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DISTRIBUTED OPTIMIZATION VIA ACTIVE DISTURBANCE REJECTION CONTROL: A NABLA FRACTIONAL DESIGN

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This paper studies distributed optimization problems of a class of agents with fractional order dynamics and unknown external disturbances. Motivated by the celebrated active disturbance rejection control (ADRC) method, a fractional order extended state observer (Frac-ESO) is first constructed, and an ADRC-based PI-like protocol is then proposed for the target distributed optimization problem. It is rigorously shown that the decision variables of the agents reach a domain of the optimal solution when the external disturbance is bounded. In particular, for constant disturbances, the Frac-ESO is Mittag-Leffler convergent and the optimization problem can be solved exactly. Finally, numerical simulations are presented to validate the effective properties of the proposed algorithm.

Keywords: distributed optimization, nabla fractional difference, active disturbance rejection control, Lyapunov method

Classification: 68W15, 26A33, 93D05, 93D21, 49N15

1. INTRODUCTION

In recent years, distributed optimization algorithms have undergone comprehensive and profound scrutiny. Diverse algorithms have been meticulously crafted to cater to distinct convergence profiles, corresponding to various communication topologies. Among these contributions, [7, 17, 29] elucidated distributed optimization algorithms tailored for achieving asymptotic convergence with undirected graphs and directed graphs. [13] devised a distributed optimization algorithm to attain exponential convergence when conmunication is directed graphs. With the same convergence rate, [3] constructed a distributed optimization algorithm over unbalanced graphs for muti-agent system. [2] proposed a distributed quantized algorithm named Q-DGAT which used gradient descent and variables tracking methods to estimate the global aggregative variable making Q-DGAT algorithm achieve linear convergence. [14] designed a distributed ADMM algorithm called QP-ADMM, which can achieve a linear convergence rate. [9] designed a kind of zero-gradient-sum algorithm for distributed optimization which makes the sum of local gradients comes to zero within a fixed time. [5, 19, 20] provided many studies in prescribe-time distributed optimization, which are very popular in recent years. [26]

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contributed a distributed resource allocation algorithm for a multiagent network with a time-varying digraph. However, there are few studies on distributed optimization conbined with active disturbance rejection control (ADRC). [6] designed a continuous-time distributed optimization algorithm for first order multi-agent disturbance systems conbined ADRC.

ADRC, an innovation attributed to Han [10], is a control method based on PID control algorithm. It is independent of the model of controlled plant and does not distinguish internal and external disturbances. In general, ADRC includes tracking differentiators (TD), extended state observers (ESO), and nonlinear state error feedback (NLSEF). In this paper, a Frac-ESO is designed to observe and estimate unknown disturbances and it is used as compensation in feedback control.

In addition, fractional calculus, an increasingly prominent mathematical tool, has gained considerable traction within contemporary control theory. Over the past few decades, fractional calculus has been found widespread applications in modeling the behavior of dynamic system, spanning domains such as circuit systems, mechanical systems and biological systems. Distinguishing itself from integer calculus, fractional calculus can better describe the non-local and non-Markov phenomena of complex systems. Its applicability extends across an array of disciplines, including finance, physics, and biology. In [15], a fractional-order gradient algorithm for quadratic objective functions was proposed, but it cannot guarantee convergence to the global optimal solution. For such convergence problem, [25] analyzed the specific reasons and proposed a variety of solutions. Among these advancements, [1] ingeniously extended the conventional gradient optimization algorithm by replacing the gradient term with a fractional gradient, thereby introducing the novel concept of a fractional gradient optimization algorithm. [24] poposed a class of gradient algorithms with nabla fractional-order system, including three kinds of convergence rates. It's worth noting that [4] was the pioneer in introducing fractional gradients into distributed optimization algorithms. However, this algorithm can only deal with the objective function of quadratic form. In addition, [27] studied the distributed optimization problem of fractional-order nonlinear uncertain multi-agent systems with unmeasured states, employing a neural network-based adaptive optimization control strategy. [28] addressed the distributed optimization problem of fractional-order non-strict-feedback multi-agent systems using neural networks and event-triggered schemes for optimization control. [18] investigated the fixed-time distributed time-varying optimization problem of nonlinear fractional-order multi-agent systems on unbalanced directed graphs, employing innovative distributed control methods. [16] examined the multiple Mittag-Leffler stability and almost periodic solutions of fractional-order delayed neural networks, using a distributed optimization model and neural dynamic solving methods. However, these studies primarily center on continuous-time algorithms, potentially bringing about increased computational complexity and communication costs. Then, [11, 12] extended the objective function to general convex functions or strongly convex functions and designed nabla fractional distributed optimization algorithms for different kinds of graphs. Through these studies, it is found that fractional calculus can improve the performance of optimization algorithm, and its non-Markov property can make reasonable use of the past information and avoid the real-time calculation of gradient.

This paper designs a creative optimization algorithm, which solves the problem caused by disturbance by introducing ADRC and improves the performance of the algorithm by introducing fractional calculus. Section 2 gives some basic knowledge and the problem statement. Section 3 develops an algorithm and analyzes its main properties. Section 4 provides two numerical examples to test the proposed algorithm. Finally, a concluding discussion is given in Section 5.

2. PRELIMINARY

In this section, the main problem and some basic knowledge will be illustrated.

2.1. Nabla fractional calculus

For function $f : \mathbb{N}_{a+1-n} \to \mathbb{R}$, its nth integer order backward difference is defined by [8]

$$\nabla^n f(k) := \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(k-i), \tag{1}$$

where $n \in \mathbb{Z}_+$, $k \in \mathbb{N}_{a+1} := \{a+1, a+2, a+3, \cdots\}$, $a \in \mathbb{R}$, $\binom{p}{q} := \frac{\Gamma(p+1)}{\Gamma(q+1)\Gamma(p-q+1)}$ is the generalized binomial coefficient and $\Gamma(\cdot)$ represents the Gamma function.

For function $f : \mathbb{N}_{a+1} \to \mathbb{R}$, its αth Grünwald–Letnikov fractional sum is defined by [8]

$${}_a^{\mathrm{G}}\nabla_k^{-\alpha} f(k) := \sum_{i=0}^{k-a-1} (-1)^i \binom{-\alpha}{i} f(k-i), \tag{2}$$

where $\alpha \in \mathbb{R}_+$, $k \in \mathbb{N}_{a+1}$ and $a \in \mathbb{R}$.

For function $f : \mathbb{N}_{a+1-n} \to \mathbb{R}$, its αth Caputo fractional difference is defined by [8]

$${}_a^{\mathrm{C}}\nabla_k^{\alpha} f(k) := {}_a^{\mathrm{G}}\nabla_k^{\alpha-n} \nabla^n f(k), \tag{3}$$

where $\alpha \in (n-1, n)$, $n \in \mathbb{Z}_+$, $k \in \mathbb{N}_{a+1}$, $a \in \mathbb{R}$.

At this point, some lemmas need to be introduced.

Lemma 2.1. (Goodrich and Peterson [8]) For any $f : \mathbb{N}_{a+1-n} \to \mathbb{R}$, $\alpha \in (n-1, n)$, $n \in \mathbb{Z}_+$, $k \in \mathbb{N}_{a+1}$, C is any constant in $\mathbb{R}$, there exists ${}_a^{\mathrm{C}}\nabla_k^{\alpha}{}_a^{\mathrm{G}}\nabla_k^{-\alpha} f(k) = f(k)$ and ${}_a^{\mathrm{C}}\nabla_k^{\alpha} C = 0$.

