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A NEW APPROACH FOR FUZZY GYRONORMS ON GYROGROUPS AND ITS FUZZY TOPOLOGIES

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In this paper, a new approach for fuzzy gyronorms on gyrogroups is presented. The relations between fuzzy metrics(in the sense of Morsi), fuzzy gyronorms, gyronorms on gyrogroups are studied. Also, some sufficient conditions, which can make a fuzzy normed gyrogroup to be a topological gyrogroup and a fuzzy topological gyrogroup, are found. Meanwhile, the relations between topological gyrogroups, fuzzy topological gyrogroups and stratified fuzzy topological gyrogroups are studied. Finally, the properties of fuzzifying topological gyrogroups are studied.

Keywords: fuzzy metric, fuzzy gyronorm, gyrogroup, topological gyrogroup, fuzzy topological gyrogroup, fuzzifying topological gyrogroup

Classification: 47H10, 58E30

1. INTRODUCTION

In [13], Ungar introduced the notion of a gyrogroups, arising from the parameterization of the Lorentz transformation group. This leads to the formation of gyrogroup theory, a rich subject in mathematics. Loosely speaking, a gyrogroup is a group-like structure in which the associative law fails to satisfy. It turns out that gyrogroups share remarkable analogies with groups. Suksumran extended several well-known results in group theory to the case of gyrogroups such as the Lagrange theorem [10], the fundamental isomorphism theorems, the Cayley theorem [11], the orbit-stabilizer theorem, the class equation, and the Burnside lemma [12]. From the topological aspect, Atiponrat [2] introduced the notion of topological gyrogroups, in spite of having a weaker algebraic form, topological gyrogroups carry almost the same basic properties owned by topological groups. After this, Cai et al. [4] extened the famous Birkhoff-Kakutani theorem by proving that every first-countable Hausdorff topological gyrogroup is metrizable.

In fuzzy topological algebra, the combination of fuzzy metric structure and algebraic structure is a noteworthy subject. Fuzzy normed spaces and fuzzy metric spaces are the most frequently studied structures. Several scholars studied fuzzy metrics on group. They find some sufficient conditions to make nonsymmetric topological algebraic structure become stronger topological structure. In particular, Salvador Romaguera [7] introduced the notion of fuzzy metric groups and investigated properties of the quotient

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subgroups of a fuzzy metric group. Sánchez and Sanchis [8] proved that the completion of a fuzzy metric group $(G, M, *)$ such that $(M, *)$ is invariant on G. Recently, Suksumran [9, 10] introduced the normed gyrogroup and proved that the normed gyrogroups are homogeneous. Also, Suksumran form left invariant metric spaces and derive a version of the Mazur-Ulam theorem. Specially, Suksumran gave certain sufficient conditions, involving the right-gyrotranslation inequality and Klee's condition, for a normed gyrogroup to be a topological gyrogroup. Base the notion of fuzzy norm which is introduced by Bag and Samanta [3], Xie [16] introduced the notion of fuzzy gyronorms on gyrpgroup and discussed the fuzzy metric structures on fuzzy normed gyrogroups in 2022. Also, some sufficient conditions, which make a fuzzy normed gyrogroup to be a topological gyrogroup, were found.

As we have known, the fuzzy norm in the sense of Morsi is also a very widely used norm, especially in the field of fuzzy topology. For example, Yan [17] discussed the fuzzifying topological structure in Morsi's normed spaces. Influenced by [16] and [17], we intend to give a new approach for fuzzy gyronorms on gyrogroups in this paper, some properties of fuzzy normed gyrogroups are studied. Also, the relations between fuzzy normed gyrogroups and normed gyrogroups, fuzzy metrics(in the sence of Morsi) and fuzzy gyronorms on gyrogroups are studied. Then we find the conditions, which make fuzzy normed gyrogroup to be fuzzy topological gyrogroup and fuzzifying topological gyrogroup. The paper is organized as follows. In section 2, some basic facts and definitions are stated. In section 3, we mainly show the relations between a family of ordinary gyronorms and a fuzzy gyronorm. Section 4 is devoted to study the relations between fuzzy metrics in the sense of Morsi and fuzzy gyronorms on gyrogroups. In section 5, It is shown that some sufficient conditions can make a fuzzy normed gyrogroup to be a topological gyrogroup and a fuzzy topological gyrogroup. In addition, the relations between topological gyrogroups, fuzzy topological gyrogroups and stratified fuzzy topological gyrogroups are discussed. In section 6, we study the properties of fuzzifying topological gyrogroups.

