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CONVEX (L, M)-FUZZY REMOTE NEIGHBORHOOD OPERATORS

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In this paper, two kinds of remote neighborhood operators in (L, M)-fuzzy convex spaces are proposed, which are called convex (L, M)-fuzzy remote neighborhood operators. It is proved that these two kinds of convex (L, M)-fuzzy remote neighborhood operators can be used to characterize (L, M)-fuzzy convex structures. In addition, the lattice structures of two kinds of convex (L, M)-fuzzy remote neighborhood operators are also given.

Keywords: convex (L, M)-fuzzy remote neighborhood operator, (L, M)-fuzzy convex structure, complete lattice

Classification: 03E72, 52A01, 54A40

1. INTRODUCTION

Convex sets widely exist in various research areas of mathematics, such as metric spaces, lattices, graphs, and topological spaces (see, for example, [4, 5, 13, 14]). In order to deal with set-theoretic structures satisfying several axioms which are satisfied by usual convex sets, abstract convex structures [15] were established, which provides a more general framework for studying convex sets.

Many scholars generalized convex structures to fuzzy context from different viewpoints. Generally speaking, there are three approaches to extensions of convex structures to the fuzzy context, they are called L-convex structures (see, for example, [5, 9, 10]), M-fuzzying convex structures (see, for example,[11, 17]), and (L, M)-fuzzy convex structures (see, for example,[12, 22, 23, 24]), respectively. Actually, both L-convex structures and M-fuzzifying convex structures can be regarded as special cases of (L, M)-fuzzy convex structures (see [12]). At present, many researchers studied fuzzy convex structures from different aspects, such as fuzzy hull operators, fuzzy interval operators, bases and subbases, product and coproduct structures, fuzzy betweenness relations, and so on (see, for example [7, 8, 12, 25, 26, 27]).

Wang [16] established the theory of remote neighborhood systems, which played important role in TML. Subsequently, Yue and Fang [21] proposed the concept of fuzzy remote neighborhood systems in FTML, and studied the connections between fuzzy co-topologies and topological fuzzy remote neighborhood systems. More generally, Yang and Li [20] introduced the concept of topological remote neighborhood systems of fuzzy

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points in the (L, M)-fuzzy setting, and gave the relationship between (L, M)-fuzzy co-topologies and topological (L, M)-fuzzy remote neighborhood systems. Using the idea of remote neighborhood systems, Yang and Li [18] firstly proposed the concept of convex L-remote neighborhood systems in L-convex spaces, then Yang and Pang [19] studied the relationship between convex L-remote neighborhood systems and L-betweenness relations. Note that (L, M)-fuzzy convex structure is the more general framework than L-convex structures and M-fuzzying convex structures. Inspired by this, we will study convex remote neighborhood operators in (L, M)-fuzzy setting.

The contents are organized as follows. In Section 2, we recall some necessary concepts and results. In Section 3, we will propose the concept of the first kind of convex (L, M)-fuzzy remote neighborhood operators, and discuss the categorical relationship between this kind of operators and (L, M)-fuzzy convex structures. In Section 4, we will propose the concept of the second kind of convex (L, M)-fuzzy remote neighborhood operators, and also establish the categorical relationship between this kind of operators and (L, M)-fuzzy convex structures.

2. PRELIMINARIES

In this paper, let $M(L)$ be a complete lattice with the smallest element $\perp_M(\perp_L)$ and the largest element $\top_M(\top_L)$. $M_{\perp_M} = M - \{\perp_M\}$. An element u in a complete lattice M is said to be coprime if $u \leq s \vee t$ implies that $u \leq s$ or $u \leq t$. The set of all coprimes in $M_{\perp_M}$ is denoted by $J(M)$. $\forall\ u, s \in M$, we say that u is wedge below s in M (in symbols, $u \prec s$) if for all subsets $D \subseteq M, s \leq \bigvee D$ always implies that $u \leq d$ for some $d \in D$. We denote $\beta(x) = \{y \in M \mid y \prec x\}$. A complete lattice M is said to be completely distributive iff for each $x \in M, x = \bigvee \beta(x) = \bigvee \beta^*(x)$, where $\beta^*(x) = \beta(x) \cap J(M)$ is called the standard greatest minimal family of x (see [16]).

$\forall u, s \in L$, we say that u is way below s in L (in symbols, $u \ll s$) if for all directed subsets $D \subseteq L, s \leq \bigvee D$ always implies that $u \leq d$ for some $d \in D$. A complete lattice L is said to be continuous if for all $x \in L, \Downarrow x$ is directed and $x = \bigvee \Downarrow x$, where $\Downarrow x = \{y \in L \mid y \ll x\}$. For a directed subset $D \subseteq L$, we use $\bigvee^d D$ to denote its supremum. Let L be a continuous lattice, then way below relations have some properties as follows : (1) $u \ll s$ implies $u \leq s$; (2) $u \ll \bigvee^d D$ implies $u \ll d$ for some $d \in D$; (3) If u is a coprime, we have $u \ll s$ if and only if $u \prec s$ (see [2]).

For a nonempty set X, L^X denotes the set of all L-subsets on X. The operators on L can be translated onto L^X in a pointwise way. If L is a continuous lattice, then L^X is also a continuous lattice. The way below relation on L^X is still denoted by $\ll$. The smallest element and the largest element in L^X are denoted by $\perp_L^X$ and $\top_L^X$, respectively. For each $x \in X$ and $\mu \in L - \{\perp_L\}$, the L-subset x_μ is called a fuzzy point when $x_\mu(y) = \mu$ if $y = x$ and $x_\mu(y) = \perp_L$ if $y \neq x$. The set of all fuzzy points in L^X is denoted by $J(L^X)$. For each $\gamma \in L$, let $\underline{\gamma}$ denote the constant L-fuzzy subset of X with the value γ. We say $\{C_i\}_{i\in I}$ is directed subset of L^X, if for each $C_{i_1}, C_{i_2} \in \{C_i\}_{i\in I}$, there exists $C_{i_3} \in \{C_i\}_{i\in I}$ such that $C_{i_1} \leq C_{i_3}$ and $C_{i_2} \leq C_{i_3}$. We usually use the symbol $\bigvee_{i\in I}^d C_i$ to represent the supremum of a directed subset $\{C_i\}_{i\in I} \subseteq L^X$. Let X, Y be two nonempty sets. For an ordinary mapping $h : X \longrightarrow Y$, then the preimage $h^{\leftarrow}(S)$ of $S \in L^Y$ and the image $h^{\rightarrow}(U)$ of $U \in L^X$ are defined by:$h^{\leftarrow}(S)(x) = S(h(x))$, and $h^{\rightarrow}(U)(y) = \bigvee\{U(x) \mid x \in X, h(x) = y\}$, respectively It can be verified that the pair $(h^{\rightarrow}, h^{\leftarrow})$ is a Galois connection on $(L^X, \leq)$ and $(L^Y, \leq)$. It is easy to check that $h^{\rightarrow}(x_\gamma) = h(x)_\gamma \in J(L^Y)$ ($\forall\ x_\gamma \in J(L^X)$).

Definition 2.1. (Fang and Yue [1]) A mapping $\mathfrak{C} : L^X \longrightarrow M$ is called an (L, M)-fuzzy closure system on X if it satisfies:

(LMC1) $\mathfrak{C}(\bot_L^X) = \mathfrak{C}(\top_L^X) = \top_M$;

(LMC2) $\bigwedge_{i\in I} \mathfrak{C}(U_i) \leq \mathfrak{C}(\bigwedge_{i\in I} U_i)$.

Definition 2.2. (Pang [7], Pang [8], Shi and Xiu [12]) A closure system $\mathfrak{C}$ is called (L, M)-fuzzy convex structure, if one of the following conditions hold (the second then following as a consequence):

(LMC3) If $\{U_k\}_{k\in K} \subseteq L^X$ is totally ordered, then $\bigwedge_{k\in K} \mathfrak{C}(U_k) \leq \mathfrak{C}(\bigvee_{k\in K} U_k)$.

(LMC3)* If $\{U_i\}_{i\in I} \subseteq L^X$ is directed, then $\bigwedge_{i\in I} \mathfrak{C}(U_i) \leq \mathfrak{C}(\bigvee^d_{i\in I} U_i)$.

If $\mathfrak{C}$ is an (L, M)-fuzzy convex structure on X, then the pair $(X, \mathfrak{C})$ is called an (L, M)-fuzzy convex space. Let $(X, \mathfrak{C}_X)$ and $(Y, \mathfrak{C}_Y)$ be (L, M)-fuzzy convex spaces and $g : X \longrightarrow Y$ be a mapping. We say g is called (L, M)-convexity preserving ((L, M)-CP, in short) (see [7, 12]) if $\mathfrak{C}_Y(S) \leq \mathfrak{C}_X(g^{\leftarrow}(S))$ for all $S \in L^Y$. It is easy to check that all (L, M)-fuzzy convex spaces as objects and all corresponding (L, M)-CP mappings as morphisms form a category, denoted by (L, M)-**FC**.

Remark 2.3. (Zhao et al. [25]) The set of all (L, M)-fuzzy convex structrues on X is denoted by $\mathbf{FC}(X, L, M)$. Define a relation $\leq$ on $\mathbf{FC}(X, L, M)$ as follows: $\forall\ U \in L^X, \mathfrak{C}_1 \leq \mathfrak{C}_2 \Longleftrightarrow \mathfrak{C}_1(U) \leq \mathfrak{C}_2(U)$. Then $(\mathbf{FC}(X, L, M), \leq)$ is a complete lattice, where $\mathfrak{C}^1 : L^X \longrightarrow M$ defined by $\forall A \in L^X$, $\mathfrak{C}^1(A) = \top_M$ is the greatest element in $(\mathbf{FC}(X, L, M), \leq)$, and $\mathfrak{C} : L^X \longrightarrow M$ defined by $\mathfrak{C}(A) = \bigwedge_{j\in J} \mathfrak{C}_j(A)$ is the infimum of $\{\mathfrak{C}_j\}_{j\in J} \subseteq L^X$.

3. THE FIRST KIND OF CONVEX (L, M)-FUZZY REMOTE NEIGHBORHOOD OPERATORS

In this section, L denotes a completely distributive De Morgan algebra and M denotes a completely distributive lattice. We will give the concept of the first kind of convex (L, M)-fuzzy remote neighborhood operators, and discuss categorical relationship between this kind of operators and (L, M)-fuzzy convex structures.

Definition 3.1. Let X be a set. A mapping $\mathfrak{R} : L^X \times M_{\bot_M} \times X \longrightarrow L$ is called a convex (L, M)-fuzzy remote neighborhood operator iff satisfies the following seven axioms: for each $x \in X$, $A, B \in L^X$ and $a, b \in M_{\bot_M}$,

(CFR1) $\mathfrak{R}(\bot_L^X, a, x) = \top_L$.