Lemma 2.2. (Wei et al. [23]) For any $\alpha \in (0,1)$, $y(k) \in \mathbb{R}^p$, $p \in \mathbb{Z}_+$, $k \in \mathbb{N}_{a+1}$, $a \in \mathbb{R}$ and the positive definite matrix $P \in \mathbb{R}^{p \times p}$, one has the following inequality

$${}_a^{\mathrm{C}}\nabla_k^{\alpha} y^\top(k) P y(k) \leq 2 y^\top(k) P {}_a^{\mathrm{C}}\nabla_k^{\alpha} y(k). \tag{4}$$

Lemma 2.3. (Wei [22]) If $\alpha \in (0,1)$, $\lambda < 0$, $a \in \mathbb{R}$, $\beta \in [\alpha, \alpha+1)$, $k \in \mathbb{N}_{a+1}$, for Mittag–Leffler function $\mathcal{F}_{\alpha,\beta}(\lambda, k, a)$, the following equation holds

$$\lim_{k \to +\infty} \mathcal{F}_{\alpha,\beta}(\lambda, k, a) = 0. \tag{5}$$

Lemma 2.4. (Wei [21]) For system ${}_a^{\mathrm{C}}\nabla_k^{\alpha} x(k) = f(k, x(k))$, if there exist parameters $\beta \in (0,1)$, b, c, α_1, α_2, α_3, $\alpha_4 > 0$ and the Lyapunov function $V(k, x(k))$ such that

$$\alpha_1 \|x(k)\|^b \leq V(k, x(k)) \leq \alpha_2 \|x(k)\|^{bc}, \tag{6}$$

$$ {}_{a}^{\mathrm{C}}\nabla_{k}^{\beta}V(k,x(k)) \le -\alpha_3\|x(k)\|^{bc} + \alpha_4, \tag{7} $$

where $k \in \mathbb{N}_{a+1}$, $a \in \mathbb{R}$, $V(k,x(k))$ is Lipschitz continuous with regard to $x(k)$, then system is uniformly ultimate bounded at $x_e = 0$.

Lemma 2.5. (Wei [22]) If $\alpha \in (0,1)$, $\lambda < 0$, $p^{\bar{q}}$ is rising function writen as $\frac{\Gamma(p+q)}{\Gamma(p)}$, there exists $\lim_{p\to+\infty}\frac{p^{\bar{q}}}{p^{q}} = 1$ and $\lim_{k\to+\infty}\frac{\mathcal{F}_{\alpha,1}(\lambda,k,a)\Gamma(1-\alpha)}{(k-a)^{\overline{-\alpha}}} = 1$.

Lemma 2.6. (Wei [21]) For system ${}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha}x(k) = f(k,x(k))$, if there exist parameters $\beta \in (0,1)$, $\gamma \in (0,1)$, b, c, α_1, α_2, α_3, $\sigma > 0$ and the Lyapunov function $V(k,x(k))$ such that

$$ \alpha_1\|x(k)\|^{b} \le V(k,x(k)) \le \alpha_2\|x(k)\|^{bc}, \tag{8} $$

$$ {}_{a}^{\mathrm{C}}\nabla_{k}^{\beta}V(k,x(k)) \le -\alpha_3\|x(k)\|^{bc} + h(k), \tag{9} $$

$$ \sum\nolimits_{j=a+1}^{+\infty}\left|{}_{a}^{\mathrm{G}}\nabla_{j}^{\gamma}h(j)\right| = \sigma < +\infty, \tag{10} $$

$$ \lim_{k\to+\infty} h(k) = 0, \tag{11} $$

where $k \in \mathbb{N}_{a+1}$, $a \in \mathbb{R}$, $V(k,x(k))$ is Lipschitz continuous with regard to $x(k)$, then system is uniformly attractive at $x_e = 0$.

2.2. Convexity and smoothness

If a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is m-strongly convex, then there is $[\nabla f(y) - \nabla f(x)]^{\top}(y-x) \ge m\|y-x\|^2$, $\forall x,\ y \in \mathbb{R}^n$. If a function $f : \mathbb{R}^n \to \mathbb{R}$ is l-Lipschitz continuous, then there is $\|f(x) - f(y)\| \le l\|x-y\|$, where $l > 0$ is the Lipschitz coefficient. Especially, if a function $f : \mathbb{R}^n \to \mathbb{R}$ is l-smooth, then there is $\|\nabla f(x) - \nabla f(y)\| \le l\|x-y\|$.

2.3. Graph theory

The information-sharing relationship between agents can be characterized through the language of graphs. The graph $\mathcal{G} = (\mathcal{V},\mathcal{E},A)$ represents the information-sharing relationship between agents, where $\mathcal{V}$ is the set of points, i. e., the set of N agents. $\mathcal{E}$ is the set of edges, which is a subset of $\mathcal{V}\times\mathcal{V}$. If $(i,j) \in \mathcal{E}$, it means that the ith agent is connected to the jth agent. The matrix $A \in \mathbb{R}^{N\times N}$ is the adjacency matrix of this graph. If agent i is adjacent to agent j, then $A_{ij} = 1$, otherwise $A_{ij} = 0$. In particular, $A_{ii} = 0$ is equal to 0 for all $i \in \mathcal{V}$. The Laplacian matrix of graph $\mathcal{G}$ is denoted by $L \in \mathbb{R}^{N\times N}$, and each of its element satisfies $L_{ij} = -A_{ij}$ for $i \ne j$ and $L_{ii} = \sum_{j=1,j\ne i}^{N} A_{ij}$.

2.4. Problem description

Considering a system consisted by N agents, their information is shared and the relationship is represented by an undirected graph $\mathcal{G}$. Each agent satisfies the following dynamics system

$$ {}_{a}^{\mathrm{C}}\nabla_{k}^{\alpha}x_i(k) = u_i(k) + q_i(k), i = 1,\ldots,N, \tag{12} $$

where $x_i \in \mathbb{R}^n$ is the state of agent i, $u_i \in \mathbb{R}^n$ is the input of agent i and $q_i(k) \in \mathbb{R}^n$ is the unknown external disturbance of agent i.
The main problem can be described as

$$\min_{x\in\mathbb{R}^n} \tilde{f}(x) = \sum\nolimits_{i=1}^{N} \tilde{f}_i(x), \tag{13}$$

where $x \in \mathbb{R}^n$ is a global variable, $\tilde{f}(\cdot)$ is the global objective function, $\tilde{f}_i(\cdot)$ is the local objective function assigned to the ith agent and N is the number of agents. Let $x_i = [x_{i1}, x_{i2}, \dots, x_{in}]^\top \in \mathbb{R}^n$ denote the local optimization variable of the ith agent. Then the original problem can be transformed into an optimization problem with a consensus constraint

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} f(\boldsymbol{x}) = \sum\nolimits_{i=1}^{N} f_i(x_i), \\ &\text{s.t. } x_1 = x_2 = \cdots = x_N. \end{aligned} \tag{14}$$

Aiming at the problem in (14) and considering agents in (12), an algorithm based on ADRC is designed. Before moving on this, the following assumptions are given.

Assumption 1. The undirected graph $\mathcal{G}$ is connected.

Lemma 2.7. Under Assumption 1, 1 and 0 are eigenvalues of matrix L and there is $\mathbf{1}_N^\top L = \mathbf{0}_N^\top$. Furthermore, there exists a matrix $Q \in \mathbb{R}^{N\times(N-1)}$, which satisfies $\mathbf{1}_N^\top Q = \mathbf{0}_{N-1}$, $Q^\top Q = I_{N-1}$, $QQ^\top = I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top$. Moreover, $Q^\top LQ$ is a positive definite matrix.

Assumption 2. The local objective function f_i is differentiable and m_i-strongly convex, and ∇f_i is l_i-Lipschitz continuous on $\mathbb{R}^n$. In other words, f_i is differentiable, m_i-strongly convex and l-smooth.

