KYBERNETIKA — VOLUME *60* (2024), NUMBER 3, PAGES 394–411

ASYMPTOTIC FUZZY CONTRACTIVE MAPPINGS IN FUZZY METRIC SPACES

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Fixed point theory in fuzzy metric spaces has grown to become an intensive field of research. However, due to the complexity involved in the nature of fuzzy metrics, the authors need to develop innovative machinery to establish new fixed point theorems in such kind of spaces. In this paper, we propose the concepts of asymptotic fuzzy ψ-contractive and asymptotic fuzzy Meir–Keeler mappings, and describe some new machinery by which the corresponding fixed point theorems are proved. In this sense, the techniques used for the proofs in Section 5 are completely new.

Keywords: fuzzy metric space, asymptotic fuzzy ψ-contractive mapping, asymptotic fuzzy Meir–Keeler mapping, fixed point

Classification: 54H25, 47H10

1. INTRODUCTION

Asymptotic fixed point theory deals with the conditions describing the behavior of the iterates of a mapping. In 2003, Kirk [13] obtained a result which is an asymptotic version of the fixed point theorem by Boyd and Wong [2] (see also [12]), whereas Suzuki introduced in [25] the notion of asymptotic contraction of Meir–Keeler type and generalized Kirk results. The beauty of asymptotic contractions is that they are not even non-expansive and, hence, they are not iteratively equivalent to any class of contractions. However, at the same time, the most arduous part of research in this topic is to give constructive proofs of metrical fixed point theorems for this kind of mappings (see, for example, [12, 17]).

On the other hand, fuzzy metric fixed point theory seems to be more diverse than the regular metric fixed point theory. This is due to the pliability exhibited in the concept of fuzzy metric. Therefore, to study fixed point theory in such spaces, one might need to use or develop new fuzzy mathematical tools (see, for example, [4, 5, 6, 8, 22, 23, 24], in order of appearance). This aspect might be the reason that no results on asymptotic mappings in the fuzzy setting have appeared in the literature, and even there are just few results on the topic in the context of classical metric spaces. In this paper, we make an attempt to introduce the fuzzy version of asymptotic contractions and to give constructive proofs of the corresponding fixed point theorems.

DOI: 10.14736/kyb-2024-3-0394

The structure of this paper is as follows. In Section 2, we show some preliminary concepts and results that will be of interest for the remainder of the paper. In Section 3, we introduce the concept of asymptotic fuzzy ψ-contractive mappings and, then, we establish a fixed point theorem for such mappings by proving a cycle of auxiliary results. In Section 4, we first prove a characterization theorem for asymptotic fuzzy ψ-contractive mappings on compact spaces. Consequently, an example is constructed to show that the class of fuzzy asymptotic contractions is not equivalent to any known classes of fuzzy contractive mappings (e. g., fuzzy ψ-contractions [19], fuzzy Meir–Keeler contractions [26], fuzzy-$\mathcal{Z}$-contractions [24]). In Section 5, we introduce the concepts of asymptotic fuzzy contractions of Meir–Keeler type of first and second kinds, and prove the corresponding fixed point theorems. The techniques used for the proofs in this section are completely new. Finally, in Section 6, some conclusions are presented.

2. PRELIMINARIES

In this section, we state some basic concepts which will be needed in the sequel.

Definition 2.1. (Schweizer and Sklar [21]) A mapping $*\colon [0,1]\times[0,1]\to[0,1]$ is called a continuous triangular norm (t-norm, for short) if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative, i. e., $a*b=b*a$ and $a*(b*c)=(a*b)*c$, for all $a,b,c\in[0,1]$;

(ii) $*$ is continuous;

(iii) $1*a=a$, for all $a\in[0,1]$;

(iv) $a*b\le c*d$, whenever $a\le c$ and $b\le d$, with $a,b,c,d\in[0,1]$.

Some basic examples of t-norms are the minimum t-norm $*_m$, $a*_m b=\min\{a,b\}$, the product t-norm $*_p$, $a*_p b=ab$, or the Łukasiewicz t-norm $*_L$, $a*_L b=\max\{a+b-1,0\}$, for all $a,b\in[0,1]$.

We say that the t-norm $*$ satisfies the cancellation law if $a*b=a*c$ implies $a=0$, or $b=c$.

In the following definition, we denote $*^m a := a*\overset{m}{\cdots}*a$, for $m\in\mathbb{N}$.

Definition 2.2. (Hadžić and Pap [11]) A t-norm $*$ is said to be of *H-type* if the sequence $\{*^m a\}_{m=1}^{\infty}$ is equicontinuous at $a=1$, i. e., for all $\varepsilon\in(0,1)$, there exists $\eta\in(0,1)$ such that, if $a\in(1-\eta,1]$, then $*^m a>1-\varepsilon$, for all $m\in\mathbb{N}$.

The most important and well known continuous t-norm of H-type is $*=\min$. Other examples can be found in [11].

Definition 2.3. (George and Veeramani [3]) A fuzzy metric space (for short, GV-fuzzy metric space) is an ordered triple $(X,M,*)$ such that X is a (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X\times X\times(0,\infty)$ satisfying the following conditions, for all $x,y,z\in X$ and $t,s>0$:

(GV1) $M(x,y,t) > 0$;

(GV2) $M(x,y,t) = 1$ if and only if $x = y$;

(GV3) $M(x,y,t) = M(y,x,t)$;

(GV4) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$;

(GV5) $M(x,y,\cdot)\colon (0,\infty) \to (0,1]$ is continuous.

Remark 2.4. Note that, in this context, condition (GV2) in Definition 2.3 has the following meaning:

$$M(x,x,t) = 1 \text{ for all } x \in X \text{ and } t > 0, \text{ and } M(x,y,t) < 1 \text{ for all } x \neq y \text{ and } t > 0.$$

For the topological properties of a fuzzy metric space, the reader is referred to [3].

Definition 2.5. (George and Veeramani [3]; Schweizer and Sklar [21]) Let $(X, M, *)$ be a fuzzy metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called an M-Cauchy sequence if, for each $\varepsilon \in (0,1)$ and each $t > 0$, there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, for all $n, m \geq n_0$.

The sequence $\{x_n\}$ is called convergent, and it converges to x, if, for each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$, for all $n \geq n_0$.

We say that the space $(X, M, *)$ is M-complete if every M-Cauchy sequence in X is convergent to some $x \in X$.

