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SOME RESULTS ON THE WEAK DOMINANCE RELATION BETWEEN ORDERED WEIGHTED AVERAGING OPERATORS AND T-NORMS

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Aggregation operators have the important application in any fields where the fusion of information is processed. The dominance relation between two aggregation operators is linked to the fusion of fuzzy relations, indistinguishability operators and so on. In this paper, we deal with the weak dominance relation between two aggregation operators which is closely related with the dominance relation. Weak domination of isomorphic aggregation operators and ordinal sum of conjunctors is presented. More attention is paid to the weak dominance relation between ordered weighted averaging operators and Łukasiewicz t-norm. Furthermore, the relationships between weak dominance and some functional inequalities of aggregation operators are discussed.

Keywords: domination, OWA operators, ordinal sum, t-norm

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1. INTRODUCTION

Aggregation operators [20] have been applied in many fields, such as decision-making, image processing, machine learning and many others. It is well known that there exist four main classes of aggregation operators: conjunctive operators, disjunctive operators, internal operators and hybrid operators. The main representative of conjunctive and disjunctive aggregation operators is undoubtedly the class of triangular norms (t-norms) and triangular conorms (t-conorms) [21]. Moreover, arithmetic means (AM) and ordered weighted averaging operators (OWA) [7, 34] are the important members of internal operators. In order to select the suitable aggregation operators for the problem to be solved, the aggregation operators often need to satisfy some constraints. In general, these constraints consists of some functional equations (or inequalities) such as the distributive equation [9] (or subdistributivity, superdistributivity inequality [14, 15]), modular equation [18, 32] (or submodular inequality [6]), migrative equation [8] (or supermigrative inequality [17]), dominance relation inequality [1] and so on.

The dominance relation between two binary operators was firstly introduced in probabilistic metric spaces [31, 33]. Later, the dominance relation of t-norms (or copulas,

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t-conorms) [28, 29] had been widely investigated in the areas of fuzzy relations [5, 13, 19, 26] and the open problem about its transitivity [30]. With the growing application areas of the dominance relation, more general classes of aggregation operators were considered in [3, 4, 27]. Especially, Mesiar and Saminger [23] discussed the domination of ordered weighted averaging operators over t-norms and proved that OWA operators with nonincreasing weighting vectors dominate Łukasiewicz t-norm which has the wide application in fuzzy set theory [2, 11, 12, 24, 25, 35].

In [1], the weak dominance relation between two binary operators was introduced. It can be treated as a generalization of the dominance relation. Moreover, the weak dominance can also be viewed as an inequality generalization of modularity equation [18] which is in connection with some associative equations and often used in fuzzy theory. Note that some results about the weak dominance between t-norms and t-conorms have been given in [1, 22]. In this paper, along the line of study in [22, 23], we continue the analysis of weak dominance relations for the other classes of aggregation operators, especially the internal operators. We will focus on the weak dominance relation between ordered weighted averaging operators (OWA) and t-norms.

Sections 2 provides some preliminary concepts and results about aggregation operators and their several representatives including conjunctors, t-norms, t-conorms and the ordered weighted averaging operators. Section 3 includes the main results of this paper. Firstly, we deal with the weak domination of isomorphic aggregation operators and ordinal sum of conjunctors. Then, we prove that every ordered weighted averaging operator weakly dominates Łukasiewicz t-norm. Finally, we consider the relationships between weak dominance and the functional inequalities including superdistributivity, submodular inequality.

2. PRELIMINARIES

In this section we recall the definitions and some basic results of aggregation operators including conjunctors, t-norms, t-conorms and the ordered weighted averaging operators. Moreover, the weak dominance relation between aggregation operators is introduced, too.

Definition 2.1. (Grabisch et al. [20]) An (extended) aggregation operator is $A : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$, which is increasing with respect to the variables, $A(x) = x$ for all $x \in [0,1]$ and fulfills the conditions: $A(0,\ldots,0) = 0$ and $A(1,\ldots,1) = 1$, where $\mathbb{N}$ is the set of strictly positive integers.

It is obvious that aggregation operator A can be identified with a family $(A_{(n)})_{n\in\mathbb{N}} : [0,1]^n \to [0,1]$ of $n-$ary operations, i.e., $A_{(n)}(x_1,\ldots,x_n) = A(x_1,\ldots,x_n)$. Note that $n-$ary operations $A_{(n)} : [0,1]^n \to [0,1]$ which is increasing with respect to both variables and fulfills the conditions: $A(\underbrace{0,\ldots,0}_{n}) = 0$ and $A(\underbrace{1,\ldots,1}_{n}) = 1$ are referred to as $n-$ary aggregation operators. Moreover, aggregation operators can be defined on any closed interval $[a,b] \subseteq [-\infty,+\infty]$ with the simple modifications. Unless explicitly mentioned otherwise, we will focus on the aggregation operators acting on the unit interval.

Definition 2.2. (Grabisch et al. [20]) Let $A : [0,1]^n \to [0,1]$ be a $n-$ary aggregation operator. Let φ be a monotone bijection on $[0,1]$. The operator $A_\varphi : [0,1]^n \to [0,1]$ defined by

$$A_\varphi(x_1, \dots, x_n) = \varphi^{-1}(A(\varphi(x_1), \dots, \varphi(x_n)))$$

for all $x_1, \dots, x_n \in [0,1]$ is called the φ conjugate of A.

The (id-lower) ordinal sum of a family of aggregation operators [20] is defined in the following.