(CFR2) $\mathfrak{R}(A, a, x) \leq A'(x)$.

(CFR3) If $A \leq B$, then $\mathfrak{R}(B, a, x) \leq \mathfrak{R}(A, a, x)$.

(CFR4) If $a \leq b$, then $\mathfrak{R}(A, b, x) \leq \mathfrak{R}(A, a, x)$.

(CFR5) $\mathfrak{R}(A, a, x) = \bigvee \Big\{\mathfrak{R}(B, a, x) | B'(y) \leq \mathfrak{R}(A, a, y), \forall y \in X\Big\}$.

(CFR6) $\mathfrak{R}(\bigvee^{d}_{j\in J} A_j, a, x) = \bigwedge_{j\in J} \mathfrak{R}(A_j, a, x)$.

(CFR7) For any nonempty subset $\{a_j\}_{j\in J}$ of $M_{\perp_M}$ the implication holds:

$$A'(x) = \mathfrak{R}(A, a_j, x) \quad \forall j \in J,\ \forall x \in X \Longrightarrow\ A'(x) = \mathfrak{R}\Big(A, \bigvee\nolimits_{j\in J} a_j, x\Big).$$

If $\mathfrak{R}$ is a convex (L, M)-fuzzy remote neighborhood operator on X, then the pair $(X, \mathfrak{R})$ is called a convex (L, M)-fuzzy remote neighborhood space. Let $(X, \mathfrak{R}_X)$ and $(Y, \mathfrak{R}_Y)$ be two convex (L, M)-fuzzy remote neighborhood spaces, then a function $g : X \longrightarrow Y$ is called a convex (L, M)-fuzzy remote neighborhood preserving (hereinafter referred to as (L, M)-RNP^1) if $\mathfrak{R}_Y(A, a, g(x)) \leq \mathfrak{R}_X(g^{\leftarrow}(A), a, x)$ for each $x \in X$, $A \in L^Y$ and $a \in M_{\perp_M}$. The category of all convex (L, M)-fuzzy remote neighborhood spaces as objects and all their (L, M)-RNP^1 mappings as morphisms is denoted by (L, M)-FR^1. The set of all convex (L, M)-fuzzy remote neighborhood operators on X is denoted by $\mathbf{CFR^1}(X, L, M)$. Define a relation $\leq$ on $\mathbf{CFR^1}(X, L, M)$ as follows: $\forall x \in X, A \in L^X$ and $a \in M_{\perp_M}$, $\mathfrak{R}_1 \leq \mathfrak{R}_2 \Longleftrightarrow \mathfrak{R}_1(A, a, x) \leq \mathfrak{R}_2(A, a, x)$. It is easy to check that $(\mathbf{CFR^1}(X, L, M), \leq)$ is a poset.

Remark 3.2. (1) As we know, convex structures are topological-like structures. So, if one replace conditions (CFR3) and (CFR6) in Definition 3.1 with the following condition:

$$\mathfrak{R}(A \vee B, a, x) = \mathfrak{R}(A, a, x) \wedge \mathfrak{R}(B, a, x)\ (\forall x \in X, A, B \in L^X \text{and } a \in M_{\perp_M}).$$

Then $\mathfrak{R}$ is called a topological (L, M)-fuzzy remote neighborhood operator. We can show that topological (L, L)-fuzzy remote neighborhood operators and Hohle's L-fuzzy neighborhood systems (cf., e. g., [[3], Definition 8.1.8]) are one-to-one correspondence. In particular, if $L = M = \{0, 1\}$, then topological (L, M)-fuzzy remote neighborhood operators can degenerate to remote neighborhood systems, and there is a one-to-one correspondence between topological $(\{0, 1\}, \{0, 1\})$-fuzzy remote neighborhood operators and neighborhood systems (cf., e. g., [[6]]).

(2) $(\mathbf{CFR^1}(X, L, M), \leq)$ is a complete lattice, where $\mathfrak{R}_1 : L^X \times M_{\perp_M} \times X \longrightarrow L$ defined by $\mathfrak{R}_1(A, a, x) = A'(x)$ $(\forall x \in X, A \in L^X,\ a \in M_{\perp_M})$ is the greatest element of $(\mathbf{CFR^1}(X, L, M), \leq)$. And, let $\emptyset \neq \{\mathfrak{R}_i\}_{i\in I} \subseteq (\mathbf{CFR^1}(X, L, M), \leq)$ and I is an index set. The mapping $\mathfrak{R} : L^X \times M_{\perp_M} \times X \longrightarrow L$ defined by

$$\mathfrak{R}(A, a, x) = \bigvee \Big\{C^{'}(x) \in L \mid A \leq C,\ \bigwedge_{i\in I} \bigvee_{\substack{\forall y\in X,\\ \mathfrak{R}_i(C,r,y)=C^{'}(y)}} r \geq a\Big\}$$

is the infimum of $\{\mathfrak{R}_i\}_{i\in I}$. In this case, we denote it as $\bigcap_{i\in I} \mathfrak{R}_i$.

Next, let's establish a one-to-one correspondence between $\mathbf{CFR^1}(X, L, M)$ and $\mathbf{FC}$ (X, L, M).

Theorem 3.3. Let $\mathfrak{C} \in \mathbf{FC}(X, L, M)$. Define a mapping $\mathfrak{R}_{\mathfrak{C}} : L^X \times M_{\perp_M} \times X \longrightarrow L^X$ as follows: for each $x \in X, A \in L^X$ and $a \in M_{\perp_M}$,

$$\mathfrak{R}_{\mathfrak{C}}(A, a, x) = \bigvee \Big\{ D'(x) \in L : D \geq A,\ \mathfrak{C}(D) \geq a \Big\}.$$

Then $\mathfrak{R}_{\mathfrak{C}} \in \mathbf{CFR^1}(X, L, M)$.

Proof. For each $A \in L^X$ and $a \in M_{\perp_M}$, define $\mathfrak{R}_{\mathfrak{C}}(A, a) \in L^X$ as folllows:

$$\mathfrak{R}_{\mathfrak{C}}(A, a) = \bigvee \Big\{ D' \in L^X : D \geq A,\ \mathfrak{C}(D) \geq a \Big\}.$$

Then, for each $x \in X$,

$$\mathfrak{R}_{\mathfrak{C}}(A, a, x) = \mathfrak{R}_{\mathfrak{C}}(A, a)(x).$$

(CFR1) $\forall a \in M_{\perp_M}$, we have $\mathfrak{C}(\perp_L^X) = \top_L \geq a$. So,

$$\mathfrak{R}_{\mathfrak{C}}(\perp_L^X, a, x) = \mathfrak{R}_{\mathfrak{C}}(\perp_L^X, a)(x) = \top_L^X(x) = \top_L.$$

(CFR2) is satisfied from the definition of $\mathfrak{R}_{\mathfrak{C}}$. So we omit it.

(CFR3) If $A \leq B$, then

$$\begin{aligned} \mathfrak{R}_{\mathfrak{C}}(A, a) &= \bigvee \Big\{ D' \in L^X : D \geq A,\ \mathfrak{C}(D) \geq a \Big\} \\ &\geq \bigvee \Big\{ D' \in L^X : D \geq B,\ \mathfrak{C}(D) \geq a \Big\} \\ &= \mathfrak{R}_{\mathfrak{C}}(B, a). \end{aligned}$$

So, $\mathfrak{R}_{\mathfrak{C}}(B, a, x) \leq \mathfrak{R}_{\mathfrak{C}}(A, a, x)$.

(CFR4) If $a \leq b$, then by (CFR2) and (CFR3), we have $\mathfrak{R}_{\mathfrak{C}}(\mathfrak{R}'_{\mathfrak{C}}(A, b), a) \leq \mathfrak{R}_{\mathfrak{C}}(A, b)$, and $\mathfrak{R}_{\mathfrak{C}}(\mathfrak{R}'_{\mathfrak{C}}(A, b), a) \leq \mathfrak{R}_{\mathfrak{C}}(A, a)$. Thus,

$$\mathfrak{C}(\mathfrak{R}'_{\mathfrak{C}}(A, b)) = \mathfrak{C}\Big(\bigwedge \Big\{ D \in L^X : D \geq A,\ \mathfrak{C}(D) \geq b \Big\} \Big) \geq b \geq a.$$

It implies that

$$\mathfrak{R}_{\mathfrak{C}}(\mathfrak{R}'_{\mathfrak{C}}(A, b), a) = \bigvee \Big\{ D' \in L^X : D \geq \mathfrak{R}'_{\mathfrak{C}}(A, b),\ \mathfrak{C}(D) \geq a \Big\} \geq \mathfrak{R}_{\mathfrak{C}}(A, b).$$

So, $\mathfrak{R}_{\mathfrak{C}}(\mathfrak{R}'_{\mathfrak{C}}(A, b), a) = \mathfrak{R}_{\mathfrak{C}}(A, b)$. Hence, $\mathfrak{R}_{\mathfrak{C}}(A, b) \leq \mathfrak{R}_{\mathfrak{C}}(A, a)$. It follows that

$$\mathfrak{R}_{\mathfrak{C}}(A, b, x) \leq \mathfrak{R}_{\mathfrak{C}}(A, a, x).$$

(CFR5) Let $B \in L^X$ and $B' \leq \mathfrak{R}_{\mathfrak{C}}(A,a)$, then $B \geq A$. By (CFR3), we have $\mathfrak{R}_{\mathfrak{C}}(B,a) \leq \mathfrak{R}_{\mathfrak{C}}(A,a)$. It implies that

$$\bigvee \Big\{\mathfrak{R}_{\mathfrak{C}}(B,a) | B' \leq \mathfrak{R}_{\mathfrak{C}}(A,a)\Big\} \leq \mathfrak{R}_{\mathfrak{C}}(A,a).$$

On the other hand, by the definition of $\mathfrak{R}_{\mathfrak{C}}$, we have $\mathfrak{C}(\mathfrak{R}'_{\mathfrak{C}}(A,a)) \geq a$. It implies that

$$\mathfrak{R}_{\mathfrak{C}}(\mathfrak{R}'_{\mathfrak{C}}(A,a),a) = \bigvee \Big\{D' \in L^X : D \geq \mathfrak{R}'_{\mathfrak{C}}(A,a),\ \mathfrak{C}(D) \geq a\Big\} \geq \mathfrak{R}_{\mathfrak{C}}(A,a).$$