Definition 2.6. (George and Veeramani [3]) Let $(X, M, *)$ be a fuzzy metric space. A collection of non-empty sets $\{A_i\}_{i \in I}$ in X is said to have fuzzy diameter zero if, for each $r \in (0,1)$ and $t > 0$, we can find $i_{r,t} \in I$ (depending on r and t) such that $M(x,y,t) > 1 - r$ for all $x, y \in A_{i_{r,t}}$.

Definition 2.7. (Gregori et al. [9]) Let $(X, M, *)$ be a fuzzy metric space. The fuzzy diameter of a (non-empty) set A of X, with respect to t, is the function $\phi_A : (0, +\infty) \to [0,1]$, also denoted by $diam(A)$, given by $\phi_A(t) := \inf\{M(x,y,t) : x, y \in A\}$, for each $t > 0$.

Proposition 2.8. (Gregori et al. [9]) The function ϕ_A is well-defined and, in addition, it satisfies the following properties:

(i) If $s < t$, then $\phi_A(s) \leq \phi_A(t)$.

(ii) If $A \subseteq B$, then $\phi_A(t) \geq \phi_B(t)$.

(iii) $\phi_A(t) = 1$, for some t if, and only if, A is a singleton set.

We say that a mapping $T : X \to X$ has a contractive fixed point x^*, if $x^* = Tx^*$ and $T^n x \to x^*$ for all $x \in X$. Similarly, we say that a mapping $T : X \to X$ has an approximate fixed point if there exists $x \in X$ such that $x \in Fix(T)$ and, given a sequence $\{x_n\}$, if $d(x_n, Tx_n) \to 0$, then $x_n \to x$.

3. MAIN RESULTS

Now, we state the main results of this paper.

Definition 3.1. Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ is said to be an asymptotic fuzzy ψ-contractive mapping if there exists a sequence of functions $\psi_n : [0,1] \to [0,1]$ such that

(A_1) $M(x,y,t) > 0 \Rightarrow M(T^n x, T^n y, t) \geq \psi_n(M(x,y,t))$,
for all $x, y \in X$ and for each $t > 0$,

(A_2) ψ_n converges to ψ uniformly on $[0,1]$, and

(A_3) ψ is continuous, non-decreasing, and $\psi(t) > t$, for every $t \in (0,1)$.

The following theorem is our main result.

Theorem 3.2. Let $(X, M, *)$ be an M-complete fuzzy metric space, where $*$ is an H-type t-norm, and let $T : X \to X$ be a uniformly continuous asymptotic fuzzy ψ-contractive mapping such that the mapping $\psi : [0,1] \to [0,1]$ satisfies the condition:

$$\limsup_{s \to 0^+} \psi(s) > 0. \tag{1}$$

Then T has a unique fixed point x^*. Moreover, x^* is both contractive and approximate fixed point.

The proof of Theorem 3.2 will be preceded by a cycle of auxiliary results.

First, we establish the following notation. Given a nondecreasing sequence $\{b_n\} \subset (0,1]$ such that $b_n \to 1$ as $n \to \infty$, we define the family of sets:

$$\mathcal{A}_n = \{x \in X : M(x, Tx, t) \geq b_n\},\ n \in \mathbb{N}. \tag{2}$$

Lemma 3.3. Let T be a continuous asymptotic fuzzy ψ-contractive mapping from a fuzzy metric space $(X, M, *)$ into itself. Then

$$\limsup_{n \to \infty} M(T^n x, T^n y, t) = 1, \text{ for every } x, y \in X \text{ and } t > 0. \tag{3}$$

P r o o f. The condition (3) is trivially true if $x = y$. So let $x \neq y$ and assume that $0 < l := \limsup_{n\to\infty} M(T^n x, T^n y, t) < 1$. In the case where $T^k x = T^k y$ for some $k \in \mathbb{N}$ and $x, y \in X$, since T is continuous, we have $T^{k+1}x = T(T^k x) = T(T^k y) = T^{k+1} y$. This implies that $T^n x = T^n y$ for every $n \in \mathbb{N}$ large enough and, hence,

$$\lim_{n \to \infty} M(T^n x, T^n y, t) = 1.$$

So let $T^k x \neq T^k y$ for any $k \in \mathbb{N}$ and $x, y \in X$, then, by (A_1), we have

$$M(T^{n+k} x, T^{n+k} y, t) \geq \psi_n(M(T^k x, T^k y, t)), \text{ for every } k \in \mathbb{N} \text{ and } t > 0,$$

i. e.,

$$l = \limsup_{n\to\infty} M(T^{n+k}x, T^{n+k}y, t) \geq \limsup_{n\to\infty} \psi_n(M(T^kx, T^ky, t)) = \psi(M(T^kx, T^ky, t)).$$

In particular, for every $t > 0$, $l \geq \psi(M(T^{k_m}x, T^{k_m}y, t))$, where $\{k_m\}$ is an increasing sequence of positive integers such that $M(T^{k_m}x, T^{k_m}y, t) \to l$ as $m \to \infty$.

Therefore, using the continuity of ψ and taking the limit as $m \to \infty$ in the previous inequality, we have $l \geq \psi(l)$, which contradicts (A_3). This proves that

$$\limsup_{n\to\infty} M(T^nx, T^ny, t) = 1,$$

for every $x, y \in X$ and $t > 0$. □

Lemma 3.4. Let $(X, M, *)$ be a fuzzy metric space, $T : X \to X$ be an asymptotic fuzzy ψ-contractive mapping with $\lim_{s\to 1} \psi_1(s) = 1$, and $\{x_n\}$ be a sequence in X such that $M(x_n, Tx_n, t) \to 1$ as $n \to \infty$ for every $t > 0$. Then, for all $k \in \mathbb{N}$,

$$\lim_{n\to\infty} M(x_n, T^kx_n, t) = 1, \text{ for every } t > 0. \tag{4}$$

P r o o f. We apply the induction principle. By hypothesis, condition (4) is satisfied for $k = 1$. Assume that (4) holds for some $k \in \mathbb{N}$. Then, by the asymptotic fuzzy ψ-contractivity and the assumption on ψ_1, we get

$$\begin{aligned} M(x_n, T^{k+1}x_n, t) &\geq M(x_n, Tx_n, t/2) * M(Tx_n, T(T^kx_n), t/2) \\ &\geq M(x_n, Tx_n, t/2) * \psi_1(M(x_n, T^kx_n, t/2)) \to 1 \text{ as } n \to \infty. \end{aligned}$$

