\left(\bar{A}_d(\chi, \hat{\chi}, u) - \bar{N}_d(z)\bar{C}_d(\chi, \hat{\chi})\right)^T \bar{P}_d(z) + \bar{P}_d^T(z)\left(\bar{A}_d(\chi, \hat{\chi}, u) - \bar{N}_d(z)\bar{C}_d(\chi, \hat{\chi})\right) < 0. \tag{11}$$

In order to obtain LMI conditions, the previous expression is rewritten in a convex form (see Section 2.2) with

$$\bar{P}_d(z) = \sum_{k=1}^{r} \mathbf{w}_k(z)\bar{P}_{dk}, \ \bar{A}_d(\chi, \hat{\chi}, u) = \sum_{i=1}^{r} \sum_{j=1}^{\rho} \mathbf{w}_i(z)\boldsymbol{\omega}_j(\zeta)\bar{A}_{dij},$$

$$\bar{C}_d(\chi,\hat{\chi}) = \sum_{i=1}^{r}\sum_{j=1}^{\rho}\mathbf{w}_i(z)\boldsymbol{\omega}_j(\zeta)\bar{C}_{dij};$$

then (11) is equivalent to

$$\sum_{i=1}^{r}\sum_{k=1}^{r}\sum_{j=1}^{\rho}\mathbf{w}_i(z)\mathbf{w}_k(z)\boldsymbol{\omega}_j(\zeta)\left(\bar{P}_{dk}^T\bar{A}_{dij} - \bar{L}_{dk}\bar{C}_{dij} + (*)\right) < 0,$$

where

$$\sum_{k=1}^{r}\mathbf{w}_k(z)\bar{L}_{dk} = \begin{bmatrix}\sum_{k=1}^{r}\mathbf{w}_k(z)L_{d1k}\\ \sum_{k=1}^{r}\mathbf{w}_k(z)L_{d2k}\end{bmatrix} \text{ and } \bar{N}_d(z) = \left(\sum_{k=1}^{r}\mathbf{w}_k(z)\bar{P}_{dk}\right)^{-T}\begin{bmatrix}\sum_{k=1}^{r}\mathbf{w}_k(z)L_{d1k}\\ \sum_{k=1}^{r}\mathbf{w}_k(z)L_{d2k}\end{bmatrix},$$

have been substituted. The proposed LMI conditions are obtained once Lemma 2.1 is applied. The final observer form (10) can be derived by writing (8) as

$$\dot{\hat{\chi}} = N_{d1}(z)\,(y-\hat{y}) + \beta \tag{12}$$

$$E_d(y)\beta = f_d(\hat{\chi},u) + N_{d2}(z)(y-\hat{y}). \tag{13}$$

From (12) we have $\beta = \dot{\hat{\chi}} - N_{d1}(z)\,(y-\hat{y})$ and substituting it in (13) yields:

$$E_d(y)\dot{\hat{\chi}} = f_d(\hat{\chi},u) + E_d(y)N_{d1}(z)(y-\hat{y}) + N_{d2}(z)(y-\hat{y}),$$

which after regrouping terms gives (10). Thus, concluding the proof. □

Remark 3.2. The performance of the UI observer can be improved in terms of decay rate, i. e., the convergence rate of the error towards the origin. This can be guaranteed by $\dot{V}(\bar{e}_d) \le 2\alpha V(\bar{e}_d)$, $\alpha > 0$, which in turn is implied by LMIs (3) with

$$\Upsilon_{ik}^{j} := \begin{bmatrix} P_{d3k}^T A_{dij} - L_{d1k}C_{dij} + (*) + 2\alpha P_{d1} & (*) \\ P_{d4k}^T A_{dij} - L_{d2k}C_{dij} + P_{d1} - E_{di}^T P_{d3k} & -E_{di}^T P_{d4k} + (*)\end{bmatrix},$$

for $(i,k) \in \{1,2,\ldots,r\}^2$, $j \in \{1,2,\ldots,\rho\}$.