So,

$$\bigvee \Big\{\mathfrak{R}_{\mathfrak{C}}(B,a) | B' \leq \mathfrak{R}_{\mathfrak{C}}(A,a)\Big\} \geq \mathfrak{R}_{\mathfrak{C}}(\mathfrak{R}'_{\mathfrak{C}}(A,a),a) \geq \mathfrak{R}_{\mathfrak{C}}(A,a).$$

Hence,

$$\mathfrak{R}_{\mathfrak{C}}(A,a) = \bigvee \Big\{\mathfrak{R}_{\mathfrak{C}}(B,a) | B' \leq \mathfrak{R}_{\mathfrak{C}}(A,a)\Big\}.$$

It follows that

$$\mathfrak{R}_{\mathfrak{C}}(A,a,x) = \bigvee \Big\{\mathfrak{R}_{\mathfrak{C}}(B,a,x) | B'(y) \leq \mathfrak{R}_{\mathfrak{C}}(A,a,y), \forall y \in X\Big\}.$$

(CFR6) By (CFR3), we easily obtain

$$\mathfrak{R}_{\mathfrak{C}}\Big(\bigvee^d_{j\in J} A_j, a\Big) \leq \bigwedge_{j\in J} \mathfrak{R}_{\mathfrak{C}}(A_j,a).$$

On the other hand, let $\{A_j\}_{j\in J}$ is a directed subfamily of L^X, then $\{\mathfrak{R}'_{\mathfrak{C}}(A_j,a)\}_{j\in J}$ is a directed subfamily of L^X. By (CFR2), we have $\bigvee^d_{j\in J} \mathfrak{R}'_{\mathfrak{C}}(A_j,a) \geq \bigvee^d_{j\in J} A_j$. Then by the definition of $\mathfrak{C}$ and $\mathfrak{R}_{\mathfrak{C}}$, we have

$$\mathfrak{C}\Big(\bigvee^d_{j\in J} \mathfrak{R}'_{\mathfrak{C}}(A_j,a)\Big) \geq \bigwedge_{j\in J} \mathfrak{C}\Big(\mathfrak{R}'_{\mathfrak{C}}(A_j,a)\Big) \geq a.$$

So,

$$\mathfrak{R}_{\mathfrak{C}}\Big(\bigvee^d_{j\in J} A_j, a\Big) = \bigvee \Big\{D' \in L^X : D \geq \bigvee^d_{j\in J} A_j, \mathfrak{C}(D) \geq a\Big\} \geq \bigwedge_{j\in J} \mathfrak{R}_{\mathfrak{C}}(A_j,a).$$

It implies that

$$\mathfrak{R}_{\mathfrak{C}}\Big(\bigvee^d_{j\in J} A_j, a\Big) \geq \bigwedge_{j\in J} \mathfrak{R}_{\mathfrak{C}}(A_j,a).$$

Hence,

$$\mathfrak{R}_{\mathfrak{C}}\Big(\bigvee^d_{j\in J} A_j, a\Big) = \bigwedge_{j\in J} \mathfrak{R}_{\mathfrak{C}}(A_j,a).$$

It follows that

$$\mathfrak{R}_{\mathfrak{C}}(\bigvee^d_{j\in J} A_j, a, x) = \bigwedge_{j\in J} \mathfrak{R}_{\mathfrak{C}}(A_j,a,x).$$

(CFR7) Let $\{a_j\}_{j\in J}$ be a nonempty subset of $M_{\perp_M}$ and satisfy $A' = \mathfrak{R}_{\mathfrak{C}}(A,a_j)$ for each $j \in J$. Obviously, for each $j \in J$,

$$\mathfrak{C}(A) = \mathfrak{C}(\mathfrak{R}'_{\mathfrak{C}}(A,a_j)) = \mathfrak{C}\Big(\bigwedge \Big\{D \in L^X : D \geq A,\ \mathfrak{C}(D) \geq a_j\Big\}\Big) \geq a_j.$$

It follows that $\mathfrak{C}(A) \geq \bigvee_{j\in J} a_j \geq a_j$. By (CFR4), we have

$$\begin{aligned} A' = \bigwedge_{j\in J} \mathfrak{R}_{\mathfrak{C}}(A, a_j) &\geq \mathfrak{R}_{\mathfrak{C}}\Big(A, \bigvee_{j\in J} a_j\Big) \\ &= \bigvee \Big\{D' \in L^X : D \geq A,\ \mathfrak{C}(D) \geq \bigvee_{j\in J} a_j\Big\} \geq A'. \end{aligned}$$

It implies that $\mathfrak{R}_{\mathfrak{C}}\Big(A, \bigvee_{j\in J} a_j\Big) = A'$. Hence,

$$A' = \mathfrak{R}_{\mathfrak{C}}(A, a_j) \quad \forall j \in J \Longrightarrow A' = \mathfrak{R}_{\mathfrak{C}}(A, \bigvee_{j\in J} a_j).$$

It follows that

$$A'(x) = \mathfrak{R}_{\mathfrak{C}}(A, a_j, x) \quad \forall j \in J,\ \forall x \in X \Longrightarrow A'(x) = \mathfrak{R}_{\mathfrak{C}}\Big(A, \bigvee_{j\in J} a_j, x\Big).$$

□

Conversely, we can construct an (L, M)-fuzzy convex structure via a convex (L, M)-fuzzy remote neighborhood operator.

Theorem 3.4. Let $\mathfrak{R} \in \mathbf{CFR^1}(X, L, M)$. Define a mapping $\mathfrak{C}_{\mathfrak{R}} : L^X \longrightarrow M$ as follows: for each $A \in L^X$ and $a \in M_{\perp_M}$,

$$\mathfrak{C}_{\mathfrak{R}}(A) = \bigvee \Big\{a \in M_{\perp_M} : A'(y) = \mathfrak{R}(A, a, y), \forall y \in X\Big\}.$$

Then $\mathfrak{C}_{\mathfrak{R}} \in \mathbf{FC}(X, L, M)$.

Proof. For each $A \in L^X$ and $a \in M_{\perp_M}$, define $\mathfrak{R}(A, a) \in L^X$ as folllows: for each $y \in X$, $\mathfrak{R}(A, a)(y) = \mathfrak{R}(A, a, y)$. Then, $\mathfrak{C}_{\mathfrak{R}}(A) = \bigvee \Big\{a \in M_{\perp_M} : A' = \mathfrak{R}(A, a)\Big\}$.

(LMC1) By (CFR1) and (CFR2), for each $a \in M_{\perp_M}$, we have $\mathfrak{R}(\perp_L^X, a) = \top_L^X$ and $\mathfrak{R}(\top_L^X, a) = \perp_L^X$, so we obtain $\mathfrak{C}_{\mathfrak{R}}(\perp_L^X) = \mathfrak{C}_{\mathfrak{R}}(\top_L^X) = \top_M$.

(LMC2) By (CFR2), we easily obtain

$$\mathfrak{R}\Big(\bigwedge_{j\in J} A_j, a_0\Big) \leq \Big(\bigwedge_{j\in J} A_j\Big)'.$$

On the other hand, let $b \in \beta^*\Big(\bigwedge_{j\in J} \mathfrak{C}_{\mathfrak{R}}(A_j)\Big)$, then $b \prec \bigwedge_{j\in J} \mathfrak{C}_{\mathfrak{R}}(A_j)$ and $b \in J(M)$. Thus, $b \prec \mathfrak{C}_{\mathfrak{R}}(A_j)$ for each $j \in J$. By the definition of $\mathfrak{C}_{\mathfrak{R}}$, there exists $a_j \in M_{\perp_M}$ such that $A_j^{'} = \mathfrak{R}(A_j, a_j)$ and $b \leq a_j$. Let $a_0 = \bigwedge_{j\in J} a_j$, then we have $b \leq a_0$. By (CFR4) and (CFR3), we obtain

$$\mathfrak{R}\Big(\bigwedge_{j\in J} A_j, a_0\Big) \geq \mathfrak{R}\Big(\bigwedge_{j\in J} A_j, a_j\Big) \geq \mathfrak{R}(A_j, a_j).$$

So,

$$\mathfrak{R}\Big(\bigwedge_{j\in J} A_j, a_0\Big) \geq \bigvee_{j\in J} \mathfrak{R}(A_j, a_j) = \bigvee_{j\in J} A_j^{'} = \Big(\bigwedge_{j\in J} A_j\Big)',$$

i. e.,

$$\mathfrak{R}\Big(\bigwedge_{j\in J} A_j, a_0\Big) \geq \Big(\bigwedge_{j\in J} A_j\Big)^{'}.$$

Therefore,

$$\mathfrak{R}\Big(\bigwedge_{j\in J} A_j, a_0\Big) = \Big(\bigwedge_{j\in J} A_j\Big)^{'}.$$

It implies that $\mathfrak{C}_{\mathfrak{R}}\Big(\bigwedge_{j\in J} A_j\Big) \geq a_0 \geq b$. Hence

$$\mathfrak{C}_{\mathfrak{R}}\Big(\bigwedge_{j\in J} A_j\Big) \geq \bigwedge_{j\in J} \mathfrak{C}_{\mathfrak{R}}(A_j).$$

(LMC3)* Let $\{A_j\}_{j\in J}$ be a directed subfamily of L^X and $b \in \beta^*\Big(\bigwedge_{j\in J} \mathfrak{C}_{\mathfrak{R}}(A_j)\Big)$, then $b \prec \bigwedge_{j\in J} \mathfrak{C}_{\mathfrak{R}}(A_j)$ and $b \in J(M)$. Thus, $b \prec \mathfrak{C}_{\mathfrak{R}}(A_j)$ for each $j \in J$. There exists $a_j \in M_{\perp_M}$ such that $A_j^{'} = \mathfrak{R}(A_j, a_j)$ and $b \leq a_j$. Let $a_0 = \bigwedge_{i\in J} a_j$, then $b \leq a_0$. By (CFR2), (CFR4) and (CFR6), we have $\bigwedge_{j\in J} A_j^{'} = \Big(\bigvee^d_{j\in J} A_j\Big)^{'} \geq \mathfrak{R}\Big(\bigvee^d_{j\in J} A_j, a_0\Big) = \bigwedge_{j\in J} \mathfrak{R}(A_j, a_0) \geq \bigwedge_{j\in J} \mathfrak{R}(A_j, a_j) = \bigwedge_{j\in J} A_j^{'}$. So, $\mathfrak{R}\Big(\bigvee^d_{j\in J} A_j, a_0\Big) = \Big(\bigvee^d_{j\in J} A_j\Big)^{'}$. It implies that

$$\mathfrak{C}_{\mathfrak{R}}\Big(\bigvee^d_{j\in J} A_j\Big) \geq a_0 \geq b.$$

Hence

$$\mathfrak{C}_{\mathfrak{R}}\Big(\bigvee^d_{j\in J} A_j\Big) \geq \bigwedge_{j\in J} \mathfrak{C}_{\mathfrak{R}}(A_j).$$

□

Proposition 3.5. (1) $\mathfrak{C}_{\mathfrak{R}_{\mathfrak{C}}} = \mathfrak{C}$ $(\forall \mathfrak{C} \in \mathbf{FC}(X, L, M))$;

(2) $\mathfrak{R}_{\mathfrak{C}_{\mathfrak{R}}} = \mathfrak{R}$ $(\forall \mathfrak{R} \in \mathbf{CFR^1}(X, L, M))$.