Definition 2.3. Let $A_i : \bigcup_{n\in\mathbb{N}}[a_i, b_i]^n \to [a_i, b_i], i \in \mathbb{I} = \{1, \dots, k\}$ be a family of aggregation operators defined on nonoverlapping domains $[a_i, b_i], i = 1, \dots, k, 0 \leq a_1 < b_1 \leq a_2 \langle b_2 \leq \dots \rangle b_k \leq 1$. Then the ordinal sum $A = (\langle a_i, b_i, A_i \rangle)_{n\in\mathbb{I}} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is the aggregation operator defined by

$$A(x_1, \dots, x_n) = \begin{cases} A_i\left(\min(x_1, b_i), \dots, \min(x_n, b_i)\right), & \min(x_1, \dots, x_n) \in [a_i, b_i[\\ \min(x, y), & \text{otherwise}, \end{cases} \tag{1}$$

for every $n \in \mathbb{N}$.

Some important classes of aggregation operators are recalled [20].

Definition 2.4. If an aggregation operator $A : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ satisfies the condition: $A_{(n)}(x_1, \dots, x_i, \dots, x_n) \leq \min(x_1, \dots, x_i, \dots, x_n)$ for every $n \in \mathbb{N}, x_i \in [0,1], i \in \{1, \dots, n\}$, then A is said to be conjunctive; moreover, if

$$A_{(n)}(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) = 0, A_{(n)}(1, \dots, 1, x_i, 1, \dots, 1) = x_i$$

for every $n \in \mathbb{N}, x_i \in [0,1], i \in \{1, \dots, n\}$, then A is said to be a conjunctor or semi-copula.

Definition 2.5. (Klement et al. [21]) A t-norm is a commutative, associative, increasing function $T : [0,1]^2 \to [0,1]$ such that $T(1, x) = x$ for all $x \in [0,1]$.

In literature, the four t-norms: T_M, T_P, T_L, and T_D are often discussed, which are given by, for $x, y \in [0,1]$,

$$\begin{aligned} T_M(x,y) &= \min(x,y), \\ T_P(x,y) &= x \cdot y, \\ T_L(x,y) &= \max(x+y-1, 0), \\ T_D(x,y) &= \begin{cases} 0 & (x,y) \in [0,1[^2, \\ \min(x,y) & \text{otherwise}. \end{cases} \end{aligned}$$

It is obvious that $T(x,y) \leq T_M(x,y), x, y \in [0,1]$ for any t-norm T. If a continuous t-norm T satisfies $T(x,x) < x$ for all $x \in]0,1[$, then it is called a continuous Archimedean t-norm. Moreover, if T is continuous, Archimedean and for all $x \in]0,1], 0 < y < z < 1$ implies $T(x,y) < T(x,z)$, then T is called strict. If T is continuous, Archimedean and for all $x \in]0,1[$, there exists $y \in]0,1[$ such that $T(x,y) = 0$, then T is called nilpotent. It is well known that every strict (nilpotent) t-norm is isomorphic to product t-norm T_P (Łukasiewicz t-norm T_L).

Each continuous t-norm can be represented as an ordinal sum of continuous Archimedean t-norms.

Definition 2.6. (Klement et al. [21]) A t-conorm is a commutative, associative, increasing function $S : [0,1]^2 \to [0,1]$ such that $S(0,x) = x$ for all $x \in [0,1]$.

T-conorms $S_L(x,y) = \min(x+y,1), S_M(x,y) = \max(x,y), x,y \in [0,1]$. It is obvious that $S_L(x,y) = 1 - T_L(1-x,1-y)$. Any t-conorm S satisfies $S(x,y) \geq S_M(x,y)$ for all $x,y \in [0,1]$.

More information concerning t-norms and t-conorms can be found in [21].

The ordered weighted averaging operators (OWA) were introduced in [34] and had the close relationship with Choquet integral [7].

Definition 2.7. The operator $A : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ given by

$$A(x_1,\ldots,x_n) = \sum_{i=1}^{n} w_i \cdot x_{(i)} \tag{2}$$

where $(x_{(1)},\ldots,x_{(n)})$ is a non-decreasing permutation of the $n-$tuple $(x_1,\ldots,x_n)$, is called an OWA operator with the weight $w = (w_1,\ldots,w_n) \in [0,1]^n, \sum_{i=1}^{n} w_i = 1$.

Now we recall the definition of (weak) dominance about two binary operations [31, 33].

Definition 2.8. Let A and B be two binary operations defined on the unit interval $[0,1]$. Then we say that A dominates B, and denoted by $A \gg B$, if for each $x_1,x_2,y_1,y_2 \in [0,1]$,

$$A(B(x_1,y_1),B(x_2,y_2)) \geq B(A(x_1,x_2),A(y_1,y_2)). \tag{3}$$

It has been shown in [26], the preservation of B-transitivity of fuzzy relations during an aggregation process is guaranteed if the involved aggregation operator A dominates the corresponding B for the case B is t-norm.

Definition 2.9. ([1]) Let A and B be two binary operations defined on the unit interval $[0,1]$. Then we say that A weakly dominates B, denoted by $A \gg B$, if

$$A(B(x_1,y_1),x_2) \geq B(A(x_1,x_2),y_1). \tag{4}$$

for all $x_1,x_2,y_1 \in [0,1]$.

Note that the weak dominance is introduced in the discussion of the dominance between two strict t-norms. Thus, the study of weak dominance relations of the more general class of operations demands more attentions. It is obvious that the following result holds.

Proposition 2.10. (Proposition 11 in Li et al. [22]) Let A and B be two binary aggregation operators defined on the unit interval $[0,1]$, having a common neutral element $e \in [0,1]$. If $A \gg B$ then $A \gg B$.