3.2. The controller

Let us consider a nonlinear control law:

$$u(t) = K(x)x(t), \tag{14}$$

with $K(x) \in \mathbb{R}^{m\times n}$ as the nonlinear controller gain. Thus, without the actuator fault $f_a(t)$, the closed-loop system yields

$$\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}\\ \ddot{x}\end{bmatrix} = \begin{bmatrix}\dot{x}\\ f(x) + g(x)K(x)x - E(y)\dot{x}\end{bmatrix}. \tag{15}$$

In order to design the control law by means of LMIs, a convex rewriting of the closed-loop system is required (see Section 2.2); thus assuming that $f(x) = A(x)x$, where the nonlinearities in $A(x)$ are well-defined in Ω_x, then (15) gives

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \sum_{i=1}^{r}\sum_{k=1}^{r} \mathbf{w}_i(x)\mathbf{w}_k(x) \begin{bmatrix} 0 & I \\ A_i + B_i K_k & -E_i \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix}, \tag{16}$$

where it has been considered:

$$E(y) = \sum_{i=1}^{r} \mathbf{w}_i(x) E_i, \;\; f(x) = A(x)x = \sum_{i=1}^{r} \mathbf{w}_i(x) A_i x,$$
$$g(x) = \sum_{i=1}^{r} \mathbf{w}_i(x) B_i, \;\; K(x) = \sum_{k=1}^{r} \mathbf{w}_k(x) K_k;$$

the vertex matrices are computed such that

$$E_i = E(y)|_{\mathbf{w}_i=1}, \; A_i = A(x)|_{\mathbf{w}_i=1}, \; B_i = g(x)|_{\mathbf{w}_i=1}, \; K_i \in \mathbb{R}^{m\times n}, i \in \{1, 2, \ldots, r\},$$

the latter being matrices to be designed. The following result is adopted from the developments in [16]:

Theorem 3.3. The origin of the system (15) is asymptotically stable if there exist matrices $X_1 = X_1^T$, X_{3k}, $X_{4k} \in \mathbb{R}^{n\times n}$ and $M_k \in \mathbb{R}^{m\times n}$, $k \in \{1, 2, \ldots, r\}$ such that $X_1 > 0$ and LMIs

$$\frac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ik} + \Upsilon_{ki} < 0,$$

hold for all $(i, k) \in \{1, 2, \ldots, r\}^2$ with

$$\Upsilon_{ik} := \begin{bmatrix} X_{3k} + X_{3k}^T & (*) \\ A_i X_1 + B_i M_k - E_i X_{3k} + X_{4k}^T & -E_i X_{4k} - X_{4k}^T E_i^T \end{bmatrix},$$

for $(i, k) \in \{1, 2, \ldots, r\}^2$. If found feasible, the controller gains are $K_k = M_k X_1^{-1}$, $k \in \{1, 2, \ldots, r\}$, then $u = \sum_{k=1}^{r} \mathbf{w}_k(x) K_k x = K(x)x$.

Proof. It follows similar lines as the ones given in [16] and therefore not repeated here. □

Recall that the control law (2) is constituted by the term $K(\tilde{x})\hat{x}$, where $\tilde{x} \in \mathbb{R}^n$ denotes that the gain only depends on available signals ($\hat{x}$ and y), for instance, if $x \in \mathbb{R}^4$ and $y^T = [x_1 \; x_3]$, then $\tilde{x}^T = [y_1 \; \hat{x}_2 \; y_2 \; \hat{x}_4] = [x_1 \; \hat{x}_2 \; x_3 \; \hat{x}_4]$. In what follows a mathematical proof is given to show that the separation principle holds.

3.3. The fault-tolerant control scheme

As mentioned in Section 2, the control law under design is (2) for the faulty system (1), that is, $u = K(\tilde{x})\hat{x} - \hat{f}_a$, with $K(\tilde{x})$ being the controller gain that uses only available signals $\hat{x}$ and y. Thus, the closed-loop system is

$$\begin{aligned} E(y)\dot{x} &= f(x)+g(x)K(\tilde{x})\hat{x}+ g(x)f_a-g(x)\hat{f}_a \\ &= f(x)+g(x)K(\tilde{x})x- g(x)[K(\tilde{x}) \;\; W]\begin{bmatrix} x-\hat{x} \\ d-\hat{d} \end{bmatrix}, \end{aligned} \tag{17}$$

which combined with the UI observer (10):

$$\begin{bmatrix} E(y) & 0 \\ 0 & I_{mp} \end{bmatrix}\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{d}} \end{bmatrix} = \begin{bmatrix} f(\hat{x})+g(\hat{x})u-g(\hat{x})W\hat{d} \\ S\hat{d} \end{bmatrix} + \begin{bmatrix} E_d(y) & I_{n+mp} \end{bmatrix}\begin{bmatrix} N_{d1}(\tilde{x}) \\ N_{d2}(\tilde{x}) \end{bmatrix}(y-\hat{y})$$

yields the following error system in terms of $x - \hat{x}$ and $d - \hat{d}$ (the dynamics of d are taken from (5)), where the error signal is $e_\chi = [x^T - \hat{x}^T \;\; d^T - \hat{d}^T]^T$:

$$\begin{aligned} E_d(y)\dot{e}_\chi = &\begin{bmatrix} f(x)-f(\hat{x})+g(x)Wd-g(\hat{x})W\hat{d}+(g(x)-g(\hat{x}))u \\ S(d-\hat{d}) \end{bmatrix} \\ &-\begin{bmatrix} E(y) & I_{n+mp} \end{bmatrix}\begin{bmatrix} N_{d1}(\tilde{x}) \\ N_{d2}(\tilde{x}) \end{bmatrix}(h(x)-h(\hat{x})). \end{aligned}$$

