KYBERNETIKA — VOLUME *60* (2024), NUMBER 3, PAGES 412–425

ON SPARSITY OF APPROXIMATE SOLUTIONS TO MAX-PLUS LINEAR SYSTEMS

Pingke Li

When a system of one-sided max-plus linear equations is inconsistent, the approximate solutions within an admissible error bound may be desired instead, particularly with some sparsity property. It is demonstrated in this paper that obtaining the sparsest approximate solution within a given L_∞ error bound may be transformed in polynomial time into the set covering problem, which is known to be NP-hard. Besides, the problem of obtaining the sparsest approximate solution within a given L_1 error bound may be reformulated as a polynomial-sized mixed integer linear programming problem, which may be regarded as a special scenario of the facility location-allocation problem. By this reformulation approach, this paper reveals some interesting connections between the sparsest approximate solution problems in max-plus algebra and some well known problems in discrete and combinatorial optimization.

Keywords: max-plus algebra, max-plus linear systems, sparsity, set covering, mixed integer linear programming

Classification: 15A80, 90C24, 90C11

1. INTRODUCTION

The theory of max-plus algebra provides an attractive approach to handling some nonlinear problems in a linear-like manner and has been applied to solve many real-world problems in scheduling, production, transportation, etc. See, e. g., Baccelli et al. [1], Heidergott et al. [6], Gondran and Minoux [4], Butkovič [2], and Joswig [7].

Let $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ and

$$a \oplus b = \max\{a, b\}, \quad a \otimes b = a + b, \quad \forall a, b \in \mathbb{R}_{\max}.$$

Max-plus algebra is the commutative idempotent semiring $(\mathbb{R}_{\max}, \oplus, \otimes)$, also called as a dioid, endowed with "addition" $\oplus$ and "multiplication" $\otimes$ defined over $\mathbb{R}_{\max}$. Analogously, min-plus algebra is the commutative idempotent semiring $(\mathbb{R}_{\min}, \oplus', \otimes')$ with $\mathbb{R}_{\min} = \mathbb{R} \cup \{+\infty\}$ and

$$a \oplus' b = \min\{a, b\}, \quad a \otimes' b = a + b, \quad \forall a, b \in \mathbb{R}_{\min}.$$

DOI: 10.14736/kyb-2024-3-0412

The pair of operations $(\oplus, \otimes)$, as well as that of $(\oplus', \otimes')$, can be extended to vectors and matrices of compatible sizes in the same way as in linear algebra, preserving the analogous commutative, associative, and distributive properties.

A finite system of one-sided max-plus linear equations is defined as

$$\max_{j \in N}\{a_{ij} + x_j\} = b_i, \quad \forall i \in M,$$

where $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$ are two index sets. It may be called a system of max-plus linear equations or a max-plus linear system for short and expressed in its matrix form

$$A \otimes \boldsymbol{x} = \boldsymbol{b}$$

with $A = (a_{ij})_{m \times n} \in \mathbb{R}_{\max}^{m \times n}$, $\boldsymbol{b} = (b_i)_{m \times 1} \in \mathbb{R}_{\max}^{m}$, and $\boldsymbol{x} = (x_j)_{n \times 1} \in \mathbb{R}_{\max}^{n}$, respectively. It is a fundamental problem in max-plus algebra to determine the solution set

$$S(A, \boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R}_{\max}^{n} \mid A \otimes \boldsymbol{x} = \boldsymbol{b}\}$$

for a system of max-plus linear equations with the coefficient matrix A and right-hand side vector $\boldsymbol{b}$. The system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ is called consistent if $S(A, \boldsymbol{b}) \neq \emptyset$ and inconsistent otherwise.

To check whether $S(A, \boldsymbol{b})$ is empty or not, a principal solution $\widehat{\boldsymbol{x}} = (\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n)^T$ may be constructed as

$$\widehat{x}_j = \min_{i \in M}\{-a_{ij} + b_i\}, \quad \forall j \in N,$$

that is,

$$\widehat{\boldsymbol{x}} = A^\sharp \otimes' \boldsymbol{b},$$

where $A^\sharp = -A^T$ is the conjugate matrix of A in the context of max-plus algebra. Without loss of generality, one may assume that the right-hand side vector $\boldsymbol{b}$ consists of only finite elements and the coefficient matrix A is doubly $\mathbb{R}$-astic, i. e., A has at least one finite element in each row and in each column. Under this assumption, the values in the principal solution $\widehat{\boldsymbol{x}}$ are all finite. By the residuation theory, $A \otimes \widehat{\boldsymbol{x}} \leq \boldsymbol{b}$ and hence, $S(A, \boldsymbol{b}) \neq \emptyset$ if and only if $A \otimes \widehat{\boldsymbol{x}} = \boldsymbol{b}$, which means that the consistency of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ is fully characterized by its principal solution.

It is clear that the principal solution $\widehat{\boldsymbol{x}}$ is also the maximum solution in $S(A, \boldsymbol{b})$ whenever $S(A, \boldsymbol{b}) \neq \emptyset$ because $A \otimes \boldsymbol{x} \leq \boldsymbol{b}$ if and only if $\boldsymbol{x} \leq \widehat{\boldsymbol{x}}$. Furthermore, $S(A, \boldsymbol{b})$ can be determined along with a finite number of minimal solutions so that

$$S(A, \boldsymbol{b}) = \bigcup_{\check{\boldsymbol{x}} \in \check{S}(A, \boldsymbol{b})} \left\{\boldsymbol{x} \in \mathbb{R}_{\max}^{n} \mid \check{\boldsymbol{x}} \leq \boldsymbol{x} \leq \widehat{\boldsymbol{x}}\right\},$$

where $\check{S}(A, \boldsymbol{b})$ is the set of minimal solutions. A minimal solution in $\check{S}(A, \boldsymbol{b})$ possesses the number of finite elements as few as possible, of which the values are the same with those in the maximum solution. To determine $\check{S}(A, \boldsymbol{b})$ is therefore equivalent to the enumeration of all minimal solutions of an associated set covering problem, which implies that the number of minimal solutions could be exponentially large with respect to the instance size. For practical purposes, the minimal solutions with some particular features may be desired, e. g., those with the least number of finite elements.