Assumption 3. For external disturbance $q_i(k)$, there exists $M>0$ satisfying

$$\sup_{k\in\mathbb{N}_{a+1}} \left\| {}_a^{\mathrm{C}}\nabla_k^\alpha q_i(k)\| + \|q_i(k)\right\| \le M. \tag{15}$$

3. MAIN RESULTS

In this section, the design of fractional distributed optimization algorithm and a series of theorems will be given.

3.1. The design of Frac-ESO

For the system in (12), Frac-ESO is designed as follows

$$\begin{cases} {}_a^{\mathrm{C}}\nabla_k^\alpha \hat{x}_i(k) = \hat{q}_i(k) + 2\omega\left[x_i(k) - \hat{x}_i(k)\right] + u_i(k), \\ {}_a^{\mathrm{C}}\nabla_k^\alpha \hat{q}_i(k) = \omega^2\left[x_i(k) - \hat{x}_i(k)\right], \end{cases} \tag{16}$$

where $\omega > 0$, $\hat{x}_i(k) \in \mathbb{R}^n$ is the estimation of $x_i(k)$ and $\hat{q}_i(k) \in \mathbb{R}^n$ is the estimation of $q_i(k)$. Therefore, it can be observed that both external disturbance and agents' state could be estimated by designed Frac-ESO.

Theorem 3.1. With Assumption 3, there exists $r_1 > 0$ such that the error between $q_i(k)$ and $\hat{q}_i(k)$ satisfies

$$\|q_i(k) - \hat{q}_i(k)\| \leq r_1 \mathcal{F}_{\alpha,1}(-\omega, k, a) + M, i = 1, \ldots, N, k \in \mathbb{N}_{a+1}. \tag{17}$$

P r o o f. For the convenience of calculation, the error variables are defined as

$$\begin{cases} \tilde{x}_i(k) = x_i(k) - \hat{x}_i(k), \\ \tilde{q}_i(k) = q_i(k) - \hat{q}_i(k). \end{cases} \tag{18}$$

By combining (12), (16) and (18), it follows

$$\begin{cases} {}_a^{\mathrm{C}}\nabla_k^{\alpha} \tilde{x}_i(k) = \tilde{q}_i(k) - 2\omega \tilde{x}_i(k), \\ {}_a^{\mathrm{C}}\nabla_k^{\alpha} \tilde{q}_i(k) = -\omega^2 \tilde{x}_i(k) + {}_a^{\mathrm{C}}\nabla_k^{\alpha} q_i(k). \end{cases} \tag{19}$$

In the case of Assumption 3, after defining $z_i(k) = [\tilde{x}_i(k), \tilde{q}_i(k)]^\top$, there is ${}_a^{\mathrm{C}}\nabla_k^{\alpha} z_i(k) \leq Az_i(k) + \begin{bmatrix} 0 \\ M \end{bmatrix}$, where $A = \begin{bmatrix} -2\omega I_n & I_n \\ -\omega^2 I_n & 0 \end{bmatrix}$, $z_i(k) \in \mathbb{R}^{2n}$. The eigenvalues of A are calculated to be $\lambda_{1,2} = -\omega < 0$. As the largest eigenvalue of matrix A, $-\omega$ satisties ${}_a^{\mathrm{C}}\nabla_k^{\alpha} \tilde{q}_i(k) \leq -\omega \tilde{q}_i(k) + M$. By applying Lemma 2.4, there exists a parameter $r_1 > 0$ such that $\|\tilde{q}_i(k)\| \leq \|\tilde{q}_i(a)\mathcal{F}_{\alpha,1}(-\omega, k, a) + M\| \leq r_1 \mathcal{F}_{\alpha,1}(-\omega, k, a) + M$. All of these complete the proof. □

Corollary 3.2. When $q_i(k) = d_n$, there exists $r_2 > 0$ such that the following inequalitity holds

$$\|q_i(k) - \hat{q}_i(k)\| \leq r_2 \mathcal{F}_{\alpha,1}(-\omega, k, a), i = 1, \ldots, N, k \in \mathbb{N}_{a+1}. \tag{20}$$

Remark 3.3. Using the Frac-ESO designed above, the state of agents and external disturbance can be estimated in real-time. Moreover, when the external disturbance is bounded as the shape of Assumption 3, the error estimated by Frac-ESO is also bounded. In particular, when the external disturbance is constant, the error is Mittag–Leffler convergence.

3.2. Algorithm construction

Based on the designed Frac-ESO in (16), the following fractional PI control can be designed as

$$\begin{cases} {}_a^{\mathrm{C}}\nabla_k^{\alpha} u_i(k) = -\rho \sum_{j=1}^{N} L_{ij} x_j(k) - \rho v_i(k) - \rho \nabla f_i\left(x_i(k)\right) - \hat{q}_i(k), \\ {}_a^{\mathrm{C}}\nabla_k^{\alpha} v_i(k) = \rho \sum_{j=1}^{N} L_{ij} x_j(k). \end{cases} \tag{21}$$

Here, parameter $\rho > 0$, $-\rho \sum_{j=1}^{N} L_{ij} x_j(k)$ is the consensus term that causes all agents to converge to the same point. $-\rho \nabla f_i\left(x_i(k)\right)$ is a negative gradient term to ensure that the state of each agent is updated iteratively in the direction of minimizing the objective function f_i. $-\hat{q}_i(k)$ is the compensation term for external disturbance. $-\rho v_i(k)$ is the integral feedback term which is used to eliminate the steady-state error of the algorithm. Combining (12), (16) and (21), integrating the designed Frac-ESO and controller into the model of controlled plant, a fractional distributed optimization algorithm with the idea of ADRC is developed as

$$\begin{cases} {}_a^{\mathrm{C}}\nabla_k^{\alpha} v_i(k) = \rho \sum_{j=1}^{N} L_{ij} x_j(k), \\ {}_a^{\mathrm{C}}\nabla_k^{\alpha} x_i(k) = -\rho \sum_{j=1}^{N} L_{ij} x_j(k) - \rho v_i(k) - \rho \nabla f_i\left(x_i(k)\right) + q_i(k) - \hat{q}_i(k), \\ {}_a^{\mathrm{C}}\nabla_k^{\alpha} \hat{x}_i(k) = \hat{q}_i(k) + 2\omega[x_i(k) - \hat{x}_i(k)] + u_i(k), \\ {}_a^{\mathrm{C}}\nabla_k^{\alpha} \hat{q}_i(k) = \omega^2[x_i(k) - \hat{x}_i(k)], \end{cases} \tag{22}$$

where $x_i \in \mathbb{R}^n$ is the state of agent i, $L \in \mathbb{R}^{N\times N}$ is the Laplacian matrix of the graph $\mathcal{G}$ and L_{ij} is the (i,j) elements of L. Since (22) can be regarded as a closed-loop control system, the convergence of the algorithm can be regarded as the stability of this system.

Theorem 3.4. For algorithm (22), if the external disturbance is bounded as the form of (17), then there exsit parameters $r_3 > 0$, $M_1 > 0$ such that

$$\|x_i(k) - x^*\| \le M_1 + r_3 \mathcal{F}_{\alpha,1}^{\frac{1}{2}}(-\frac{\rho^2}{2\lambda_{\mathcal{H}}}, k, a), \tag{23}$$

where $\lambda_{\mathcal{H}}$ is related to ρ and Lipschize coefficient l which will be defined later.