Once the factorization of error signals is performed, we have $A_o(x, \hat{x}, u, d)(x - \hat{x}) + B_o(x, \hat{x})W(d - \hat{d}) = f(x)-f(\hat{x})+g(x)Wd-g(\hat{x})W\hat{d}+(g(x)-g(\hat{x}))u$ and $C_o(x,\hat{x})(x - \hat{x}) = h(x) - h(\hat{x})$, so the previous error system results in

$$E_d(y)\dot{e}_\chi = \left(\begin{bmatrix} A_o(x,\hat{x},u,d) & B_o(x,\hat{x})W \\ 0 & S \end{bmatrix} - \begin{bmatrix} E(y) & I_{n+mp} \end{bmatrix}\begin{bmatrix} N_{d1}(\tilde{x}) \\ N_{d2}(\tilde{x}) \end{bmatrix}\begin{bmatrix} C(x,\hat{x}) & 0 \end{bmatrix}\right)e_\chi.$$

In order to drive the trajectories of $x(t)$ and $e_\chi(t)$ asymptotically to zero, the following augmented system is built (recall that $E(y)$ and $E_d(y)$ are invertible):

$$\begin{bmatrix} \dot{x} \\ \dot{e}_\chi \end{bmatrix} = \begin{bmatrix} A_c(x)+B_c(x)K(\tilde{x}) & -B_c(x)\mathcal{K}(\tilde{x}) \\ 0_{(n+mp)\times n} & \mathcal{A}_o(\chi,\hat{\chi},u)-\mathcal{L}(\tilde{x})\mathcal{C}(x,\hat{x}) \end{bmatrix}\begin{bmatrix} x \\ e_\chi \end{bmatrix}, \tag{18}$$

where

$$\begin{gathered} A_c(x)x = E^{-1}(y)f(x), \quad B_c(x) = E^{-1}(y)g(x), \\ \mathcal{A}_o(\chi,\hat{\chi},u) = \begin{bmatrix} E^{-1}(y)A_o(x,\hat{x},u,d) & E^{-1}(y)B_o(x,\hat{x})W \\ 0 & S \end{bmatrix}, \\ \mathcal{K}(\tilde{x}) = [K(\tilde{x}) \;\; W], \quad \mathcal{C}(x,\hat{x}) = [C_o(x,\hat{x}) \;\; 0], \\ \mathcal{L}(\tilde{x}) = E_d^{-1}(y)\begin{bmatrix} E(y) & I_{n+mp} \end{bmatrix}\begin{bmatrix} N_{d1}(\tilde{x}) \\ N_{d2}(\tilde{x}) \end{bmatrix}. \end{gathered}$$

The following theoretical result is based on the developments found in [6, Section 7.6], it shows the possibility of designing the complete fault tolerant controller scheme by means of LMI conditions in theorems 3.1 (for the UI observer) and 3.3 (for the fault-free control law).

Theorem 3.4. The origin of the augmented system (18) is asymptotically stable if there exist matrices $X = X^T \in \mathbb{R}^{n\times n}$, $Q = Q^T \in \mathbb{R}^{(n+mp)\times(n+mp)}$ such that $X > 0$, $Q > 0$ and the inequalities

$$X\left(A_c(x){+}B_c(x)K(\tilde{x})\right) + (*) < 0, \tag{19}$$

$$\left(\mathcal{A}_o(\chi,\hat{\chi},u) - \mathcal{L}(\tilde{x})\mathcal{C}(x,\hat{x})\right) Q + (*) < 0, \tag{20}$$

hold in $\Omega \subset \mathbb{R}^{2n+mp}$, $0 \in \Omega$.

