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FINITE-TIME OBSERVABILITY OF PROBABILISTIC BOOLEAN MULTIPLEX CONTROL NETWORKS

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This paper investigates the finite-time observability of probabilistic Boolean multiplex control networks (PBMCNs). Firstly, the finite-time observability of the PBMCNs is converted into the set reachability issue according to the parallel interconnection technique (a minor modification of the weighted pair graph method in the literature). Secondly, the necessary and sufficient condition for the finite-time observability of PBMCNs is presented based on the set reachability. Finally, the main conclusions are substantiated by providing illustrative examples.

Keywords: finite-time observability, semi-tensor product, probabilistic Boolean multiplex control networks, set reachability

Classification: 93B07,93C10,93E03

1. INTRODUCTION

In 1969, Kauffman [13, 14] introduced the Boolean networks (BNs) model, which elucidated the underlying mechanism of order generation and pioneered a novel research domain for investigating complex systems. Through the BNs, the macro-behavior and micro-mechanism of complex systems are combined, which not only gives us methodological enlightenment to study complex systems, but also presents a novel approach to address the complexity issues in the real world, specifically in systems biology [15, 1], chemistry [12], engineering [17], social networks [31], etc. Recently, the introduction of the semi-tensor product (STP) has led to significant advancements in addressing various theoretical challenges associated with BNs, such as disturbance decoupling [3], stability and stabilization [2, 32, 43, 29, 36], reachability [45, 28], controllability [30, 19, 35], synchronization [48], optimal control [34, 39, 40] and other related problems [37, 7, 10].

Observability, as one of the most important concepts in control theory, has always been the focus of researchers. At present, there have been many achievements about the observability of BNs/Boolean control networks (BCNs). For example, the main result of [16] is that the problem of determining whether a BN/BCN/ABN is observable is NP-hard. In [46], Zhang et al. successfully addressed the problem of determining the observability of BCNs using finite automata technology, and in [47], they solved the observability problem of switched BCNs by employing both finite automata theory and

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formal language theory. Subsequently, inspired by [46] and [47], Cheng et al. [5] proposed an equivalent method to study the observability of BCNs. Driven by the above researches, Zhu et al. [50] gived some results related to the observability of BCNs based on the knowledge of graph theory. In [11], Guo et al. gave the necessary and sufficient conditions of several different types of observability. In [4], Cheng et al. obtained sufficient and necessary conditions for the observability of Boolean control networks by using set controllability. In fact, in essence, these methods in literature [50],[11],[4] are equivalent to the method proposed by [46]. Li et al. respectively discussed the observability of BNs/BCNs with impulsive effect, state delay and redundant channels [22, 23, 21]. In [44], Zhu et al. studied the controllability and observability of sampled-data BCNs, and gave the necessary and sufficient conditions for controllability and observability. Besides, some recent developments about BNs/BCNs are shown in [24, 25, 26, 42, 38, 18]. In [49], Guo et al. investigated the set reachability and observability of probabilistic Boolean networks (PBNs). In [20], Fornasini et al. addressed the observability and reconfiguration of PBNs within finite time intervals. In [27], Li et al. studied the observability of PBNs on the premise that the initial state is not clearly known.

Compared to the extensive research conducted on with BNs, BCNs, and PBNs, studies related to the observability of probabilistic Boolean control networks (PBCNs) are still in their nascent stage. In [33], several types of observability of PBCNs have been studied. On this basis, in order to understand the evolution of complex biological systems with many levels and interactions, multi-layer networks have been proposed as a new description. It is worth noting that the multi-layer network not only provides a multi-level model for constructing biochemical systems, but also can better describe richer interaction structures, and motivated by the work [41] of Wu et.al, we aim to consider integrally the interaction among nodes across different layers by constructing a global state layer in this paper. Indeed, numerous distinct signal channels do work in parallel in cellular biochemical networks. Besides cellular biochemical network, multilayer network has extensive applications in natural science, social science and information science. Based on the aforementioned discussion, it becomes evident that studying the observability of PBMCNs is both meaningful and challenging. Although the literature [33] has already explored the observability of PBCNs, a significant distinction remains: Even for the degenerated PBMCNs, their observability differs from the single-layer PBCNs' due to the presence of a global state layer in our system, which deviates from conventional coupling mechanisms.

The following are the primary contributions of this paper:

1) In this paper, the PBMCNs with global state layer is proposed, which can simulate more complex dynamic systems. The conclusion of this paper is exemplified by a case study conducted in a chain supermarket.

2) Through parallel interconnection technology, the observability of PBMCNs can be equivalently converted into the set reachability issue of augmented interconnected PBMCNs, and the necessary and sufficient condition for the finite-time observability is provided.

The rest of this paper is arranged as follows. In Section 2, necessary preliminaries used in the paper are provided, the definitions of observability and the necessary and sufficient condition for the set reachability of PBMCNs are introduced. In Section 3, the finite-time observability is investigated by parallel interconnection technology. Section 4 illustrates the proposed approach with a case study on chain supermarket. Finally, Section 5 provides a brief conclusion.

Notation: I_k: $k \times k$ identity matrix. $\mathcal{D} = \{0, 1\}$: the logic domain, $\mathcal{D}^k$: the set of k-dimensional column vector with entries logical values $0, 1$. Δ_k denotes the set of all of the columns of I_k. δ_n^i denotes i-th column of I_n, δ_n^0 denotes n-dimensional vector $[0\ 0\ \dots\ 0]^T$, and $\delta_n[i_1, i_2, \dots, i_m]$ denotes matrix H with $Col_s(H) = \delta_n^{i_s}$. $[H]_{i,j}$ is the element at ith row jth column of matrix H, $Col_i(H)$ denotes ith column of matrix H. $\mathbb{R}^m$ denotes the set of m-dimensional column vectors, $\mathcal{L}_{n\times m}$ denotes the set of logic matrices with $n \times m$ dimensions (For a matrix H, if each column of it belongs to Δ_n, then the matrix H is called a logic matrix), $\mathbb{R}_{n\times m}$ denotes the set of real matrices with $n \times m$ dimensions. $[U : V]$: $\{U, U + 1, \dots, V\}$, where $U \leq V$ and U, V are positive integers. $\mathbf{0}_{m\times n}$: $m \times n$ null matrix.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1. Preliminaries

Some preliminaries from the sequel are provided in this section.