Thus, $\lim_{n\to\infty} M(x_n, T^{k+1}x_n, t) = 1$, and (4) holds for $k + 1$. □

Remark 3.5. The conclusion of Lemma 3.4 is still valid if we replace the condition $\lim_{s\to 1} \psi_1(s) = 1$ by the uniform continuity of the mapping T, in the sense that

$$\text{if } \lim_{n\to\infty} M(x_n, y_n, t) = 1, \ \forall t > 0, \text{ then } \lim_{n\to\infty} M(Tx_n, Ty_n, t) = 1, \ \forall t > 0.$$

Lemma 3.6. Let $(X, M, *)$ be a fuzzy metric space and let the sets $\mathcal{A}_n$ be defined by (2). Then $diam(\mathcal{A}_n) \to 1$ as $n \to \infty$ iff given arbitrary sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{A}_n$, conditions $M(x_n, Tx_n, t) \to 1$ and $M(y_n, Ty_n, t) \to 1$ as $n \to \infty$ imply that $M(x_n, y_n, t) \to 1$ as $n \to \infty$.

P r o o f. The necessity part is obvious because of the triangular property of the fuzzy metric. To show the sufficiency, since $\mathcal{A}_{n+1} \subseteq \mathcal{A}_n$, by Proposition 2.8(ii), we get $diam(\mathcal{A}_{n+1}) \geq diam(\mathcal{A}_n)$. Since we have a monotonic sequence and bounded from above (by 1), hence $diam(\mathcal{A}_n) \to l$, as $n \to \infty$, for some $l \in [0, 1]$. Suppose that $l \neq 1$, i. e., $0 \leq l < 1$, then, there exists $0 < s \leq l$ such that $0 \leq diam(\mathcal{A}_n) < 1 - s$ for every $n \in \mathbb{N}$. Hence, there are $x_n, y_n \in \mathcal{A}_n$ such that

$$0 \leq M(x_n, y_n, t) < 1 - s$$

so that $M(x_n, y_n, t) \not\to 1$ as $n \to \infty$. On the other hand, by (2), we get that both the sequences $M(x_n, Tx_n, t) \to 1$ and $M(y_n, Ty_n, t) \to 1$ as $n \to \infty$. So, by hypothesis, $M(x_n, y_n, t) \to 1$, which is a contradiction. Thus, $l = 1$. □

Lemma 3.7. Let T be an asymptotic fuzzy ψ-contractive mapping with either T uniformly continuous, or $\lim_{s\to 1} \psi_1(s) = 1$. Let $\mathcal{A}_n$ be defined by (2). Then $diam(\mathcal{A}_n) \to 1$, as $n \to \infty$.

Proof. We apply Lemma 3.6. So let $\{x_n\}$ and $\{y_n\}$ be such that both sequences $\{M(x_n, Tx_n, t)\}$ and $\{M(y_n, Ty_n, t)\}$ converge to 1, for all $t > 0$. Let $a_n(t) = M(x_n, y_n, t)$ and $b_{n,k}(t) = M(x_n, T^k x_n, t) * M(y_n, T^k y_n, t)$ for every $k \in \mathbb{N}$. Let $\{\psi_n\}$ and ψ be as in Definition 3.1. Then, given $\varepsilon \in (0, 1)$, there exists $k \in \mathbb{N}$ such that

$$\psi_k(t) > \psi(t) * \varepsilon, \quad \forall\ t > 0. \tag{5}$$

So, by (A_1) and (5), we have

$$\begin{aligned} a_n(3t) &\geq M(x_n, T^k x_n, t) * M(y_n, T^k y_n, t) * M(T^k x_n, T^k y_n, t) \\ &\geq b_{n,k}(t) * \psi_k(M(x_n, y_n, t)) \\ &= b_{n,k}(t) * \psi_k(a_n(t)) \\ &\geq b_{n,k}(t) * \psi(a_n(t)) * \varepsilon. \end{aligned} \tag{6}$$

Hence, by Remark 3.5 (or Lemma 3.4),

$$\lim_{n\to\infty} M(Tx_n, T^2 x_n, t) = 1,\ \forall t > 0$$

and, similarly,

$$\lim_{n\to\infty} M(x_n, T^2 x_n, t) = 1,\ \forall t > 0.$$

Analogously,

$$\lim_{n\to\infty} M(x_n, T^k x_n, t) = 1,\ \forall t > 0,$$

for every $k \in \mathbb{N}$, and, analogously for $\{y_n\}$. This proves that $\lim_{n\to\infty} b_{n,k}(t) = 1$, for every $k \in \mathbb{N}$. So, taking the limit on both sides of (6), we get

$$\lim_{n\to\infty} a_n(3t) \geq \varepsilon * \lim_{n\to\infty} \psi(a_n(t)). \tag{7}$$

Since $\varepsilon \in (0, 1)$ is arbitrary, we have

$$\lim_{n\to\infty} a_n(3t) \geq \lim_{n\to\infty} \psi(a_n(t)).$$

Suppose that $\lim_{n\to\infty} a_n(t) \to s \neq 1$. Then, the above inequality reduces to $\lim_{n\to\infty} a_n(3t) \geq \lim_{n\to\infty} \psi(a_n(t))$ i.e. $s \geq \psi(s) > s$, a contradiction. Thus $\lim_{n\to\infty} a_n(t) \to 1$ and it suffices to apply Lemma 3.6 to completes the proof. □

P r o o f . [Proof of Theorem 3.2] Let the sets A_n is defined by (2). Lemma 3.3 yields that

$$\lim_{n\to\infty} M(T^n x, T^{n+1}x, t) = 1 \text{ for every } x \in X \text{ and for every } t > 0,$$

which implies that the sets A_n are non-empty.