P r o o f. The asymptotic stability at the origin of (18) is implied by the existence of $\mathcal{P} = \begin{bmatrix} X & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}$ being positive definite and such that

$$\mathcal{P}\begin{bmatrix} A_c(x){+}B_c(x)K(\tilde{x}) & -B_c(x)\mathcal{K}(\tilde{x}) \\ 0_{(n+mp)\times n} & \mathcal{A}_o(\chi,\hat{\chi},u) - \mathcal{L}(\tilde{x})\mathcal{C}(x,\hat{x}) \end{bmatrix} + (*) < 0, \tag{21}$$

holds in $\Omega \subset \mathbb{R}^{2n+mp}$, $0 \in \Omega$. The first block-position (1,1) of the previous inequality yields the one in (19). Now, if applying congruence on (21) with

$$\mathcal{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q \end{bmatrix} = \begin{bmatrix} X & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix}^{-1},$$

results in

$$\begin{bmatrix} A_c(x){+}B_c(x)K(\tilde{x}) & -B_c(x)\mathcal{K}(\tilde{x}) \\ 0_{(n+mp)\times n} & \mathcal{A}_o(\chi,\hat{\chi},u) - \mathcal{L}(\tilde{x})\mathcal{C}(x,\hat{x}) \end{bmatrix}\mathcal{Q} + (*) < 0, \tag{22}$$

which implies (20) by the block position (2,2). Now, considering that conditions (19) and (20) are satisfied, then

$$V = \begin{bmatrix} x \\ e_\chi \end{bmatrix}^T \begin{bmatrix} \lambda X & 0 \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} x \\ e_\chi \end{bmatrix} \tag{23}$$

is a valid Lyapunov function candidate for system (18), with some $\lambda > 0$ to be specified later on. The time-derivative of (23) is negative definite if (arguments omitted)

$$\begin{bmatrix} \lambda X(A_c(\cdot){+}B_c(\cdot)K(\cdot)) + (*) & -\lambda X B_c(\cdot)\mathcal{K}(\cdot) \\ (*) & Q^{-1}(\mathcal{A}_o(\cdot) - \mathcal{L}(\cdot)\mathcal{C}(\cdot)) + (*) \end{bmatrix} < 0, \tag{24}$$

holds in $\Omega \subset \mathbb{R}^{2n+mp}$, $0 \in \Omega$. Applying Schur complement yields

$$\lambda R(\cdot) - S(\cdot) > 0, \tag{25}$$

with

$$S(\cdot) = X\left(A_c(\cdot){+}B_c(\cdot)K(\cdot)\right) + (*) \quad \text{and}$$

$$R(\cdot) = X B_c(\cdot) K(\cdot)\left(Q^{-1}\left(\mathcal{A}_o(\cdot) - \mathcal{L}(\cdot)\mathcal{C}(\cdot)\right) + (*)\right)^{-1} K^T(\cdot) B_c^T(\cdot) X.$$

Conditions (19) and (20) assure that $\lambda_{\min}(R(\cdot)) < 0$ and $\lambda_{\max}(S(\cdot)) < 0$; therefore, λ can be chosen such that $\lambda\left(\lambda_{\min}(R(\cdot))\right) > \lambda_{\max}(S(\cdot))$ to guarantee (25). □

Remark 3.5. As the previous theorem shows, the controller and the UI observer can be designed independently. Moreover, as LMIs in theorems 3.1 and 3.3 can be verified simultaneously [6, 48], the numerical complexity of the entire LMI problem can be approximated by $\log_{10}(n_d^3 n_l)$, where n_d is the number of scalar decision variables and n_l is the number LMI rows in both theorems [55]. Hence, we have $n_d = 0.5(n + mp)(n + mp + 1) + 2r(n + mp)^2 + 2r(n + mp)o + 0.5n(n + 1) + 2rn^2 + rmn$ and $n_l = r^2\rho(2n + 2mp) + (n + mp) + r^2 2n + n$. Thus, the numerical complexity is linked to the number of states as well as the number of nonlinearities s and σ as the number of vertex exponentially increases, i. e., $r = 2^s$ and $\rho = 2^\sigma$; additionally, inverting the descriptor matrix $E(y)$ may introduce more nonlinearities [10].

In contrast with other methodologies, the LMI framework provides conditions numerically tractable [6, 3]; additionally, closed-loop performances such as speed converge (see Remark 3.2), input bounds, disturbance attenuation via H_∞ can be directly added within the same framework. On the other hand, observers based-on sliding mode [8] or high-gain [38] approaches might be applied too; nonetheless, the former requires to compute a change of variables and to satisfy matching conditions that might be hard to satisfy while for the later the observer gain might be too large and it may destabilize the closed-loop system because of the well-known peaking phenomena [27]. Therefore, the fault-tolerant control scheme hereby presented is fully in terms of LMIs, which are solvable in polynomial time via commercially available software [14].