Proof. (1) Assume that $b \in \beta^*\Big(\mathfrak{C}_{\mathfrak{R}_{\mathfrak{C}}}(A)\Big)$, then $b \in J(M)$ and

$$b \prec \mathfrak{C}_{\mathfrak{R}_{\mathfrak{C}}}(A) = \bigvee \Big\{a \in M_{\perp_M} : A' = \mathfrak{R}_{\mathfrak{C}}(A, a)\Big\},$$

there exists $a_0 \in M_{\perp_M}$ such that $A' = \mathfrak{R}_{\mathfrak{C}}(A, a_0)$ and thus $b \leq a_0$. By the definition of $\mathfrak{R}_{\mathfrak{C}}$, we have $\mathfrak{C}(A) = \mathfrak{C}(\mathfrak{R}_{\mathfrak{C}}^{'}(A, a_0)) \geq a_0 \geq b$. Hence, $\mathfrak{C}_{\mathfrak{R}_{\mathfrak{C}}}(A) \leq \mathfrak{C}(A)$.

On the other hand, if $\mathfrak{C}(A) = \perp_M$, we obtain $\mathfrak{C}_{\mathfrak{R}_{\mathfrak{C}}}(A) \geq \perp_M = \mathfrak{C}(A)$. If $\mathfrak{C}(A) \in M_{\perp_M}$, then

$$A' \geq \mathfrak{R}_{\mathfrak{C}}(A, \mathfrak{C}(A)) = \bigvee \Big\{D' \in L^X : D \geq A,\ \mathfrak{C}(D) \geq \mathfrak{C}(A)\Big\} \geq A'.$$

Therefore, we obtain $A' = \mathfrak{R}_{\mathfrak{C}}(A, \mathfrak{C}(A))$. Then by the definition of $\mathfrak{C}_{\mathfrak{R}_{\mathfrak{C}}}$, we have $\mathfrak{C}_{\mathfrak{R}_{\mathfrak{C}}}(A) \geq \mathfrak{C}(A)$. Hence, $\mathfrak{C}_{\mathfrak{R}_{\mathfrak{C}}} = \mathfrak{C}$.

(2) Let $D \geq A$, and $\mathfrak{C}_{\mathfrak{R}}(D) \geq a$. By (CFR7), we have

$$D' \geq \mathfrak{R}(D, a) \geq \mathfrak{R}(D, \mathfrak{C}_{\mathfrak{R}}(D)) = \mathfrak{R}\Big(D, \bigvee \Big\{a \in M_{\perp_M} : D' = \mathfrak{R}(D, a)\Big\}\Big) = D'.$$

So, $D' = \mathfrak{R}(D,a) \le \mathfrak{R}(A,a)$. It follows that for each $x \in X$, we obtain

$$\mathfrak{R}_{\mathfrak{C}_{\mathfrak{R}}}(A,a) = \bigvee \left\{ D' \in L^X : D \ge A, \mathfrak{C}_{\mathfrak{R}}(D) \ge a \right\} \le \mathfrak{R}(A,a).$$

Therefore, $\mathfrak{R}_{\mathfrak{C}_{\mathfrak{R}}}(A,a) \le \mathfrak{R}(A,a)$.

On the other hand, by (CFR2), we easily obtain $\mathfrak{R}(\mathfrak{R}'(A,a),a) \le \mathfrak{R}(A,a)$. By (CFR2) and (CFR5), we have $\mathfrak{R}(A,a) = \bigvee \left\{ \mathfrak{R}(B,a) | B' \le \mathfrak{R}(A,a) \right\} \le \mathfrak{R}(\mathfrak{R}'(A,a),a)$. So, $\mathfrak{R}(\mathfrak{R}'(A,a),a) = \mathfrak{R}(A,a)$. It implies that

$$\begin{aligned} \mathfrak{R}_{\mathfrak{C}_{\mathfrak{R}}}(A,a) &= \bigvee \left\{ D' \in L^X : D \ge A, \mathfrak{C}_{\mathfrak{R}}(D) \ge a \right\} \\ &\ge \bigvee \left\{ D' \in L^X : A' \ge D' = \mathfrak{R}(D,a) \right\} \\ &\ge \mathfrak{R}(A,a). \end{aligned}$$

Hence, $\mathfrak{R}_{\mathfrak{C}_{\mathfrak{R}}}(A,a) \ge \mathfrak{R}(A,a)$. Therefore, $\mathfrak{R}_{\mathfrak{C}_{\mathfrak{R}}} = \mathfrak{R}$. □

Proposition 3.6. (1) If $g : (X, \mathfrak{C}_X) \longrightarrow (Y, \mathfrak{C}_Y)$ is an (L, M)-CP, then $g : (X, \mathfrak{R}_{\mathfrak{C}_X}) \longrightarrow (Y, \mathfrak{R}_{\mathfrak{C}_Y})$ is an (L, M)-RNP^1.

(2) If $g : (X, \mathfrak{R}_X) \longrightarrow (Y, \mathfrak{R}_Y)$ is an (L, M)-RNP^1, then $g : (X, \mathfrak{C}_{\mathfrak{R}_X}) \longrightarrow (Y, \mathfrak{C}_{\mathfrak{R}_Y})$ is an (L, M)-CP.

Proof. (1) Since $g : (X, \mathfrak{C}_X) \longrightarrow (Y, \mathfrak{C}_Y)$ is an (L, M)-CP, it follows that $\mathfrak{C}_X(g^{\leftarrow}(A)) \ge \mathfrak{C}_Y(A)$ for each $A \in L^Y$. So,

$$\begin{aligned} \mathfrak{R}_{\mathfrak{C}_Y}(A,a,g(x)) &= \bigvee \left\{ D'(g(x)) \in L : D \ge A,\ \mathfrak{C}_Y(D) \ge a \right\} \\ &\le \bigvee \left\{ (g^{\leftarrow}(D))'(x) \in L : g^{\leftarrow}(D) \ge g^{\leftarrow}(A), \mathfrak{C}_X(g^{\leftarrow}(D)) \ge a \right\} \\ &\le \bigvee \left\{ C'(x) \in L : C \ge g^{\leftarrow}(A),\ \mathfrak{C}_X(C) \ge a \right\} \\ &= \mathfrak{R}_{\mathfrak{C}_X}(g^{\leftarrow}(A),a,x). \end{aligned}$$

Hence, $g : (X, \mathfrak{R}_{\mathfrak{C}_X}) \longrightarrow (Y, \mathfrak{R}_{\mathfrak{C}_Y})$ is an (L, M)-RNP^1.

(2) Since $g : (X, \mathfrak{R}_X) \longrightarrow (Y, \mathfrak{R}_Y)$ is an (L, M)-RNP^1, it follows that

$$\mathfrak{R}_Y(A,a,g(x)) \le \mathfrak{R}_X(g^{\leftarrow}(A),a,x)$$

for all $x \in X$, $A \in L^Y$ and $a \in M_{\perp_M}$. So,

$$\begin{aligned} \mathfrak{C}_{\mathfrak{R}_Y}(A) &= \bigvee \left\{ a \in M_{\perp_M} : A'(y) = \mathfrak{R}_Y(A,a,y), \forall y \in Y \right\} \\ &\le \bigvee \left\{ a \in M_{\perp_M} : A'(g(x)) = \mathfrak{R}_Y(A,a,g(x)), \forall x \in X \right\} \\ &\le \bigvee \left\{ a \in M_{\perp_M} : (g^{\leftarrow}(A))'(x) = \mathfrak{R}_X(g^{\leftarrow}(A),a,x), \forall x \in X \right\} \\ &= \mathfrak{C}_{\mathfrak{R}_X}(g^{\leftarrow}(A)). \end{aligned}$$

Hence, $g : (X, \mathfrak{C}_{\mathfrak{R}_X}) \longrightarrow (Y, \mathfrak{C}_{\mathfrak{R}_Y})$ is an (L, M)-CP. □

By Propositions 3.5 and 3.6, we obtain the following theorem:

Theorem 3.7. $\mathbf{R^C}$ and $\mathbf{C^R}$ are isomorphic functors, where two functors are given as follows:

$$\mathbf{R^C}: \begin{cases} (L,M)\text{-}\mathbf{FC} \longrightarrow (L,M)\text{-}\mathbf{FR^1}, \\ (X,\mathfrak{C}) \longmapsto (X,\mathfrak{R}_{\mathfrak{C}}), \\ g \longmapsto g. \end{cases} \qquad \mathbf{C^R}: \begin{cases} (L,M)\text{-}\mathbf{FR^1} \longrightarrow (L,M)\text{-}\mathbf{FC}, \\ (X,\mathfrak{R}) \longmapsto (X,\mathfrak{C}_{\mathfrak{R}}), \\ g \longmapsto g. \end{cases}$$

By Proposition 3.5, Remarks 2.3 and 3.2, we have the following result.

Corollary 3.8. $(\mathbf{CFR^1}(X, L, M), \leq)$ and $(\mathbf{FC}(X, L, M), \leq)$ are complete lattice isomorphic.

4. THE SECOND KIND OF CONVEX (L, M)-FUZZY REMOTE NEIGHBORHOOD OPERATORS

In this section, L denotes a continuous lattice and M denotes a completely distributive lattice. We will give the concept of the second kind of convex (L, M)-fuzzy remote neighborhood operators, study some its fundamental properties and discuss the categorical relationship between this kind of operators and (L, M)-fuzzy convex structures.

4.1. The lattice structures of convex (L, M)-fuzzy remote neighborhood operators

In this subsection, we will give the concept of convex (L, M)-fuzzy remote neighborhood operators and study the lattice structures of this kind of operators.