2. PRELIMINARIES

Throughout this paper, $I = [0, 1]$, the family of all fuzzy sets on X is denoted by I^X. Recall that a binary operation $S : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous s-norm if S satisfies the following conditions:

(i) S is associative and commutative;

(ii) S is continuous;

(iii) $S(x, 0) = x, \forall x \in [0, 1]$;

(iv) $S(a, b) = S(c, d)$ whenever $a \leq c, b \leq d$, with $a, b, c, d \in [0, 1]$.

The basic s-norms may be as follows:

(1) $S_M(a, b) = \max\{a, b\}$;

(2) $S_P(a, b) = a + b - a \cdot b$;

(3) $S_L(a, b) = \min\{a + b, 1\}$.

The s-norm is said to be strictly monotone if $S(a_1, b_1) < S(a, b)$ for all $a_1 < a, b_1 < b$. The example of strictly monotone and continuous s-norm is max. One can easily show that $S \geq max$ for every continuous s-norm S.

Definition 2.1. (Morsi [6]) Let η be a nonascending function $\mathbb{R} \to I$. The notations $\eta(b-), \eta(b+)$ denote the limit of η from the left at b and the limit of η from the right at b for all $b \in \mathbb{R}$, respectively. An equivalence relation $\sim$ is defined on the collection of nonascending functions $\mathbb{R} \to I$ as follows: For two such functions η and ζ, $\eta \sim \zeta$ iff for all $b \in \mathbb{R}$, $\eta(b-) = \zeta(b-)$ and $\eta(b+) = \zeta(b+)$.

A fuzzy real number is an equivalence class, under the above equivalence relation $\sim$, of nonascending left continuous functions $\eta : \mathbb{R} \to I$ with $\eta(-\infty+) = 1$ and $\eta(+\infty-) = 0$. The set of all fuzzy real numbers is denoted by $\mathbb{R}(I)$.

We shall not distinguish between a fuzzy real number and any of its representative functions $\mathbb{R} \to I$.

Definition 2.2. (Morsi [6])

(1) The relation smaller than or equal to $\preceq$ is defined on $\mathbb{R}(I)$ by $\eta \preceq \zeta$ iff $\eta(t) \leq \zeta(t)$ for all $t \in \mathbb{R}$.

(2) A fuzzy real number η is said to be positive if $\eta(r) = 1$ for some real $r > 0$, and $\mathbb{R}^+(I)$ is the collection of all positive fuzzy real numbers.

(3) $\mathbb{R}^*(I)$ is the collection of all fuzzy real numbers η with $\eta(0) = 1$.

For every $r \in \mathbb{R}$, the fuzzy real number $\widetilde{r} : \mathbb{R} \to I$ is given as follows:

$$\widetilde{r}(s) = \begin{cases} 1, & s \leq r, \\ 0, & s > r. \end{cases}$$

Definition 2.3. (Morsi [6]) Let $\eta, \xi \in \mathbb{R}(I)$, $\eta + \xi$ is the fuzzy real number defined by $(\eta + \xi)(t) = \bigvee_{r+s=t} (\eta(r) \wedge \xi(s))$ for all $t \in \mathbb{R}$.

Lemma 2.4. (Morsi [6]) Let $\eta, \xi \in \mathbb{R}(I)$ and $r \in \mathbb{R}$, then $(\eta + \xi)(t) = \bigwedge_{r+s=t} (\eta(r) \vee \xi(s))$.