4. EXAMPLES

In this section, the advantages of the proposal are illustrated via both academic as well as physical examples. The first one is intended to show how the UI observer works on nonlinear descriptor systems where the considered fault is adequately estimated. The second example illustrates the actuator fault estimation problem occurring on the inverted pendulum on a car. The LMI conditions as well as simulations have been carried out in the LMIToolbox [14] and Simulink within MATLAB.

Example 4.1. Consider a nonlinear descriptor (1) with:

$$E(y) = \begin{bmatrix} 0.87 & 0.33+0.5(1-2\eta) \\ 0.53-\delta(1-2\eta) & 0.95 \end{bmatrix},$$
$$f(x) = \begin{bmatrix} -0.81x_1 + 0.83x_2 + x_2\delta\sin x_1 \\ -0.74x_1 + 0.57x_2 \end{bmatrix},\ g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where the output is $y = h(x) = x_2$, the term $\eta = 1/(1 + x_2^2)$ depends exclusively on available signals, $\delta > 0$ is a real-valued parameter, the actuator faults are

$$f_a(t) = \begin{cases} 1.5(t-5) + 0.3\sin(5t), & 5 \le t < 10 \\ 1.5(15-t) + 0.3\sin(5t), & 10 \le t \le 15 \\ 0.5\sin(5t) + 0.3\sin(5t), & 20 \le t < 37 \\ 0.5(t-45 + 0.3\sin(5t)), & 45 \le t \le 50 \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

The task is to design a nonlinear fault tolerant control law (2), to this end let us first design an UI observer (10) such that x_1 and the fault f_a are reconstructed, under the assumption that $f_a^{(3)}(t) \approx 0$; hence the augmented error system (9) can be obtained by means of the factorization method firstly presented in [43] and summarized in section 2.1.

Notice that $g(x) = [0\ 1]^T$, then let us focus on the difference

$$f(x) - f(\hat{x}) = \begin{bmatrix} -0.81e_1 + 0.83e_2 + \delta(x_2 \sin x_1 - \hat{x}_2 \sin \hat{x}_1) \\ -0.74e_1 + 0.57e_2 \end{bmatrix};$$

by using a third order approximation ($\sin x_1 \approx x_1 - x_1^3/6$) and grouping terms yields

$$\begin{bmatrix} -0.81 + \delta x_2 - \frac{1}{6}\delta x_2(x_1^2 + x_1\hat{x}_1 + \hat{x}_1) & 0.83 + \delta\hat{x}_1 - \frac{1}{6}\delta\hat{x}_1^3 \\ -0.74 & 0.57 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Now, we are ready to express the error system (9) with matrices:

$$A_d(\chi, \hat{\chi}, u)e_\chi = \left[\begin{array}{cc|ccc} \multicolumn{2}{c|}{\multirow{2}{*}{$f(x) - f(\hat{x})$}} & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and $E_d(y) = \left[\begin{array}{c|c} E(y) & 0_{2\times 3} \\ \hline 0_{3\times 2} & I_3 \end{array}\right]$ with $\chi_1 = x_1$, $\chi_2 = x_2$, $\chi_3 = f_a$, $\chi_4 = f_a^{(1)}$ and $\chi_5 = f_a^{(2)}$. Hence, we have 5 measurable nonlinearities ($x_2, \hat{x}_1, \hat{x}_1^2, \hat{x}_1^3$ and $1/(1 + x_2^2)$) and 2 unmeasurable ones (x_1 and x_1^2); a convex representation in the region of interest $\Omega_x \times \Omega_{\hat{x}} = \{|x_1| \leq 1, |\hat{x}_1| \leq 1, |x_2| \leq 1\}$ gives 2^7 vertex models (omitted here due to space reasons). LMI conditions in Theorem 3.1 are found feasible for $\delta = 0.6$, some of the computed gains are:

$$L_{d11} = 1 \times 10^6 \begin{bmatrix} 1.6099 \\ 7.1595 \\ 0.8550 \\ 1.8139 \\ 1.2205 \end{bmatrix}, \quad L_{d21} = 1 \times 10^6 \begin{bmatrix} 3.4452 \\ 6.6859 \\ -4.9892 \\ -1.6365 \\ 0.5400 \end{bmatrix}.$$

Now, under the assumption that $f_a(t) = 0$ and all states are measurable, the controller (14) will be designed. To do this, a new convex representation of the system is needed; by considering the same region $\Omega_x \times \Omega_{\hat{x}}$ we have $z_1(x) = 1/(1 + x_2^2) \in [0.5\ 1]$ and $z_2(x) = \sin x_1 \in [-0.8415\ 0.8415]$. LMIs in Theorem 3.3 are feasible, the resulting gains are:

$$K_1 = \begin{bmatrix} 4.0105 & -13.1415 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 4.0363 & -12.8426 \end{bmatrix},$$
$$K_3 = \begin{bmatrix} 2.4243 & -9.6254 \end{bmatrix}, \quad K_4 = \begin{bmatrix} 2.2045 & -8.0947 \end{bmatrix},$$
$$X_1 = \begin{bmatrix} 120.0703 & 22.2918 \\ 22.2918 & 16.5766 \end{bmatrix}.$$