Let $x, y \in A_n$. Then, we have

$$\begin{aligned} M(x,y,t) &\geq M(x,Tx,t/3) * M(y,Ty,t/3) * M(Tx,Ty,t/3) \\ &\geq b_n * b_n * \psi(b_n) \\ &> b_n * b_n = b_n \end{aligned}$$

(since $*$ is continuous H-type t-norm, so there exists a sequence $\{b_n\}$ such that $b_n \in (0,1]$, $b_n \to 1$ and $b_n * b_n = b_n$ (see [11] Theorem 8 p. 366)), i.e., $\{A_n\}$ has fuzzy diameter zero. So we conclude that

$$\bigcap_{n\in N} A_n = x^* = Fix(T).$$

We show that x^* is an approximate fixed point. Assume that $M(x_n, Tx_n, t) \to 1$. Set $y_n = x^*$ for every $n \in \mathbb{N}$. Then, by Lemma 3.6, we get $M(x_n, x^*, t) \to 1$. In particular, given $x \in X$ if $x_n = T^n x$, then $M(x_n, Tx_n, t) \to 1$ and the above arguments gives $T^n x \to x^*$. i.e. x^* is a contractive fixed point. □

Remark 3.8. Mimicking Remark 4 in ([12], p. 154), the following question is worthy to consider:

Question: Does Theorem 3.2 remain true if "uniform continuity of T" is replaced by continuity?

The following result will be used in the next section.

Proposition 3.9. Let $\psi(t) = 1$ for every $t \in (0,1)$. Then T is an asymptotic fuzzy ψ-contraction iff fuzzy diam of $T^n(X)$ is zero.

P r o o f . To prove the necessity, fix $\epsilon > 0$. Then as $\psi(t) = 1$ for every $t \in (0,1)$ and from (A_3) $\psi(t) > t$, for every $t \in (0,1)$. So there is an $m \in \mathbb{N}$ such that $\psi_m(t) > 1-\epsilon$, for every $t \in (0,1)$. Let $x, y \in X$, as T is an asymptotic fuzzy ψ-contraction. We have

$$M(T^n x, T^n y, s) \geq \psi_n(M(x,y,s)) > 1-\epsilon, \text{ for every } n \geq m; \text{ for every } s > 0.$$

Hence we get $\phi(T^n(X)) > 1-\epsilon$,. That means the fuzzy diam of $T^n(X)$ is zero. For the sufficiency it is enough to set $\psi_n(s) =$ fuzzy diam of $T^n(X)$, for every $n \in \mathbb{N}$. □

Proposition 3.10. Let $(X, M, *)$ be a fuzzy metric space and T be an asymptotic fuzzy-ψ-contraction. Then T is surjective iff X is a singleton.

P r o o f . Part 'if' is trivial. So let T be a surjective and suppose, on the contrary, that fuzzy cardinality of $X \neq 1$. Then the fuzzy diam of $X = r < 1$. Since T is an asymptotic fuzzy-ψ-contraction. i.e. $\phi_X(t) := \inf\{M(x,y,t) : x, y \in X, \text{ for each } t > 0\} = r < 1$.

Then, we have $\psi(M(x,y,t)) \geq \psi(r) > r$. Set $\epsilon = \frac{r}{\psi(r)}$. By the hypothesis that T is an asymptotic fuzzy-ψ-contraction, there is a $k \in \mathbb{N}$ such that

$$\psi_k(t) > \psi(t) \cdot \epsilon, \text{ for every } t \in (0,1).$$

Given $x, y \in X, s > 0$,

$$M(T^k x, T^k y, s) \geq \psi_k(M(x,y,s)) > \psi(M(x,y,s)) \cdot \epsilon = r.$$

Hence the fuzzy diam of $T^k(X) > r$ i.e. fuzzy diam of $T^k(X) >$ fuzzy diam of X.

On the other hand T is surjective, so is T^k. Thus $T^k(X) = X$ which violates the fuzzy dimensionality of $T^k(X)$. So X must be a singleton. □

4. ASYMPTOTIC CONTRACTION ON COMPACT SPACES

Theorem 4.1. Assume that $(X, M, *)$ is a compact and T is a continuous self map on X. The following statements are equivalent:

(i) T is an asymptotic fuzzy-ψ-contraction;

(ii) the core $Y = \cap_{n\in\mathbb{N}} T^n(X)$ is a singleton;

(iii) T is an asymptotic fuzzy ψ-contraction, where $\psi_1(t) = 1$ for every $t \in (0,1)$.

Proof. (i) ⇒ (iii) following [16, proposition 2], T map Y onto Y and the restriction $T_{|Y}$ is also an asymptotic fuzzy contraction, Proposition 3.10 yields Y is a singleton. (ii) ⇒ (iii) [16, proposition 2] also ensures that the fuzzy diam of $T^n(X) \to$ fuzzy diam of Y. Hence, we get that the fuzzy diam of $T^n(X) \to 1$, since Y is singleton. So it is suffices to apply Proposition 3.9. (iii) ⇒ (i) a fortiori. Thus, condition (i)=(iii) are equivalent. □

The following example (inspired from Jachymaki [12]) shows that there exits asymptotic fuzzy-ψ-contraction which are not fuzzy non-expansive. i.e. there are $x, y \in X$ and $t > 0$ such that $M(Tx, Ty, t) < M(x, y, t)$. This shows that the class of all asymptotic fuzzy-contraction is not equivalent to any class of known fuzzy contractive mappings (e.g. fuzzy ψ-contraction [19], fuzzy Meir–Keeler contraction [26], fuzzy-Z-contraction [24]).

Example 4.2. Let $X = \{x_n : n \in \mathbb{N}\} \cup \{1\}$, where $\{x_n\}$ is an arbitrary sequence such that $x_n \in (0,1), x_n < x_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = 1$. Define a fuzzy set M on $X \times X \times (0, \infty)$ by

$$M(x,y,t) = \begin{cases} 1, & \text{if } x = y; \\ min\{x,y\}, & \text{otherwise.} \end{cases} \quad \text{for all } x, y \in X,\ t \in (0,\infty).$$

Then (X, M, x_m) is an M-complete fuzzy metric space. We define a mapping T as follows: we consider the subsequence $(x_1, x_{1+2}, x_{1+2+3}, x_{1+2+3+4}, \ldots) = (x_{n(n+1)/2})_{n=1}^{\infty}$. Set $k_n = \frac{n(n+1)}{2}$ and define

$$Tx_0 = x_0, \text{ where } x_0 = 1,$$

$$Tx_{k_n} = x_{k_{n+2}-1}, n \in \mathbb{N},$$
$$Tx_i = x_{i-1}, i \in \mathbb{N} \setminus \{k_n : n \in \mathbb{N}\}.$$

Then T is obviously continuous. Next, it can be verified that $T^{k_{n+1}-2}(X) \subset [x_{k_n}, x_0], n \in \mathbb{N}$. Since $(T^n(X))_{n=1}^{\infty}$ is increasing, we get

$$\bigcap_{n\in\mathbb{N}} T^n(X) = \bigcap_{n\in\mathbb{N}} T^{k_{n+1}-2}(X) \subset \bigcap_{n\in\mathbb{N}} [x_{k_n}, x_0] = \{x_0\}.$$

This yields $\bigcap_{n\in\mathbb{N}} T^n(X) = \{x_0\}$, since x_0 is a core of T as a fixed point of it. Thus by Theorem 4.1, T is an asymptotic fuzzy ψ-contraction. On the other hand, since $M(Tx_2, Tx_0, t) = min\{x_1, x_0\} = x_1 < x_2 = min\{x_2, x_0\} = M(x_2, x_0, t)$, i. e., T is not fuzzy non-expansive and hence not any type of fuzzy contractive mappings.