Proof. Defining the operation $\mathrm{col}(z_1, \cdots, z_m) := \left[z_1^\top, \ldots, z_m^\top\right]^\top$, according to (18), one has

$$\begin{cases} x(k) = \mathrm{col}(x_1(k), \ldots, x_N(k)), \\ v(k) = \mathrm{col}(v_1(k), \ldots, v_N(k)), \\ \tilde{q}(k) = \mathrm{col}(\tilde{q}_1(k), \ldots, \tilde{q}_N(k)), \\ \nabla f(x(k)) = \mathrm{col}(\nabla f_1\left(x_1(k)\right), \ldots, \nabla f_N\left(x_N(k)\right)). \end{cases} \tag{24}$$

Then the variables $v(k)$ and $x(k)$ can be governed by

$$\begin{cases} {}_a^{\mathrm{C}}\nabla_k^{\alpha} v(k) = \rho(L \otimes I_n)x(k), \\ {}_a^{\mathrm{C}}\nabla_k^{\alpha} x(k) = -\rho \nabla f(x) - \rho v(k) - \rho(L \otimes I_n)x(k) + \tilde{q}(k), \end{cases} \tag{25}$$

where $\otimes$ represents the Kronecker product. When the external disturbance $q_i(k) \equiv \mathbf{0}_n$, then $\tilde{q}_i(k) \equiv \mathbf{0}_n$. Therefore, the system (25) is converted to

$$\begin{cases} {}_a^{\mathrm{C}}\nabla_k^{\alpha} v(k) = \rho(L \otimes I_n)x(k), \\ {}_a^{\mathrm{C}}\nabla_k^{\alpha} x(k) = -\rho \nabla f(x(k)) - \rho v(k) - \rho(L \otimes I_n)x(k). \end{cases} \tag{26}$$

Since there is no external disturbance, algorithm (26) converges without Frac-ESO. According to Assumption 1 and Assumption 2, it has a unique equilibrium point (x^*, v^*), where $v^* = -\nabla f(x^*)$. When considering Lemma 2.1, the following equations hold

$$\sum_{i=1}^{N} v_i(k) = \mathbf{0}_n, \sum_{i=1}^{N} \nabla f_i\left(x^*\right) = \mathbf{0}_n. \tag{27}$$

According to the equivalence between equilibrium point (x^*, v^*) in (26) and the optimal value x^* in (14), the optimization problem can be transformed into a stability problem to solve. In order to facilitate analysis, the following variables are defined as

$$\begin{cases} X_i(k) = x_i(k) - x^*, \\ V_i(k) = v_i(k) - v^*, \\ F_i\left(X_i(k)\right) = \nabla f_i\left(x_i(k)\right) - \nabla f_i\left(x^*\right), \\ X(k) = \operatorname{col}\left(X_1(k), \ldots, X_N(k)\right), \\ V(k) = \operatorname{col}\left(V_1(k), \ldots, V_N(k)\right), \\ F(X(k)) = \operatorname{col}\left(F_1\left(X_1(k)\right), \ldots, F_N(k)\left(X_N(k)\right)\right), \end{cases} \tag{28}$$

where $X(k) \in \mathbb{R}^{nN}$, $V(k) \in \mathbb{R}^{nN}$, $F(X(k)) \in \mathbb{R}^{nN}$. By transformation (28), the system (26) is converted into the error system as follows

$$\begin{cases} {}_a^{\mathrm{C}}\nabla_k^\alpha V(k) = \rho\left(L \otimes I_n\right) X(k), \\ {}_a^{\mathrm{C}}\nabla_k^\alpha X(k) = -\rho F(X(k)) - \rho V(k) - \rho\left(L \otimes I_n\right) X(k) + \tilde{q}(k). \end{cases} \tag{29}$$

In order to study system (29) more clearly, the following variables are defined as

$$\begin{cases} \chi = \left(T^\top \otimes I_n\right) X(k), \\ \vartheta = \left(T^\top \otimes I_n\right), \\ T = \left[\dfrac{\mathbf{1}_N}{\sqrt{N}}, Q\right]. \end{cases} \tag{30}$$

For the sake of analysis, $\chi = \operatorname{col}\left(\chi_1, \chi_2\right)$ and $\vartheta = \operatorname{col}\left(\vartheta_1, \vartheta_2\right)$ are defined, where $\chi_1 \in \mathbb{R}^n$, $\vartheta_1 \in \mathbb{R}^n$, $\chi_2 \in \mathbb{R}^{n(N-1)}$, $\vartheta_2 \in \mathbb{R}^{n(N-1)}$. It's important to note that Q is a matrix defined in Lemma 2.7. On this basis, one has

$$\begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} \left(\frac{\mathbf{1}_N^\top}{\sqrt{N}} \otimes I_n\right) X(k) \\ \left(Q^\top \otimes I_n\right) X(k) \end{bmatrix}, \tag{31}$$

$$\begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} = \begin{bmatrix} \left(\frac{\mathbf{1}_N^\top}{\sqrt{N}} \otimes I_n\right) V(k) \\ \left(Q^\top \otimes I_n\right) V(k) \end{bmatrix}. \tag{32}$$

Calculating the fractional difference on hands of (31) and (32) yields the following equations

$$\begin{bmatrix} {}_a^{\mathrm{C}}\nabla_k^\alpha \chi_1 \\ {}_a^{\mathrm{C}}\nabla_k^\alpha \chi_2 \end{bmatrix} = \begin{bmatrix} \left(\frac{\mathbf{1}_N^\top}{\sqrt{N}} \otimes I_n\right) {}_a^{\mathrm{C}}\nabla_k^\alpha X(k) \\ \left(Q^\top \otimes I_n\right) {}_a^{\mathrm{C}}\nabla_k^\alpha X(k) \end{bmatrix}, \tag{33}$$

$$\begin{bmatrix} {}_a^{\mathrm{C}}\nabla_k^\alpha \vartheta_1 \\ {}_a^{\mathrm{C}}\nabla_k^\alpha \vartheta_2 \end{bmatrix} = \begin{bmatrix} \left(\frac{\mathbf{1}_N^\top}{\sqrt{N}} \otimes I_n\right) {}_a^{\mathrm{C}}\nabla_k^\alpha V(k) \\ \left(Q^\top \otimes I_n\right) {}_a^{\mathrm{C}}\nabla_k^\alpha V(k) \end{bmatrix}. \tag{34}$$

Substituting algorithm (29) into equation (33), it follows

$$\begin{bmatrix} {}_a^{\mathrm{C}}\nabla_k^\alpha \chi_1 \\ {}_a^{\mathrm{C}}\nabla_k^\alpha \chi_2 \end{bmatrix} = \begin{bmatrix} \left(\frac{\mathbf{1}_N^\top}{\sqrt{N}} \otimes I_n\right) \left[-\rho F(X(k)) - \rho V(k) - \rho\left(L \otimes I_n\right) X(k) + \tilde{q}(k)\right] \\ \left(Q^\top \otimes I_n\right) \left[-\rho F(X(k)) - \rho V(k) - \rho\left(L \otimes I_n\right) X(k) + \tilde{q}(k)\right] \end{bmatrix}. \tag{35}$$

Because of (27) and the property of $\mathbf{1}_N^\top L = \mathbf{0}_N$, one has

$$\left(\frac{\mathbf{1}_N^\top}{\sqrt{N}} \otimes I_n\right) V(k) = \mathbf{0}_n, \left(\frac{\mathbf{1}_N^\top}{\sqrt{N}} \otimes I_n\right) (L \otimes I_n) = \mathbf{0}_n. \tag{36}$$

Then, χ_1 can be described as

$${}_a^{\mathrm{C}}\nabla_k^\alpha \chi_1 = -\frac{\rho}{\sqrt{N}} \left(\mathbf{1}_N \otimes I_n\right)^\top F(X(k)) + \frac{1}{\sqrt{N}} \left(\mathbf{1}_N \otimes I_n\right)^\top \tilde{q}(k). \tag{37}$$

Multiplying $\left[\frac{\mathbf{1}_N}{\sqrt{N}} \otimes I_n, Q \otimes I_n\right]$ both sides of (31), it follows

$$X(k) = \frac{\mathbf{1}_N}{\sqrt{N}} \otimes I_n \chi_1 + Q \otimes I_n \chi_2. \tag{38}$$