Definition 2.1. [6] Given two matrices $X \in \mathbb{R}_{n\times m}$, $Y \in \mathbb{R}_{p\times q}$, the semi-tensor product of X and Y is

$$X \ltimes Y = (X \otimes I_{\alpha/m})(Y \otimes I_{\alpha/p}),$$

where α is the least common multiple of m and p, $\otimes$ is the Kronecker product. The STP of matrices may be regarded as an extension of the standard matrix product, since $X \ltimes Y = XY$ when $m = p$.

Lemma 2.2. [6] Let $A \in \mathbb{R}^m$, $B \in \mathbb{R}^n$, then $W_{[m,n]} \ltimes A \ltimes B = B \ltimes A$, where $W_{[m,n]}$ is a swap matrix with indices m and n, and is defined as $W_{[m,n]} = [I_n \otimes \delta_m^1\ I_n \otimes \delta_m^2\ \dots\ I_n \otimes \delta_m^m]$. Besides, M is an arbitrary matrix, then, $AM = (I_m \otimes M)A$.

Lemma 2.3. [6] Let $f(X_1, X_2, \dots, X_n) : \mathcal{D}^n \to \mathcal{D}$ be a Boolean function. Then, there exists a unique matrix $F_f \in \mathcal{L}_{2\times 2^n}$, known as the structure matrix of f, such that

$$f(X_1, X_2, \dots, X_n) = F_f \ltimes_{i=1}^n x_i,$$

where $x_i \in \Delta_2$, and $Col_i(F_f) = f(\delta_{2^n}^i), i = 1, 2, \dots, 2^n$.

Lemma 2.4. [6] If $X \in \Delta_k, X^2 = M_{r,k}X$, where $M_{r,k}$ is a power-reducing matrix with index k defined as $M_{r,k} = [\delta_k^1 \otimes \delta_k^1\ \delta_k^2 \otimes \delta_k^2\ \dots\ \delta_k^k \otimes \delta_k^k]$.

Proposition 2.5. [41] For any node $i \in \{1, 2, \dots, n\}$, if $\Phi_{\text{in}(i)} = \{\epsilon_1, \epsilon_2, \dots, \epsilon_k\} (k \leqslant n)$, we can find a suitable dimension matrix Π_i such that

$$\Pi_i \ltimes_{i=1}^n x_i = x_{\epsilon_1} x_{\epsilon_2} \dots x_{\epsilon_k},$$

where $\Phi_{\text{in}(i)}$ denotes the set of incoming neighbor of node i.

2.2. Problem formulation

This section introduces the model of PBMCNs, and converts the network to an algebraic form using the STP approach.

On the premise of the PBMCNs with J layers and N nodes in each layer, the total number of different nodes is n ($N \leq n \leq NJ$), which is described as

$$\begin{cases} X(t+1) = f^{\theta(t)}(X(t), U(t)), \\ \widetilde{X}(t+1) = \widetilde{f}^{\theta(t)}(X(t), U(t)), \\ Y(t+1) = h(\widetilde{X}(t+1)), \end{cases} \tag{1}$$

where $X(t) = [X_1(t), X_2(t), \ldots, X_{JN}(t)]^T \in \mathcal{D}^{JN}$, $\widetilde{X}(t) = [\widetilde{X}_1(t), \widetilde{X}_2(t), \ldots, \widetilde{X}_n(t)]^T \in \mathcal{D}^n$, $Y(t) = [Y_1(t), Y_2(t), \ldots, Y_n(t)]^T \in D^n$, and $U(t) = [U_1(t), U_2(t), \ldots, U_m(t)]^T \in \mathcal{D}^m$ are logical vectors, and X_i, $\widetilde{X}_i$,Y_j,$U_k \in \mathcal{D}$ represent the state of $i-$th node, $i-$th global state node, $j-$th output node, and $k-$th control input, respectively. Assume that $\widetilde{X}(0) = \widetilde{X}(1)$ and $y(0) = y(1)$. The stochastic switching signal is denoted by $\theta(t) \in [1{:}\xi]$, where ξ denotes the number of candidates for Boolean multiplex control networks. Furthermore, for any $\theta(t) = q \in [1 : \xi]$, f^q: $\mathcal{D}^{JN+m} \to \mathcal{D}^{JN}$ and $\widetilde{f}^q$: $\mathcal{D}^{JN+m} \to \mathcal{D}^n$ denote the logical functions of the qth overall and global subnetwork of PBMCNs, respectively. In addition, $\widetilde{f}^q$ also known as the canalizing function, and $h : \mathcal{D}^n \to \mathcal{D}^n$ denotes the output function.

Remark 2.6. The global state $\widetilde{X}(t+1)$ of the network (1) was proposed by Wu et al. [41], whose purpose is to describe the state of the network from the perspective of systems biology, where $\widetilde{X}(t+1) = \ltimes_{i=1}^{n} \widetilde{X}_i(t+1)$, and $\widetilde{X}_i(t+1)$ is written as

$$\widetilde{X}_i(t+1) = \widetilde{f}_i(X_i^{l_{i_1}}(t), X_i^{l_{i_2}}(t), \ldots, X_i^{l_{i_s}}(t), U_1(t), U_2(t), \ldots, U_m(t)). \tag{2}$$

Here, the state of node i is represented as $x_i^{l_{i_s}}$ when the node i appears in the $l_{i_s}-$th layer, and $\{l_{i_1}, l_{i_2}, \ldots, l_{i_s}\} \subseteq \{1, 2, \ldots, J\}$.