For illustration purposes, the time-evolution of the states without a fault tolerant control ($\hat{f}_a = 0$) is depicted on the top of Figure 1 for initial conditions $x(0) = [0.5 \;\; -0.5]^T$; as expected when the fault (26) occurs the states are seriously affected and driven to outside of designing region. Additionally, the fault and its estimation is shown on the bottom of Figure 1, as expected, It can be clearly appreciated that the fault f_a is adequately reconstructed.

On the other hand, under the fault tolerant scheme (2), simulation results are shown in Figure 2. It can be clearly appreciated that the closed-loop system has a good performance under the signal fault (26), as the system states are always closed to the origin, see the top of Figure 2. The evolution in time of control law (2) is displayed on the bottom of Figure 2.

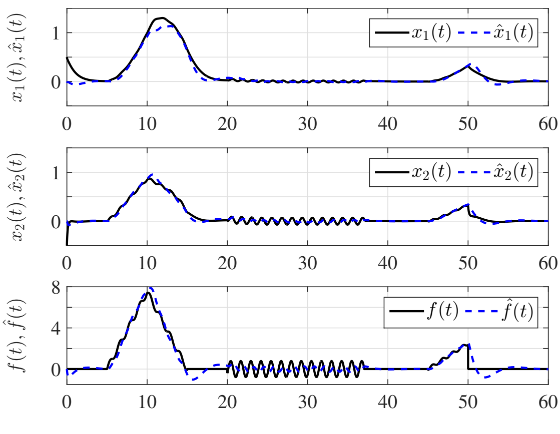

Fig. 1. Time-evolution of the states and fault versus their estimation in Example 4.1.

Finally, it is important to stress that:

- The system under study is in descriptor form; the proposed FTC scheme begins by constructing an UI observer (10); in this sense we compare with works focused on such observers. Hence, approaches [19, 20, 25, 36] cannot deal directly with the system under study, because they only consider standard systems and the scheduling parameters are assumed to be available, which is not the case for this example. Approaches in [37, 41, 56] deal with unmesaurable variables but they are not in the descriptor form. As for approaches concerning UI observers for descriptor systems, [9] cannot deal with unavailable premise variables. Nonetheless, from system (1), it is always possible to obtain $\dot{x}(t) = E^{-1}(y)\left(f(x) + g(x)\left(u(t) + f_a(t)\right)\right)$ as to compute an error system via factorization (this step might be more difficult due to complex nonlinear expressions). In order to perform a "fair" comparison, we employ LMIs in Theorem 3.1 with $E_i = I$, $r = 2^6 = 64$, $\rho = 2^2 = 4$ and adequate vertex matrices A_{ij}. The LMIs for the descriptor as well for the standard forms are run for positive values of δ, for the descriptor we have feasible solutions up to

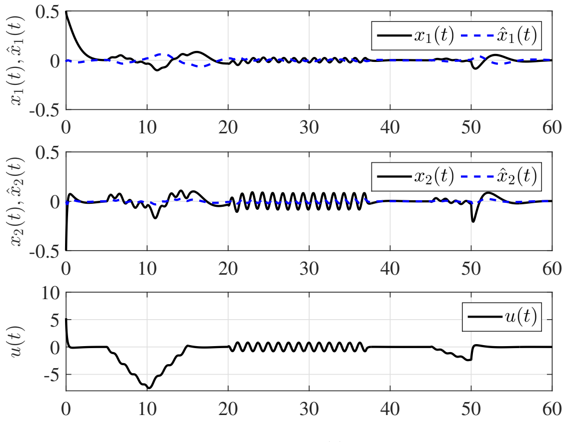

Fig. 2. Closed-loop trajectories under the FTC scheme (2) for Example 4.1.

$\delta = 0.683$ while the standard configuration is feasible until $\delta = 0.446$; the larger the value of δ is, the less conservative are the conditions. In terms of numerical complexity (see Remark 3.5), our descriptor FTC approach has 14.7149 while for the standard setup is 16.2148.