After multipling $\left(Q^\top \otimes I_n\right) (L \otimes I_n)$ on both sides of equation (38), the following equation holds

$$\left(Q^\top \otimes I_n\right) (L \otimes I_n) X(k) = \left(Q^\top L Q \otimes I_n\right) \chi_2. \tag{39}$$

By combining (32), (35) and (39), χ_2 can be rewritten as

$${}_a^{\mathrm{C}}\nabla_k^\alpha \chi_2 = -\left(Q \otimes I_n\right)^\top \left[\rho F(X(k)) - \tilde{q}(k)\right] - \rho \bar{L} \chi_2 - \rho \vartheta_2, \tag{40}$$

where $\bar{L} = Q^\top L Q \otimes I_n$.
Substituting equation (29) into equation (34) yields the following equation

$$\begin{bmatrix} {}_a^{\mathrm{C}}\nabla_k^\alpha \vartheta_1 \\ {}_a^{\mathrm{C}}\nabla_k^\alpha \vartheta_2 \end{bmatrix} = \begin{bmatrix} \rho\left(\frac{\mathbf{1}_N^\top}{\sqrt{N}} \otimes I_n\right) (L \otimes I_n) X(k) \\ \rho \left(Q^\top \otimes I_n\right) (L \otimes I_n) X(k) \end{bmatrix}. \tag{41}$$

Because of (36) and (27), there must be ${}_a^{\mathrm{C}}\nabla_k^\alpha \vartheta_1 = \mathbf{0}_n$, ${}_a^{\mathrm{C}}\nabla_k^\alpha \vartheta_2 = \rho \bar{L} \chi_2$ and $\vartheta_1(k) \equiv \mathbf{0}_n$. Then the system in (29) will be transformed to

$$\begin{cases} {}_a^{\mathrm{C}}\nabla_k^\alpha \chi_1 = -\frac{\rho}{\sqrt{N}} \left(\mathbf{1}_N \otimes I_n\right)^\top F(X(k)) + \frac{1}{\sqrt{N}} \left(\mathbf{1}_N \otimes I_n\right)^\top \tilde{q}(k), \\ {}_a^{\mathrm{C}}\nabla_k^\alpha \chi_2 = -\left(Q \otimes I_n\right)^\top \left[\rho F(X(k)) - \tilde{q}(k)\right] - \rho \bar{L} \chi_2 - \rho \vartheta_2, \\ {}_a^{\mathrm{C}}\nabla_k^\alpha \vartheta_2 = \rho \bar{L} \chi_2. \end{cases} \tag{42}$$

The Lyapunov function is selected as

$$\begin{aligned} W(k) = & \frac{1}{2}\rho(\phi+1)\chi_1^\top \chi_1 + \frac{1}{2}\rho(\chi_2 + \vartheta_2)^\top (\chi_2 + \vartheta_2) \\ & + \frac{1}{2}\rho\phi\chi_2^\top \chi_2 + \frac{1}{2}\rho(\phi+1)\vartheta_2^\top \bar{L}^{-1} \vartheta_2, \end{aligned} \tag{43}$$

where $\phi \geq l^2 + 1 > 0$, $m = \min\{m_1, \ldots, m_N\}$, $l = \max\{l_1, \ldots, l_N\}$. Since $\bar{L}$ is a positive definite matrix, $W(k)$ is positive definite and radially unbounded. Besides, $\bar{\lambda}_{\mathcal{H}} \|p\|^2 \leq W(k) \leq \lambda_{\mathcal{H}} \|p\|^2$, where $p = \mathrm{col}\,(\chi, \vartheta_2)$, $\mathcal{H} = \frac{1}{2} \begin{bmatrix} \iota I_n & \mathbf{0}_{n \times n(N-1)} & \mathbf{0}_{n \times n(N-1)} \\ \mathbf{0}_{n(N-1) \times n} & \iota I_{n(N-1)} & \rho I_{n(N-1)} \\ \mathbf{0}_{n(N-1) \times n} & \rho I_{n(N-1)} & \rho I_{n(N-1)} + \iota \bar{L}^{-1} \end{bmatrix}$,

$\iota = \rho(\phi+1)$, $\lambda_{\mathcal{H}} > 0$ is the maximum eigenvalue of $\mathcal{H}$, and $\bar{\lambda}_{\mathcal{H}} > 0$ is the minimum eigenvalue of $\mathcal{H}$. According to Lemma 2.2, the fraction difference of $W(k)$ can be expressed as

$$\begin{aligned}
&{}_a^{\mathrm{C}}\nabla_k^{\alpha} W(k) \\
&\leq \rho(\phi+1)\chi_1^{\top}{}_a^{\mathrm{C}}\nabla_k^{\alpha}\chi_1 + \rho\phi\chi_2^{\top}{}_a^{\mathrm{C}}\nabla_k^{\alpha}\chi_2 \\
&\quad +\rho\left(\chi_2+\vartheta_2\right)^{\top}{}_a^{\mathrm{C}}\nabla_k^{\alpha}\left(\chi_2+\vartheta_2\right) \\
&\quad +\rho(\phi+1)\vartheta_2^{\top}\bar{L}^{-1}{}_a^{\mathrm{C}}\nabla_k^{\alpha}\vartheta_2 \\
&= -\rho^2(\phi+1)X(k)^{\top}\left(\tfrac{1_N}{\sqrt{N}}\otimes I_n\right)\left(\tfrac{1_N^{\top}}{\sqrt{N}}\otimes I_n\right)F(X(k)) \\
&\quad +\rho(\phi+1)X(k)^{\top}\left(\tfrac{1_N}{\sqrt{N}}\otimes I_n\right)\left(\tfrac{1_N^{\top}}{\sqrt{N}}\otimes I_n\right)\tilde{q}(k) \\
&\quad -\rho^2\phi\chi_2^{\top}\left(Q\otimes I_n\right)^{\top}F(X(k)) + \rho\phi\chi_2^{\top}\left(Q\otimes I_n\right)^{\top}\tilde{q}(k) \\
&\quad -\rho^2\phi\chi_2^{\top}\bar{L}\chi_2 - \rho^2\phi\chi_2^{\top}\vartheta_2 + \rho^2(\phi+1)\vartheta_2^{\top}\chi_2 \\
&\quad +\rho\left(\chi_2+\vartheta_2\right)^{\top}\left(Q\otimes I_n\right)^{\top}\tilde{q}(k) - \rho^2\left(\chi_2+\vartheta_2\right)^{\top}\vartheta_2 \\
&\quad -\rho^2\left(\chi_2+\vartheta_2\right)^{\top}\left(Q\otimes I_n\right)^{\top}F(X(k)) \\
&= -\rho^2(\phi+1)X(k)^{\top}F(X(k)) + \rho(\phi+1)X(k)^{\top}\tilde{q}(k) \\
&\quad -\rho^2\vartheta_2^{\top}\left(Q\otimes I_n\right)^{\top}F(X(k)) + \rho\vartheta_2^{\top}\left(Q\otimes I_n\right)^{\top}\tilde{q}(k) \\
&\quad -\rho^2\phi\chi_2^{\top}\bar{L}\chi_2 - \rho^2\vartheta_2^{\top}\vartheta_2.
\end{aligned} \tag{44}$$

Since f_i is a m_i-strongly convex function and $\|X(k)\| = \|\chi\|$, the following equation holds

$$-\rho^2(\phi+1)X(k)^{\top}F(X(k)) \leq -\rho^2(\phi+1)m\|\chi\|^2. \tag{45}$$

Since ∇f_i is l_i-Lipschitz on $\mathbb{R}^n$, the following inequality holds by applying $\|X(k)\| = \|\chi\|$, $\|(Q\otimes I_n)\| = 1$ and Young's inequality

$$-\vartheta_2^{\top}\left(Q\otimes I_n\right)^{\top}F(X(k)) \leq \frac{1}{4}\vartheta_2^{\top}\vartheta_2 + l^2\|\chi\|^2. \tag{46}$$