Remark 2.11. **(i)** In Proposition 2.10, the condition that A and B have the common neutral element is essential. For example, by the monotonicity of t-norm and t-conorm, we can easily demonstrate that $T_M \gg S_L$. However, T_M does not weakly dominate S_L by taking $x_1 = \frac{1}{2}, y_1 = \frac{3}{4}, x_2 = 0$ in (4).

(ii) A may be not dominates B when A weakly dominates B (see Remark 3.16 below).

The above definition can be easily generalized to the general aggregation operators.

Definition 2.12. Consider an $n-$ary aggregation operator $A_{(n)}$ and an $m-$ary aggregation operator $B_{(m)}$ defined on the unit interval $[0,1]$. Then we say that $A_{(n)}$ weakly dominates $B_{(m)}$, denoted by $A_{(n)} \gg B_{(m)}$, if

$$A_{(n)}(B_{(m)}(x_1, y_1, \ldots, y_{m-1}), x_2, \ldots, x_n) \geq B_{(m)}(A_{(n)}(x_1, x_2, \ldots, x_n), y_1, \ldots, y_{m-1}) \tag{5}$$

for all $x_i, y_j \in [0,1]$ with $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

Let A and B be two aggregation operators. Then we say that A weakly dominates B if $A_{(n)}$ weakly dominates $B_{(m)}$ for all $m, n \in \mathbb{N}$.

3. THE MAIN RESULTS

In this section, we will focus on the weak dominance relation between t-norms and the ordered weighted averaging operators. Firstly, we present some general results about the weak dominance between two aggregation operators.

3.1. General properties of weak dominance between two aggregation operators

Proposition 3.1. Let A, B be two aggregation operators defined on the unit interval $[a, b]$. Then the following statements hold:

(i) $A \gg B$ if and only if $A_\varphi \gg B_\varphi$ for all strictly increasing bijections $\varphi : [c, d] \to [a, b]$.

(ii) $A \gg B$ if and only if $B_\varphi \gg A_\varphi$ for all strictly decreasing bijections $\varphi : [c, d] \to [a, b]$.

Proof. Suppose that $A \gg B$, that is,

$$A_{(n)}(B_{(m)}(x_1, y_1, \ldots, y_{m-1}), x_2, \ldots, x_n) \geq B_{(m)}(A_{(n)}(x_1, x_2, \ldots, x_n), y_1, \ldots, y_{m-1}),$$

for all $m, n \in \mathbb{N}, x_i, y_j \in [a, b]$ with $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Now, we prove that $A_\varphi \gg B_\varphi$ for all strictly increasing bijections $\varphi : [c, d] \to [a, b]$. For arbitrary strictly increasing bijection φ, we have $\varphi(x_i), \varphi(y_j) \in [a, b], i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$ and

$$\begin{aligned}
&A_\varphi(B_\varphi(x_1, y_1, \ldots, y_{m-1}), x_2, \ldots, x_n)\\
&= \varphi^{-1} \circ A(\varphi \circ B_\varphi(x_1, y_1, \ldots, y_{m-1}), \varphi(x_2), \ldots, \varphi(x_n))\\
&= \varphi^{-1} \circ A(\varphi \circ \varphi^{-1} \circ B(\varphi(x_1), \varphi(y_1), \ldots, \varphi(y_{m-1})), \varphi(x_2), \ldots, \varphi(x_n))\\
&= \varphi^{-1} \circ A(B(\varphi(x_1), \varphi(y_1), \ldots, \varphi(y_{m-1})), \varphi(x_2), \ldots, \varphi(x_n))\\
&\geq \varphi^{-1} \circ B(A(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n)), \varphi(y_1), \ldots, \varphi(y_{m-1}))\\
&= \varphi^{-1} \circ B(\varphi \circ \varphi^{-1} \circ A(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n)), \varphi(y_1), \ldots, \varphi(y_{m-1}))\\
&= B_\varphi(A_\varphi(x_1, x_2, \ldots, x_n), y_1, \ldots, y_{m-1}).
\end{aligned}$$

Hence, $A_\varphi \gg B_\varphi$.

Conversely, if $A_\varphi \gg B_\varphi$ then $(A_\varphi)_{\varphi^{-1}} \gg (B_\varphi)_{\varphi^{-1}}$ by the above arguments. Due to $(A_\varphi)_{\varphi^{-1}} = A, (B_\varphi)_{\varphi^{-1}} = B$, $A \gg B$.

The case of strictly decreasing bijections can be proved analogously. □

Remark 3.2. Note that the above results still hold when the weak dominance is replaced with dominance in Proposition 3.1 (Proposition 4.11 in [26]).

Proposition 3.3. Let A, B be two aggregation operators. Then the following holds:

(i) If B is associative and $A_{(n)} \gg B_{(2)}$ for all $n \in \mathbb{N}$, then $A \gg B$.

(ii) If A is associative and $A_{(2)} \gg B_{(m)}$ for all $m \in \mathbb{N}$, then $A \gg B$.

Proof. (i) If B is associative and $A_{(n)} \gg B_{(2)}$ for all $n \in \mathbb{N}$, then for all $m \in \mathbb{N}$, we have

$$\begin{aligned}
&A_{(n)}(B_{(m)}(x_1, y_1, \ldots, y_{m-1}), x_2, \ldots, x_n)\\
&= A_{(n)}(B_{(2)}(x_1, B_{(m-1)}(y_1, \ldots, y_{m-1})), x_2, \ldots, x_n)\\
&\geq B_{(2)}(A_{(n)}(x_1, x_2, \ldots, x_n), B_{(m-1)}(y_1, \ldots, y_{m-1}))\\
&= B_{(m)}(A_{(n)}(x_1, x_2, \ldots, x_n), y_1, \ldots, y_{m-1}).
\end{aligned}$$

Thus, for all $m, n \in \mathbb{N}$, $A_{(n)} \gg B_{(m)}$ and $A \gg B$.