Definition 4.1. A mapping $\mathfrak{R} : L^X \longrightarrow M^{J(L^X)}$ is called a convex (L, M)-fuzzy remote neighborhood operator on X if it satisfies: for any $U, S \in L^X$,

(LMR1) $\mathfrak{R}(\perp_L^X)(x_\lambda) = \top_M$;

(LMR2) $\mathfrak{R}(U)(x_\lambda) = \perp_M$ for each $x_\lambda \leq U$;

(LMR3) $\mathfrak{R}(U)(x_\lambda) = \bigvee_{x_\lambda \not\leq S \geq U} \bigwedge_{y_\mu \not\leq S} \mathfrak{R}(S)(y_\mu)$;

(LMR4) For each directed subfamily $\{U_i\}_{i\in I} \subseteq L^X$,

$$\mathfrak{R}\left(\bigvee\nolimits_{i\in I}^{d} U_i\right)(x_\lambda) = \bigvee\nolimits_{\mu \ll \lambda} \bigwedge\nolimits_{i\in I} \mathfrak{R}(U_i)(x_\mu).$$

If $\mathfrak{R}$ is a convex (L, M)-fuzzy remote neighborhood operator on X, then the pair $(X, \mathfrak{R})$ is called a convex (L, M)-fuzzy remote neighborhood space. The set of all convex (L, M)-fuzzy remote neighborhood operators on X is denoted by $\mathbf{CFR^2}(X, L, M)$. Define a relation $\leq$ on $\mathbf{CFR^2}(X, L, M)$ as follows: $\forall\, U \in L^X, x_\lambda \in J(L^X), \mathfrak{R}_1 \leq \mathfrak{R}_2 \Longleftrightarrow \mathfrak{R}_1(U)(x_\lambda) \leq \mathfrak{R}_2(U)(x_\lambda)$, then $(\mathbf{CFR^2}(X, L, M), \leq)$ is a poset.

Let $(X, \mathfrak{R}_X)$ and $(Y, \mathfrak{R}_Y)$ be convex (L, M)-fuzzy remote neighborhood spaces and $g : X \longrightarrow Y$ be a mapping. We say g is called convex (L, M)-fuzzy remote neighborhood preserving ((L, M)-RNP^2, in short) if $\mathfrak{R}_Y(g^{\rightarrow}(U))(g(x)_\lambda) \leq \mathfrak{R}_X(U)(x_\lambda)$, for all $U \in L^X, x_\lambda \in J(L^X)$. It is easy to check that all convex (L, M)-fuzzy remote neighborhood spaces as objects and all their (L, M)-RNP^2 mappings as morphisms form acategory, denoted by (L, M)-$\mathbf{FR^2}$. It is easy to know that convex (L, M)-fuzzy remote neighborhood operator can degenerate to convex L-remotehood system by restricting $M = \{0, 1\}$ (see [18, 19]).

Example 4.2. Let $X = \{x, y, z\}$, $L = \{0, 1\}$ and $M = \{\bot_M, a, b, c, d, \top_M\}$ (see Figure 1). Then $L^X = \{\underline{0}, x_1, y_1, z_1, A, B, C, \underline{1}\}$ and $J(L^X) = \{x_1, y_1, z_1\}$, where $A = x_1 \vee y_1$, $B = x_1 \vee z_1$, and $C = y_1 \vee z_1$. Define a mapping $\mathfrak{R} : L^X \longrightarrow M^{J(L^X)}$ as follows: for each $x_\lambda \in J(L^X)$, $\mathfrak{R}(\underline{0})(x_\lambda) = \top_M, \mathfrak{R}(\underline{1})(x_\lambda) = \bot_M$, and

$$\mathfrak{R}(x_1)(x_\lambda) = \begin{cases} \bot_M, & \text{if } x_\lambda = x_1, \\ a, & \text{if } x_\lambda = y_1, \\ b, & \text{otherwise,} \end{cases} \qquad \mathfrak{R}(y_1)(x_\lambda) = \begin{cases} a, & \text{if } x_\lambda = x_1, \\ \bot_M, & \text{if } x_\lambda = y_1, \\ d, & \text{otherwise,} \end{cases}$$

$$\mathfrak{R}(z_1)(x_\lambda) = \begin{cases} c, & \text{if } x_\lambda = x_1, \\ a, & \text{if } x_\lambda = y_1, \\ \bot_M, & \text{otherwise,} \end{cases} \qquad \mathfrak{R}(A)(x_\lambda) = \begin{cases} \bot_M, & \text{if } x_\lambda = x_1, \\ \bot_M, & \text{if } x_\lambda = y_1, \\ b, & \text{otherwise ,} \end{cases}$$

$$\mathfrak{R}(B)(x_\lambda) = \begin{cases} \bot_M, & \text{if } x_\lambda = x_1, \\ a, & \text{if } x_\lambda = y_1, \\ \bot_M, & \text{otherwise ,} \end{cases} \qquad \mathfrak{R}(C)(x_\lambda) = \begin{cases} c, & \text{if } x_\lambda = x_1, \\ \bot_M, & \text{if } x_\lambda = y_1, \\ \bot_M, & \text{otherwise.} \end{cases}$$

Then we can easily verify that $\mathfrak{R} \in \mathbf{CFR^2}(X, L, M)$.

Proposition 4.3. Let $\mathfrak{R} : L^X \longrightarrow M^{J(L^X)}$ be a mapping. Then (LMR3) implies (LMR5) and (LMR6).

(LMR5) $\mathfrak{R}(U)(x_\lambda) = \bigvee_{\mu \ll \lambda} \mathfrak{R}(U)(x_\mu)$;

(LMR6) $U \leq S$ implies $\mathfrak{R}(S) \leq \mathfrak{R}(U)$.

Proof. (LMR5) Since L is a continuous lattice, it follows that $\lambda = \bigvee \Downarrow \lambda$. Then,

$$\{S \in L^X \mid x_\lambda \not\leq S\} = \bigcup_{\mu \ll \lambda} \{S \in L^X \mid x_\mu \not\leq S\}.$$

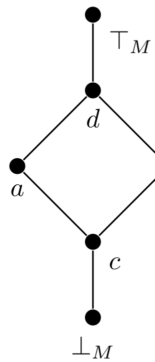

Fig. 1. Hasse diagram of M.

By (LMR3), we have

$$\begin{aligned}\mathfrak{R}(U)(x_\lambda) &= \bigvee\nolimits_{x_\lambda \not\leq S \geq U} \bigwedge\nolimits_{y_\nu \not\leq S} \mathfrak{R}(S)(y_\nu)\\ &= \bigvee\nolimits_{\mu \ll \lambda} \bigvee\nolimits_{x_\mu \not\leq S \geq U} \bigwedge\nolimits_{y_\nu \not\leq S} \mathfrak{R}(S)(y_\nu)\\ &= \bigvee\nolimits_{\mu \ll \lambda} \mathfrak{R}(U)(x_\mu).\end{aligned}$$

(LMR6) If $U \leq S$, then

$$\mathfrak{R}(U)(x_\lambda) = \bigvee\nolimits_{x_\lambda \not\leq C \geq U} \bigwedge\nolimits_{y_\nu \not\leq C} \mathfrak{R}(C)(y_\nu) \quad \geq \quad \bigvee\nolimits_{x_\lambda \not\leq D \geq S} \bigwedge\nolimits_{y_\nu \not\leq D} \mathfrak{R}(D)(y_\nu) = \mathfrak{R}(S)(x_\lambda).$$

Hence, $\mathfrak{R}(S) \leq \mathfrak{R}(U)$. □

Proposition 4.4. Let $\mathfrak{R} : L^X \longrightarrow M^{J(L^X)}$ be a mapping satisfying (LMR1)-(LMR3). Then the following statements are equivalent.

(LMR4) For each directed subfamily $\{U_i\}_{i \in I} \subseteq L^X$,

$$\mathfrak{R}\left(\bigvee\nolimits^{d}_{i \in I} U_i\right)(x_\lambda) = \bigvee\nolimits_{\mu \ll \lambda} \bigwedge\nolimits_{i \in I} \mathfrak{R}(U_i)(x_\mu).$$

(LMR4)* $\mathfrak{R}(U)(x_\lambda) = \bigvee_{\mu \ll \lambda} \bigwedge_{F \ll U} \mathfrak{R}(F)(x_\mu)$.

Proof. (LMR4)$\Longrightarrow$(LMR4)*. For each $U \in L^X, \Downarrow U = \{F \in L^X \mid F \ll U\}$ is directed. Then it follows from (LMR4) that

$$\mathfrak{R}(U)(x_\lambda) = \mathfrak{R}\left(\bigvee\nolimits^{d}_{F \ll U} F\right)(x_\lambda) = \bigvee\nolimits_{\mu \ll \lambda} \bigwedge\nolimits_{F \ll U} \mathfrak{R}(F)(x_\mu).$$

(LMR4)*$\Longrightarrow$(LMR4). By Proposition 4.3, we have

$$\mathfrak{R}\left(\bigvee\nolimits^{d}_{i\in I} U_i\right)(x_\lambda) = \bigvee\nolimits_{\mu\ll\lambda} \mathfrak{R}\left(\bigvee\nolimits^{d}_{i\in I} U_i\right)(x_\mu) \leq \bigvee\nolimits_{\mu\ll\lambda} \bigwedge\nolimits_{i\in I} \mathfrak{R}(U_i)(x_\mu).$$

Furthermore, it follows from (LMR4)* and (LMR6) that

$$\begin{aligned}\mathfrak{R}\left(\bigvee\nolimits^{d}_{i\in I} U_i\right)(x_\lambda) &= \bigvee\nolimits_{\mu\ll\lambda} \bigwedge\nolimits_{F\ll\bigvee^{d}_{i\in I} U_i} \mathfrak{R}(F)(x_\mu) \\ &= \bigvee\nolimits_{\mu\ll\lambda} \bigwedge\nolimits_{i\in I} \bigwedge\nolimits_{F\ll U_i} \mathfrak{R}(F)(x_\mu) \\ &\geq \bigvee\nolimits_{\mu\ll\lambda} \bigwedge\nolimits_{i\in I} \mathfrak{R}(U_i)(x_\mu).\end{aligned}$$

This shows that

$$\mathfrak{R}(\bigvee\nolimits^{d}_{i\in I} U_i)(x_\lambda) = \bigvee\nolimits_{\mu\ll\lambda} \bigwedge\nolimits_{i\in I} \mathfrak{R}(U_i)(x_\mu).$$

□

Now, we will study the lattice structures of convex (L, M)-fuzzy remote neighborhood operators.

Theorem 4.5. $(\mathbf{CFR^2}(X, L, M), \leq)$ is a complete lattice.

Proof. We need to prove the following two steps.

Step 1: Define $\mathfrak{R}_1 : L^X \longrightarrow M^{J(L^X)}$ as follows: for any $U \in L^X, x_\lambda \in J(L^X)$,

$$\mathfrak{R}_1(U)(x_\lambda) = \begin{cases} \top_M, & \text{if } x_\lambda \nleq U, \\ \bot_M, & \text{if } x_\lambda \leq U, \end{cases}$$

then $\mathfrak{R}_1$ is the greatest element of $(\mathbf{CFR^2}(X, L, M), \leq)$. Indeed, (LMR1) and (LMR2) are trivial, so we omit them.