Next, let $x(t) = \ltimes_{l=1}^{J} x^l(t)$ represents the vector form of $X(t)$, where $x^l(t) = \ltimes_{a_{il}=1} x_i^l(t)$ represents the overall state of the lth layer, and $a_{il} = 1$ if node i in the lth layer, x_i^l denotes the state of node i in the lth layer. $\widetilde{x}(t) = \ltimes_{i=1}^{n} \widetilde{x}_i(t)$, $y(t) = \ltimes_{j=1}^{n} y_j(t)$ and $u(t) = \ltimes_{k=1}^{m} u_k(t)$ represent the vector forms of $\widetilde{X}(t)$, $Y(t)$ and $U(t)$, respectively. Then, according to Lemma 2.3, PBMCNs (1) can be rewritten as

$$\begin{cases} x(t+1) = L_{\theta(t)} u(t) x(t), \\ \widetilde{x}(t+1) = \widetilde{L}_{\theta(t)} u(t) x(t), \\ y(t+1) = H \widetilde{x}(t+1), \end{cases} \tag{3}$$

where $x(t) \in \Delta_{2^{NJ}}$, $\widetilde{x}(t) \in \Delta_{2^n}$, $y(t) \in \Delta_{2^n}$, $u(t) \in \Delta_{2^m}$, $L_j \in \mathcal{L}_{2^{NJ} \times 2^{m+NJ}}, j = \theta(t) \in [1 : \xi]$ and $\widetilde{L}_j \in \mathcal{L}_{2^n \times 2^{m+NJ}}, j = \theta(t) \in [1 : \xi]$ denote the overall and global structure matrices of $j-$th sub-network, respectively. $H \in \mathcal{L}_{2^n \times 2^n}$ denotes the output structure matrix. Next, we use $\phi(t) := \delta_{\xi}^{\theta(t)} \in \Delta_{\xi}$ to denote the vector form of $\theta(t)$. As a result, the PBMCNs (3) can be expressed as

$$\begin{cases} x(t+1) = L\phi(t)u(t)x(t), \\ \widetilde{x}(t+1) = \widetilde{L}\phi(t)u(t)x(t), \\ y(t+1) = H\widetilde{x}(t+1), \end{cases} \tag{4}$$

where $L = [L_1, L_2, \dots, L_{\xi}]$, $\widetilde{L} = [\widetilde{L}_1, \widetilde{L}_2, \dots, \widetilde{L}_{\xi}]$. In this study, we apply the algebraic form (4) to show our method and results. The overall and global state trajectories of system under the stochastic switching signal $\phi(t)$ and control sequence $\mathbf{u} = \{u(t)\}$ are represented by $x(t; x_0, \phi, \mathbf{u})$ and $\widetilde{x}(t; x_0, \phi, \mathbf{u})$, respectively. Let $y(t; x_0, \phi, \mathbf{u}) = H\widetilde{x}(t; x_0, \phi, \mathbf{u})$ represents the corresponding output.

Assumed that $\theta(t)$ has the following probability distribution: $\Pr\{\theta(t) = i\} = p_i^{\theta}, i \in [1 : \xi]$, where $0 \le p_i^{\theta} \le 1$ and $\sum_{i=1}^{\xi} p_i^{\theta} = 1$. For simplicity, let $\mathrm{p}^{\theta} := [p_1^{\theta}, p_2^{\theta}, \dots, p_{\xi}^{\theta}]^T$. The vector p^{θ} is referred to as a probability distribution vector of $\theta(t)$. The one-step overall transition probability matrix (TPM) P and global TPM $\widetilde{P}$ can therefore be expressed as follows.

$$\begin{array}{l} P = \sum_{i=1}^{\xi} p_i^{\theta} L_i = L \ltimes \mathrm{p}^{\theta}, \\ \widetilde{P} = \sum_{i=1}^{\xi} p_i^{\theta} \widetilde{L}_i = \widetilde{L} \ltimes \mathrm{p}^{\theta}, \end{array}$$

Here, P and $\widetilde{P}$ can be expressed as $P = [P_1 \ P_2 \ \dots \ P_{2^m}]$ and $\widetilde{P} = [\widetilde{P}_1 \ \widetilde{P}_2 \ \dots \ \widetilde{P}_{2^m}]$, $P_k \in \mathbb{R}_{2^{JN} \times 2^{JN}}$ and $\widetilde{P}_k \in \mathbb{R}_{2^n \times 2^{JN}}$ denote the kth block of P and $\widetilde{P}$, respectively. Then, we have

$$p_{ij} = [P_k]_{ij} = \Pr\{x(t+1) = \delta_{2^{NJ}}^{i} \mid x(t) = \delta_{2^{NJ}}^{j}, u(t) = \delta_{2^m}^{k}\}$$

and

$$\widetilde{p}_{ij} = [\widetilde{P}_k]_{ij} = \Pr\{\widetilde{x}(t+1) = \delta_{2^n}^{i} \mid x(t) = \delta_{2^{NJ}}^{j}, u(t) = \delta_{2^m}^{k}\}.$$

Afterwards, $P(k)$ and $\widetilde{P}(k)$ denote the k-step overall and global TPM of PBMCNs (4), respectively. Consequently, it holds that $P(k) = (L \ltimes \mathrm{p}^{\theta})^k$, $\widetilde{P}(k) = (\widetilde{L} \ltimes \mathrm{p}^{\theta})^k$. In early work [9], the state transfer graph reconstruction technology was used to study the finite-time set reachability of PBMCNs (4). This method will transform the PBMCNs (4) into a random logic dynamical systems by the extended indicator matrices $\hat{\boldsymbol{D}}_{\mathcal{S}_d^c}$ and $\hat{\boldsymbol{D}}_{\widetilde{\mathcal{S}}_d^c}$, specific descriptions are as follows

$$\hat{\boldsymbol{D}}_{\mathcal{S}_d^c} := \begin{bmatrix} \boldsymbol{D}_{\mathcal{S}_d^c} & \mathbf{0}_{2^{JN} \times 2^{JN}} & \cdots & \mathbf{0}_{2^{JN} \times 2^{JN}} \\ \mathbf{0}_{2^{JN} \times 2^{JN}} & \boldsymbol{D}_{\mathcal{S}_d^c} & \cdots & \mathbf{0}_{2^{JN} \times 2^{JN}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{2^{JN} \times 2^{JN}} & \mathbf{0}_{2^{JN} \times 2^{JN}} & \cdots & \boldsymbol{D}_{\mathcal{S}_d^c} \end{bmatrix}_{2^{(m+JN)} \times 2^{(m+JN)}},$$