(ii) It can be proven analogously as (i). □

Remark 3.4. Note that the above results still hold for the dominance case in Proposition 3.3 (Proposition 2.8 in [26]).

Proposition 3.5. Consider two ordinal sum aggregation operators $A^1 = (\langle a_i, b_i, A_{1,i}\rangle)_{i\in I}$ and $A^2 = (\langle a_i, b_i, A_{2,i}\rangle)_{i\in I}$ where $A_{1,i}, A_{2,i}$ are the conjuctors defined on $[a_i, b_i]$ for all $i \in I$. Then A^1 weakly dominates A^2 if and only if $A_{1,i}$ weakly dominates $A_{2,i}$, for all $i \in I$.

Proof. Suppose that A^1 weakly dominates A^2, that is,

$$A^1_{(n)}(A^2_{(m)}(x_1, y_1, \ldots, y_{m-1}), x_2, \ldots, x_n) \geq A^2_{(m)}(A^1_{(n)}(x_1, \ldots, x_n), y_1, \ldots, y_{m-1}) \quad (6)$$

for all $m, n \in \mathbb{N}, x_i, y_j \in [0, 1]$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

Consider $x_i, y_j \in [a_i, b_i]$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. There are three cases to be discussed.

(i) $\min(x_1, x_2, \ldots, x_n) \in [a_i, b_i[$ and $\min(x_1, y_1, \ldots, y_{m-1}) \in [a_i, b_i[$. Then by the ordinal sum structure of A^1 and A^2, Eq. (6) can be equivalently expressed as

$$A^1_{1,i}(A^2_{2,i}(x_1, y_1, \ldots, y_{m-1}), x_2, \ldots, x_n) \geq A^2_{2,i}(A^1_{1,i}(x_1, \ldots, x_n), y_1, \ldots, y_{m-1}).$$

(ii) $\min(x_1, x_2, \ldots, x_n) = b_i$ and $\min(x_1, y_1, \ldots, y_{m-1}) \in [a_i, b_i[$. Then $x_1 = x_2 = \ldots = x_n = b_i$. There exist two distinguished subcases.

– $a_{i+1} = b_i$. By Eq. (6) can be equivalently expressed as

$$A^1_{1,i}(A^2_{2,i}(b_i, y_1, \dots, y_{m-1}), b_i, \dots, b_i) \geq A^2_{2,i}(A^1_{1,i+1}(b_i, \dots, b_i), y_1, \dots, y_{m-1}).$$

Due to $A^1_{1,i+1}(b_i, b_i, \dots, b_i) = b_i = A^1_{1,i}(b_i, b_i, \dots, b_i)$, we have

$$A^1_{1,i}(A^2_{2,i}(b_i, y_1, \dots, y_{m-1}), b_i, \dots, b_i) \geq A^2_{2,i}(A^1_{1,i}(b_i, \dots, b_i), y_1, \dots, y_{m-1}).$$

– $a_{i+1} > b_i$. Eq. (6) can be equivalently expressed as

$$A^1_{1,i}(A^2_{2,i}(b_i, y_1, \dots, y_{m-1}), b_i, \dots, b_i) \geq A^2_{2,i}(\min(b_i, \dots, b_i), y_1, \dots, y_{m-1}).$$

Since $\min(b_i, b_i, \dots, b_i) = b_i = A^1_{1,i}(b_i, b_i, \dots, b_i)$, we have

$$A^1_{1,i}(A^2_{2,i}(b_i, y_1, \dots, y_{m-1}), b_i, \dots, b_i) \geq A^2_{2,i}(A^1_{1,i}(b_i, \dots, b_i), y_1, \dots, y_{m-1}).$$

(iii) $\min(x_1, x_2, \dots, x_n) \in [a_i, b_i[$ and $\min(x_1, y_1, \dots, y_{m-1}) = b_i$. The proof is similar to that of case (ii).

Hence, $A_{1,i}$ weakly dominates $A_{2,i}$, for all $i \in I$ if A^1 weakly dominates A^2.

Conversely, suppose that for all $i \in I$ it holds that $A_{1,i} \gg A_{2,i}$. We need to prove that A^1 weakly dominates A^2, that is, for all $m, n \in \mathbb{N}$,

$$A^1(A^2(x_1, y_1, \dots, y_{m-1}), x_2, \dots, x_n) \geq A^2(A^1(x_1, \dots, x_n), y_1, \dots, y_{m-1}) \tag{7}$$

where $x_i, y_j \in [0, 1]$ for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

Note that the ordinal sum conjunctors $A^1 = (\langle a_i, b_i, A_{1,i}\rangle)_{i\in I}$ can be reformed as

$$A^1(x_1, \dots, x_n) = \begin{cases} A_{1,i}(\min(x_1, b_i), \dots, \min(x_n, b_i)), & \min(x_1, \dots, x_n) \in [a_i, b_i], \\ \min(x, y), & \text{otherwise.} \end{cases} \tag{8}$$

We can distinguish the following six cases.