- Comparisons with respect to observer-based FTC: the algorithm proposed in [21, Section 4] only considers available scheduling variables and does not contemplate descriptors models; similar shortcoming presents [47]. Conditions in [1, 31] cannot be directly employed since they are for standard state space representations. As for descriptor approaches, [45] conditions cannot be used because the scheduling vector must be available. Regarding approaches based-on sliding modes, [7] provides conditions to estimate actuator faults for LPV systems; nevertheless it cannot be used for the fault estimation part as the scheduling vector must be known and the considered system is not in a descriptor form.

Example 4.2. Consider the inverted pendulum on car with dynamics [12]:

$$(m_l+M)\dot{x}_3+m_lL\dot{x}_4\cos x_2+bx_3-m_lLx_4^2\sin x_2=F,$$
$$m_lL\dot{x}_3\cos x_2+(J+m_lL^2)\dot{x}_4-m_lgL\sin x_2+\kappa x_4=0,$$

where x_1 (m) and x_3 (m/s) are the position an velocity of the car, respectively; x_2 (rad) and x_4 (rad/s) are the position and velocity of the pendulum, respectively; the parameters are $L = 0.3$ m, $M = 2.3$ Kg, $m_l = 0.2$ Kg, $J = 0.0099$ Kg·m^2, $b = 5 \times 10^{-5}$ N·s·m^{-1}, $g = 9.81$ m·s^{-2}, and $\kappa = 0.005$ N·m·s·rad^{-1}; $F = -9.6(u + f_a)$ with u (V) being the control signal and f_a is an actuator fault; positions x_1 and x_2 are measured with encoders, i. e., the output is $y = [x_1\ x_2]^T$. The state-space model in descriptor form can be written as (1) with

$$E(y)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & m_l+M & m_lL\cos x_2\\ 0 & 0 & m_lL\cos x_2 & J+m_lL^2\end{bmatrix},$$

$$f(x)=\begin{bmatrix}x_3\\ x_4\\ m_lLx_4^2\sin x_2-bx_3\\ m_lL\sin x_2-\kappa x_4\end{bmatrix},\quad g(x)=\begin{bmatrix}0\\ 0\\ -9.6\\ 0\end{bmatrix}.$$

The fault is defined as

$$f_a(t)=\begin{cases}0.08(t-5), & 5\le t\le 10\\ 0.08(15-t), & 10\le t\le 15\\ 0.05\sin(0.75t-4)+0.05, & 20\le t\le 37\\ 0.07(t-45), & 45\le t\le 50\\ 0, & \text{otherwise.}\end{cases}$$

Now, by considering the measurable signals (output) in the descriptor model, matrix $E_d(y)$ and vector $f_d(\chi,u)$ with $p=3$ (impliying $f_a^{(3)}\approx 0$) can be easily identified. The states of the extended system (6) are $\chi_1=x_1$, $\chi_2=x_2$, $\chi_3=x_3$, $\chi_4=x_4$, $\chi_5=f_a$, $\chi_6=f_a^{(1)}$, and $\chi_7=f_a^{(2)}$. Once an unknown input nonlinear descriptor observer in (8) is considered, the error system (9) can be obtained with

$$E_d(y)=\left[\begin{array}{c|c}E(y) & 0_{4\times 3}\\ \hline 0_{3\times 4} & I_3\end{array}\right]\quad\text{and}\quad A_d(\chi,\hat\chi)=\begin{bmatrix}0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & -b & m_lL(\chi_4+\hat\chi_4)\sin y_2 & -9.6 & 0 & 0\\ 0 & 0 & 0 & -\kappa & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

where factorization $m_lL\chi_4^2\sin y_2-m_lL\hat\chi_4^2\sin y_2=m_lL(\chi_4+\hat\chi_4)\sin y_2$ in $A_d(\chi,\hat\chi)e_\chi=f_d(\chi,u)-f_d(\hat\chi,u)$ has been employed, with $\hat\chi_1=y_1$ and $\hat\chi_2=y_2$ in $f_d(\hat\chi,u)$. The non-constant terms in the nonlinear error system are: $z_1=\hat x_4=\hat\chi_4\in[-3,3]$, $z_2=\sin y_2\in[-0.383,0.383]$, and $z_3=\cos y_2\in[0.924,1]$ (available); $\zeta_1=x_4=\chi_4\in[-3,3]$ (unavailable); the bounds have been induced from the desired operation region $\Omega_x\times\Omega_{\hat x}=\{|y_2|\le\pi/8,|x_4|\le 3,|\hat x_4|\le 3\}$. In order to increase the converge rate of the UI observer, LMIs with $\alpha=5$ together with simple matrices $P_{d3k}=P_{d3}$ and $P_{d4k}=P_{d4}$ are found feasible (see Remark 3.2).