$$\hat{\boldsymbol{D}}_{\widetilde{\mathcal{S}}_d^c} := \begin{bmatrix} \boldsymbol{D}_{\widetilde{\mathcal{S}}_d^c} & \mathbf{0}_{2^n \times 2^n} & \cdots & \mathbf{0}_{2^n \times 2^n} \\ \mathbf{0}_{2^n \times 2^n} & \boldsymbol{D}_{\widetilde{\mathcal{S}}_d^c} & \cdots & \mathbf{0}_{2^n \times 2^n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{2^n \times 2^n} & \mathbf{0}_{2^n \times 2^n} & \cdots & \boldsymbol{D}_{\widetilde{\mathcal{S}}_d^c} \end{bmatrix}_{2^{(m+JN)} \times 2^{(m+JN)}},$$

where $\boldsymbol{D}_{\mathcal{S}_d^c}$ and $\boldsymbol{D}_{\widetilde{\mathcal{S}}_d^c}$ are defined as follows

$$Col_j(\boldsymbol{D}_{\mathcal{S}_d^c}) := \begin{cases} \delta_{2^{JN}}^j, & \text{if } \delta_{2^{JN}}^j \in \mathcal{S}_d^c, \\ \delta_{2^{JN}}^0, & \text{otherwise,} \end{cases}$$

$$Col_j(\boldsymbol{D}_{\widetilde{\mathcal{S}}_d^c}) := \begin{cases} \delta_{2^n}^j, & \text{if } \delta_{2^n}^j \in \widetilde{\mathcal{S}}_d^c, \\ \delta_{2^n}^0, & \text{otherwise,} \end{cases}$$

where $\mathcal{S}_d^c$ and $\widetilde{\mathcal{S}}_d^c$ are the complements of overall target subset $\mathcal{S}_d$ and global target subset $\widetilde{\mathcal{S}}_d$. As a result, the one-step overall TPM $\hat{P}$ and global TPM $\check{P}$ are denoted as

$$\begin{aligned} \hat{P} &= L \ltimes \mathrm{p}^\theta \ltimes \hat{\boldsymbol{D}}_{\mathcal{S}_d^c} = P\hat{\boldsymbol{D}}_{\mathcal{S}_d^c}, \\ \check{P} &= \widetilde{L} \ltimes \mathrm{p}^\theta \ltimes \hat{\boldsymbol{D}}_{\widetilde{\mathcal{S}}_d^c} = \widetilde{P}\hat{\boldsymbol{D}}_{\widetilde{\mathcal{S}}_d^c}. \end{aligned} \tag{5}$$

Afterwards, in [9], we define $\hat{Q}$ and $\check{Q}$ as η-step overall and global state TPMs, respectively.

$$\begin{aligned} \hat{Q} &= \big(\hat{P}W_{[2^{JN},2^m]}\big)^\eta W_{[2^{m\eta},2^{JN}]}, \\ \check{Q} &= \big(\check{P}W_{[2^{JN},2^m]}\big)\big(\hat{P}W_{[2^{JN},2^m]}\big)^{\eta-1} W_{[2^{m\eta},2^{JN}]}. \end{aligned} \tag{6}$$

The η-step overall and global TPM of random logic dynamical systems are splited up into $2^{m\eta}$ matrices, $\hat{Q}$ and $\check{Q}$, as follows

$$\begin{aligned} \hat{Q} &= [\hat{Q}_1, \hat{Q}_2, \dots, \hat{Q}_{2^{m\eta}}], \\ \check{Q} &= [\check{Q}_1, \check{Q}_2, \dots, \check{Q}_{2^{m\eta}}]. \end{aligned} \tag{7}$$

The symbol $\bigwedge\limits_s$ has the following definition:

$$\begin{cases} Col_j(\hat{Q}_q) \bigwedge\limits_s \delta_{2^{JN}}^0 = \delta_{2^{JN}}^0, \\ Col_j(\hat{Q}_q) \bigwedge\limits_s \delta_{2^{JN}}^k = Col_j(\hat{Q}_q), \quad k \in [1:2^{JN}]. \end{cases}$$

Thus, we have

$$\begin{cases} Col_j(\check{Q}_q) \bigwedge\limits_s \delta_{2^n}^0 = \delta_{2^n}^0, \\ Col_j(\check{Q}_q) \bigwedge\limits_s \delta_{2^n}^k = Col_j(\check{Q}_q), \quad k \in [1:2^n]. \end{cases}$$

Definition 2.7. For the PBMCNs (4), suppose that the initial subset $\mathcal{S}_0 \subseteq \Delta_{2^{NJ}}$ is the set of initial states, the overall target subset $\mathcal{S}_d \subseteq \Delta_{2^{NJ}}$ is the set of overall final states, and the global target subset $\widetilde{\mathcal{S}}_d \subseteq \Delta_{2^n}$ is the set of global final states.

(i) $\mathcal{S}_d$ is said to be overall set reachable with probability one from any initial state $x_0 \in \mathcal{S}_0$ on $[0:\eta-1]$ if, one can find a positive integer $t \geq 1$ and an input sequence $\mathbf{u}$ such that

$$\Pr\{\exists t \in [0:\eta-1], s.t.\ x(t; x_0, \phi, \mathbf{u}) \in \mathcal{S}_d\} = 1.$$

(ii) $\widetilde{\mathcal{S}}_d$ is said to be global set reachable with probability one from any initial state $x_0 \in \mathcal{S}_0$ on $[0:\eta-1]$ if, one can find a positive integer $t\geq 1$ and an input sequence $\mathbf{u}$ such that

$$\Pr\{\exists t \in [0:\eta-1], s.t.\ \widetilde{x}(t; x_0, \phi, \mathbf{u}) \in \widetilde{\mathcal{S}}_d\} = 1.$$

Theorem 2.8. [9] Considering PBMCNs (4), suppose that $\mathcal{S}_0 = \{\delta^j_{2^{JN}} \mid j \in \Theta_0\}$, where Θ_0 is a subset of $[1 : 2^{JN}]$. Then, the following two statements hold.