(i) $x_1 = \min(x_1, \dots, x_n, y_1, \dots, y_{m-1}) \in [a_i, b_i]$ for some $i \in I$. Then $\min(x_1, \dots, x_n) \in [a_i, b_i]$ and $\min(x_1, y_1, \dots, y_{m-1}) \in [a_i, b_i]$.

$$\begin{aligned} &A^2(A^1(x_1, \dots, x_n), y_1, \dots, y_{m-1}) \\ &= A^2(A^1_{1,i}(\min(x_1, b_i), \dots, \min(x_n, b_i), y_1), \dots, y_{m-1}) \\ &= A^2_{2,i}(A^1_{1,i}(\min(x_1, b_i), \dots, \min(x_n, b_i)), \min(y_1, b_i), \dots, \min(y_{m-1}, b_i)) \end{aligned}$$

and

$$\begin{aligned} &A^1(A^2(x_1, y_1, \dots, y_{m-1}), x_2, \dots, x_n) \\ &= A^1(A^2_{2,i}(\min(x_1, b_i), \min(y_1, b_i) \dots, \min(y_{m-1}, b_i)), x_2, \dots, x_n) \\ &= A^1_{1,i}(A^2_{2,i}(\min(x_1, b_i), \dots, \min(y_{m-1}, b_i)), \min(x_2, b_i), \dots, \min(x_n, b_i)). \end{aligned}$$

Hence, (7) holds by $A_{1,i} \gg A_{2,i}$.

(ii) $x_1 = \min(x_1, \ldots, x_n, y_1, \ldots, y_{m-1}) \notin [a_i, b_i]$ for any $i \in I$. Then

$$A^1(A^2(x_1, y_1, \ldots, y_{m-1}), x_2, \ldots, x_n) = x_1 = A^2(A^1(x_1, \ldots, x_n), y_1, \ldots, y_{m-1}).$$

(iii) $x_2 = \min(x_1, \ldots, x_n, y_1, \ldots, y_{m-1}) \in [a_i, b_i]$ and $\min(x_1, y_1, \ldots, y_{m-1}) \in [a_i, b_i]$ for some $i \in I$. The proof is similar to that of case (i).

(iv) $x_2 = \min(x_1, \ldots, x_n, y_1, \ldots, y_{m-1}) \in [a_i, b_i]$ and $\min(x_1, y_1, \ldots, y_{m-1}) \in [a_j, b_j]$ for some $i, j \in I$ and $j > i$. Then

$$\begin{aligned}
&A^2(A^1(x_1, \ldots, x_n), y_1, \ldots, y_{m-1}) \\
&= A^2(A^1_{1,i}(\min(x_1, b_i), \ldots, \min(x_n, b_i)), y_1, \ldots, y_{m-1}) \\
&= A^2_{2,i}(A^1_{1,i}(\min(x_1, b_i), \ldots, \min(x_n, b_i)), \min(y_1, b_i), \ldots, \min(y_{m-1}, b_i))
\end{aligned}$$

and

$$\begin{aligned}
&A^1(A^2(x_1, y_1, \ldots, y_{m-1}), x_2, \ldots, x_n) \\
&= A^1(A^2_{2,j}(\min(x_1, b_j), \min(y_1, b_j) \ldots, \min(y_{m-1}, b_j), x_2, \ldots, x_n) \\
&= A^1_{1,i}(\min(A^2_{2,j}(\min(x_1, b_j), \ldots, \min(y_{m-1}, b_j)), b_i), \min(x_2, b_i), \ldots, \min(x_n, b_i)) \\
&= A^1_{1,i}(b_i, \min(x_2, b_i), \ldots, \min(x_n, b_i)) \\
&= A^1_{1,i}(\min(x_1, b_i), \min(x_2, b_i), \ldots, \min(x_n, b_i)).
\end{aligned}$$

Hence, (7) holds since $A^2_{2,i}$ is conjunctive.

(v) $x_2 = \min(x_1, \ldots, x_n, y_1, \ldots, y_{m-1}) \in [a_i, b_i]$ for some $i \in I$ and $\min(x_1, y_1, \ldots, y_{m-1}) \notin [a_k, b_k]$ for any $k \in I$. Then $\min(x_1, y_1, \ldots, y_{m-1}) > b_i$. We have

$$\begin{aligned}
&A^2(A^1(x_1, \ldots, x_n), y_1, \ldots, y_{m-1}) \\
&= A^2(A^1_{1,i}(\min(x_1, b_i), \ldots, \min(x_n, b_i)), y_1, \ldots, y_{m-1}) \\
&= A^2_{2,i}(A^1_{1,i}(\min(x_1, b_i), \ldots, \min(x_n, b_i)), \min(y_1, b_i), \ldots, \min(y_{m-1}, b_i))
\end{aligned}$$

and

$$\begin{aligned}
&A^1(A^2(x_1, y_1, \ldots, y_{m-1}), x_2, \ldots, x_n) \\
&= A^1(\min(x_1, y_1, \ldots, y_{m-1}), x_2, \ldots, x_n) \\
&= A^1_{1,i}(\min(x_1, y_1, \ldots, y_{m-1}, b_i), \min(x_2, b_i), \ldots, \min(x_n, b_i)) \\
&= A^1_{1,i}(b_i, \min(x_2, b_i), \ldots, \min(x_n, b_i)).
\end{aligned}$$

Hence, (7) holds since $A^2_{2,i}$ is conjunctive and $A^1_{1,i}$ is increasing.

(vi) The other cases. The proof is similar to those of cases (iii)-(v).

This completes the proof that A^1 weakly dominates A^2. □

Remark 3.6. The above result holds for the dominance case in Proposition 3.5 (Proposition 4 in [27]).

3.2. OWA operator and Łukasiewicz t-norm

In this subsection, we deal with the weak dominance between OWA operator and Łukasiewicz t-norm. Firstly, the weak domination of Łukasiewicz t-norm over OWA operator can not hold.

Proposition 3.7. There exists no $n-$ary OWA operator $O_{(n)}$ such that $T_L \gg O_{(n)}$.