Definition 2.5. (Morsi [6]) For every $b \in \mathbb{R}$, the fuzzy subsets R_b and L_b of $\mathbb{R}(I)$ are defined as follows: for all $\eta \in \mathbb{R}(I)$,

$$R_b(\eta) = \eta(b+) \text{ and } L_b(\eta) = 1 - \eta(b).$$

Definition 2.6. (Morsi [6]) A fuzzy pseudo-metric on a nonempty set X is a mapping $d : X \times X \to \mathbb{R}^*(I)$ which satisfies: for all $x, y, z \in X$,

(FM1) $d(x, x) = \widetilde{0}$;

(FM2) $d(x, y) = d(y, x)$;

(FM3) $d(x, z) \preceq d(x, y) + d(y, z)$.

(X, d) is called a fuzzy pseudo-metric space. If, in addition, d satisfies:

(FM4) $x \neq y \Rightarrow d(x, y) \in \mathbb{R}^+(I)$.

Definition 2.7. (Morsi [6]) Let (X, d) be a fuzzy pseudo-metric space, $x \in X$, and $\eta \in \mathbb{R}^*(I) - \{\widetilde{0}\}$. The fuzzy open ball in (X, d) with center x and radius η is the fuzzy subset $B(x;\eta) \in I^X$ defined as follows: for all $y \in X$,

$$B(x;\eta)(y) = \sup\{R_s(\eta) \wedge L_s[d(x, y)] : s \in \mathbb{R}\}.$$

The notation $B(;)$ for fuzzy open balls will be maintained in the sequel.

Let $\underline{c} : X \to [0, 1]$ be a mapping defined by $\underline{c}(x) = c, \forall x \in X$, i.e, $\underline{c} \in I^X$ is a constant fuzzy set.

Definition 2.8. (Abdulkadir [1]) A fuzzy topological space is an ordered pair $(X, \mathscr{T})$ such that X is a set and $\mathscr{T} \subset I^X$ satisfies the following conditions

(i) $\underline{0}, \underline{1} \in \mathscr{T}$;

(ii) If $\lambda_1, \lambda_2 \in \mathscr{T}$, then $\lambda_1 \wedge \lambda_2 \in \mathscr{T}$;

(iii) If $\lambda_i \in \mathscr{T}$ for all $i \in J$, then $\bigvee_{i\in J} \lambda_i \in \mathscr{T}$.

In [5], Lowen proposed a more natural definition of fuzzy topology, called stratified fuzzy topology, since constant functions may not be open in general with Chang's definition.

Definition 2.9. (Lowen [5]) A stratified fuzzy topological space is an ordered pair $(X, \mathscr{T})$ such that X is a set and $\mathscr{T} \subset I^X$ satisfies the following conditions

(i) $\underline{c} \in \mathscr{T}$ for all $c \in [0, 1]$;

(ii) If $\lambda_1, \lambda_2 \in \mathscr{T}$, then $\lambda_1 \wedge \lambda_2 \in \mathscr{T}$;

(iii) If $\lambda_i \in \mathscr{T}$ for all $i \in J$, then $\bigvee_{i\in J} \lambda_i \in \mathscr{T}$.

Lowen [5] gave a relation between the category of classical topologies and the category of fuzzy topologies by introducing the following two mappings ω and ι.

Definition 2.10. (Lowen [5]) Let X be a nonempty set, τ be a topology on X. Define $\omega(\tau) = \{\lambda : \lambda \text{ is lower semicotinuous}\}$.

Proposition 2.11. (Lowen [5]) Let X be a nonempty set, τ be a topology on X. Then $\omega(\tau)$ is a stratified fuzzy topology on X.

Proposition 2.12. (Lowen [5]) Let $(X, \mathscr{T})$ be a fuzzy topological space and $0 \leq \alpha < 1$. The family $\iota_\alpha(\mathscr{T}) = \{[\lambda]^\alpha : \lambda \in \mathscr{T}\}$ is a topology on X, which is called the α-level topology of $\mathscr{T}$, where $[\lambda]^\alpha = \{x \in X : \lambda(x) > \alpha\}$. On the other hand, $\bigcup\{\iota_\alpha(\mathscr{T}) : \alpha \in [0, 1)\}$ is a subbase for the topology $\iota(\mathscr{T})$.

Let $\{\tau_\alpha : \alpha \in [0, 1)\}$ be a family of topologies on X. In order to guarantee the existence of at least one fuzzy topology $\mathscr{T}$ on X such that $\iota_\alpha(\mathscr{T}) = \tau_\alpha$ for all $\alpha \in [0, 1)$, a necessary and sufficient condition was given in [14, 15].