According to Young's inequality, following inequalities can be given

$$\rho(\phi+1)X(k)^{\top}\tilde{q}(k) \leq \frac{1}{2}\rho^2\|X(k)\|^2 + \frac{1}{2}(\phi+1)^2\|\tilde{q}(k)\|^2, \tag{47}$$

$$\rho\vartheta_2^{\top}\left(Q\otimes I_n\right)^{\top}\tilde{q}(k) \leq \frac{1}{4}\rho^2\left\|\vartheta_2\right\|^2 + \|\tilde{q}(k)\|^2. \tag{48}$$

Therefore, the fractional difference of $W(k)$ can be simplified as

$$\begin{aligned}
{}_a^{\mathrm{C}}\nabla_k^{\alpha} W(k) \leq & -\frac{\rho^2}{2}(2m+1)\|\chi\|^2 \\
& -\rho^2\frac{l^2+1}{m}X(k)^{\top}\left(L\otimes I_n\right)X(k) \\
& -\frac{\rho^2}{2}\left\|\vartheta_2\right\|^2 + \left[\frac{1}{2}(\phi+1)^2+1\right]\|\tilde{q}(k)\|^2 \\
\leq & -\frac{\rho^2}{2}\|p\|^2 + \left[\frac{1}{2}(\phi+1)^2+1\right]\|\tilde{q}(k)\|^2.
\end{aligned} \tag{49}$$

According to Theorem 3.1, there exists $M_2 > 0$ which satisfies $\left[\frac{1}{2}(\phi+1)^2+1\right]\|\tilde{q}(k)\|^2 \leq M_2$. Consequently, one has

$$\begin{cases} \bar{\lambda}_{\mathcal{H}}\|p\|^2 \leq W(k) \leq \lambda_{\mathcal{H}}\|p\|^2, \\ {}_a^{\mathrm{C}}\nabla_k^{\alpha} W(k) \leq -\dfrac{\rho^2}{2}\|p\|^2 + M_2. \end{cases} \tag{50}$$

Applying Lemma 2.4, there exists $r_3 > 0$, $r_4 > 0$, $M_1 > 0$, satisfying $\|x_i(k) - x^*\| \leq \|M_2 + r_4\mathcal{F}_{\alpha,1}(-\frac{\rho^2}{2\lambda_{\mathcal{H}}}, k, a)\|^{\frac{1}{2}} \leq M_1 + r_3\mathcal{F}_{\alpha,1}^{\frac{1}{2}}(-\frac{\rho^2}{2\lambda_{\mathcal{H}}}, k, a)$. Proof is thus completed. □

Theorem 3.5. For algorithm (22), if the disturbance $q_i(k) = d_{n\times 1}$, the corresponding system is uniformily attractive at $x_e = 0$, i. e., $\lim_{k\to+\infty} x_i(k) = 0$.

Proof. Referring to the previous proof, (49) also holds. With Corollary 3.2, there exsits $M_3 > 0$ satisfying

$$\begin{aligned} {}_a^{\mathrm{C}}\nabla_k^{\alpha} W(k) &\leq -\frac{\rho^2}{2}\|p\|^2 + \left[\frac{1}{2}(\phi+1)^2+1\right]\|\tilde{q}(k)\|^2 \\ &\leq -\frac{\rho^2}{2}\|p\|^2 + M_3\mathcal{F}_{\alpha,1}^2(-\omega, k, a). \end{aligned} \tag{51}$$

Defining $h(k) = M_3\mathcal{F}_{\alpha,1}^2(-\omega, k, a)$, $\lim_{k\to+\infty} h(k) = 0$ can be obtained by Lemma 2.3. According to Lemma 2.5, there exsits $M_4 > 0$ satisfying

$$\begin{aligned} \sum\nolimits_{j=a+1}^{+\infty} \left|{}_a^{\mathrm{G}}\nabla_j^{\gamma} h(j)\right| &\approx M_4 \sum\nolimits_{j=a+1}^{+\infty} \left|{}_a^{\mathrm{G}}\nabla_j^{\gamma} \frac{(j-a)^{\overline{-2\alpha}}}{\Gamma(1-2\alpha)}\right| \\ &= M_4 \sum\nolimits_{j=a+1}^{+\infty} \left|\frac{(j-a)^{\overline{-2\alpha-\gamma}}}{\Gamma(1-2\alpha-\gamma)}\right| \\ &= M_4 \frac{(k-a)^{\overline{1-2\alpha-\gamma}}}{\Gamma(2-2\alpha-\gamma)}. \end{aligned} \tag{52}$$

Since there exists $\gamma \in (0, 1)$ satisfying $1 - 2\alpha - \gamma < 0$, $\sum_{j=a+1}^{+\infty} \left|{}_a^{\mathrm{G}}\nabla_j^{\gamma} h(j)\right| < +\infty$ holds. By applying Lemma 2.6, $\lim_{k\to+\infty} x_i(k) = 0$ holds. □

Remark 3.6. This paper designs a nabla fractional distributed optimization algorithm. By using the ADRC method, the Frac-ESO is designed to estimate the disturbance of the system and eliminate the static error of the system. For different types of disturbance, the convergence performance of the proposed algorithm is also different. When the disturbance is time-varying, the estimation error of the disturbance is bounded, and the optimization error is uniformly bounded. When the disturbance is constant, the estimation error of the disturbance is Mittag–Leffler convergent, and the optimization error is uniformly attractive.

Remark 3.7. In addition, compared with the previous continuous-time algorithm [6], the proposed algorithm is a discrete-time algorithm, which avoids communicating informaton and calculating the gradient of the objective function in real time. Besides, nabla fractional calculus improves the dynamic performance of the algorithm since its non-Markov and non-local properties make reasonable use of historical information. These changes make it possible to obtain better properties.

4. SIMULATION STUDY

In this section, two numerical examples will be given to verify the feasibility and effectiveness of the algorithm. The external disturbance is three-dimensional and bounded which satisfies

$$q_{ik}(t) = q_{ik0} + Eq_{ik1}\sin\left(\omega_{ik1}t + \phi_{ik1}\right) + Eq_{ik2}\sin\left(\omega_{ik2}t + \phi_{ik2}\right), k = 1, 2, 3, \tag{53}$$

where $[\omega_{i11}, \omega_{i21}, \omega_{i31}] = [\frac{\pi}{2(i+3)}, \frac{\pi}{i+4}, \frac{\pi}{2(i+4)}]$, $[\omega_{i12}, \omega_{i22}, \omega_{i32}] = [\frac{\pi}{3(i+3)}, \frac{2\pi}{3(i+3)}, \frac{\pi}{2(i+4)}]$, $[q_{i10}, q_{i20}, q_{i30}] = \left[i+2, i+2, \frac{1}{3}\right]$, $[q_{i11}, q_{i21}, q_{i31}] = \left[i+1, \frac{1}{2}, \frac{1}{5}\right]$, $[q_{i12}, q_{i22}, q_{i32}] = \left[i, \frac{1}{2}, \frac{1}{3}\right]$, $[\phi_{i11}, \phi_{i21}, \phi_{i31}] = [0, 0, 0]$, $[\phi_{i12}, \phi_{i22}, \phi_{i32}] = [0, 0, 0]$, $i = 1, 2, \ldots, N$, $E = 0, 1, 100, 10^4$. E is the expansion of disturbance. Especially, when $E = 0$, the external disturbance degenerates to a constant.