(LMR3) We distinguish the following two cases:

Case 1: if $x_\lambda \leq U$, then $\mathfrak{R}_1(U)(x_\lambda) = \bot_M$, and $\{S \in L^X \mid x_\lambda \nleq S \geq U\} = \emptyset$. Hence,

$$\bigvee\nolimits_{x_\lambda\nleq S\geq U} \bigwedge\nolimits_{y_\mu\nleq S} \mathfrak{R}_1(S)(y_\mu) = \bigvee \emptyset = \bot_M = \mathfrak{R}_1(U)(x_\lambda).$$

Case 2: if $x_\lambda \nleq U$, then $\mathfrak{R}_1(U)(x_\lambda) = \top_M$, and

$$\top_M \geq \bigvee\nolimits_{x_\lambda\nleq S\geq U} \bigwedge\nolimits_{y_\mu\nleq S} \mathfrak{R}_1(S)(y_\mu) \geq \bigwedge\nolimits_{y_\mu\nleq U} \mathfrak{R}_1(U)(y_\mu) = \bigwedge\nolimits_{y_\mu\nleq U} \top_M = \top_M.$$

Hence,

$$\bigvee\nolimits_{x_\lambda\nleq S\geq U} \bigwedge\nolimits_{y_\mu\nleq S} \mathfrak{R}_1(S)(y_\mu) = \top_M = \mathfrak{R}_1(U)(x_\lambda).$$

(LMR4) We also distinguish the following two cases:

Case 1: if $x_\lambda \leq \bigvee^d_{i\in I} U_i$, then $\mathfrak{R}_1(\bigvee^d_{i\in I} U_i)(x_\lambda) = \perp_M$, and for any $\mu \ll \lambda$, we have $\mu \ll \lambda \leq \bigvee^d_{i\in I} U_i(x)$. There exists $i_0 \in I$ such that $\mu \leq U_{i_0}(x)$, i. e., $x_\mu \leq U_{i_0}$. It implies that $\mathfrak{R}_1(U_{i_0})(x_\mu) = \perp_M$. So,

$$\perp_M \leq \bigwedge_{i\in I} \mathfrak{R}_1(U_i)(x_\mu) \leq \mathfrak{R}_1(U_{i_0})(x_\mu) = \perp_M.$$

Hence, $\bigwedge_{i\in I} \mathfrak{R}_1(U_i)(x_\mu) = \perp_M$ for any $\mu \ll \lambda$. Therefore,

$$\bigvee_{\mu\ll\lambda} \bigwedge_{i\in I} \mathfrak{R}_1(U_i)(x_\mu) = \perp_M = \mathfrak{R}_1\left(\bigvee^d_{i\in I} U_i\right)(x_\lambda).$$

Case 2: if $x_\lambda \not\leq \bigvee^d_{i\in I} U_i$, then $\mathfrak{R}_1\left(\bigvee^d_{i\in I} U_i\right)(x_\lambda) = \top_M$, and there exists $\mu_0 \ll \lambda$ such that $x_{\mu_0} \not\leq \bigvee^d_{i\in I} U_i$, which means $x_{\mu_0} \not\leq U_i$ for each $i \in I$, So, $\bigwedge_{i\in I} \mathfrak{R}_1(U_i)(x_{\mu_0}) = \top_M$. Hence,

$$\top_M = \bigwedge_{i\in I} \mathfrak{R}_1(U_i)(x_{\mu_0}) \leq \bigvee_{\mu\ll\lambda} \bigwedge_{i\in I} \mathfrak{R}_1(U_i)(x_\mu) \leq \top_M,$$

Therefore,

$$\bigvee_{\mu\ll\lambda} \bigwedge_{i\in I} \mathfrak{R}_1(U_i)(x_\mu) = \mathfrak{R}_1\left(\bigvee^d_{i\in I} U_i\right)(x_\lambda).$$

From the above proof, we know that $\mathfrak{R}_1$ is a convex (L, M)-fuzzy remote neighborhood operator, i. e., $\mathfrak{R}_1 \in \mathbf{CFR^2}(X, L, M)$. For any $\mathfrak{R} \in \mathbf{CFR^2}(X, L, M)$, according to the definition of $\mathfrak{R}_1$, for each $U \in L^X$ and $x_\lambda \in J(L^X)$, we have $\mathfrak{R}(U)(x_\lambda) \leq \mathfrak{R}_1(U)(x_\lambda)$. It implies that $\mathfrak{R} \leq \mathfrak{R}_1$. Hence $\mathfrak{R}_1$ is the greatest element of $(\mathbf{CFR^2}(X, L, M), \leq)$.

Step 2: $\forall\ \emptyset \neq \{\mathfrak{R}_i\}_{i\in I} \subseteq (\mathbf{CFR^2}(X, L, M), \leq)$, where I be an index set. Define $\mathfrak{R} : L^X \longrightarrow M^{J(L^X)}$ as follows: for each $U \in L^X, x_\lambda \in J(L^X)$,

$$\mathfrak{R}(U)(x_\lambda) = \bigvee_{x_\lambda \not\leq S \geq U} \bigwedge_{y_\mu \not\leq S} \bigwedge_{i\in I} \mathfrak{R}_i(S)(y_\mu).$$

Then, we can verify that $\mathfrak{R} \in \mathbf{CFR^2}(X, L, M)$. Meanwhile,

$$\begin{aligned} \mathfrak{R}(U)(x_\lambda) &= \bigvee_{x_\lambda \not\leq S \geq U} \bigwedge_{y_\mu \not\leq S} \bigwedge_{i\in I} \mathfrak{R}_i(S)(y_\mu) \\ &\leq \bigwedge_{i\in I} \bigvee_{x_\lambda \not\leq S \geq U} \bigwedge_{y_\mu \not\leq S} \mathfrak{R}_i(S)(y_\mu) \\ &= \bigwedge_{i\in I} \mathfrak{R}_i(U)(x_\lambda) \\ &\leq \mathfrak{R}_i(U)(x_\lambda). \end{aligned}$$

It implies that $\mathfrak{R}$ is a lower bound of $\{\mathfrak{R}_{\mathrm{i}}\}_{i\in I}$.

If $\mathfrak{R}^\star$ is another convex (L, M)-fuzzy remote neighborhood operator, such that $\mathfrak{R}^\star \leq \mathfrak{R}_i$ for any $i \in I$. Then, for any $U \in L^X$ and $x_\lambda \in J(L^X)$, we have $\mathfrak{R}^\star(U)(x_\lambda) \leq \bigwedge_{i\in I} \mathfrak{R}_i(U)(x_\lambda)$. In particular, $\forall S \in \{S \in L^X | x_\lambda \not\leq S \geq U\}$, we obtain

$$\bigwedge_{y_\mu \not\leq S} \mathfrak{R}^\star(S)(y_\mu) \leq \bigwedge_{y_\mu \not\leq S} \bigwedge_{i\in I} \mathfrak{R}_i(S)(y_\mu).$$

Hence,

$$\begin{aligned}\mathfrak{R}^{\star}(U)(x_\lambda) &= \bigvee_{x_\lambda \nleq S \geq U} \bigwedge_{y_\mu \nleq S} \mathfrak{R}^{\star}(S)(y_\mu) \\ &\leq \bigvee_{x_\lambda \nleq S \geq U} \bigwedge_{y_\mu \nleq S} \bigwedge_{i \in I} \mathfrak{R}_i(S)(y_\mu) \\ &= \mathfrak{R}(U)(x_\lambda).\end{aligned}$$

It follows that $\mathfrak{R}$ is the infimum of the $\{\mathfrak{R}_i\}_{i\in I}$, and denoted by $\bigwedge_{i\in I}\mathfrak{R}_{\mathrm{i}}$. Therefore, $(\mathbf{CFR^2}(X, L, M), \leq)$ is a complete lattice. □

4.2. The category of convex (L, M)-fuzzy remote neighborhood operators

In this subsection, we will discuss the categorical relationship between convex (L, M)-fuzzy remote neighborhood operators and (L, M)-fuzzy convex structures.

Theorem 4.6. Let $\mathfrak{C} \in \mathbf{FC}(X, L, M)$. Define a mapping $\mathfrak{R}^{\mathfrak{C}} : L^X \longrightarrow M$ as follows: for any $U \in L^X, x_\lambda \in J(L^X)$,

$$\mathfrak{R}^{\mathfrak{C}}(U)(x_\lambda) = \bigvee_{x_\lambda \nleq S \geq U} \mathfrak{C}(S).$$

Then $\mathfrak{R}^{\mathfrak{C}} \in \mathbf{CFR^2}(X, L, M)$.

Proof. (LMR1) Since $\top_M = \mathfrak{C}(\perp_L^X) \leq \bigvee_{x_\lambda \nleq S \geq \perp_L^X} \mathfrak{C}(S) = \mathfrak{R}^{\mathfrak{C}}(\perp_L^X)(x_\lambda) \leq \top_M$. It follows that $\mathfrak{R}^{\mathfrak{C}}(\perp_L^X)(x_\lambda) = \top_M$.

(LMR2) If $x_\lambda \leq U$, then $\{S \in L^X \mid x_\lambda \nleq S \geq U\} = \emptyset$. So, we have

$$\mathfrak{R}^{\mathfrak{C}}(U)(x_\lambda) = \bigvee_{x_\lambda \nleq S \geq U} \mathfrak{C}(S) = \bigvee \emptyset = \perp_M.$$

(LMR3) If $U \leq V$, by the definition of $\mathfrak{R}^{\mathfrak{C}}$, we have

$$\mathfrak{R}^{\mathfrak{C}}(V)(x_\lambda) = \bigvee_{x_\lambda \nleq S \geq V} \mathfrak{C}(S) \leq \bigvee_{x_\lambda \nleq S \geq U} \mathfrak{C}(S) = \mathfrak{R}^{\mathfrak{C}}(U)(x_\lambda).$$

It implies that $\mathfrak{R}^{\mathfrak{C}}(V)(x_\lambda) \leq \mathfrak{R}^{\mathfrak{C}}(U)(x_\lambda)$. For any $x_\lambda \nleq V \geq U$,

$$\mathfrak{C}(V) \leq \bigwedge_{y_\mu \nleq V} \mathfrak{R}^{\mathfrak{C}}(V)(y_\mu) \leq \mathfrak{R}^{\mathfrak{C}}(V)(x_\lambda) \leq \mathfrak{R}^{\mathfrak{C}}(U)(x_\lambda).$$