(i) $\mathcal{S}_d$ is overall set reachable with probability one from $\mathcal{S}_0$ on $[0 : \eta - 1]$ if and only if there exists an input sequence $\mathbf{u}$ such that

$$Col_j\left[\bigwedge_{s\ q=1}^{2^{m\eta}} \hat{Q}_q\right] = \delta^0_{2^{JN}}, \qquad \forall j \in \Theta_0.$$

(ii) $\widetilde{\mathcal{S}}_d$ is global set reachable with probability one from $\mathcal{S}_0$ on $[0 : \eta - 1]$ if and only if there exists an input sequence $\mathbf{u}$ such that

$$Col_j\left[\bigwedge_{s\ q=1}^{2^{m\eta}} \check{Q}_q\right] = \delta^0_{2^n}, \qquad \forall j \in \Theta_0.$$

Before giving the theorem of the finite-time observability of PBMCNs (4), we introduce several sigificant definitions.

Definition 2.9. State $x_0 \neq x'_0$ are distinguishable, if exists an input sequence $\mathbf{u}$ and an integer $t \geq 0$ such that $y(t; x_0, \phi, \mathbf{u}) \neq y(t; x'_0, \phi, \mathbf{u})$.

Definition 2.10. [41] The PBMCNs (4) is considered to be observable with probability one on $[0 : \eta]$ if there exists an input sequence $\mathbf{u}$, such that for any two distinct initial states $x_0, x'_0 \in \Delta_{2^{JN}}$, we have $\Pr\{y(\eta; x_0, \phi, \mathbf{u}) \neq y(\eta; x'_0, \phi, \mathbf{u})\} = 1$.

Remark 2.11. Different from the definition of observability of PBCNs in [33], the output y of the PBMCNs (4) in this paper is related to the global state $\widetilde{x}$, i.e. $y = h(\widetilde{x})$ rather than $y = h(x)$ as defined in [33]. Meanwhile, although the output state $y = h(\widetilde{x})$ can be determined by the global state, conversely, the network structure of a PBMCNs with global state layer cannot be obtained. Therefore, it is meaningful to study PBMCNs with global state layers.

3. FINITE-TIME OBSERVABILITY WITH PROBABILITY ONE

The finite-time observability of PBMCNs is studied in this section. First, the finite-time observability of PBMCNs is equivalently converted into the finite-time set reachability of an augmented interconnected PBMCNs. Subsequently, we interconnect PBMCNs (4) with a duplicate

$$\begin{cases} x'(t+1) = L\phi(t)u(t)x'(t), \\ \widetilde{x}'(t+1) = \widetilde{L}\phi(t)u(t)x'(t), \\ y'(t+1) = H\widetilde{x}'(t+1), \end{cases} \tag{8}$$

in parallel. By this means, the observability issue can be transformed into a set reachability issue of the interconnected PBMCNs. Afterwards, let $\tau(t) := x(t) \ltimes x'(t)$, $\widetilde{\tau}(t) := \widetilde{x}(t) \ltimes \widetilde{x}'(t)$, $\kappa(t) = y(t) \ltimes y'(t)$ and $\phi_\tau(t) := \phi(t)$, then the state-space of the interconnected PBMCNs is expressed as

$$\begin{cases} \tau(t+1) = L_\tau \phi_\tau(t) u(t) \tau(t), \\ \widetilde{\tau}(t+1) = \widetilde{L}_\tau \phi_\tau(t) u(t) \tau(t), \\ \kappa(t+1) = \Gamma \widetilde{\tau}(t+1). \end{cases} \tag{9}$$

For convenience, let $W_1 = W_{[2^{JN},(2^{JN+m})\times\xi]}$, $W_2 = W_{[(2^{JN+m})\times\xi,(2^{JN+m})\times\xi]}$, $W_3 = W_{[(2^m)\times\xi,(2^{JN+m})\times\xi]}$ and $W_4 = W_{[2^n,(2^{JN+m})\times\xi]}$. Then, L_τ, $\widetilde{L}_\tau$ and Γ in (9) can be expressed as

$$\begin{aligned} L_\tau &= L W_1 L W_2 W_3 M_{r,(2^m)\times\xi}, \\ \widetilde{L}_\tau &= \widetilde{L} W_4 \widetilde{L} W_2 W_3 M_{r,(2^m)\times\xi}, \\ \Gamma &= H W_{[2^n,2^n]} H W_{[2^n,2^n]}. \end{aligned}$$

Notably, for any $\tau_0 = x_0 \ltimes x_0'$, we have $\tau(t;\tau_0,\phi_\tau,\mathbf{u}) = x(t;x_0,\phi,\mathbf{u}) \ltimes x(t;x_0',\phi,\mathbf{u})$, and $\widetilde{\tau}(t;\tau_0,\phi_\tau,\mathbf{u}) = \widetilde{x}(t;x_0,\ \phi,\mathbf{u}) \ltimes \widetilde{x}(t;x_0',\phi,\mathbf{u})$, where $x(t;x_0',\phi,\mathbf{u})$ and $\widetilde{x}(t;x_0',\phi,\mathbf{u})$ denote the overall and global solutions to PBMCNs (8) starting from x_0', respectively. $\tau(t;\tau_0,\phi_\tau,\mathbf{u})$ and $\widetilde{\tau}(t;\tau_0,\phi_\tau,\mathbf{u})$ denote the overall and global solutions to PBMCNs (9) starting from τ_0, respectively. The probability distribution vector of $\phi_\tau(t)$ is also p^θ since $\phi(t) := \phi_\tau(t)$, and the one-step overall and global TPMs of PBMCNs (9) can be calculated by

$$\begin{aligned} P^\tau &= L_\tau \ltimes \mathrm{p}^\theta, \\ \widetilde{P}^\tau &= \widetilde{L}_\tau \ltimes \mathrm{p}^\theta. \end{aligned}$$