Example 4.1. This example considers a multi-agent system with an undirected connected topology of five individuals and the dynamics of each agent is described by the system as shown in Figure 1. The object function of the system is $f(x) = \sum_{i=1}^{5} f_i(x)$, where the local object function of the system is $f_i(x) = \|x - x_i(a)\|^2$. The initial value of each agent is $x_i(a) = [0.5 \times i, 0.5 \times (i-2), 0.5 \times (i-4)]^\top$. The initial values of estimation are $\hat{q}_i(a) = [0, 0, 0]^\top$ and $\hat{x}_i(a) = [0, 0, 0]^\top$. The fractional order used in this example is chosen as $\alpha = 0.9$. When the external disturbance takes different value of E, different results are obtained. By taking different values of E, it is found that when the value of E decreases, the estimation effect of Frac-ESO on the time-varying disturbance of system becomes better. Figure 2(a) to Figure 2(d) are the results of $E = 10^4$. It can be seen that each error is large. Figure 3(a) to Figure 3(d) are the results of $E = 100$ and Figure 4(a) to Figure 4(d) are the results of $E = 1$. As E decreases, the effect of each estimation error and optimization error gradually becomes better. The optimization error and the estimation error are uniformly bounded. When $E = 0$, the time-varying disturbance degenerates to a constant. The result are shown from Figure 5(a) to Figure 5(d). At this point, uniformly boundedness becomes Mittag–Leffler convergent for the estimation of disturbance and uniformly boundedness becomes uniformly attractive behavior for the agents' states.

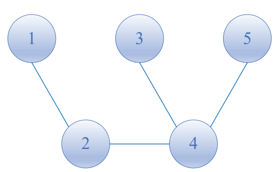
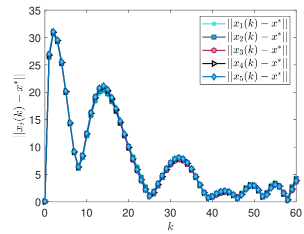
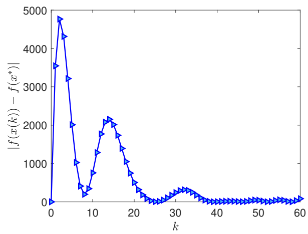


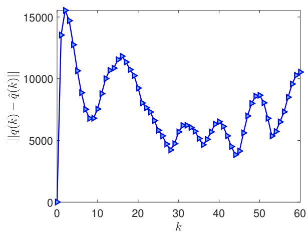
Fig. 1. The communication topology.



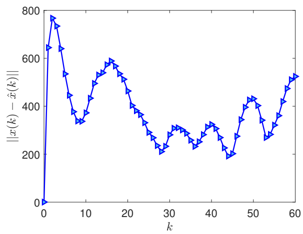
(a) The evolution of $\|x_i(k) - x^*\|$.



(b) The evolution of $|f(x(k)) - f(x^*)|$.

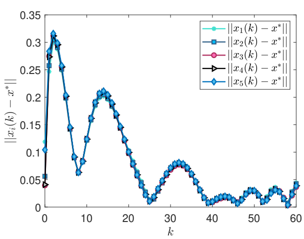


(c) The evolution of $\|q(k)-\hat{q}(k)\|$.

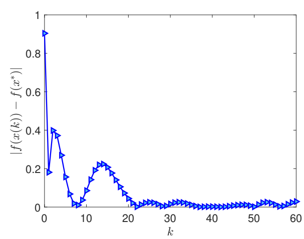


(d) The evolution of $\|x(k)-\hat{x}(k)\|$.

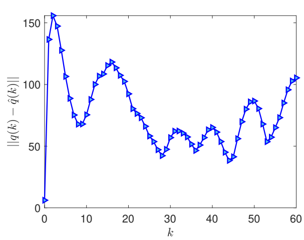
Fig. 2. Results of Example 4.1 with $E = 10^4$.



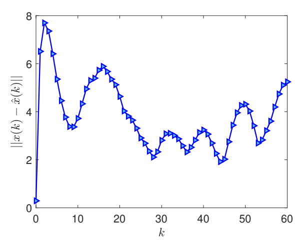
(a) The evolution of $\|x_i(k) - x^*\|$.



(b) The evolution of $|f(x(k)) - f(x^*)|$.

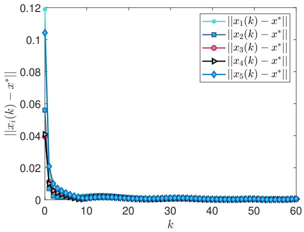


(c) The evolution of $\|q(k)-\hat{q}(k)\|$.

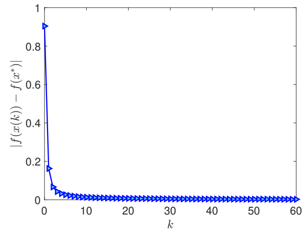


(d) The evolution of $\|x(k)-\hat{x}(k)\|$.

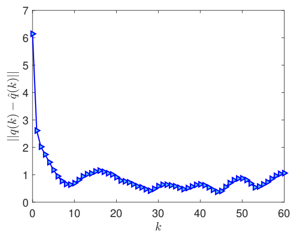
Fig. 3. Results of Example 4.1 with $E = 100$.



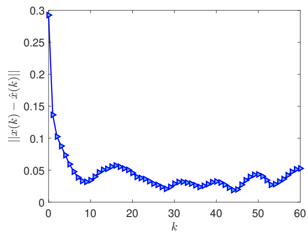
(a) The evolution of $\|x_i(k) - x^*\|$.



(b) The evolution of $|f(x(k)) - f(x^*)|$.

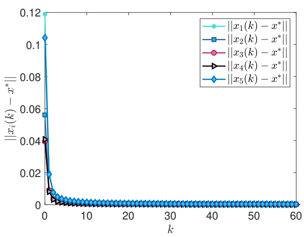


(c) The evolution of $\|q(k)-\hat{q}(k)\|$.

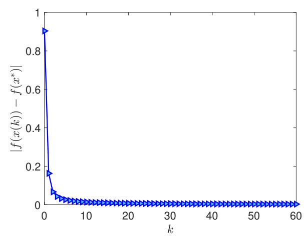


(d) The evolution of $\|x(k)-\hat{x}(k)\|$.

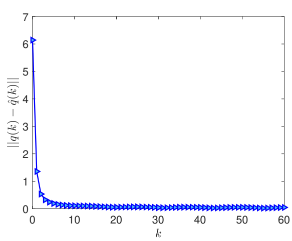
Fig. 4. Results of Example 4.1 with $E = 1$.



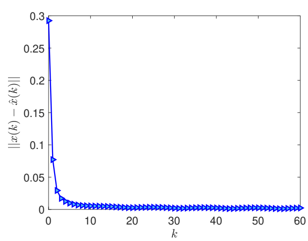
(a) The evolution of $\|x_i(k) - x^*\|$.



(b) The evolution of $|f(x(k)) - f(x^*)|$.



(c) The evolution of $\|q(k)-\hat{q}(k)\|$.



(d) The evolution of $\|x(k)-\hat{x}(k)\|$.

Fig. 5. Results of Example 4.1 with $E = 0$.