Define the support of any vector $\boldsymbol{x} \in \mathbb{R}^n_{\max}$ as

$$\operatorname{supp}(\boldsymbol{x}) = \{j \in N \mid x_j \neq -\infty\},$$

that is, the index set corresponding to its finite elements. This is analogous to the counterpart in conventional linear algebra because $-\infty$ is regarded as the zero element with respect to "addition" $\oplus$ in max-plus algebra. A solution in $S(A, \boldsymbol{b})$ is called the sparsest solution, which is necessarily a minimal solution, if it has the minimum number of finite elements. To find the sparsest solution to the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ is to solve the optimization problem

$$\begin{aligned} \min \quad & |\operatorname{supp}(\boldsymbol{x})| \\ \text{s.t.} \quad & A \otimes \boldsymbol{x} = \boldsymbol{b}, \end{aligned}$$

where $|\operatorname{supp}(\boldsymbol{x})|$ denotes the cardinality of $\operatorname{supp}(\boldsymbol{x})$, i. e., the number of elements in $\operatorname{supp}(\boldsymbol{x})$. This problem is NP-hard and can be reduced to the classical set covering problem in polynomial time.

However, the consistency of max-plus linear equations is sensitive to noise or perturbation in the data. When the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ is inconsistent, the approximate solutions may be considered by minimizing $\|A \otimes \boldsymbol{x} - \boldsymbol{b}\|_p$ where $\|\cdot\|_p$ denotes the conventional L_p vector norm, e. g., $p = 1$, 2, or ∞. A closed form optimal solution for the L_∞ scenario may be directly constructed as $\widehat{\boldsymbol{x}} + \Delta$ where $2\Delta = \|A \otimes \widehat{\boldsymbol{x}} - \boldsymbol{b}\|_\infty$ and the conventional addition is conducted componentwise, see, e. g., Cuninghame-Green [3], Krivulin [8, 9], and Butkovič [2]. The L_1 scenario, as demonstrated by Li [10], may be reformulated into a polynomial-sized mixed integer linear programming problem and solved by an off-the-shelf optimization solver.

When the sparsest approximate solution is concerned, it may be obtained by solving the optimization problem

$$\begin{aligned} \min \quad & |\operatorname{supp}(\boldsymbol{x})| \\ \text{s.t.} \quad & \|A \otimes \boldsymbol{x} - \boldsymbol{b}\|_p \leq \epsilon, \end{aligned}$$

where $\epsilon > 0$ is an admissible error bound, not necessarily the minimum one. This problem has been investigated on the L_p scenario with $p < \infty$ by Tsiamis and Maragos [14] and Tsilivis et al. [15] with the additional "lateness" constraint $A \otimes \boldsymbol{x} \leq \boldsymbol{b}$. The constraint $A \otimes \boldsymbol{x} \leq \boldsymbol{b}$ implies $\boldsymbol{x} \leq \widehat{\boldsymbol{x}} = A^\sharp \otimes' \boldsymbol{b}$ and makes the corresponding optimization problem somewhat more tractable by imposing the supermodular properties so that a greedy algorithm can be implemented. The L_∞ scenario without the lateness constraint has also been tackled by Tsilivis et al. [15] with a two-stage procedure based on the approximate solutions of the L_p scenarios.

This paper focuses on the sparsest approximate solutions to inconsistent max-plus linear equations without the lateness constraint, particularly on the scenarios of the L_∞ norm and the L_1 norm. It turns out obtaining the sparsest approximate solution within a given L_∞ error bound is equivalent to solving a set covering problem defined by the characteristic matrix of associated max-plus linear inequalities. Consequently, it can be solved directly without recourse to the two-stage procedure developed by Tsilivis et al. [15]. Furthermore, the problem of obtaining the sparsest approximate solution within a given L_1 error bound can be transformed into a polynomial-sized mixed integer linear programming problem, which may be tackled directly by some well developed optimization solvers.

The rest of this paper is organized as follows. Section 2 introduces the well known connections between the max-plus linear system and the set covering problem. Section 3 deals with the sparsest approximate solutions for the scenarios of the L_∞ and the L_1 admissible error bounds, respectively, based on a reformulation approach. The conclusions are addressed in Section 4.

2. MAX-PLUS LINEAR SYSTEMS

Recall that a max-plus linear system $A\otimes\boldsymbol{x}=\boldsymbol{b}$ is consistent if and only if $A\otimes\widehat{\boldsymbol{x}}=\boldsymbol{b}$ with $\widehat{\boldsymbol{x}}=A^\sharp\otimes'\boldsymbol{b}$. Furthermore, it can be reduced into its characteristic matrix $\tilde{Q}=(\tilde{q}_{ij})_{m\times n}$ such that

$$\tilde{q}_{ij}=\begin{cases}\widehat{x}_j, & \text{if } a_{ij}+\widehat{x}_j=b_i\\ \emptyset, & \text{otherwise.}\end{cases}$$

It is straightforward that the system $A\otimes\boldsymbol{x}=\boldsymbol{b}$ is consistent if and only if no row of $\tilde{Q}$ consists of only empty elements, or alternatively, $\bigcup_{j\in N}M_j=M$ where $M_j=\{i\in M\mid \tilde{q}_{ij}=\widehat{x}_j\}$, $\forall j\in N$. Consequently, the characteristic matrix $\tilde{Q}$ may be further simplified as a 0-1 matrix $Q=(q_{ij})_{m\times n}$ such that

$$q_{ij}=\begin{cases}1, & \text{if } \tilde{q}_{ij}=\widehat{x}_j\\ 0, & \text{otherwise}\end{cases}$$

and hence, explicitly defines an instance of the set covering problem corresponding to the system $A\otimes\boldsymbol{x}=\boldsymbol{b}$.