Proposition 2.13. (Pwuyts [14]) Let $\{\tau_\alpha : \alpha \in [0,1)\}$ be a family of topologies on X. Then the following are equivalent:

(a) There exists at least a fuzzy topology $\mathscr{T}$ on X such that $\forall \alpha \in [0,1) : \iota_\alpha(\mathscr{T}) = \tau_\alpha$;

(b) LT-property: $\forall \alpha \in [0,1), \forall G \in \tau_\alpha, \exists (G_\beta)_{\beta \in (\alpha,1)} \in \prod_{\beta \in (\alpha,1)} \tau_\beta$ descending and $G = \bigcup_{\beta \in (\alpha,1)} G_\beta$.

Proposition 2.14. (Pwuyts [15]) Let $\mathscr{F} = \{\tau_\alpha : \alpha \in [0,1)\}$ be a family of topologies on X. Then there exist a fuzzy topology $\mathscr{T}$ on X having $\mathscr{F}$ as their level topologies, I. e. such that

$$\iota_\alpha(\mathscr{T}) = \tau_\alpha, \forall \alpha \in [0,1) \qquad (\star)$$

if and only if $\mathscr{F}$ has the LT-property. Moreover, the fuzzy topology $\mathscr{T}(\mathscr{F}) = \{\lambda : [\lambda]^\alpha \in \tau_\alpha, \forall \alpha \in [0,1)\}$ is the finest of all stratified I-topologies on X if $(\star)$ holds.

Theorem 2.15. (Morsi [6]) Let (X, d) be a fuzzy pseudo-metric space. Let $x, y \in X$ and r be a positive real number. Then

$$B(x;\widetilde{r})(y) = L_r[d(x,y)] = 1 - d(x,y)(r).$$

Definition 2.16. (Morsi [6]) Let (X, d) be a fuzzy pseudo-metric space. The fuzzy topology on (X, d) is the fuzzy topology on X with subbase the collection of all fuzzy open balls in (X, d).

This fuzzy topology is also called the fuzzy topology associated with d. When d is a fuzzy metric, its associated fuzzy topology will be called a fuzzy metric topology.

Definition 2.17. (Morsi [6]) Let $\alpha \in [0,1)$. A fuzzy topological space (X, τ) is said to be

(i) $\alpha - T_0$ if for every $x \neq y$ in X there is $U \in \tau$ such that $U(x) > \alpha$ and $U(y) = 0$ or $U(y) > \alpha$ and $U(x) = 0$;

(ii) $\alpha - T_1$ if for every $x \neq y$ in X there is $U \in \tau$ such that $U(x) > \alpha$ and $U(y) = 0$;

(iii) $\alpha - T_2$ if for every $x \neq y$ in X there are disjoint $U, V \in \tau$ such that $U(x) \wedge V(y) > \alpha$.

Definition 2.18. (Ying [18]) Given a set X, a mapping $\mathcal{T} : 2^X \to [0,1]$ is called a fuzzifying topology on X if it satisfies the following axioms:

(FY1) $\mathcal{T}(\varnothing) = \mathcal{T}(X) = 1$;

(FY2) $\mathcal{T}(U \cap V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V), \forall U, V \in 2^X$;

(FY3) $\mathcal{T}(\bigcup_{j \in J} U_j) \geq \bigwedge_{j \in J} \mathcal{T}(U_j), \forall U_j \subseteq 2^X (j \in J)$.

Definition 2.19. (Zhang [19]) A fuzzifying neighborhood structure on a set X is a family of functions $P = \{p_x : 2^X \to [0,1] | x \in X\}$ with the following conditions. For all $x \in X, U, V \subseteq X$:

(N1) $p_x(X) = 1$;

(N2) $p_x(U) > 0 \Rightarrow x \in U$;

(N3)] $p_x(U \cap V) = p_x(U) \wedge p_x(V)$.

The pair (X, P) is called a fuzzifying neighborhood space, and it will be called topological if it additionally satisfies the following condition, for all $x \in X, U, V \subseteq X$:

(N4) $p_x(U) = \bigvee_{V \in \dot{x}|U} \bigwedge_{y \in V} p_y(V)$.