5. ASYMPTOTIC FUZZY CONTRACTION OF MEIR–KEELER OF FIRST KIND

Definition 5.1. Let $(X, M, \star)$ be a fuzzy metric space. A mapping $T : X \to X$ is said to be an asymptotic fuzzy contraction of Meir–Keeler of first kind (AFCMKFK, for short) if there exists a sequence $\psi_n : [0,1] \to [0,1]$ and the following conditions hold:

(C_1) $\liminf_{n\to\infty} \psi_n(\epsilon) \geq \epsilon, \ \forall\ \epsilon \in (0,1)$;

(C_2) $\forall \epsilon \in (0,1), \exists$ a $\delta \in (0,1)$ and $v \in \mathbb{N}$ such that,

$$\epsilon \star \delta \leq s \leq \epsilon \Rightarrow \psi_v(s) \geq \epsilon; \quad \forall s \in [\epsilon \star \delta,\ \epsilon];$$

(C_3) $M(T^n x, T^n y) > \psi_n(M(x,y,t)),\ \forall n \in \mathbb{N}$ and $\forall x, y \in X$ with $x \neq y$.

Proposition 5.2. Let $(X, M, \star)$ be a fuzzy metric space and T be an asymptotic fuzzy ψ- contractive mapping on X. Then T is also AFCMKFK.

P r o o f. Define a sequence (ψ_n) of functions from $[0,1]$ into itself by

$$\phi_n(s) = \psi_n(s) \cdot \frac{1}{e^{\frac{s}{n}}},\ \forall s \in (0,1) \text{ and } n \in \mathbb{N}.$$

We shall show that (ψ_n) satisfies $(C_1 - C_3)$. Since $\psi_n \to \psi$ and $\psi(s) > s$, $\forall s \in (0,1)$, we have

$$\liminf_{n\to\infty} \phi_n(s) = \liminf_{n\to\infty} [\psi_n(s) \cdot \frac{1}{e^{\frac{s}{n}}}] = \psi(s) \geq s.$$

Thus (C_1) holds.

Fix $n \in \mathbb{N}$ and $x, y \in X$ with $x \neq y$. Since $M(x,y,t) > 0$, we have

$$\begin{aligned} M(T^n x, T^n y) &> \psi_n(M(x,y,t)) \\ &> \psi_n(M(x,y,t)) \cdot \frac{1}{e^{\frac{M(x,y,t)}{n}}} \\ &= \phi_n(M(x,y,t)). \end{aligned}$$

Thus (C_3) holds.

To prove (C_2) assume that $\forall \epsilon \in (0,1)$ there exists a $\delta \in (0,1)$ such that $\epsilon \star \delta \leq t$. Since $\psi(\epsilon) > \epsilon$ and ψ is continuous, we can choose $\delta \in (0,1)$ and $v \in \mathbb{N}$ such that

$$\psi(t) \geq \psi(\epsilon) \cdot \frac{\epsilon \star \delta}{\delta} \cdot e^{\frac{2t}{v}}, \ \forall\ t \in [\epsilon \star \delta, \epsilon],$$

and

$$\psi_v(t) \geq \psi(t) \cdot \frac{\delta}{(\epsilon \star \delta) \cdot e^{\frac{t}{v}}}.$$

Then

$$\begin{aligned}\phi_v(t) = \psi_v(t) \cdot \frac{1}{e^{\frac{t}{v}}} &\geq \psi(t) \cdot \frac{\delta}{(\epsilon \star \delta) \cdot e^{\frac{t}{v}}} \cdot \frac{1}{e^{\frac{t}{v}}} \\ &\geq \psi(\epsilon) \cdot \frac{\epsilon \star \delta}{\delta} \cdot e^{\frac{2t}{v}} \frac{\delta}{(\epsilon \star \delta)} \cdot \frac{1}{e^{\frac{2t}{v}}} \\ &\geq \psi(\epsilon).\end{aligned}$$

Thus (C_2) holds. □

Theorem 5.3. Let $(X, M, \star)$ (where the t-norms $\star$ satisfies cancellation law) be a fuzzy metric space. Let T be an AFCMKFK on X. Assume that T^l is continuous for some $l \in \mathbb{N}$. Then there exists a unique fixed point $z \in X$. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.

Proof. We note that

$$M(T^n x, T^n y, t) \geq \psi_n(M(x,y,t))\ \forall\ t > 0 \text{ and } x, y \in X.$$

We first show that

$$\lim_{n\to\infty} M(T^n x, T^n y, t) = 1,\ \forall\ x, y \in X \text{ and } \forall t > 0.$$

Assume $\alpha = \alpha(t) = \liminf_n M(T^n x, T^n y, t) < 1$. Then from (C_2), there exists $v_1 \in \mathbb{N}$ such that

$$\psi_{v_1}(M(x,y,t)) \geq M(x,y,t).$$

So we have

$$M(T^{v_1} x, T^{v_1} y, t) > \psi_{v_1}(M(x,y,t)) \geq M(x,y,t).$$

By (C_3) and (C_1), we get

$$\begin{aligned}\alpha &= \liminf_{n\to\infty} M(T^n T^{v_1} x, T^n T^{v_1} y, t) \\ &\geq \liminf_{n\to\infty} \psi_n M(T^{v_1} x, T^{v_1} y, t) \\ &\geq M(T^{v_1} x, T^{v_1} y, t) > M(x,y,t).\end{aligned}$$

By similar argument, we get $\alpha > M(T^l x, T^l y, t)\ \forall l \in \mathbb{N} \cup \{0\}$. Thus $\{M(T^l x, T^l y, t)\}$ converges to α.