By this means, a feasible cover is coded as a vector $\boldsymbol{y}\in\{0,1\}^n$ such that $Q\boldsymbol{y}\geq\boldsymbol{e}$, where $\boldsymbol{e}$ is the vector of all ones with its length determined by the context. According to the construction of Q, any feasible cover $\boldsymbol{y}$ induces a solution $\boldsymbol{x}\in S(A,\boldsymbol{b})$ by

$$x_j=\begin{cases}\widehat{x}_j, & \text{if } y_j=1\\ -\infty, & \text{otherwise.}\end{cases}$$

The transformation also works reversely to convert a solution $\boldsymbol{x}\in S(A,\boldsymbol{b})$, whenever it is nonempty, to a feasible cover $\boldsymbol{y}$ by considering only those elements with the values of $\widehat{x}_j$'s. This reveals the well-known one-to-one correspondence between the minimal solutions to max-plus linear equations and the minimal covers of its associated set covering problem. Therefore, a solution $\boldsymbol{x}\in S(A,\boldsymbol{b})$ with the minimum cardinality of $\mathrm{supp}(\boldsymbol{x})$ defines a cover of the minimum cardinality in the conventional sense, and vice versa. It follows that the optimization problem

$$\min \quad |\mathrm{supp}(\boldsymbol{x})|$$

$$\text{s.t.} \quad A \otimes \boldsymbol{x} = \boldsymbol{b}$$

is equivalent to its corresponding set covering problem, defined by the simplified characteristic matrix Q,

$$\begin{aligned} \min \quad & \boldsymbol{e}^T \boldsymbol{y} \\ \text{s.t.} \quad & Q\boldsymbol{y} \geq \boldsymbol{e} \\ & \boldsymbol{y} \in \{0,1\}^n. \end{aligned}$$

Example 2.1. Consider the following system of max-plus linear equations

$$\begin{pmatrix} 6 & 9 & 9 \\ 6 & 6 & 8 \\ 8 & 7 & 4 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

The associated principal solution is $\widehat{\boldsymbol{x}} = (-7,-6,-6)^T$ and the system is consistent. The corresponding characteristic matrix and its simplified version are, respectively,

$$\tilde{Q} = \begin{pmatrix} \emptyset & -6 & -6 \\ \emptyset & \emptyset & -6 \\ -7 & -6 & \emptyset \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

There are only two minimal covers associated with Q, i. e.,

$$\check{\boldsymbol{y}}_1 = (0,1,1)^T, \qquad \check{\boldsymbol{y}}_2 = (1,0,1)^T,$$

which induce two minimal solutions to the max-plus linear equations as

$$\check{\boldsymbol{x}}_1 = (-\infty,-6,-6)^T, \qquad \check{\boldsymbol{x}}_2 = (-7,-\infty,-6)^T.$$

The solution set $S(A,\boldsymbol{b})$ is therefore determined as

$$S(A,\boldsymbol{b}) = \bigcup_{k=1,2} \left\{\boldsymbol{x} \in \mathbb{R}^3_{\max} \mid \check{\boldsymbol{x}}_k \leq \boldsymbol{x} \leq \widehat{\boldsymbol{x}}\right\}.$$

The both minimal solutions are also the sparsest solutions for this instance.

An essentially same procedure may be applied to handle a system of max-plus linear inequalities in the form

$$\boldsymbol{\ell} \leq A \otimes \boldsymbol{x} \leq \boldsymbol{u},$$

where $\boldsymbol{\ell} = (\ell_1, \ell_2, \ldots, \ell_m)^T$ and $\boldsymbol{u} = (u_1, u_2, \ldots, u_m)^T$ are the lower bound and the upper bound vectors, respectively. It is consistent if and only if $A \otimes \widehat{\boldsymbol{x}} \geq \boldsymbol{\ell}$ with $\widehat{\boldsymbol{x}} = A^\sharp \otimes' \boldsymbol{u}$. Analogously, its characteristic matrix $\tilde{Q} = (\tilde{q}_{ij})_{m\times n}$ is constructed as

$$\tilde{q}_{ij} = \begin{cases} [-a_{ij} + \ell_i, \widehat{x}_j], & \text{if } a_{ij} + \widehat{x}_j \geq \ell_i \\ \emptyset, & \text{otherwise.} \end{cases}$$

The solution set $S(A, \boldsymbol{\ell}, \boldsymbol{u})$, whenever it is nonempty, is still determined by the maximum solution $\widehat{\boldsymbol{x}}$ and a finite set $\check{S}(A, \boldsymbol{\ell}, \boldsymbol{u})$ of minimal solutions in the same manner for max-plus linear equations. However, there is no longer a direct one-to-one correspondence between the minimal solutions and the minimal covers as in the case of max-plus linear equations. This is because the nonempty elements of $\tilde{Q}$ are of interval type so that a minimal solution to max-plus linear inequalities may assume the values other than $\widehat{x}_j$'s and $-\infty$. By some routinely used reformulation tricks, the problem of determining all minimal solutions can be transformed to the enumeration of all minimal covers of an augmented set covering problem with some additional cardinality constraints, as illustrated by Li and Fang [13] and Li [11].