In [19], it is shown that $P^{\mathcal{T}} = \{p_x^{\mathcal{T}} | x \in X\}$ is a generalized neighborhood system if $\mathcal{T}$ is a fuzzifying topology, where $p_x^{\mathcal{T}} : 2^X \to [0,1]$ is defined by $p_x^{\mathcal{T}}(U) = \bigvee_{x \in V \subseteq U} \mathcal{T}(V)$ and $\mathcal{T}^p(U) = \bigwedge_{x \in U} p_x(U)$ is a fuzzifying topology if $P = \{p_x | x \in X\}$ is a generalized neighborhood system.

Let G be a nonempty set, and let $\oplus : G \times G \to G$ be a binary operation on G. Then the pair $(G, \oplus)$ is called a groupoid or a magma. A function f from a groupoid $(G_1, \oplus_1)$ to a groupoid $(G_2, \oplus_2)$ is said to be a groupoid homomorphism if $f(x \oplus_1 y) = f(x) \oplus_2 f(y)$ for any elements $x, y \in G_1$. In addition, a bijective groupoid homomorphism from a groupoid $(G, \oplus)$ to itself will be called a groupoid automorphism. We will write $Aut(G, \oplus)$ for the set of all automorphisms of a groupoid $(G, \oplus)$.

Definition 2.20. (Suksumran [9]) Let $(G, \oplus)$ be a nonempty groupoid. We say that $(G, \oplus)$ or just G (when it is clear from the context) is a gyrogroup if the following holds:

(1) There is an identity element $e \in G$ such that $e \oplus x = x$ for all $x \in G$.

(2) For each $x \in G$, there exists an inverse element $\ominus x \in G$ such that $\ominus x \oplus x = e$.

(3) For any $x, y \in G$, there exists an gyroautomorphism $gyr[x, y] \in Aut(G, \oplus)$ such that $x \oplus (y \oplus z) = (x \oplus y) \oplus gyr[x, y](z)$ for all $z \in G$.

(4) For any $x, y \in G$, $gyr[x \oplus y, y] = gyr[x, y]$.

One can easily show that any gyrogroup has a unique two-sided identity e, and an element a of the gyrogroup has a unique two-sided inverse $\ominus a$. Let $(G, \cdot)$ be a group. It is clear that G with the trivial gyroautomorphisms, that is, $gyr[x, y]$ is the identity map for all $x, y \in G$, is a gyrogroup. Conversely, every gyrogroup with trivial gyroautomorphisms forms a group. From this point of view, gyrogroups naturally generalize groups.

Proposition 2.21. (Suksumran [9]) Let $(G, \oplus)$ be a gyrogroup and $a, b, c \in G$. Then

(1) $\ominus(\ominus a) = a$;

(2) $\ominus a \oplus (a \oplus b) = b$;

(3) $gyr[a, b](c) = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c))$;

(4) $(a \oplus b) \oplus c = a \oplus (b \oplus gyr[b, a](c))$;

(5) $a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b](c))$;

(6) $\ominus(a \oplus b) = gyr[a, b](\ominus b \ominus a)$;

(7) $(\ominus a \oplus b) \oplus gyr[\ominus a, b](\ominus b \oplus c) = \ominus a \oplus c$;

(8) $gyr[a, b] = gyr[\ominus a, \ominus b]$;

(9) $gyr[a, b] = gyr^{-1}[b, a]$, the inverse of $gyr[b, a]$;

(10) $gyr[gyr[a, b](a), gyr[a, b](b)] = gyr[a, b]$.

Atiponrat extended the idea of topological groups to topological gyrogroups as gyrogroups with a topology such that its binary operation is jointly continuous and the operation of taking the inverse is continuous.

Definition 2.22. (Atiponrat [2]) A triple $(G, \tau, \oplus)$ is called a topological gyrogroup if the following hold:

(1) (G, τ) is a topological space;

(2) $(G, \oplus)$ is a gyrogroup;

(3) The binary operation $\oplus : G \times G \to G$ is continuous, where $G \times G$ is endowed with the product topology;

(4) The operation of taking the inverse $\ominus(\cdot) : G \to G$, I. e. $x \to \ominus x$, is also continuous.