Since $0 < M(x,y,t) < 1$, there exists $\delta_2 \in (0,1)$ and $v_2 \in \mathbb{N}$ such that

$$\alpha \star \delta_2 < t < \alpha \Rightarrow \psi_{v_2}(t) \geq \alpha.$$

Then, we choose $v_3 \in \mathbb{N}$ such that $M(T^{v_3}x, T^{v_3}y, t) > \alpha \star \delta_2$. Then, we have

$$M(T^{v_2+v_3}x, T^{v_2+v_3}y, t) > \psi_{v_2} M(T^3x, T^3y, t) \geq \alpha.$$

This is a contradiction. Thus

$$\lim_{n\to\infty} M(T^nx, T^ny, t) = 1. \tag{8}$$

Let $u \in X$ and define $\{u_n\}$ by $u_n = T^nu$. From (8), we have $\liminf_{n\to\infty} M(u_n, u_{n+1}, t) = 1$.

We shall show that $\lim_{n\to\infty} \inf_{m>n} M(u_n, u_m, t) = 1$. So let $\epsilon \in (0,1)$ be fixed. Then there exists $\delta_4 \in (\epsilon, 1)$ and $v_4 \in \mathbb{N}$ such that $\epsilon \star \delta_4 \leq t \leq \epsilon \Rightarrow \psi_{v_4}(t) \geq \epsilon$ and there exist $v_5 \in \mathbb{N}$ such that

$$M(u_n, u_{n+1}, t)^{v_4} > \delta_4, \quad \forall n \geq v_5. \tag{9}$$

Arguing by contradiction, we assume that there exists $l \in \mathbb{N}$ such that $m > l \geq v_5$ and $M(u_l, u_m, t) < \epsilon^2 = \epsilon \star \epsilon$. Then, we put

$$k = \min\{j \in \mathbb{N} : l < j, \epsilon \star \delta_4 \geq M(u_l, u_j, t)\}, \tag{10}$$

and

$$M(u_l, u_m, t) < \epsilon \star \epsilon < \epsilon \star \delta_4.$$

It is obvious that $k \leq m$.
Since

$$\begin{aligned} \delta_4^{k-l} &\leq \star_{j=l}^{k-l} M(u_j, u_{j+1}, \frac{t}{k-l})^{v_4} \\ &\leq M(u_l, u_k, t)^{v_4} \\ &\leq (\epsilon \star \delta_4)^{v_4} \\ &< (\delta_4 \star \delta_4)^{v_4} \\ &= \delta_4^{2v_4}, \end{aligned} \tag{11}$$

so, we have $2v_4 < k - l$ and hence $l < k - 2v_4 < k - v_4$.

We have

$$\begin{aligned} (\delta_4 \star M(u_l, u_{k-v_4}, t))^{v_4} &= \delta_4^{v_4} \star M(u_l, u_{k-v_4}, t)^{v_4} \\ &\leq (\star_{j=0}^{v_4-1} M(u_{k-j-1}, u_{k-j}, t))^{v_4} \star M(u_l, u_{k-v_4}, t)^{v_4} \\ &\leq (M(u_{k-v_4}, u_k, v_4 \cdot t))^{v_4} \star (M(u_l, u_{k-v_4}, t))^{v_4} \\ &= (M(u_{k-v_4}, u_k, v_4 \cdot t) \star M(u_l, u_{k-v_4}, t))^{v_4} \\ &\leq (M(u_l, u_k, (v_4+1)t))^{v_4} \leq (\epsilon \star \delta_4)^{v_4}, \end{aligned}$$

i. e.,

$$\delta_4 \star M(u_l, u_{k-v_4}, t) \leq \epsilon \star \delta_4,$$

and so, using cancellation law of t-norms $\star$, we get

$$\epsilon \star \delta_4 \leq M(u_l, u_{k-v_4}, t) \leq \epsilon.$$

Then, by (C_2), we have

$$\begin{aligned} M(u_{l+v_4}, u_k, t) &= M(T^{v_4} u_l, T^{v_4} u_{k-v_4}, t) \\ &> \psi_{v_4} M(u_l, u_{k-v_4}, t) \geq \epsilon. \end{aligned} \tag{12}$$

This contradicts the definition of k. Therefore, $m > n \geq v_5$ implies $M(u_n, u_m, t) \geq \epsilon^2 = \epsilon \star \epsilon$. So, $\{u_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\{u_n\}$ converges to z.

Then from the continuity of T^l, we have

$$z = \lim_{n\to\infty} T^{l+n} u = \lim_{n\to\infty} T^l \circ T^n u = T^l(\lim_{n\to\infty} T^n u) = T^l z,$$

i. e., z is a fixed point of T^l.

Since

$$\lim_{n\to\infty} M(T^{nl+1}u, Tz, t) = \lim_{n\to\infty} M(T^{nl+1}u, T^{nl+1}z, t) = \lim_{n\to\infty} M(T^n u, T^n z, t) = 1,$$

by (8), we have

$$Tz = \lim_{n\to\infty} T^{nl+1} u = \lim_{n\to\infty} T^n u = z,$$

i. e., z is a fixed point of T.

Further, if $Tx = x$, then $M(z, x, t) = \lim_{n\to\infty}(T^n z, T^n x, t) = 1$ and hence $x = z$. Therefore, a fixed point of T is unique. Since u is arbitrary, $\lim_{n\to\infty} T^n u = z$ hold for every $u \in X$. □

In order to avoid the assumption of cancellation property on t-norm in Theorem 5.3, we introduce the following:

(C_2') $\forall \epsilon \in (0,1), \exists$ a $\delta \in (0,1)$ and $\forall v \in \mathbb{N}$ such that $\epsilon \star \delta \leq t$ and $\delta \star t \leq \delta \star \epsilon \Rightarrow \psi_v(t) \geq \epsilon$.

It is obvious from definition (C_2') and (C_2) that $(C_2') \Rightarrow (C_2)$.

Instead of steps (11) up to (12) in the proof of Theorem 5.3, we prove it using new condition (C_2') as follows: we have

$$\begin{aligned} M(u_l, u_k, t)^{v_4} &\geq \left(\star_{j=1}^{v_4} M(u_{i+j-1}, u_{i+j}, \frac{t}{2v_4})\right)^{v_4} \star M\left(u_{i+v_4}, u_k, \frac{t}{2}\right)^{v_4} > \delta_4^{v_4} \star \epsilon^{v_4} \\ &= (\delta_4 \star \epsilon)^{v_4}, \end{aligned}$$

i. e., $M(u_l, u_k, t) > \delta_4 \star \epsilon$, a contradiction of the definition of k.