Nevertheless, if a solution with the minimum cardinality of its support is desired, it suffices, taking advantage of the structure of the solution set, to construct the simplified characteristic matrix $Q = (q_{ij})_{m\times n}$ as

$$q_{ij} = \begin{cases} 1, & \text{if } \tilde{q}_{ij} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

and solve the corresponding set covering problem defined by Q. Any optimal solution $\check{\boldsymbol{y}}$, a minimal cover of the minimum cardinality, induces the sparsest approximate solution $\bar{\boldsymbol{x}}$, not necessarily minimal, by

$$\bar{x}_j = \begin{cases} \widehat{x}_j, & \text{if } y_j = 1 \\ -\infty, & \text{otherwise.} \end{cases}$$

To make it a minimal solution to the system of max-plus linear inequalities, a modification may be performed by either a depth-first or a breadth-first search procedure over its support using the information recorded in $\tilde{Q}$.

Example 2.2. Consider the following system of max-plus linear inequalities

$$\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \leq \begin{pmatrix} 6 & 9 & 9 \\ 6 & 6 & 8 \\ 8 & 7 & 4 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}.$$

The associated principle solution is $\widehat{\boldsymbol{x}} = (-6, -5, -5)^T$ and the system is consistent. Since only the sparsest solutions are concerned in this context, the characteristic matrix and its simplified version are therefore constructed, respectively, as

$$\tilde{Q} = \begin{pmatrix} \emptyset & [-7,-5] & [-7,-5] \\ \emptyset & \emptyset & [-6,-5] \\ [-8,-6] & [-7,-5] & \emptyset \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Consequently, two minimal covers of Q can be identified as

$$\check{\boldsymbol{y}}_1 = (0, 1, 1)^T, \qquad \check{\boldsymbol{y}}_2 = (1, 0, 1)^T,$$

and the induced solutions to the max-plus linear inequalities are, respectively,

$$\bar{\boldsymbol{x}}_1 = (-\infty, -5, -5)^T, \qquad \bar{\boldsymbol{x}}_2 = (-6, -\infty, -5)^T.$$

However, the both solutions, although the sparsest, are not minimal. They can be modified to be the minimal solutions, according to the information in $\tilde{Q}$, as

$$\check{\boldsymbol{x}}_1 = (-\infty, -7, -6)^T, \qquad \check{\boldsymbol{x}}_2 = (-8, -\infty, -6)^T.$$

It can be verified that they are the only two minimal solutions for this particular instance and are the sparsest solutions as well. The solution set $S(A, \boldsymbol{\ell}, \boldsymbol{u})$ of the system of max-plus linear inequalities under consideration is therefore determined as

$$S(A, \boldsymbol{\ell}, \boldsymbol{u}) = \bigcup_{k=1,2} \left\{ \boldsymbol{x} \in \mathbb{R}^3_{\max} \mid \check{\boldsymbol{x}}_k \leq \boldsymbol{x} \leq \widehat{\boldsymbol{x}} \right\}.$$

Roughly speaking, as illustrated by Example 2.1 and Example 2.2, there is no essential difference in the solution methods between max-plus linear equations and max-plus linear inequalities as long as the sparsest solutions are concerned, although the patterns of characteristic matrices are somewhat more complicated for max-plus linear inequalities.

3. SPARSEST APPROXIMATE SOLUTIONS

When a system of max-plus linear equations $A \otimes \boldsymbol{x} = \boldsymbol{b}$ is inconsistent, the approximate solutions may be considered by minimizing the residual error $\|A \otimes \boldsymbol{x} - \boldsymbol{b}\|_p$, with respect to the L_p norm of vectors.

For the L_∞ scenario, the minimum residual error Δ can be calculated directly by $2\Delta = \|A \otimes \widehat{\boldsymbol{x}} - \boldsymbol{b}\|_\infty$ with $\widehat{\boldsymbol{x}} = A^\sharp \otimes' \boldsymbol{b}$. If the sparsest approximate solution is desired with respect to an admissible error bound ϵ, necessarily $\epsilon \geq \Delta$, it needs to solve the optimization problem

$$\begin{aligned} \min \quad & |\mathrm{supp}(\boldsymbol{x})| \\ \text{s.t.} \quad & \|A \otimes \boldsymbol{x} - \boldsymbol{b}\|_\infty \leq \epsilon. \end{aligned}$$

Note that the constraint $\|A \otimes \boldsymbol{x} - \boldsymbol{b}\|_\infty \leq \epsilon$ is equivalent to the system of max-plus linear inequalities

$$\boldsymbol{b} - \epsilon \leq A \otimes \boldsymbol{x} \leq \boldsymbol{b} + \epsilon,$$

of which the corresponding principal solution shifts right by ϵ, that is,

$$A^\sharp \otimes' (\boldsymbol{b} + \epsilon) = \widehat{\boldsymbol{x}} + \epsilon.$$

The characteristic matrix $\tilde{Q} = (\tilde{q}_{ij})_{m \times n}$ can be constructed accordingly as

$$\tilde{q}_{ij} = \begin{cases} [-a_{ij} + b_i - \epsilon, \widehat{x}_j + \epsilon], & \text{if } a_{ij} + \widehat{x}_j + 2\epsilon \geq b_i \\ \emptyset, & \text{otherwise} \end{cases}$$

and further reduced to the 0-1 matrix $Q = (q_{ij})_{m \times n}$ as demonstrated in Section 2 in order to obtain the sparsest solutions.