Definition 2.23. (Atiponrat [2]) Let $(G, \tau, \oplus)$ be a topological gyrogroup, and let A, B be subsets of G. Defining $A \oplus B$ and $\ominus A$ to be the following sets:

$A \oplus B = \{a \oplus b : a \in A, b \in B\}$;

$\ominus A = \{\ominus a : a \in A\} = (\ominus)^{-1}(A)$.

A is said to be symmetric if $\ominus A = A$. By abuse of notion, we write $x \oplus A$ and $A \oplus x$ to mean $\{x\} \oplus A$ and $A \oplus \{x\}$, respectively, for any $x \in G$.

Suksumran introduce the notion of gyronorms on gyrogroups.

Definition 2.24. (Suksumran [9]) Let $(G, \oplus)$ be a gyrogroup. A function $p : G \to \mathbb{R}$ is called a gyronorm on G if the following properties hold: For all $x, y \in G$,

(G1) $p(x) \geq 0$ and $p(x) = 0$ if and only if $x = e$;

(G2) $p(x) = p(\ominus x)$;

(G3) $p(x \oplus y) \leq p(x) + p(y)$;

(G4) $p(x) = p(gyr[a, b](x))$ for all $a, b \in G$.

3. THE RELATIONS BETWEEN A FAMILY OF ORDINARY GYRONORMS AND A FUZZY GYRONORM

In this section we shall introduce a new notion of fuzzy gyronorms on gyrogroups. Also, the relationships between a family of ordinary gyronorms and a fuzzy gyronorm are studied.

In 2018, Suksumran introduced the notions of gyronorms on gyrogroups and said that any gyrogroup with a gyronorm is called a normed gyrogyoup. In 1988, Morsi provide a method for introducing fuzzy pseudo-metrics for pairs of crisp points, and fuzzy pseudo-norms for crisp points, as fuzzy real numbers $\geq \widetilde{0}$. Then we introduce the notion of fuzzy gyronorms on gyrogroups as follows.

Definition 3.1. Let $(G, \oplus)$ be a gyrogroup and S be a continuous s-norm. The pair $(\|\cdot\|, S)$ is called a fuzzy gyronorm on G if $\|\cdot\| : G \to \mathbb{R}^*(I)$ satisfying the following conditions: For all $x, y \in G$ and for all $s, t \in (0, +\infty)$,

(FG1) $\|e\| = \widetilde{0}$;

(FG2) $x \neq e \Rightarrow \|x\| \in \mathbb{R}^+(I)$;

(FG3) $\|x\| = \|\ominus x\|$;

(FG4) $\|x \oplus y\|(s+t) \leq S(\|x\|(s), \|y\|(t))$;

(FG5) $\|x\| = \|gyr[a, b](x)\|$ for all $a, b \in G$.

Then the ordered triple $(G, \|\cdot\|, S)$ is called a fuzzy normed gyrogroup if $(\|\cdot\|, S)$ is a fuzzy gyronorm on G.

The relations between a family of ordinary gyronorms and a fuzzy gyronorm are shown as follows:

Theorem 3.2. Let $(G, \|\cdot\|, S)$ be a fuzzy normed gyrogroup and S be a strictly monotone. For each $a \in (0, 1]$, define the mapping as $p_a(x) = \bigwedge\{t > 0 : \|x\|(t) < a\}$ for all $x \in G$. Then $\{p_a(\cdot)\}_{a \in (0,1]}$ is a family of left continuous and nonascending gyronorms on G.

Proof. First, we have one conclusion: $p_a(x) < s$ if and only if $\|x\|(s) < a$. If $p_a(x) < s$, then there exists $t_0 < s$ such that $\|x\|(t_0) < a$. Then $\|x\|(s) \leq \|x\|(t_0) < a$. on the contrary, if $a > \|x\|(s) = \lim_{t \to s-} \|x\|(t)$, then exists $t < s$ such that $\|x\|(t) < a$, thus $p_a(x) < t < s$.