For logical matrix $H \in \Delta_{2^n\times 2^n}$, we define the H-distinguishable subset as follows:

$$\Omega_H := \{\delta_{2^{2n}}^i = x \ltimes x' | H\widetilde{P}x \neq H\widetilde{P}x'\}.$$

Meanwhile, we let $\Omega_I = \{x \ltimes x' | x \neq x'\}$. Further, the H-indistinguishable subset $\Omega_{I\setminus H}$ is defined as follows:

$$\Omega_{I\setminus H} = \{\delta_{2^{JN}}^i \ltimes \delta_{2^{JN}}^j | Col_i(H\widetilde{P}) = Col_j(H\widetilde{P}) \text{ and } i<j\}.$$

In other words, every state $\tau \in \Omega_{I\setminus H}$ corresponds to a state pair (x_0, x_0') of PBMCNs (4) that satisfies $H\widetilde{x}_0 = H\widetilde{x}_0'$, that is $H\widetilde{x}(1;x_0,\phi,\mathbf{u}) = H\widetilde{x}(1;x_0',\phi,\mathbf{u})$.

Theorem 3.1. Assume that the PBMCNs (4) has two distinct initial states $x_0, x_0' \in \Delta_{2^{JN}}$ and an input sequence $\mathbf{u}$. Then, x_0 and x_0' are distinguishable by if and only if one of the following conditions holds:

(i) $x_0 \ltimes x_0' \in \Omega_H$.

(ii) $x_0 \ltimes x_0' \in \Omega_{I\setminus H}$ and there exists a positive integer $t \geq 1$ such that $\widetilde{\tau}(t;x_0 \ltimes x_0',\phi_\tau,\mathbf{u}) \in \Omega_H$.

P r o o f. (Sufficiency) According to the definition of Ω_H, $x_0 \ltimes x_0' \in \Omega_H$ is equivalent to $H\widetilde{P}x_0 \neq H\widetilde{P}x_0'$, that is $y(1) \neq y(1)'$. According to the Definition 2.10, x_0 and x_0' are distinguishable by $\mathbf{u}$. Besides, if $x_0 \ltimes x_0' \in \Omega_{I\setminus H}$, and there exists a positive integer $t \geq 1$ such that $\widetilde{\tau}(t; x_0 \ltimes x_0', \phi_\tau, \mathbf{u}) \in \Omega_H$, it means that there exists a positive integer $t \geq 1$ such that $H\widetilde{x}(t; x_0, \phi, \mathbf{u}) \neq H\widetilde{x}(t; x_0', \phi, \mathbf{u})$. Then, by the Definition 2.10, x_0 and x_0' are distinguishable by $\mathbf{u}$.

(Necessity) According to Definition 2.10, one have x_0 and x_0' are distinguishable by $\mathbf{u}$. That is to say, $y(t; x_0, \phi, \mathbf{u}) \neq y(t; x_0', \phi, \mathbf{u})$ at some later instant $t \geq 0$. Then, one have $y(0) = y(1) = H\widetilde{x}(1; x_0, \phi, \mathbf{u}) \neq H\widetilde{x}(1; x_0', \phi, \mathbf{u}) = y(1)' = y(0)'$. According to the definition of H-indistinguishable subset Ω_H, it can be concluded that $x_0 \ltimes x_0' \in \Omega_H$. By the same token, when $x_0 \ltimes x_0' \in \Omega_{I\setminus H}$ and $y(t; x_0, \phi, \mathbf{u}) \neq y(t; x_0', \phi, \mathbf{u})$ for $t \geq 1$, this means that there exists a positive integer $t \geq 1$ such that $\widetilde{\tau}(t; x_0 \ltimes x_0', \phi_\tau, \mathbf{u}) \in \Omega_H$. □

According to Theorem 2.8, the observability of PBMCNs (4) is now transformed into the issue of judging whether the solution of PBMCNs (9) starting from $\tau_0 = x_0 \ltimes x_0' \in \Omega_{I\setminus H}$ can reach the Ω_H at some instant $t > 0$. Based on this, inspired by the previous work [9], the following theorem is presented.

Theorem 3.2. Considering PBMCNs (4) and interconnected PBMCNs (9).

(i) PBMCNs (4) is observable with probability one on $[0 : \eta - 1]$ if and only if the subset Ω_H is global set reachable with probability one from $\Omega_{I\setminus H}$ on $[0 : \eta - 1]$ for the interconnected PBMCNs (9).

(ii) Assume that $\Omega_H = \{\delta_{2^{2n}}^i | i \in \Theta_H\}$ and $\Omega_{I\setminus H} = \{\delta_{2^{2JN}}^j | j \in \Theta_{I\setminus H}\}$, where Θ_H is a subset of $[1 : 2^{2n}]$ and $\Theta_{I\setminus H}$ is a subset of $[1 : 2^{2JN}]$. Then, PBMCNs (4) is finite-time observable with probability one on $[0 : \eta - 1]$ iff

$$Col_j\left[\bigwedge_{s\ q=1}^{2^{m\eta}} \check{Q}_q\right] = \delta_{2^{2n}}^0, \qquad \forall j \in \Theta_{I\setminus H}.$$

P r o o f. First, we prove the sufficiency of (i). Suppose that the subset Ω_H is global set reachable with probability one from $\Omega_{I\setminus H}$ on $[0 : \eta - 1]$ for the interconnected PBMCNs (9), then for any $\tau_0 \in \Omega_{I\setminus H}$, one have

$$\Pr\{\exists t \in [0 : \eta - 1],\ s.t.\ \widetilde{\tau}(t; \tau_0, \phi_\tau, \mathbf{u}) \in \Omega_H\} = 1.$$