Subsequently, it suffices to solve the set covering problem defined by the simplified characteristic matrix Q. Any optimal solution $\check{\boldsymbol{y}}$ induces the sparsest approximate solution $\bar{\boldsymbol{x}}$, within the given L_∞ error bound, by

$$\bar{x}_j = \begin{cases} \widehat{x}_j + \epsilon, & \text{if } \check{y}_j = 1 \\ -\infty, & \text{otherwise,} \end{cases}$$

which is not necessarily a minimal solution to the system $\boldsymbol{b} - \epsilon \le A \otimes \boldsymbol{x} \le \boldsymbol{b} + \epsilon$. It can be modified to be a minimal one in polynomial time, if necessarily, according to the information recorded in the characteristic matrix $\tilde{Q}$.

Example 3.1. Consider the following system of max-plus linear equations

$$\begin{pmatrix} 6 & 9 & 9 \\ 6 & 6 & 7 \\ 8 & 7 & 4 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}.$$

The associated principal solution is $\widehat{\boldsymbol{x}} = (-7, -6, -6)^T$. The system is inconsistent and $2\Delta = \|A \otimes \widehat{\boldsymbol{x}} - \boldsymbol{b}\|_\infty = 2$. Let the admissible error bound be $\epsilon = \Delta = 1$. The constraint $\|A \otimes \boldsymbol{x} - \boldsymbol{b}\|_\infty \le 1$ for this instance is equivalent to the system of max-plus linear inequalities

$$\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \le \begin{pmatrix} 6 & 9 & 9 \\ 6 & 6 & 7 \\ 8 & 7 & 4 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \le \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}.$$

Since the shifted principle solution is $\widehat{\boldsymbol{x}} + \epsilon = (-6, -5, -5)^T$, the corresponding characteristic matrix and its simplified version can be constructed, respectively, as

$$\tilde{Q} = \begin{pmatrix} \emptyset & [-7, -5] & [-7, -5] \\ \emptyset & \emptyset & [-5, -5] \\ [-8, -6] & [-7, -5] & \emptyset \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Similar to Example 2.2, two sparsest approximate solutions, with respect to the admissible L_∞ error bound, can be identified as

$$\bar{\boldsymbol{x}}_1 = (-\infty, -5, -5)^T, \qquad \bar{\boldsymbol{x}}_2 = (-6, -\infty, -5)^T,$$

of which both can be modified to be minimal, if necessarily, as

$$\check{\boldsymbol{x}}_1 = (-\infty, -7, -5)^T, \qquad \check{\boldsymbol{x}}_2 = (-8, -\infty, -5)^T.$$

Remark 3.2. A two-stage solution method has been developed by Tsilivis et al. [15] to approximately solve the sparsity problem with respect to the admissible L_∞ error bound by first solving the L_p scenario with the lateness constraint and then performing a translation operation. This procedure involves the calculation of the L_p norm of vectors with a large value of p as suggested. However, as addressed by Gotoh and Uryasev [5], the calculation, when p is large, could suffer from some numerical issues.

For the L_1 scenario, the optimization problem

$$\begin{aligned} \min \quad & |\mathrm{supp}(\boldsymbol{x})| \\ \text{s.t.} \quad & \|A \otimes \boldsymbol{x} - \boldsymbol{b}\|_1 \leq \epsilon \end{aligned}$$

is somewhat more complicated since it is difficult to determine the minimum L_1 residual error. Although a feasible admissible error bound $\epsilon = n\Delta$ is available inherited from the L_∞ scenario, it imposes little restriction on the individual elements of the residual vector, a feature associated with the L_∞ norm. However, as demonstrated by Li [10], the problem of minimizing $\|A \otimes \boldsymbol{x} - \boldsymbol{b}\|_1$ can be reformulated into a polynomial-sized mixed integer linear programming problem and solved by some well developed optimization solvers. Once the minimum L_1 residual error δ has been obtained, the admissible error bound can be set as $\epsilon \geq \delta$. Furthermore, the reformulation approach developed by Li [10] may be applied with necessary modifications to deal with the sparsity problem with respect to the admissible L_1 error bound.

Let $\boldsymbol{t} = (t_1, t_2, \ldots, t_m)^T$ be a nonnegative vector such that $\mathrm{abs}(A \otimes \boldsymbol{x} - \boldsymbol{b}) \leq \boldsymbol{t}$, where the absolute values of a vector are taken componentwise. By this means, the constraint $\|A \otimes \boldsymbol{x} - \boldsymbol{b}\|_1 \leq \|\boldsymbol{t}\|_1$ is equivalent to

$$\boldsymbol{b} - \boldsymbol{t} \leq A \otimes \boldsymbol{x} \leq \boldsymbol{b} + \boldsymbol{t},$$

which may be regarded as a system of two-sided max-plus linear inequalities because the both sides of the inequalities have the unknown vector either $\boldsymbol{x}$ or $\boldsymbol{t}$. Besides, the vector $\boldsymbol{x}$ can be written as

$$\boldsymbol{x} = \widehat{\boldsymbol{x}} + \boldsymbol{s},$$

where $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)^T$ is nonnegative, to reduce the searching space for the approximate solutions. Consequently, as demonstrated by Li [10], by introducing an additional group of mn binary variables z_{ij}, $i \in M$, $j \in N$, and a large enough positive constant K, the system $\boldsymbol{b} - \boldsymbol{t} \leq A \otimes \boldsymbol{x} \leq \boldsymbol{b} + \boldsymbol{t}$ can be reformulated as a system of conventional linear inequalities

$$\begin{aligned} & -t_i + s_j \leq -a_{ij} - \widehat{x}_j + b_i, \quad \forall i \in M,\ j \in N, \\ & t_i + s_j + K(1 - z_{ij}) \geq -a_{ij} - \widehat{x}_j + b_i, \quad \forall i \in M,\ j \in N, \\ & \textstyle\sum_{j \in N} z_{ij} = 1, \quad \forall i \in M \end{aligned}$$