Therefore,

$$\mathfrak{R}^{\mathfrak{C}}(U)(x_\lambda) = \bigvee_{x_\lambda \nleq V \geq U} \mathfrak{C}(V) \leq \bigvee_{x_\lambda \nleq V \geq U} \bigwedge_{y_\mu \nleq V} \mathfrak{R}^{\mathfrak{C}}(V)(y_\mu) \leq \mathfrak{R}^{\mathfrak{C}}(U)(x_\lambda).$$

Hence,

$$\mathfrak{R}^{\mathfrak{C}}(U)(x_\lambda) = \bigvee_{x_\lambda \nleq V \geq U} \bigwedge_{y_\mu \nleq V} \mathfrak{R}^{\mathfrak{C}}(V)(y_\mu).$$

(LMR4) For each directed subfamily $\{U_i\}_{i\in I} \subseteq L^X$. In order to show the following equality:

$$\mathfrak{R}^{\mathfrak{C}}\left(\bigvee\nolimits^{d}_{i\in I} U_i\right)(x_\lambda) = \bigvee\nolimits_{\mu\ll\lambda} \bigwedge\nolimits_{i\in I} \mathfrak{R}^{\mathfrak{C}}(U_i)(x_\mu).$$

Notice that

$$\mathfrak{R}^{\mathfrak{C}}\left(\bigvee\nolimits^{d}_{i\in I} U_i\right)(x_\lambda) = \bigvee\nolimits_{x_\lambda \not\leqslant S \geqslant \bigvee^{d}_{i\in I} U_i} \mathfrak{C}(S) = \bigvee\nolimits_{\mu\ll\lambda} \bigvee\nolimits_{x_\mu \not\leqslant S \geqslant \bigvee^{d}_{i\in I} U_i} \mathfrak{C}(S),$$

and

$$\bigvee\nolimits_{\mu\ll\lambda} \bigwedge\nolimits_{i\in I} \mathfrak{R}^{\mathfrak{C}}(U_i)(x_\mu) = \bigvee\nolimits_{\mu\ll\lambda} \bigwedge\nolimits_{i\in I} \bigvee\nolimits_{x_\mu \not\leqslant S \geqslant U_i} \mathfrak{C}(S).$$

So, we only need to show that

$$\bigvee\nolimits_{\mu\ll\lambda} \bigvee\nolimits_{x_\mu \not\leqslant S \geqslant \bigvee^{d}_{i\in I} U_i} \mathfrak{C}(S) = \bigvee\nolimits_{\mu\ll\lambda} \bigwedge\nolimits_{i\in I} \bigvee\nolimits_{x_\mu \not\leqslant S \geqslant U_i} \mathfrak{C}(S).$$

If $S \geqslant \bigvee^{d}_{i\in I} U_i$, then $S \geqslant U_i$ for each $i \in I$. Thus,

$$\bigvee\nolimits_{x_\lambda \not\leqslant S \geqslant \bigvee^{d}_{i\in I} U_i} \mathfrak{C}(S) = \bigvee\nolimits_{\mu\ll\lambda} \bigvee\nolimits_{x_\mu \not\leqslant S \geqslant \bigvee^{d}_{i\in I} U_i} \mathfrak{C}(S) \leq \bigvee\nolimits_{\mu\ll\lambda} \bigwedge\nolimits_{i\in I} \bigvee\nolimits_{x_\mu \not\leqslant S \geqslant U_i} \mathfrak{C}(S).$$

For the inverse inequality, let

$$\alpha \prec \bigvee\nolimits_{\mu\ll\lambda} \bigwedge\nolimits_{i\in I} \bigvee\nolimits_{x_\mu \not\leqslant S \geqslant U_i} \mathfrak{C}(S).$$

Then, there exists $\mu \ll \lambda$, such that

$$\alpha \prec \bigwedge\nolimits_{i\in I} \bigvee\nolimits_{x_\mu \not\leqslant S \geqslant U_i} \mathfrak{C}(S).$$

Further, for each $i \in I$, there exists $x_\mu \not\leqslant S_{\mu i} \geqslant U_i$ such that $\alpha \leq \mathfrak{C}(S_{\mu i})$.

For each $i \in I$, define

$$S^\mu_i = \bigwedge\left\{D \in L^X | x_\mu \not\leq D \geq U_i, \alpha \leq \mathfrak{C}(D)\right\},$$

then S^μ_i is well defined, and $\alpha \leq \mathfrak{C}(S^\mu_i)$. Moreover, $\{S^\mu_i\}_{i\in I}$ is directed, and

$$x_\lambda \not\leq \bigvee\nolimits^{d}_{i\in I} S^\mu_i \geq \bigvee\nolimits^{d}_{i\in I} U_i.$$

So, by (LMC3), we have

$$\alpha \leq \bigwedge\nolimits_{i\in I} \mathfrak{C}(S^\mu_i) \leq \mathfrak{C}(\bigvee\nolimits^{d}_{i\in I} S^\mu_i) \leq \bigvee\nolimits_{x_\lambda \not\leq S \geq \bigvee^{d}_{i\in I} U_i} \mathfrak{C}(S),$$

i.e., $\alpha \leq \bigvee_{x_\lambda \not\leqslant S \geqslant \bigvee^{d}_{i\in I} U_i} \mathfrak{C}(S)$. By the arbitrariness of α, we obtain

$$\bigvee\nolimits_{x_\lambda \not\leqslant S \geqslant \bigvee^{d}_{i\in I} U_i} \mathfrak{C}(S) \geq \bigvee\nolimits_{\mu\ll\lambda} \bigwedge\nolimits_{i\in I} \bigvee\nolimits_{x_\mu \not\leqslant S \geqslant U_i} \mathfrak{C}(S).$$

Therefore,

$$\bigvee_{x_\lambda \nleq S \geqslant \bigvee^d_{i\in I} U_i} \mathfrak{C}(S) = \bigvee_{\mu \ll \lambda} \bigvee_{x_\mu \nleq S \geqslant \bigvee^d_{i\in I} U_i} \mathfrak{C}(S) = \bigvee_{\mu \ll \lambda} \bigwedge_{i\in I} \bigvee_{x_\mu \nleq S \geqslant U_i} \mathfrak{C}(S).$$

It follows from above proof that $\mathfrak{R}^{\mathfrak{C}} \in \mathbf{CFR^2}(X, L, M)$. □

Proposition 4.7. If $g : (X, \mathfrak{C}_X) \longrightarrow (Y, \mathfrak{C}_Y)$ is an (L, M)-CP, then $g : (X, \mathfrak{R}^{\mathfrak{C}_X}) \longrightarrow (Y, \mathfrak{R}^{\mathfrak{C}_Y})$ is an (L, M)-RNP^2.

Proof. Since $g : (X, \mathfrak{C}_X) \longrightarrow (Y, \mathfrak{C}_Y)$ is an (L, M)-CP, we have $\mathfrak{C}_X(g^{\leftarrow}(S)) \geq \mathfrak{C}_Y(S)$ for each $S \in L^Y$. Then for each $U \in L^X$ and $x_\lambda \in J(L^X)$, we obtain

$$\begin{aligned}\mathfrak{R}^{\mathfrak{C}_Y}(g^{\rightarrow}(U))(g(x)_\lambda) &= \bigvee_{g(x)_\lambda \nleq S \geq g^{\rightarrow}(U)} \mathfrak{C}_Y(S) \\ &\leq \bigvee_{x_\lambda \nleq g^{\leftarrow}(S) \geq U} \mathfrak{C}_X(g^{\leftarrow}(S)) \\ &\leq \bigvee_{x_\lambda \nleq D \geq U} \mathfrak{C}_X(D) \\ &= \mathfrak{R}^{\mathfrak{C}_X}(U)(x_\lambda).\end{aligned}$$

This shows that $g : (X, \mathfrak{R}^{\mathfrak{C}_X}) \longrightarrow (Y, \mathfrak{R}^{\mathfrak{C}_Y})$ is an (L, M)-RNP^2. □

By Theorem 4.6 and Proposition 4.7, we have a functor

$$\mathbf{R_C} : \begin{cases} (L, M)\text{-}\mathbf{FC} \longrightarrow (L, M)\text{-}\mathbf{FR^2}, \\ (X, \mathfrak{C}) \longmapsto (X, \mathfrak{R}^{\mathfrak{C}}), \\ g \longmapsto g. \end{cases}$$

Conversely, we can construct an (L, M)-fuzzy convex structure from a convex (L, M)-fuzzy remote neighthood operator.

Theorem 4.8. Let $\mathfrak{R} \in \mathbf{CFR^2}(X, L, M)$. Define $\mathfrak{C}^{\mathfrak{R}} : L^X \longrightarrow M$ as follows: for any $U \in L^X$,

$$\mathfrak{C}^{\mathfrak{R}}(U) = \bigwedge_{x_\lambda \nleq U} \mathfrak{R}(U)(x_\lambda),$$

then $\mathfrak{C}^{\mathfrak{R}} \in \mathbf{FC}(X, L, M)$.

Proof. (LMC1) By Definition 4.1, we have

$$\mathfrak{C}^{\mathfrak{R}}(\bot_L^X) = \bigwedge_{x_\lambda \nleq \bot_L^X} \mathfrak{R}(\bot_L^X)(x_\lambda) = \top_M,$$

$$\mathfrak{C}^{\mathfrak{R}}(\top_L^X) = \bigwedge_{x_\lambda \nleq \top_L^X} \mathfrak{R}(\top_L^X)(x_\lambda) = \bigwedge \emptyset = \top_M.$$

(LMC2) Since $\{x_\lambda \in J(L^X) \mid x_\lambda \not\leq \bigwedge_{i\in I} U_i\} = \bigcup_{i\in I}\{x_\lambda \in J(L^X) \mid x_\lambda \not\leq U_i\}$ for each $\{U_i\}_{i\in I} \subseteq L^X$, then we have

$$\begin{aligned} \mathfrak{C}^{\mathfrak{R}}\left(\bigwedge_{i\in I} U_i\right) &= \bigwedge_{x_\lambda \not\leq \bigwedge_{i\in I} U_i} \mathfrak{R}\left(\bigwedge_{i\in I} U_i\right)(x_\lambda) \\ &= \bigwedge_{i\in I} \bigwedge_{x_\lambda \not\leq U_i} \mathfrak{R}\left(\bigwedge_{i\in I} U_i\right)(x_\lambda) \\ &\geq \bigwedge_{i\in I} \bigwedge_{x_\lambda \not\leq U_i} \mathfrak{R}(U_i)(x_\lambda) = \bigwedge_{i\in I} \mathfrak{C}^{\mathfrak{R}}(U_i). \end{aligned}$$

It implies that

$$\mathfrak{C}^{\mathfrak{R}}(\bigwedge_{i\in I} U_i) \geq \bigwedge_{i\in I} \mathfrak{C}^{\mathfrak{R}}(U_i).$$