Remark 5.4.

1. From Proposition 5.2, we conclude that Theorem 3.2 becomes a particular case of Theorem 5.3.

2. In the definition 5.1, if we only consider (C_2), then we get

   (C_4) $\forall \epsilon \in (0,1), \exists$ a $\delta \in (0,1)$ and $v = 1 \in \mathbb{N}$ such that,

$$\epsilon \star \delta \leq M(x,y,t) \leq \epsilon \Rightarrow \psi_1(t) = M(Tx,Ty,t) \geq \epsilon,$$

   a fuzzy Meir–Keeler Type contractive mapping of first kind.

Indeed, our present definition of fuzzy Meir–Keeler Type contractive mapping is some what different from the definition given by Zheng et al. in [26].

5.1. Asymptotic fuzzy contraction of Meir–Keeler of second kind

Definition 5.5. Let $(X, M, \star)$ be a fuzzy metric space. A mapping $T : X \to X$ is said to be an asymptotic fuzzy contraction of Meir–Keeler of second kind (AFCMKSK, for short) if there exists a sequence $\psi_n : [0,1] \to [0,1]$ such that ψ_n converges to ψ uniformly on $[0,1]$, ψ is continuous, non-decreasing and $\psi(t) > t \; \forall, \; t \in (0,1]$ and satisfies following conditions:

(D_1) $\liminf_{n\to\infty} \psi_n(\epsilon) \geq \epsilon, \; \forall \;\; \epsilon \in (0,1)$;

(D_2) $\forall \epsilon \in (0,1)$ and $s \in (0,1)$,$\exists$ a $\delta \in (0,1)$ such that $s \star (1-\delta) = 2 - \epsilon - \delta$;

(D_3) $\forall \epsilon \in (0,1), \exists$ a $\delta \in (0,\epsilon)$ and $v \in \mathbb{N}$ such that,

$$(1-\delta) \star r \leq 2 - \epsilon - \delta \Rightarrow (1-\delta) \star \psi_v(r) \geq 2 - \epsilon - \delta; \quad \forall r \in (0,1);$$

(D_4) $M(T^n x, T^n y) > \psi_n(M(x,y,t)), \; \forall n \in \mathbb{N}$ and $\forall x, y \in X$ with $x \neq y$.

Before proving our next Theorem, we mention the following [14]:

Let $f : [0,1] \to [0,+\infty]$ be a strictly decreasing function such that $f(1) = 0$ and $f(x) + f(y)$ is in the range of f or equal to $f(0+)$ or $+\infty$ for all x, y in [0, 1]. Then the function $T : [0,1]^2 \to [0,1]$ defined as $T(x,y) = f^{-1}(f(x) + f(y))$ is a t-norm.

Theorem 5.6. Let $(X, M, \star)$(where the t-norm $\star$ with f homogeneous of order 1) be a fuzzy metric space. Let T be an AFCMKSK on X. Assume that T^l is continuous for some $l \in \mathbb{N}$. Then there exists a unique fixed point $z \in X$. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.

Proof. We note that

$$M(T^n x, T^n y, t) \geq \psi_n(M(x,y,t)) \; \forall \; t > 0 \text{ and } x, y \in X.$$

We first show that $\lim_{n\to\infty} M(T^n x, T^n y, t) = 1,\ \forall\ x, y \in X$ and $\forall t > 0$.

Assume $\alpha := \liminf_n M(T^n x, T^n y, t) < 1$. Then from (D_2) and (D_3), there exists $v_1 \in \mathbb{N}$ such that

$$(1-\delta) \star \psi_{v_1}(M(Tx, Ty, t)) \geq 2 - \epsilon - \delta.$$

So we have

$$M(T^{v_1}x, T^{v_1}y, t) > \psi_{v_1}(M(x, y, t)).$$

So

$$(1-\delta) \star M(T^{v_1}x, T^{v_1}y, t) > (1-\delta) \star \psi_{v_1}(M(x, y, t)) \geq 2 - \epsilon - \delta = (1-\delta) \star M(x, y, t).$$

Now, we have

$$\begin{aligned} \alpha &= \liminf_n M(T^n T^{v_1}x, T^n T^{v_1}y, t) \\ &\geq \liminf_n \psi_n(M(T^{v_1}x, T^{v_1}y, t)) \\ &\geq M(T^{v_1}x, T^{v_1}y, t) > M(x, y, t). \end{aligned}$$

By a similar argument, we get $\alpha > M(T^l x, T^l y, t)$, $\forall l \in \mathbb{N} \cup \{0\}$. Thus, $\{M(T^l x, T^l y, t)\}$ converges to α.

Since $0 < M(x, y, t) < 1$, there exists $\delta_2 \in (0, \alpha)$ and $v_2 \in \mathbb{N}$ such that

$$\alpha \star (1-\delta_2) < 2 - \alpha - \delta_2 \Rightarrow \psi_{v_2}(s) \star (1-\delta_2) \geq 2 - \alpha - \delta_2.$$

Then, we choose $v_3 \in \mathbb{N}$ such that

$$M(T^{v_3}x, T^{v_3}y, t) \star (1-\delta_2) \geq 2 - \alpha - \delta_2.$$

Then, we have

$$\begin{aligned} &M(T^{v_2+v_3}x, T^{v_2+v_3}y, t) \star (1-\delta_2) \\ &\qquad \geq \psi_{v_2} M(T^{v_3}x, T^{v_3}y, t) \star (1-\delta_2) \\ &\qquad \geq 2 - \alpha - \delta_2 \geq (1-\delta_2) \star \alpha \end{aligned}$$

and, by cancellation property of the t-norm $\star$, we get

$$M(T^{v_2+v_3}x, T^{v_2+v_3}y, t) \geq \alpha.$$

This is a contradiction.

Thus

$$\lim_{n\to\infty} M(T^n x, T^n y, t) = 1. \tag{13}$$

Let $u \in X$ and define $\{u_n\}$ by $u_n = T^n u$. From (13), we have $\liminf_{n\to\infty} M(u_n, u_{n+1}, t) = 1$.