Proof. Consider arbitrary OWA operator $O_{(n)}$ with weighting vector $w = (w_1, \ldots, w_n)$. Taking $x_1, y_1, \ldots, y_{n-1} \in]0,1[$ such that $0 < O_{(n)}(x_1, y_1, \ldots, y_{n-1}) < x_1$. It is obvious that there exists the maximum $x_2 \in]0,1[$ such that $T_L(O_{(n)}(x_1, y_1, \ldots, y_{n-1}), x_2) = 0$ and $T_L(x_1, x_2) > 0$. Hence, $O_{(n)}(T_L(x_1, x_2), y_1, \ldots, y_{n-1}) > 0$ and

$$T_L(O_{(n)}(x_1, y_1, \ldots, y_{n-1}), x_2) < O_{(n)}(T_L(x_1, x_2), y_1, \ldots, y_{n-1}), \tag{9}$$

which completes the proof. □

In the following, we will focus on the weak domination of OWA operators over Łukasiewicz t-norm. Note that T_M weakly dominates any t-norm by Proposition 23 in [22].

Lemma 3.8. $T_M \gg T_L$.

Remark 3.9. T_M can be interpreted as an OWA operator with weights $w = (1, 0, \ldots, 0)$. So, T_M is one of the OWA operators weakly dominating T_L.

Note that S_M weakly dominates any t-norm by Proposition 14 in [22].

Lemma 3.10. $S_M \gg T_L$.

Remark 3.11. S_M can be interpreted as an OWA operator with weights $w = (0, \ldots, 0, 1)$. So, S_M is one of the OWA operators weakly dominating T_L.

Lemma 3.12. $AM \gg T_L$.

Proof. By Proposition 3.3, we only need to prove that for the binary operators T_L, $AM_{(n)} \gg T_L$. For arbitrary $x_1, \ldots, x_n, y_1 \in [0,1], n \in \mathbb{N}$, we have

$$\begin{aligned}
&AM_{(n)} \gg T_L \\
&\Leftrightarrow AM_{(n)}(T_L(x_1, y_1), x_2, \ldots, x_n) \geq T_L(AM_{(n)}(x_1, x_2, \ldots, x_n), y_1) \\
&\Leftrightarrow AM_{(n)}\left(\max(x_1 + y_1 - 1, 0), x_2, \ldots, x_n\right) \geq \max\left(\frac{1}{n}x_1 + \ldots + \frac{1}{n}x_n + y_1 - 1, 0\right) \\
&\Leftrightarrow \max\left(\frac{1}{n}x_1 + \ldots + \frac{1}{n}x_n + \frac{1}{n}y_1 - \frac{1}{n}, \frac{1}{n}x_2 + \ldots + \frac{1}{n}x_n\right) \geq \frac{1}{n}x_1 + \ldots + \frac{1}{n}x_n + y_1 - 1.
\end{aligned}$$

Since $y_1 - 1 \leq \frac{1}{n}y_1 - \frac{1}{n} \leq 0$, the above holds. Hence, the result holds. □

Remark 3.13. *AM* can be interpreted as an OWA operator with weights $w = (\frac{1}{n}, \ldots, \frac{1}{n})$ for all $n \in \mathbb{N}$. *AM* is also one of the OWA operators weakly dominating T_L.

From above results, it can be conjectured that every OWA operator weakly dominates t-norm T_L. In the following, we prove this conjecture.

Proposition 3.14. Let $O_{(n)}$ be an arbitrary $n-$ary OWA operator. Then $S_L \gg O_{(n)}$.

Proof. By Proposition 3.3, we will show that for any $n-$ary OWA operator $O_{(n)}$ with weights $w_1, \ldots, w_n \in [0,1], \sum_{k=1}^{n} w_k = 1$ and the binary operators S_L, $S_L \gg O_{(n)}$. For arbitrary $x_1, x_2, y_1, \ldots, y_{n-1} \in [0,1], n \in \mathbb{N}$, we have

$$\begin{aligned} & S_L \gg O_{(n)} \\ & \Leftrightarrow S_L(O_{(n)}(x_1, y_1, \ldots, y_{n-1}), x_2) \geq O_{(n)}(S_L(x_1, x_2), y_1, \ldots, y_{n-1}) \\ & \Leftrightarrow \min(O_{(n)}(x_1, y_1, \ldots, y_{n-1}) + x_2, 1) \geq O_{(n)}(\min(x_1 + x_2, 1), y_1, \ldots, y_{n-1}) \\ & \Leftrightarrow O_{(n)}(x_1, y_1, \ldots, y_{n-1}) + x_2 \geq O_{(n)}(\min(x_1 + x_2, 1), y_1, \ldots, y_{n-1}). \end{aligned} \tag{10}$$

Firstly, Let us consider a simple class of $n-$ary OWA operator O_k with weighting vector

$$w = \left(\underbrace{0, \ldots, 0}_{k-1}, 1, \underbrace{0, \ldots, 0}_{n-k} \right).$$

In the following, $(y_{(1)}, y_{(2)}, \ldots, y_{(n-1)})$ denotes a non-decreasing permutation of the $(n-1)-$tuple $(y_1, \ldots, y_{n-1})$.

- $x_1 + x_2 \geq 1$. Then by Eq.(10) we have

  $$S_L \gg O_k \Leftrightarrow O_k(x_1, y_1, \ldots, y_{n-1}) + x_2 \geq O_k(1, y_1, \ldots, y_{n-1}). \tag{11}$$

  Furthermore, $O_k(1, y_1, \ldots, y_{n-1}) = y_{(k)}$ and

  $$O_k(x_1, y_1, \ldots, y_{n-1}) + x_2 = \begin{cases} y_{(k)} & y_{(k)} < x_1, \\ y_{(k-1)} + x_2 & x_1 \leq y_{(k-1)}, \\ x_1 + x_2 & y_{(k-1)} < x_1 \leq y_{(k)}. \end{cases}$$

  Hence, $S_L \gg O_k$ holds.