Note that the inequality

$$t_i + s_j + K(1 - z_{ij}) \geq -a_{ij} - \widehat{x}_j + b_i$$

means

$$a_{ij} + \widehat{x}_j + s_j + K(1 - z_{ij}) \geq b_i - t_i.$$

When $z_{ij} = 1$, it becomes

$$a_{ij} + \widehat{x}_j + s_j \geq b_i - t_i,$$

which indicates that the variable $x_j = \widehat{x}_j + s_j$ is active for the ith inequality in the system $A \otimes \boldsymbol{x} \geq \boldsymbol{b} - \boldsymbol{t}$. In other words, the 0-1 matrix $Z = (z_{ij})_{m \times n}$ records a set covering pattern under the requirement of $Z\boldsymbol{e} = \boldsymbol{e}$. Therefore, a binary vector $\boldsymbol{y} = (y_1, y_2, \dots, y_n)^T$ may be introduced to record whether the variable x_j is active or not for each $j \in N$, so that the constraints

$$\textstyle\sum_{i\in M} z_{ij} \le m y_j, \quad \forall j \in N$$

can be imposed naturally if the sparsest approximate solutions are concerned. Consequently, in order to obtain the sparsest approximate solution within the given L_1 error bound, it suffices to solve the optimization problem

$$\begin{aligned}
\min \quad & y_1 + y_2 + \cdots + y_n \\
\text{s.t.} \quad & \\
& -t_i + s_j \le -a_{ij} - \widehat{x}_j + b_i, \quad \forall i \in M,\ j \in N, \\
& t_i + s_j + K(1 - z_{ij}) \ge -a_{ij} - \widehat{x}_j + b_i, \quad \forall i \in M,\ j \in N, \\
& t_1 + t_2 + \cdots + t_n \le \epsilon, \\
& \textstyle\sum_{j\in N} z_{ij} = 1, \quad \forall i \in M, \\
& \textstyle\sum_{i\in M} z_{ij} \le m y_j, \quad \forall j \in N, \\
& t_i \ge 0, \quad \forall i \in M, \\
& s_j \ge 0, \quad \forall j \in N, \\
& y_j, z_{ij} \in \{0, 1\}, \quad \forall i \in M,\ j \in N.
\end{aligned}$$

Remark 3.3. This mixed integer linear programming problem may be regarded as a special scenario of the facility location-allocation problem with a resource capacity restriction where N is the set of candidate facilities to operate and M is the set of demand positions to serve. This reformulation approach may be applied as well when the L_2 error bound is considered.

Remark 3.4. The aggregated constraints $\sum_{i\in M} z_{ij} \le m y_j$ for all $j \in N$ in this formulation could be enforced by the disaggregated set of constraints $z_{ij} \le y_j$ for all $i \in M$ and $j \in N$, which, in theory, leads to a better formulation. However, the aggregated version is preferred in practice for a state-of-the-art optimization solver because such type of inequalities can be automatically handled by detecting violated minimal cover inequalities and the size of its linear relaxation is much smaller.

Remark 3.5. The sparsity problem with respect to the L_p residual error, $p < \infty$, has been investigated by Tsiamis and Maragos [14] and Tsilivis et al. [15] with the lateness constraint $A \otimes \boldsymbol{x} \le \boldsymbol{b}$, or equivalently, $\boldsymbol{x} \le \widehat{\boldsymbol{x}} = A^\sharp \otimes' \boldsymbol{b}$. This implies $\boldsymbol{s} = \boldsymbol{0}$ and the constraint $A \otimes (\widehat{\boldsymbol{x}} + \boldsymbol{s}) \le \boldsymbol{b} + \boldsymbol{t}$ is redundant in this context. Consequently, a greedy algorithm may be implemented to obtain nearly optimal solutions taking advantage of the supermodular properties.

Once an optimal solution $\boldsymbol{y}^*$, along with $\boldsymbol{s}^*$ and $\boldsymbol{t}^*$, has been obtained, it induces the sparsest approximate solution $\bar{\boldsymbol{x}}$ by

$$\bar{x}_j = \begin{cases} \widehat{x}_j + s_j^*, & \text{if } y_j^* = 1 \\ -\infty, & \text{otherwise.} \end{cases}$$

It is not necessarily a minimal solution to the system $\boldsymbol{b} - \boldsymbol{t}^* \leq A \circ \boldsymbol{x} \leq \boldsymbol{b} + \boldsymbol{t}^*$ but can be modified according to its characteristic matrix as demonstrated in Section 2.

Example 3.6. Consider the inconsistent system of max-plus linear equations in Example 3.1

$$\begin{pmatrix} 6 & 9 & 9 \\ 6 & 6 & 7 \\ 8 & 7 & 4 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix},$$

of which the principal solution is $\widehat{\boldsymbol{x}} = (-7, -6, -6)^T$.