We shall show that $\lim_{n\to\infty} \inf_{m>n} M(u_n, u_m, t) = 1$. So let $\epsilon \in (0, 1)$ be fixed. Then there exists $\delta_4 \in (\epsilon, 1)$ and $v_4 \in \mathbb{N}$ such that $\epsilon \star \delta_4 \leq t \leq \epsilon \Rightarrow \psi_{v_4}(t) \geq \epsilon$ and there exist $v_5 \in \mathbb{N}$ such that

$$M(u_n, u_{n+1}, t)^{v_4} > \delta_4, \quad \forall n \geq v_5. \tag{14}$$

Arguing by contradiction, we assume that there exists $l \in \mathbb{N}$ such that $m > l \geq v_5$ and $M(u_l, u_m, t) < \epsilon^2 = \epsilon \star \epsilon$. Then, we put

$$k = \min\{j \in \mathbb{N} : l < j, \epsilon \star \delta_4 \geq M(u_l, u_j, t)\}, \tag{15}$$

and

$$M(u_l, u_m, t) < \epsilon \star \epsilon < \epsilon \star \delta_4.$$

It is obvious that $k \leq m$.

Since

$$\begin{aligned} 2(1-\delta_4)v_4 > v_4(2-\epsilon-\delta_4) &\geq v_4 M(u_l, u_k, t)) \\ &\geq v_4 \star_{j=l}^{k-l} M\left(u_j, u_{j+1}, \frac{t}{k-l}\right) \\ &= \star_{j=l}^{k-l} v_4 M\left(u_j, u_{j+1}, \frac{t}{k-l}\right) \\ &> \star_{j=l}^{k-l}(1-\delta_4) = f^{-1}((k-l)f(1-\delta_4)). \end{aligned}$$

Then

$$2v_4 f(1-\delta_4) = f(2v_4(1-\delta_4)) < ((k-l)f(1-\delta_4))$$

and

$$2v_4 < (k-l),$$

and, hence,

$$l < k - 2v_4 < k - v_4.$$

We have

$$\begin{aligned} v_4(2-\epsilon-\delta_4) &\geq v_4 M(u_l, u_k, t)) \\ &\geq v_4(M(u_l, u_{k-v_4}, t/2) \star M(u_{k-v_4}, u_k, t/2)) \\ &\geq v_4 M(u_l, u_{k-v_4}, t/2) \star_{j=0}^{v_4-l} M\left(u_{k-j-1}, u_{k-j}, \frac{t}{2v_4}\right) v_4 \\ &> v_4 M(u_l, u_{k-v_4}, t/2) \star v_4(1-\delta_4) \\ &= v_4(M(u_l, u_{k-v_4}, t/2) \star (1-\delta_4)), \end{aligned}$$

i. e.,

$$(2-\epsilon-\delta_4) > M(u_l, u_{k-v_4}, t/2 \star (1-\delta_4)),$$

so, by (D_4), we have

$$\begin{aligned} (1-\delta_4) \star (M(u_{l+v_4}, u_k, t/2) &= M(T^{v_4} u_l, T^{v_4} u_{k-v_4},, t/2) \star (1-\delta_4) \\ &> \psi_{v_4} M(u_l, u_{k-v_4}, t/2) \star (1-\delta_4) \geq 2-\epsilon-\delta_4. \end{aligned} \tag{16}$$

Hence

$$\begin{aligned} v_4 M(u_l, u_k, t) &\geq \star_{j=1}^{v_4} v_4 M\left(u_{j+l-1}, u_{l+j}, \frac{t}{2v_4}\right) \star v_4 M(u_{l+v_4}, u_k, t/2) \\ &> (1-\delta_4) \star v_4 M\left(u_{l+v_4}, u_k, t/2\right) = v_4((1-\delta_4) \star M(u_{l+v_4}, u_k, t/2)). \end{aligned} \tag{17}$$

Then

$$M(u_l, u_k, t) > (1 - \delta_4) \star M(u_{l+v_4}, u_k, t/2)) \geq 2 - \epsilon - \delta_4,$$

a contradiction of the definition of k.

So $\{u_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\{u_n\}$ converges to z.

The rest of the proof follows similar lines to Theorem 5.3. □

6. CONCLUSION

Most recently, Miñana et al. [20] showed the way to obtain fuzzy versions of two well-known classical fixed point theorems in metric spaces namely- one due to Matkowski and the other one proved by Meir and Keeler. Moreover, they pointed out some inconveniences on the applicability of fixed point theorem of fuzzy Meir–Keeler contractive mappings due to Zheng and Wang [26].

These finding addon about the fact that fuzzy fixed point results are more versatile than the regular metric fixed point results. Since the flexibility which the fuzzy concepts inherently possess and therefore, it is not easy to translate the classical metric contractions and corresponding fixed point theorems in fuzzy setting. Following this direction of research, we make an attempt to intoduce a fuzzy version of asymptotic contractive mappings and forlmulted corresponding fixed point theory.

Indeed, we mentioned that (in Remark 5.4) our definition of fuzzy Meir–Keeler type contractive mapping (i. e. (C_4)) is some what different from the definition given by Zheng et al. in [26] and in the context of Miñana et al. [20] paper the following question naturally arise:

Question : is it possible to obtain fuzzy versions of classical fixed point theorems of asymptotic contractions mappings due to Kirk [13] and Suzuki [25] using the techniques suggested by Miñana et al. in [20]?

Apart from the above, it will be interesting to formulat fuzzy Caristi asymptotic and fuzzy Suzuki type asymptotic mappings and corresponding fixed point results as given in [1, 5, 18].

ACKNOWLEDGEMENTS

The authors thank Editor-in-Chief/Area Editors and Referee(s) for their valuable comments and suggestions, which were very much useful to improve the paper significantly. This work was completed while the first author (Dr. Gopal) was working as visiting Professor at the Department of Mathematics, University of Jean Spain during February 20 to July 14, 2023. He thanks administration of Guru Ghasidas Vishawavidyala for granting extraordinary leave for the mentioned period. He also thank University of Jean for their kind hospitality and support. The research of R. Rodríguez-López was partially supported by the Agencia Estatal de Investigacion (AEI) of Spain, co-financed by the European Fund for Regional Development (FEDER) corresponding to the 2021–2024 multiyear financial framework, project PID2020-113275GB-I00, and ED431C 2019/02 (GRC Xunta de Galicia).

(Received October 21, 2023)

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