- $x_1 + x_2 < 1$. Then by Eq.(10) we have

  $$S_L \gg O_k \Leftrightarrow O_k(x_1, y_1, \ldots, y_{n-1}) + x_2 \geq O_k(x_1 + x_2, y_1, \ldots, y_{n-1}). \tag{12}$$

  There exist three subcases to be discussed.

  (i) $y_{(k)} < x_1 + x_2$.
  $O_k(x_1 + x_2, y_1, \ldots, y_{n-1}) = y_{(k)}$ and

  $$O_k(x_1, y_1, \ldots, y_{n-1}) + x_2 = \begin{cases} y_{(k)} + x_2 & y_{(k)} < x_1, \\ y_{(k-1)} + x_2 & x_1 \leq y_{(k-1)}, \\ x_1 + x_2 & y_{(k-1)} < x_1 \leq y_{(k)}. \end{cases}$$

  Due to $y_{(k-1)} + x_2 \geq x_1 + x_2 > y_{(k)}$ when $x_1 \leq y_{(k-1)}$, $S_L \gg O_k$ holds.

(ii) $x_1 + x_2 \le y_{(k-1)}$. Note that $x_1 \le y_{(k-1)}$ in this case.
We have $O_k(x_1 + x_2, y_1, \ldots, y_{n-1}) = y_{(k-1)}$ and $O_k(x_1, y_1, \ldots, y_n) + x_2 = y_{(k-1)} + x_2$ So, $S_L \gg O_k$ holds.

(iii) $y_{(k-1)} < x_1 + x_2 \le y_{(k)}$. Note that $y_{(k)} < x_1$ is impossible in this case.
We have $O_k(x_1 + x_2, y_1, \ldots, y_{n-1}) = x_1 + x_2$ and

$$O_k(x_1, y_1, \ldots, y_{n-1}) + x_2 \quad = \quad \begin{cases} y_{(k-1)} + x_2 & x_1 \le y_{(k-1)}, \\ x_1 + x_2 & y_{(k-1)} < x_1 \le y_{(k)}. \end{cases}$$

Due to $y_{(k-1)} + x_2 \ge x_1 + x_2$ when $x_1 \le y_{(k-1)}$, $S_L \gg O_k$ holds.

It is obvious that $O_{(n)} = \sum_{k=1}^{n} w_k O_k$ for any $n-$ary OWA operator $O_{(n)}$ with weights $w_1, \ldots, w_n \in [0,1], \sum_{k=1}^{n} w_k = 1$. Note that all involved OWA operators are of the same arity. For arbitrary $x_1, x_2, y_1, \ldots, y_n \in [0,1], n \in \mathbb{N}$, we have

$$\begin{aligned}
& O_{(n)}(S_L(x_1, x_2), y_1, \ldots, y_{n-1}) \\
& = \sum_{k=1}^{n} w_k O_k(S_L(x_1, x_2), y_1, \ldots, y_{n-1}) \\
& \le \sum_{k=1}^{n} w_k S_L(O_k(x_1, y_1, \ldots, y_{n-1}), x_2) \\
& = \sum_{k=1}^{n} w_k \min(O_k(x_1, y_1, \ldots, y_{n-1}) + x_2, 1) \\
& = \min\left(\sum_{k=1}^{n} w_k O_k(x_1, y_1, \ldots, y_{n-1}) + \sum_{k=1}^{n} w_k x_2,\ \sum_{k=1}^{n} w_k\right) \\
& = \min\left(\sum_{k=1}^{n} w_k O_k(x_1, y_1, \ldots, y_{n-1}) + x_2,\ 1\right) \\
& = \min(O_n(x_1, y_1, \ldots, y_{n-1}) + x_2,\ 1) \\
& = S_L(O_n(x_1, y_1, \ldots, y_{n-1}) + x_2,\ 1)
\end{aligned}$$

which completes the proof. □

By Proposition 3.14 and Proposition 3.1, the following two corollaries hold obviously.

Corollary 3.15. Let $O_{(n)}$ be an arbitrary $n-$ary OWA operator. Then $O_{(n)} \gg T_L$.

Proof. It is obvious that $(O_{(n)})_\varphi$ is still a $n-$ary OWA operator and $T_L = (S_L)_\varphi$ for the strictly decreasing bijections $\varphi : [0,1] \to [0,1], \varphi(x) = 1 - x$ for all $x \in [0,1]$. Hence, the result holds by Proposition 3.14 and Proposition 3.1. □

Remark 3.16. From Corollary 3.15, we can see the difference between the weak dominance case and the dominance case, i. e., only OWA operators with non-increasing weighting vectors dominate T_L (see Corollary 1 in [23]).

It is well known that any nilpotent t-norm T is isomorphic to T_L and arbitrary $n-$ary ordered weighted quasi-arithmetic mean $O_{(n)}$ is isomorphic to an OWA operator, i. e., $T = (T_L)_\varphi$ and $O_{(n)}(x_1,\dots,x_n) = \varphi^{-1}\left(\frac{1}{n}\sum_{i=1}^n w_i\varphi\left(x_{(i)}\right)\right)$ with $\varphi : [0,1] \to [0,1]$ a strictly increasing bijection.

Corollary 3.17. Let $O_{(n)}$ be an arbitrary $n-$ary ordered weighted quasi-arithmetic mean with respect to the strictly increasing bijection $\varphi : [0,1] \to [0,1]$ with weight $w = (w_1,\dots,w_n)$. Then $O_{(n)}$ weakly dominates the nilpotent t-norm $T = (T_L)_\varphi$.

Applying the proof in Proposition 3.14, we can obtain the similar result for product t-norm T_P.