First, the UI observer is put at test in simulation, the results are shown in figures 3 and 4 for initial conditions $x(0)=[0.05\ 0.0873\ 0\ 0]^T$, $\hat x(0)=[0\ 0\ 0\ 0\ 0]^T$, under the control law $u=-6.218x_1-56.297x_2-11.676\hat x_3-14.59\hat x_4$; this controller has been obtained from Theorem 3.3 by using $z_1=x_4^2$, $z_2=(\sin x_2)/x_2$, and $z_3=\cos x_2$ to construct the corresponding convex model; for simplicity, the controller gain has been chosen constant, that is, $K=MX_1^{-1}$. Secondly, the FTC scheme (2), with $u=$

$-6.218x_1 - 56.297x_2 - 11.676\hat{x}_3 - 14.59\hat{x}_4 - \hat{f}_a$ has been implemented for the same initial conditions, simulation results are depicted in Figure 5.

Comparing results in Figure 4 with those in Figure 5 it is easy to see the advantage of using the fault tolerant control (2) in contrast with simple state feedback.

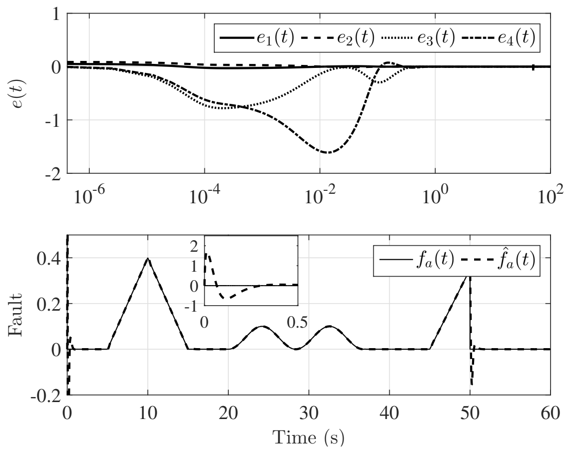

Fig. 3. Estimation error $e = x - \hat{x}$ (top); f_a and its estimation $\hat{f}_a$ (bottom).

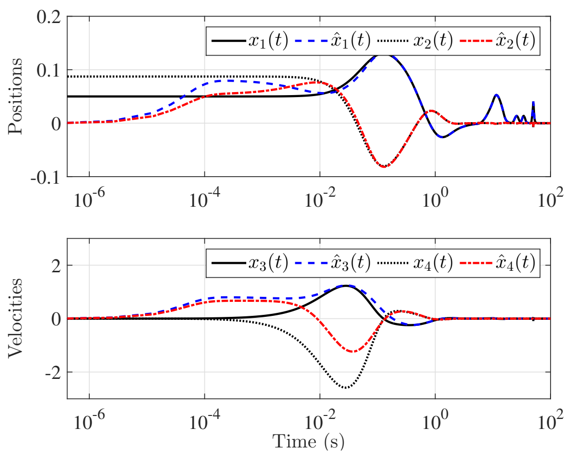

Fig. 4. Time evolution of the states $x(t)$ and its estimations $\hat{x}(t)$ under a state feedback $u = K\hat{x}$ control law.

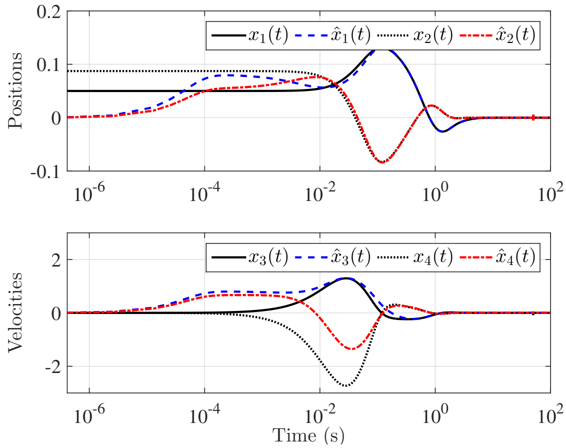

Fig. 5. Time evolution of the states $x(t)$ and their estimations $\hat{x}(t)$ under the fault tolerant control law (2).

5. CONCLUSION

An observer-based fault tolerant control scheme for nonlinear descriptor systems has been presented. The proposed scheme is able to drive the trajectories of the system asymptotically to the origin despite actuator faults. The FTC consists an UI observer whose LMI conditions have been obtained by a recently appeared factorization method; this UI observer provides estimations of the missing state variables as well as the actuator faults. Once the signals are estimated, they are fed by the fault-tolerant controller. All the conditions have been obtained in the form of LMIs. Two examples have been employed to illustrate the advantages of the proposal.

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