That is to say

$$\begin{aligned}
&\Pr\{\exists t \in [0 : \eta - 1], s.t.\ \widetilde{\tau}(t; \tau_0, \phi_\tau, \mathbf{u}) \in \Omega_H\} \\
&= \Pr\{\{\widetilde{\tau}_0 \in \Omega_H\} \cup \{\widetilde{\tau}(1; \tau_0, \phi_\tau, \mathbf{u}) \in \Omega_H\} \cup \ldots \cup \\
&\quad \{\widetilde{\tau}(\eta - 1; \tau_0, \phi_\tau, \mathbf{u}) \in \Omega_H\}\} \\
&= \Pr\{\textstyle\bigcup_{t=0}^{\eta-1}\{\widetilde{x}(t; x_0, \phi, \mathbf{u}) \ltimes \widetilde{x}(t; x_0', \phi, \mathbf{u}) \in \Omega_H\}\} \\
&= \Pr\{\textstyle\bigcup_{t=0}^{\eta-1}\{H\widetilde{x}(t; x_0, \phi, \mathbf{u}) \neq H\widetilde{x}(t; x_0', \phi, \mathbf{u})\}\} \\
&= \Pr\{H\widetilde{x}(\eta - 1; x_0, \phi, \mathbf{u}) \neq H\widetilde{x}(\eta - 1; x_0', \phi, \mathbf{u})\} \\
&= \Pr\{y(\eta - 1; x_0, \phi, \mathbf{u}) \neq y(\eta - 1; x_0', \phi, \mathbf{u})\} = 1.
\end{aligned}$$

Thus, according to the Definition 2.10, the sufficiency holds.

Next, we prove the necessity of (i). For any given states $x_0, x_0' \in \Delta_{2^{JN}}$, we can get

$$\begin{aligned}
&\Pr\{y(\eta-1;x_0,\phi,\mathbf{u}) \neq y(\eta-1;x_0',\phi,\mathbf{u})\}\\
=&\Pr\{H\widetilde{x}(\eta-1;x_0,\phi,\mathbf{u})\neq H\widetilde{x}(\eta-1;x_0',\phi,\mathbf{u})\}\\
=&\Pr\{\textstyle\bigcup_{t=0}^{\eta-1}\{H\widetilde{x}(t;x_0,\phi,\mathbf{u})\neq H\widetilde{x}(t;x_0',\phi,\mathbf{u})\}\}\\
=&\Pr\{\textstyle\bigcup_{t=0}^{\eta-1}\{\widetilde{x}(t;x_0,\phi,\mathbf{u})\ltimes\widetilde{x}(t;x_0',\phi,\mathbf{u})\in\Omega_H\}\}\\
=&\Pr\{\{\widetilde{\tau}_0\in\Omega_H\}\cup\{\widetilde{\tau}(1;\tau_0,\phi_\tau,\mathbf{u})\in\Omega_H\}\cup\ldots\cup\\
&\{\widetilde{\tau}(\eta-1;\tau_0,\phi_\tau,\mathbf{u})\in\Omega_H\}\}\\
=&\Pr\{\exists t\in[0:\eta-1], s.t.\ \widetilde{\tau}(t;\tau_0,\phi_\tau,\mathbf{u})\in\Omega_H\}.
\end{aligned} \tag{10}$$

Thereby, by (10), it can be easily proved that, for any $\tau_0 \in \Omega_{I\setminus H}$, (8) can be equivalently expressed as

$$\Pr\{\exists t \in [0:\eta-1],\ s.t.\ \widetilde{\tau}(t;\tau_0,\phi_\tau,\mathbf{u}) \in \Omega_H\} = 1.$$

Thus, the necessity holds.

Finally, we prove the conclusion (ii). Based on the conclusion (i), one has PBMCNs (4) can achieve the finite-time observability with probability one, if and only if, the subset Ω_H is global set reachable with probability one from $\Omega_{I\setminus H}$ on $[0:\eta-1]$ for the interconnected PBMCNs (9). Thereby, the claim follows claim (ii) of Theorem 2.8. □

Remark 3.3. Compared with [8], the network model studied in this paper is more complex and can be used to model gene regulatory networks with random uncertainties.

4. EXAMPLE

Example 4.1. Consider a PBMCNs with 3 layers, each layer has 1 nodes, and the total number of distinct nodes is 3, which is described as

$$l=1: x_1^1(t+1) = \begin{cases} x_1^1(t)\wedge u(t),\\ \neg x_1^1(t),\end{cases}$$

$$l=2: x_2^2(t+1) = \neg x_2^2(t) \leftrightarrow u(t),$$

$$l=3: x_3^3(t+1) = \begin{cases} \neg x_3^3(t),\\ 0,\\ x_3^3(t)\vee u(t).\end{cases}$$

The global state layer is described as

$$\begin{cases} \widetilde{x}_1(t+1) = \neg x_1^1(t),\\ \widetilde{x}_2(t+1) = x_2^2(t),\\ \widetilde{x}_3(t+1) = x_3^3(t).\end{cases}$$

Then, we can obtain the global TPM

$$\widetilde{P} = \left[\begin{array}{cccccccccccccccc}
0 & 0 & 0 & 0 & 0.15 & 0.4 & 0.15 & 0.25 & 0.15 & 0.4 & 0.15 & 0.25 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.35 & 0.1 & 0.35 & 0.25 & 0.35 & 0.1 & 0.35 & 0.25 & 0 & 0 & 0 & 0 \\
0.15 & 0.4 & 0.15 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.15 & 0.4 & 0.15 & 0.25 \\
0.35 & 0.1 & 0.35 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.35 & 0.1 & 0.35 & 0.25 \\
0 & 0 & 0 & 0 & 0.15 & 0.4 & 0.15 & 0.25 & 0.15 & 0.4 & 0.15 & 0.25 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.35 & 0.1 & 0.35 & 0.25 & 0.35 & 0.1 & 0.35 & 0.25 & 0 & 0 & 0 & 0 \\
0.15 & 0.4 & 0.15 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.15 & 0.4 & 0.15 & 0.25 \\
0.35 & 0.1 & 0.35 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.35 & 0.1 & 0.35 & 0.25
\end{array}\right].$$