The minimum L_1 error bound for this instance is $\delta = 2$ by minimizing $\|A \otimes \boldsymbol{x} - \boldsymbol{b}\|_1$ via the procedure in Li [10] with the aid of an optimization solver, e. g., `lp_solve` 5.5 in the `R` package `lpSolve`. The corresponding approximate solution is $\boldsymbol{x}^* = (-7, -6, -4)^T$ so that

$$\begin{pmatrix} 6 & 9 & 9 \\ 6 & 6 & 7 \\ 8 & 7 & 4 \end{pmatrix} \otimes \begin{pmatrix} -7 \\ -6 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}.$$

It is clear that $\|A \otimes \boldsymbol{x}^* - \boldsymbol{b}\|_1 = 2$ and also $\|A \otimes \boldsymbol{x}^* - \boldsymbol{b}\|_\infty = 2 > \Delta$. The optimal solution with respect to the minimum L_1 error bound is usually not unique. For this instance, it can be verified that, for each $\alpha \in [0, 1]$, the approximate solution

$$\boldsymbol{x}^*(\alpha) = (-7, -6, -5 + \alpha)^T$$

achieves the same error bound $\delta = 2$. Particularly, $\boldsymbol{x}^*(0) = (-7, -6, -5)^T$ also reaches the minimum L_∞ error bound $\Delta = 1$ because $A \otimes \boldsymbol{x}^*(0) = (4, 2, 1)^T$ and $\|A \otimes \boldsymbol{x}^*(0) - \boldsymbol{b}\|_\infty = 1$.

Subsequently, if the sparsest approximate solution is desired, one may set, say $\epsilon = 2$ and $K = 10^6$, and solve the following mixed integer linear programming problem

$$
\begin{aligned}
\min \quad & y_1 + y_2 + y_3 \\
\text{s.t.} \quad & \\
& -t_1 + s_1 \le 4, \quad -t_1 + s_2 \le 0, \quad -t_1 + s_3 \le 0, \\
& -t_2 + s_1 \le 4, \quad -t_2 + s_2 \le 3, \quad -t_2 + s_3 \le 2, \\
& -t_3 + s_1 \le 0, \quad -t_3 + s_2 \le 0, \quad -t_3 + s_3 \le 3, \\
& t_1 + s_1 + K(1 - z_{11}) \ge 4, \quad t_1 + s_2 + K(1 - z_{12}) \ge 0, \quad t_1 + s_3 + K(1 - z_{13}) \ge 0, \\
& t_2 + s_1 + K(1 - z_{21}) \ge 4, \quad t_2 + s_2 + K(1 - z_{22}) \ge 3, \quad t_2 + s_3 + K(1 - z_{23}) \ge 2, \\
& t_3 + s_1 + K(1 - z_{31}) \ge 0, \quad t_3 + s_2 + K(1 - z_{32}) \ge 0, \quad t_3 + s_3 + K(1 - z_{33}) \ge 3, \\
& t_1 + t_2 + t_3 \le 2, \\
& z_{11} + z_{12} + z_{13} = 1, \quad z_{21} + z_{22} + z_{23} = 1, \quad z_{31} + z_{32} + z_{33} = 1, \\
& z_{11} + z_{21} + z_{31} \le 3y_1, \quad z_{12} + z_{22} + z_{32} \le 3y_2, \quad z_{13} + z_{23} + z_{33} \le 3y_3, \\
& t_1, t_2, t_3, s_1, s_2, s_3 \ge 0, \quad y_1, y_2, y_3, z_{11}, z_{12}, \ldots, z_{33} \in \{0, 1\}.
\end{aligned}
$$

By calling `lp_solve` 5.5, an optimal solution

$$\boldsymbol{y}^* = (1, 0, 1)^T$$

can be obtained along with $\boldsymbol{s}^* = (0, 0, 2)^T$, which leads to the sparsest approximate solution

$$\bar{\boldsymbol{x}} = (-7, -\infty, -4)^T.$$

Moreover, it can be verified that for this instance the vector

$$\bar{\boldsymbol{x}}(\alpha) = (-7, -\infty, -5 + \alpha)^T$$

is also the sparsest approximate solution for each $\alpha \in [0, 1]$ with the same L_1 residual error. Note that there is another family of sparsest approximate solutions $\bar{\boldsymbol{x}}(\alpha) = (-\infty, -6, -5 + \alpha)^T$, $\forall \alpha \in [0, 1]$, for this instance.

In Example 3.6, it shows that one of the sparsest approximate solutions achieves simultaneously the minimum L_∞ error and the minimum L_1 error bounds. Such an approximate solution may not always exist. Therefore, the sparsest approximate solutions are different in general under the different criteria of admissible error bounds.

Example 3.7. Consider the following inconsistent system of max-plus linear equations

$$
\begin{pmatrix} 5 & 1 & 0 \\ 4 & 4 & 5 \\ 7 & 7 & 7 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
$$

of which the principal solution is $\widehat{\boldsymbol{x}} = (-7, -7, -7)^T$. This example has been investigated by Li [10]. It has the minimum L_∞ residual error $\Delta = 1$ and the minimum L_1 residual error $\delta = 2$.

With respect to $\Delta = 1$, the characteristic matrix of the max-plus linear system $\boldsymbol{b} - \Delta \leq A \otimes \boldsymbol{x} \leq \boldsymbol{b} + \Delta$ is

$$\tilde{Q} = \begin{pmatrix} [-6,-6] & \emptyset & \emptyset \\ \emptyset & \emptyset & [-6,-6] \\ [-8,-6] & [-8,-6] & [-8,-6] \end{pmatrix}.$$

There is only one sparsest approximate solution

$$\check{\boldsymbol{x}} = (-6, -\infty, -6)^T,$$

which is also a minimal solution, such that $A \otimes \check{\boldsymbol{x}} = (-1,-1,1)^T$ and $\|A \otimes \check{\boldsymbol{x}} - \boldsymbol{b}\|_\infty = 1$. Note that $\|A \otimes \check{\boldsymbol{x}} - \boldsymbol{b}\|_1 = 3 > \delta$. Furthermore, it can be verified that for this instance $A \otimes \check{\boldsymbol{x}} = (-1,-1,1)^T$ is the unique right-hand side vector for the consistency of max-plus linear equations within the minimum L_∞ error bound.