Proposition 3.18. Let $O_{(n)}$ be an arbitrary $n-$ary ordered weighted geometric function. Then $O_{(n)} \gg T_P$.

Proof. Consider the strictly decreasing bijection $\varphi : [0,+\infty] \to [0,1], \varphi(x) = exp(-x)$. Let ordered weighted geometric function[7] $O_{(n)}(x_1,\dots,x_n) = \prod_{i=1}^n x_{(i)}^{w_i}$ with the weight $(w_1,\dots,w_n)$. We have

$$(T_P)_\varphi (x_1,x_2) = \varphi^{-1}(\varphi(x_1\cdot\varphi(x_2))) = x_1 + x_2$$

and $\left(O_{(n)}\right)_\varphi (x_1,\dots,x_n) = \varphi^{-1}(O_{(n)}(\varphi(x_1),\dots,\varphi(x_n))) = \sum_{i=1}^n x_{(i)}w_{n+1-i}$ is an $n-$ary OWA operators with the weight $(w_n,\dots,w_1)$. Applying the similar proof in Proposition 3.14 and Corollary 3.15, we can prove that $\left(O_{(n)}\right)_\varphi$ is weakly dominated by $(T_P)_\varphi$. Hence, the result holds by Proposition 3.1. □

3.3. The relations between weak dominance and other inequalities

In this subsection, we deal with the relations between weak dominance with some functional inequalities of aggregation operators. For the sake of simplicity, we will focus on the binary aggregation operators defined on $[0,1]$.

Proposition 3.19. (Theorem 12 in [22]) Let A and B be two binary aggregation operators, having a common neutral element $e \in [0,1]$. If A or B is commutative and $A \gg B$, then $A(x,y) \geq B(x,y)$ for all $x,y \in [0,1]$.

The distributivity inequalities of aggregation operators [14, 15] play the important roles in fuzzy set theory. The weak dominance is connected with the superdistributivity of aggregation operators (A is superdistributive with respect to B if $A(B(x_1,y_1),x_2) \geq B(A(x_1,x_2),A(y_1,x_2))$ for all $x_1,x_2,y_1 \in [0,1]$).

Proposition 3.20. Let A and B be two binary aggregation operators. If A is conjunctive and $A \gg B$ then A is superdistributive with respect to B.

Proof. If $A \gg B$ then $A(B(x_1,y_1),x_2) \geq B(A(x_1,x_2),y_1)$ for all $x_1,x_2,y_1 \in [0,1]$. By the monotonicity of A, we have

$$A(B(x_1,y_1),x_2) \geq B(A(x_1,x_2),y_1) \geq B(A(x_1,x_2),A(y_1,x_2)),$$

which completes the proof. □

The modular equations and related inequalities of aggregation operators were discussed in [6, 10, 18]. The following relations between weak dominance and submodular inequalities of aggregation operators (B is submodular over A if $A(B(x_1, y_1), x_2) \geq B(A(x_1, x_2), y_1)$ for any $x_1, x_2, y_1 \in [0, 1]$ and $x_2 \leq y_1$) hold obviously.

Proposition 3.21. Let A and B be two binary aggregation operators. If $A \gg B$ then B is submodular over A.

Corollary 3.22. Let A and B be two binary aggregation operators. If $A \geq S_M, B \leq T_M$ and B is submodular over A then $A \gg B$.

Proof. We only need to prove that for any $x_1, x_2, y_1 \in [0, 1]$ and $y_1 < x_2$, we have

$$A(B(x_1, y_1), x_2) \geq B(A(x_1, x_2), y_1).$$

Indeed,

$$A(B(x_1, y_1), x_2) \geq S_M(B(x_1, y_1), x_2) \geq x_2 > y_1 \geq T_M(A(x_1, x_2), y_1) \geq B(A(x_1, x_2), y_1).$$

□

At last, note that the supermigrativity of commutative aggregation operator A (A is supermigrative if A is commutative and satisfies $A(\alpha \cdot x, y) \geq A(x, \alpha \cdot y)$ for all $\alpha \in [0, 1]$ and for all $x, y \in [0, 1]$ such that $y \leq x$) is not connected to the weak dominance of A over T_P, i. e. $A(x_1 \cdot y_1, x_2) \geq A(x_1, x_2) \cdot y_1$ for all $x_1, x_2, y_1 \in [0, 1]$. For example, the ordered weighted geometric function $O_{(2)}(x_1, x_2) = x_{(1)}^{\frac{2}{3}} \cdot x_{(2)}^{\frac{1}{3}}$ is commutative and weakly dominates T_P by Proposition 3.18, but $O_{(2)}$ is not supermigrative since $O_{(2)}(\frac{1}{4} \cdot \frac{2}{3}, \frac{1}{8}) < O_{(2)}(\frac{2}{3}, \frac{1}{4} \cdot \frac{1}{8})$. Moreover, the aggregation operator $A(x, y) = xy + \frac{1}{2}xy(1 - x)(1 - y)$ is supermigrative by Example 2.5 in [16], but $A(x_1 \cdot y_1, x_2) = \frac{11}{18} < \frac{25}{36} = A(x_1, x_2) \cdot y_1$ for $x_1 = \frac{1}{6}, x_2 = \frac{1}{3}, y_1 = \frac{1}{2}$ and A does not weakly dominate T_P.

4. CONCLUSIONS

In this paper, the weak dominance relation between two aggregation operators has been discussed. We prove that every ordered weighted averaging operator weakly dominates Łukasiewicz t-norm. Moreover, some general properties including the weak domination of isomorphic aggregation operators and ordinal sums of conjunctors are presented and the relationships between weak dominance and some functional inequalities are also discussed.

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