and the overall TPM

$$P = \left[\begin{array}{cccccccccccccccc}
0 & 0 & 0.15 & 0.4 & 0 & 0 & 0.15 & 0.4 & 0 & 0 & 0 & 0 & 0.15 & 0.25 & 0 & 0 \\
0 & 0 & 0.35 & 0.1 & 0 & 0 & 0.35 & 0.1 & 0 & 0 & 0 & 0 & 0.35 & 0.25 & 0 & 0 \\
0.15 & 0.4 & 0 & 0 & 0.15 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.15 & 0.25 \\
0.35 & 0.1 & 0 & 0 & 0.35 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.35 & 0.25 \\
0 & 0 & 0.15 & 0.4 & 0 & 0 & 0.15 & 0.4 & 0.3 & 0.5 & 0 & 0 & 0.15 & 0.25 & 0 & 0 \\
0 & 0 & 0.35 & 0.1 & 0 & 0 & 0.35 & 0.1 & 0.7 & 0.5 & 0 & 0 & 0.35 & 0.25 & 0 & 0 \\
0.15 & 0.4 & 0 & 0 & 0.15 & 0.4 & 0 & 0 & 0 & 0 & 0.3 & 0.5 & 0 & 0 & 0.15 & 0.25 \\
0.35 & 0.1 & 0 & 0 & 0.35 & 0.1 & 0 & 0 & 0 & 0 & 0.7 & 0.5 & 0 & 0 & 0.35 & 0.25
\end{array}\right].$$

On the premise of PBMCNs (4), we can obtain $\Omega_{I\setminus H}$ and Ω_H of the PBMCNs (9) as

$$\Omega_{I\setminus H} = \{\delta_{64}^{3}, \delta_{64}^{39}\},$$

$$\Omega_H = \{\delta_{64}^{2}, \delta_{64}^{4}, \delta_{64}^{5}, \delta_{64}^{6}, \delta_{64}^{7}, \delta_{64}^{8}, \delta_{64}^{9}, \delta_{64}^{11}, \ldots, \delta_{64}^{61}, \delta_{64}^{62}, \delta_{64}^{63}\}.$$

Then, we are able to obtain

$$Col_j\left[\bigwedge_{s\ q=1}^{2^2} \check{Q}_q\right] = \delta_{64}^{0}, \qquad \forall j \in \Theta_{I\setminus H}.$$

Therefore, Ω_H is global reachable with probability one from $\Omega_{I\setminus H}$ on $[0,2]$. That is to say, the PBMCNs (4) is finite-time observable.

Example 4.2. We consider using the PBMCNs to simulate a simplified DELI fresh supermarket chain to illustrate our method and conclusion. The model consists of four consitituent BMCNs

$$f^1 = \begin{cases} l = 1 : (x_1^1(t) \wedge u(t), x_2^1(t), x_1^1(t) \wedge x_2^1(t)), \\ l = 2 : (x_1^2(t), x_2^2(t) \wedge u(t), x_1^2(t) \wedge x_2^2(t)), \end{cases}$$

$$f^2 = \begin{cases} l = 1 : (x_1^1(t) \wedge u(t), x_2^1(t), x_1^1(t) \wedge x_2^1(t)), \\ l = 2 : (x_1^2(t), 0, x_1^2(t) \wedge x_2^2(t)), \end{cases}$$

$$f^3 = \begin{cases} l = 1 := (x_1^1(t) \wedge u(t), 0, x_1^1(t) \wedge x_2^1(t)), \\ l = 2 : (x_1^2(t), x_2^2(t) \wedge u(t), x_1^2(t) \wedge x_2^2(t)), \end{cases}$$

$$f^4 = \begin{cases} l = 1 := (x_1^1(t) \wedge u(t), 0, x_1^1(t) \wedge x_2^1(t)), \\ l = 2 : (x_1^2(t), 0, x_1^2(t) \wedge x_2^2(t)), \end{cases}$$

with the probability distribution vector $\mathrm{p}^\theta := [0.36,\ 0.24,\ 0.24,\ 0.16]^T$.

Here, $l = 1$ and $l = 2$ represent the DELI supermarket of the northeast and central regions, respectively. x_1^1, x_1^2, x_2^1, x_2^2, and x_3^1, x_3^2 represent the fresh quality, price advantage, favorable rate in the northeast China region and the central region, respectively. u_1 represents the path loss caused by fresh food during transportation, and the global state layer is defined as $\widetilde{x}_1(t+1), \widetilde{x}_2(t+1), \widetilde{x}_3(t+1)$, as follows

$$\begin{cases} \widetilde{x}_1(t+1) = x_1^1(t+1) \wedge x_1^2(t+1), \\ \widetilde{x}_2(t+1) = x_2^1(t+1) \vee x_2^2(t+1), \\ \widetilde{x}_3(t+1) = x_3^1(t+1) \vee x_3^2(t+1). \end{cases}$$

In the network, y_1 and y_2 represent profit growth and brand image. Thus, the output state layer is described as

$$\begin{cases} y_1(t+1) = \widetilde{x}_2(t+1), \\ y_2(t+1) = \widetilde{x}_1(t+1) \vee \widetilde{x}_3(t+1), \end{cases}$$

where profit growth y_1 is affected by price advantage, brand image y_2 is affected by the combination of fresh quality and favorable rate.

Then, using the STP method, the two-layer PBCNs can be converted into the corresponding algebraic form. According to the definition of H-distinguishable subset and H-indistinguishable subset. We can get

$$\Omega_{I\setminus H} = \{\delta_{4096}^2, \delta_{4096}^5, \delta_{4096}^6, \delta_{4096}^9, \delta_{4096}^{10}, \ldots, \delta_{4096}^{4092}, \delta_{4096}^{4095}\}$$

and

$$\Omega_H = \{\delta_{4096}^3, \delta_{4096}^4, \delta_{4096}^7, \delta_{4096}^8, \delta_{4096}^{11}, \ldots, \delta_{4096}^{4093}, \delta_{4096}^{4094}\}.$$

Then, according to the Theorem 3.1, we may conclude that Ω_H is not global reachable with probability one from $\Omega_{I\setminus H}$. That is to say, the large supermarket chains network is unobservable.

5. CONCLUSIONS

In this paper, the finite-time observability of PBMCNs was investigated by STP method. Through parallel interconnection technology, the finite-time observability of PBMCNs can be transformed into the finite-time set reachability of an augmented interconnected PBMCNs. On this basis, the theorem of finite-time observability was given. In addition, two examples were given to illustrate the effectiveness of the results obtained in this paper.

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