(LMC3)* For each directed subfamily $\{U_k\}_{k\in K} \subseteq L^X$. In order to show the following inequality:

$$\mathfrak{C}^{\mathfrak{R}}\left(\bigvee^d_{k\in K} U_k\right) \geq \bigwedge_{k\in K} \mathfrak{C}^{\mathfrak{R}}(U_k).$$

Notice that

$$\begin{aligned} \mathfrak{C}^{\mathfrak{R}}\left(\bigvee^d_{k\in K} U_k\right) &= \bigwedge_{x_\lambda \not\leq \bigvee^d_{k\in K} U_k} \mathfrak{R}\left(\bigvee^d_{k\in K} U_k\right)(x_\lambda) \\ &= \bigwedge_{x_\lambda \not\leq \bigvee^d_{k\in K} U_k} \bigvee_{\mu \ll \lambda} \bigwedge_{k\in K} \mathfrak{R}(U_k)(x_\mu) \end{aligned}$$

and

$$\bigwedge_{k\in K} \mathfrak{C}^{\mathfrak{R}}(U_k) = \bigwedge_{k\in K} \bigwedge_{x_\lambda \not\leq U_k} \mathfrak{R}(U_k)(x_\lambda).$$

So, we only need to show that

$$\bigwedge_{x_\lambda \not\leq \bigvee^d_{k\in K} U_k} \bigvee_{\mu \ll \lambda} \bigwedge_{k\in K} \mathfrak{R}(U_k)(x_\mu) \geq \bigwedge_{k\in K} \bigwedge_{x_\lambda \not\leq U_k} \mathfrak{R}(U_k)(x_\lambda).$$

Let

$$\alpha \prec \bigwedge_{k\in K} \bigwedge_{x_\lambda \not\leq U_k} \mathfrak{R}(U_k)(x_\lambda).$$

Then for each $k \in K$ and each $x_\lambda \not\leq U_k$ implies $\alpha \leq \mathfrak{R}(U_k)(x_\lambda)$. For each $x_\lambda \not\leq \bigvee^d_{k\in K} U_k$, there exists $\mu_0 \ll \lambda$ such that $x_{\mu_0} \not\leq \bigvee^d_{k\in K} U_k$. So, $x_{\mu_0} \not\leq U_k$ for each $k \in K$. This implies that $\alpha \leq \mathfrak{R}(U_k)(x_{\mu_0})$ for each $k \in K$. Thus, $\alpha \leq \bigwedge_{k\in K} \mathfrak{R}(U_k)(x_{\mu_0})$. Hence, for each $x_\lambda \not\leq \bigvee^d_{k\in K} U_k$, we know that

$$\alpha \leq \bigwedge_{k\in K} \mathfrak{R}(U_k)(x_{\mu_0}) \leq \bigvee_{\mu \ll \lambda} \bigwedge_{k\in K} \mathfrak{R}(U_k)(x_\mu).$$

Therefore,

$$\alpha \leq \bigwedge_{x_\lambda \not\leq \bigvee^d_{k\in K} U_k} \bigvee_{\mu \ll \lambda} \bigwedge_{k\in K} \mathfrak{R}(U_k)(x_\mu).$$

By the arbitrariness of α, we obtain

$$\bigwedge_{x_\lambda \not\leq \bigvee^d_{k\in K} U_k} \bigvee_{\mu \ll \lambda} \bigwedge_{k\in K} \mathfrak{R}(U_k)(x_\mu) \geq \bigwedge_{k\in K} \bigwedge_{x_\lambda \not\leq U_k} \mathfrak{R}(U_k)(x_\lambda).$$

It follows from above proof that $\mathfrak{C}^{\mathfrak{R}} \in \mathbf{FC}(X, L, M)$. □

Proposition 4.9. If $g : (X, \mathfrak{R}_X) \longrightarrow (Y, \mathfrak{R}_Y)$ is an (L, M)-RNP^2, then $g : (X, \mathfrak{C}^{\mathfrak{R}}_X) \longrightarrow (Y, \mathfrak{C}^{\mathfrak{R}}_Y)$ is an (L, M)-CP.

Proof. Since $g : (X, \mathfrak{R}_X) \longrightarrow (Y, \mathfrak{R}_Y)$ is an (L, M)-RNP^2, we have

$$\mathfrak{R}_X(U)(x_\lambda) \geq \mathfrak{R}_Y(g^{\rightarrow}(U))(g(x)_\lambda), \forall\, U \in L^X, x_\lambda \in J(L^X).$$

Then for each $S \in L^Y$, we obtain

$$\begin{aligned}
\mathfrak{C}^{\mathfrak{R}_X}(g^{\leftarrow}(S)) &= \bigwedge_{x_\lambda \not\leq g^{\leftarrow}(S)} \mathfrak{R}_X(g^{\leftarrow}(S))(x_\lambda) \\
&\geq \bigwedge_{g(x)_\lambda \not\leq S} \mathfrak{R}_Y(g^{\rightarrow}(g^{\leftarrow}(S)))(g(x)_\lambda) \\
&\geq \bigwedge_{g(x)_\lambda \not\leq S} \mathfrak{R}_Y(S)(g(x)_\lambda) \\
&\geq \bigwedge_{y_\mu \not\leq S} \mathfrak{R}_Y(S)(y_\mu) = \mathfrak{C}^{\mathfrak{R}_Y}(S).
\end{aligned}$$

This shows that $g : (X, \mathfrak{C}^{\mathfrak{R}_X}) \longrightarrow (Y, \mathfrak{C}^{\mathfrak{R}_Y})$ is an (L, M)-CP. □

By Theorem 4.8 and Proposition 4.9, we have a functor:

$$\mathbf{C_R} : \begin{cases} (L, M)\text{-}\mathbf{FR^2} \longrightarrow (L, M)\text{-}\mathbf{FC}, \\ (X, \mathfrak{R}) \longmapsto (X, \mathfrak{C}^{\mathfrak{R}}), \\ g \longmapsto g. \end{cases}$$

Now, we show that $\mathbf{R_C}$ and $\mathbf{C_R}$ are isomorphic functors.

Theorem 4.10. $\mathbf{R_C}$ and $\mathbf{C_R}$ are isomorphic functors.

Proof. It suffices to show that $\mathbf{R_C} \circ \mathbf{C_R} = \mathbf{I}_{(L,M)\text{-}\mathbf{FR^2}}$ and $\mathbf{C_R} \circ \mathbf{R_C} = \mathbf{I}_{(L,M)\text{-}\mathbf{FC}}$. That is, for each (L, M)-fuzzy convex space $(X, \mathfrak{C})$ and for each convex (L, M)-fuzzy remote neighborhood space^2 $(X, \mathfrak{R})$, it follows that

(1) $\mathfrak{R}^{\mathfrak{C}^{\mathfrak{R}}} = \mathfrak{R}$.

(2) $\mathfrak{C}^{\mathfrak{R}^{\mathfrak{C}}} = \mathfrak{C}$.

For (1), for each $U \in L^X$ and $x_\lambda \in J(L^X)$. Then by Theorems 4.6 and 4.8, we have

$$\mathfrak{R}^{\mathfrak{C}^{\mathfrak{R}}}(U)(x_\lambda) = \bigvee_{x_\lambda \not\leq S \geq U} \mathfrak{C}^{\mathfrak{R}}(S) = \bigvee_{x_\lambda \not\leq S \geq U} \bigwedge_{y_\mu \not\leq S} \mathfrak{R}(S)(y_\mu) = \mathfrak{R}(U)(x_\lambda).$$

Hence $\mathfrak{R}^{\mathfrak{C}^{\mathfrak{R}}} = \mathfrak{R}$.

For (2), for each $U \in L^X$. Then

$$\mathfrak{C}^{\mathfrak{R}^{\mathfrak{C}}}(U) = \bigwedge_{x_\lambda \not\leq U} \mathfrak{R}^{\mathfrak{C}}(U)(x_\lambda) = \bigwedge_{x_\lambda \not\leq U} \bigvee_{x_\lambda \not\leq S \geq U} \mathfrak{C}(S) \geq \bigwedge_{x_\lambda \not\leq U} \mathfrak{C}(U) = \mathfrak{C}(U).$$

On the other hand, denote $S_{x_\lambda} = \{S \in L^X \mid x_\lambda \not\leq S \geq U\}$. By the completely distributive law in [2], it follows that

$$\begin{aligned} \mathfrak{C}^{\mathfrak{R}^{\mathfrak{C}}}(U) &= \bigwedge_{x_\lambda \not\leq U} \bigvee_{x_\lambda \not\leq S \geq U} \mathfrak{C}(S) \\ &= \bigvee_{g \in \prod_{x_\lambda \not\leq U} S_{x_\lambda}} \bigwedge_{x_\lambda \not\leq U} \mathfrak{C}(g(x_\lambda)) \\ &\leq \bigvee_{g \in \prod_{x_\lambda \not\leq U} S_{x_\lambda}} \mathfrak{C}\left(\bigwedge_{x_\lambda \not\leq U} g(x_\lambda)\right). \end{aligned}$$

Since $\bigwedge_{x_\lambda \not\leq U} g(x_\lambda) = U$ for every $g \in \prod_{x_\lambda \not\leq U} S_{x_\lambda}$, we have

$$\bigvee_{g \in \prod_{x_\lambda \not\leq U} S_{x_\lambda}} \mathfrak{C}\left(\bigwedge_{x_\lambda \not\leq U} g(x_\lambda)\right) = \mathfrak{C}(U).$$

Therefore, $\mathfrak{C}^{\mathfrak{R}^{\mathfrak{C}}}(U) \leq \mathfrak{C}(U)$. This shows that $\mathfrak{C}^{\mathfrak{R}^{\mathfrak{C}}} = \mathfrak{C}$. □

By Remark 2.3 and Theorem 4.10, we easily obtain the following result.

Corollary 4.11. $(\mathbf{CFR^2}(X, L, M), \leq)$ and $(\mathbf{FC}(X, L, M), \leq)$ are complete lattice isomorphic.

By Corollaries 3.8 and 4.11, we can obtain the following result.

Corollary 4.12. If L is a completely distributive De Morgan algebra, and M is a completely distributive lattice, then $\mathbf{CFR^1}(X, L, M)$ and $\mathbf{CFR^2}(X, L, M)$ are complete lattice isomorphic.

5. CONCLUSIONS

In this paper, we proposed the concept of two kinds of convex (L, M)-fuzzy remote neighborhood operators. When L and M satisfy certain conditions, we proved that these two kind of convex (L, M)-fuzzy remote neighborhood operators and (L, M)-fuzzy convex structures are lattice (categorically) isomorphic. In the future, we will consider the compatibility of (L, M)-fuzzy convex structures and some algebraic structures.

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