Example 4.2. This example considers a multi-agent system with an undirected connected topology of six individuals and the dynamics of each agent is described by the system shown in Figure 6. The local object functions are as follows

$$
\begin{aligned}
f_1(x) &= 2x_1^2 + 4x_2 + 1, \\
f_2(x) &= (2x_2 - 3)^2 + 5x_3, \\
f_3(x) &= (x_3 + 3)^2 - 3x_1, \\
f_4(x) &= \|x - p\|_2^2, \\
f_5(x) &= x_2^2 + \mathrm{e}^{x_3}, \\
f_6(x) &= \mathrm{e}^{x_3} + 2\mathrm{e}^{-x_1},
\end{aligned}
$$

where $p = [1, -1, 1]^\top$. The initial value of each agent is $x_i(a) = [0, 0, 0]^\top$. The initial values of estimation are $\hat{q}_i(a) = [0, 0, 0]^\top$ and $\hat{x}_i(a) = [0, 0, 0]^\top$. The fractional order used in this example is chosen as $\alpha = 0.8$. Compared with Example 4.1, communication topology, local objective functions, number of agents and fractional order are changed. However, it is worth noting that both the optimization error and the estimation error have same conclusions. As the value of E decreases, the amplitude of disturbance also gradually decreases, which makes the estimation effect of Frac-ESO improve. When $E \neq 0$, the disturbance is bounded, and the error between each agent's state and the optimal value is uniformly bounded under the action of the algorithm. Figure 7(a) to Figure 7(d) are the results of $E = 10^4$, which have large deviations. When E decreases, the effect of each estimation error and optimization error are gradually improved. Figure 8(a) to Figure 8(d) are the results of $E = 100$ and Figure 9(a) to Figure 9(d) are the results of $E = 1$. The optimization error and the estimation error are uniformly bounded. When $E = 0$, the time-varying disturbance degenerates to a constant. The result are shown from Figure 10(a) to Figure 10(d). In this case, uniformly boundedness becomes Mittag–Leffler convergent for the estimation of disturbance and uniformly boundedness becomes uniformly attractive behavior for the agents' states.

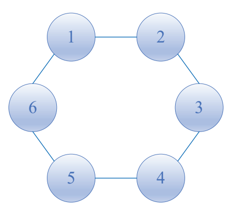
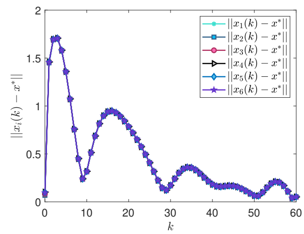
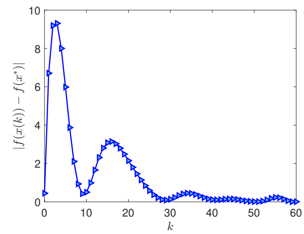


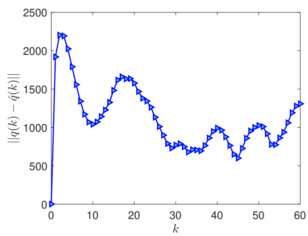
Fig. 6. The communication topology.



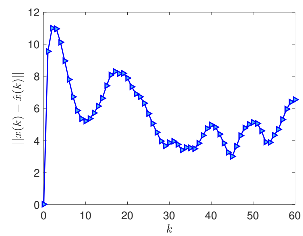
(a) The evolution of $\|x_i(k) - x^*\|$.



(b) The evolution of $|f(x(k)) - f(x^*)|$.

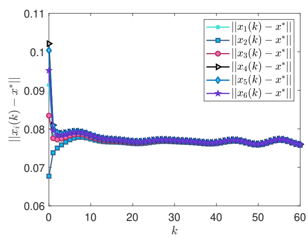


(c) The evolution of $\|q(k)-\hat{q}(k)\|$.

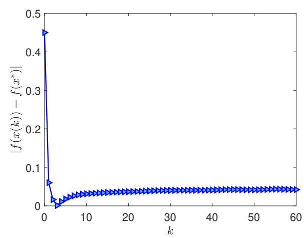


(d) The evolution of $\|x(k)-\hat{x}(k)\|$.

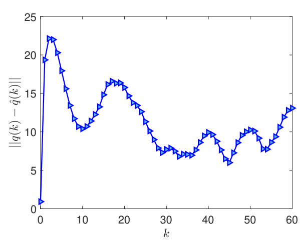
Fig. 7. Results of Example 4.2 with $E = 10^4$.



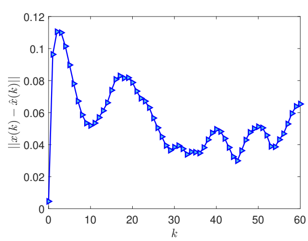
(a) The evolution of $\|x_i(k) - x^*\|$.



(b) The evolution of $|f(x(k)) - f(x^*)|$.

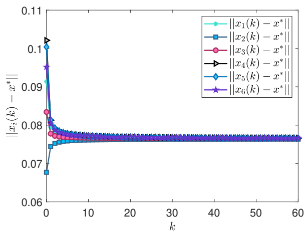


(c) The evolution of $\|q(k)-\hat{q}(k)\|$.

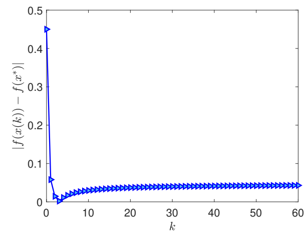


(d) The evolution of $\|x(k)-\hat{x}(k)\|$.

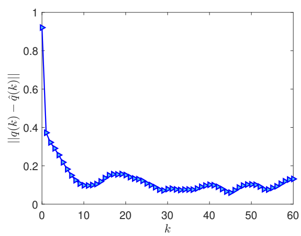
Fig. 8. Results of Example 4.2 with $E = 100$.



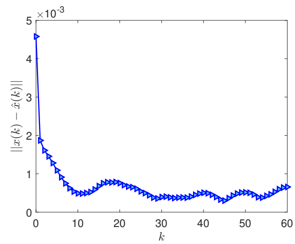
(a) The evolution of $\|x_i(k) - x^*\|$.



(b) The evolution of $|f(x(k)) - f(x^*)|$.

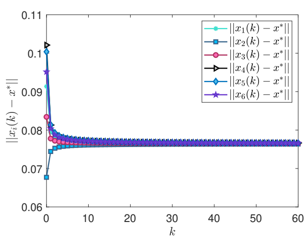


(c) The evolution of $\|q(k) - \hat{q}(k)\|$.

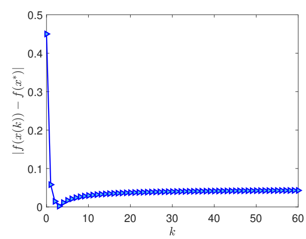


(d) The evolution of $\|x(k) - \hat{x}(k)\|$.

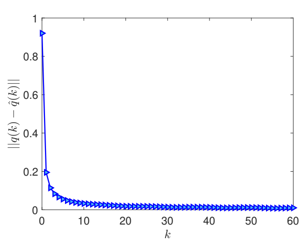
Fig. 9. Results of Example 4.2 with $E = 1$.



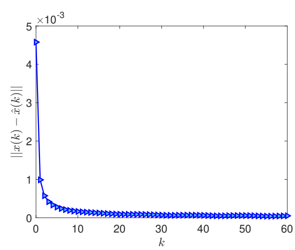
(a) The evolution of $\|x_i(k) - x^*\|$.



(b) The evolution of $|f(x(k)) - f(x^*)|$.



(c) The evolution of $\|q(k) - \hat{q}(k)\|$.



(d) The evolution of $\|x(k) - \hat{x}(k)\|$.

Fig. 10. Results of Example 4.2 with $E = 0$.

5. CONCLUSIONS

In this paper, a fractional distributed optimization algorithm based on different kinds of disturbances and the method of ADRC have been proposed. Firstly, an Frac-ESO has been constructed which is used to estimate disturbance and eliminate static errors of system. For different disturbances, the estimation performances reach different effects. The estimation error remains bounded with bounded external disturbances while constant external disturbances lead to Mittag-Leffler convergence. Using Frac-ESO for disturbance estimation, agents achieve optimal value with varying effects depending on the type of disturbance. Specifically, optimization algorithm drives convergence towards the optimal domain for bounded disturbances, while with constant disturbances, convergence leads directly to optimal values. In future, the following valuable research directions will be considered.

(i) For different communication topologies, the communication topologies of directed equilibrium graphs will be considered.

(ii) Random noise can be tried instead of fixed sinusoidal noise.

(iii) For feedback methods, nonlinear feedback may be a good choice.

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