With respect to $\delta = 2$, the optimal solution is $\boldsymbol{y}^* = (1,0,1)^T$, along with $\boldsymbol{s}^* = (2,0,2)^T$, by solving the corresponding mixed integer linear programming problem. The sparsest approximate solution is therefore

$$\bar{\boldsymbol{x}} = (-5, -\infty, -5)^T$$

such that $A \otimes \bar{\boldsymbol{x}} = (0,0,2)^T$ and $\|A \otimes \bar{\boldsymbol{x}} - \boldsymbol{b}\|_1 = 2$. Since the characteristic matrix of the max-plus linear system $A \otimes \boldsymbol{x} = A \otimes \bar{\boldsymbol{x}}$ is

$$\tilde{Q} = \begin{pmatrix} -5 & \emptyset & \emptyset \\ \emptyset & \emptyset & -5 \\ -5 & -5 & -5 \end{pmatrix},$$

the vector $\bar{\boldsymbol{x}} = (-5, -\infty, -5)^T$ is also its unique minimal solution. Besides, it can be verified that for this instance $A \otimes \bar{\boldsymbol{x}} = (0,0,2)^T$ is the unique right-hand side vector for the consistency of max-plus linear equations within the minimum L_1 error bound.

As a result, although $\check{\boldsymbol{x}} = (-6,-\infty,-6)^T$ and $\bar{\boldsymbol{x}} = (-5,-\infty,-5)^T$ have the same support, they are the sparsest approximate solutions with respect to the different criteria of residual error bounds.

4. CONCLUSIONS

The sparsity problem of approximate solutions is concerned in this paper for an inconsistent system of max-plus linear equations with respect to admissible residual error bounds. It turns out that obtaining the sparsest approximate solution within a given L_∞ error bound is essentially a set covering problem via a polynomial time transformation. It also shows that obtaining the sparsest approximate solution within a given L_1 error bound can be reformulated into a polynomial-sized mixed integer linear programming problem, which may be regarded as a special scenario of the facility location-allocation problem. Therefore, this paper reveals some interesting connections, old and new, between the sparsest approximate solution problems in max-plus algebra and some well known problems in discrete and combinatorial optimization. This also implies that the concerned problem may be solved to optimality with the aid of an off-the-shelf optimization solver and in some sense requires no tailored solving method except for very large instances. Besides, this reformulation approach can be extended, in an analogous manner as in Li [12], for the problem of minimizing a general linear objective function of the approximate solutions to max-plus linear equations.

(Received November 6, 2023)

REFERENCES

[1] F. Baccelli, G. Cohen, G. J. Olsder, and J. P. Quadrat: Synchronization and Linearity: An Algebra for Discrete Event Systems. Wiley, Chichester 1992.

[2] P. Butkovič: Max-linear Systems: Theory and Algorithms. Springer, Berlin 2010.

[3] R. A. Cuninghame-Green: Minimax Algebra, Lecture Notes in Economics and Mathematical Systems, Vol. 166. Springer, Berlin 1979.

[4] M. Gondran and M. Minoux: Graphs, Dioids and Semirings: New Models and Algorithms. Springer, New York 2008.

[5] J. Gotoh and S. Uryasev: Two pairs of families of polyhedral norms versus ℓ_p-norms: proximity and applications in optimization. Math. Program. *156* (2016), 391–431. DOI:10.1007/s10107-015-0899-9

[6] B. Heidergott, G. J. Olsder, and J. van der Woude: Max Plus at Work: Modeling and Analysis of Synchronized Systems. Princeton University Press, Princeton 2005.

[7] M. Joswig: Essentials of Tropical Combinatorics. American Mathematical Society, 2021.

[8] N. Krivulin: Methods of Idempotent Algebra for Problems in Modeling and Analysis of Complex Systems. Saint Petersburg University Press, St. Petersburg 2009. (in Russian)

[9] N. Krivulin: Solution of linear equations and inequalities in idempotent vector spaces. Int. J. Appl. Math. Inform. *7* (2013), 14–23.

[10] P. Li: A note on resolving the inconsistency of one-sided max-plus linear equations. Kybernetika *55* (2019), 531–539. DOI:10.14736/kyb-2019-3-0531

[11] P. Li: Solving the sensor cover energy problem via integer linear programming. Kybernetika *57* (2021), 568–593. DOI:10.14736/kyb-2021-4-0568

[12] P. Li: Linear optimization over the approximate solutions of a system of max-min equations. Fuzzy Sets Systems *484* (2024), 108946. DOI:10.1016/j.fss.2024.108946

[13] P. Li and S. C. Fang: On the resolution and optimization of a system of fuzzy relational equations with sup-T composition. Fuzzy Optim. Decision Making *7* (2008), 169–214. DOI:10.1007/s10700-008-9029-y

[14] A. Tsiamis and P. Maragos: Sparsity in max-plus algebra and systems. Discrete Event Dynamic Systems *29* (2019), 163–189. DOI:10.1007/s10626-019-00281-1

[15] N. Tsilivis, A. Tsiamis, and P. Maragos: Toward a sparsity theory on weighted lattices. J. Math. Imaging Vision *64* (2022), 705–717. DOI:10.1007/s10851-022-01075-1

Pingke Li, College of Economics and Management, Beijing University of Technology, Beijing, 100124. P. R. China.

e-mail: pkli@bjut.edu.cn