(G1). We have that $p_a(e) = \bigwedge\{t > 0 : \|e\|(t) < a\} = \bigwedge\{t : t > 0\} = 0$. Suppose that $x \in G$ and $x \neq e$. Then there exists $t_0 > 0$ such that $\|x\|(t_0) = 1$, which implies $\|x\|(t_0) \geq a$, hence $p_a(x) \geq t_0 > 0$.

(G2). For all $x \in G$, $p_a(x) = \bigwedge\{t > 0 : \|x\|(t) < a\} = \bigwedge\{t > 0 : \|\ominus x\|(t) < a\} = p_a(\ominus x)$.

(G3). For all $x, y, a, b \in G$ and any $\varepsilon > 0$, there are $t_1 < p_a(x) + \frac{\varepsilon}{2}$ and $t_2 < p_b(y) + \frac{\varepsilon}{2}$ such that $\|x\|(t_1) < a$ and $\|y\|(t_2) < b$. Then

$$\|x \oplus y\|(t_1 + t_2) \leq S(\|x\|(t_1), \|y\|(t_2)) < S(a, b).$$

Thus $p_{S(a,b)}(x \oplus y) < t_1 + t_2 < p_a(x) + p_b(y) + \varepsilon$. By the arbitrariness of ε, we have $p_{S(a,b)}(x \oplus y) \le p_a(x) + p_b(y)$.

(G4). For all $x, \alpha, \beta \in G$, $p_a(x) = \bigwedge\{t > 0 : \|x\|(t) < a\} = \bigwedge\{t > 0 : \|gyr[\alpha,\beta](x)\|(t) < a\} = p_a(gyr[\alpha,\beta](x))$.

For all $a \ge b$, the relation $p_a(x) \le p_b(x)$ holds trivially.

Finally, we need to show that $p_a(x)$ is left continuous for $a \in (0,1]$. For all $b > a$, we have $p_b(x) \le p_a(x)$, which implies that $p_b(x) \le \bigwedge_{a<b} p_a(x)$. Let $p_b(x) < s$, we have $\|x\|(s) < b$, then there exists $c < b$ such that $\|x\|(s) < c < b$. By conclusion, we have $p_c(x) < s$. It follows that $\bigwedge_{a<b} p_a(x) \le p_c(x) < s$. By the arbitrariness of s, hence $p_b(x) \ge \bigwedge_{a<b} p_a(x)$. □

Theorem 3.3. Let $\{p_a(\cdot)\}_{a\in(0,1]}$ be a family of left continuous and nonascending gyronorms on G, S be a continuous $s-$norm, and the mapping $\|\cdot\|_p : G \to \mathbb{R}^*(I)$ be defined as follows:

$$\|\cdot\|_p(t) = \bigwedge\{a : p_a(x) < t\}.$$

Then $(G, \|\cdot\|_p, S)$ is a fuzzy normed gyrogroup.

Proof. First we know that $\|x\|_p(t) < a$ if and only if $p_a(x) < t$. In fact, if $p_a(x) \ge t$ and $p_b(x) < t$, then $b \ge a$. Thus $\|x\|_p(t) \ge a$, which is a contradiction. Hence $p_a(x) < t$. Conversely, if $p_a(x) < t$, then $\|x\|_p(t) \le a$. We suppose that $\|x\|_p(t) = a$, in this case $p_b(x) < t \Rightarrow b \ge a$. It implies that $p_b(x) \ge t$ for all $b > a$. Then $p_a(x) = \lim_{b\to a-} p_b(x) \ge t$, which is a contradiction. Hence $\|x\|_p(t) < a$.

Then we show $\|x\|_p \in \mathbb{R}^*(I)$ for all $x \in G$. It is clear $\|x\|_p(-\infty+) = 1$, $\|x\|_p(+\infty-) = 0$ and the mapping $\|\cdot\|_p(\cdot) : \mathbb{R} \to [0,1]$ is decreasing function. For each $t < 0$, $\|x\|_p(t) = 1$, we have $\|x\|_p(0-) = \|x\|_p(0)$. For each $t \in (0, +\infty)$, let $\{t_n\}$ be an increasing sequence and it is convergent to t. If $\|x\|_p(t) < a$, then $p_a(x) < t$. Then there exists $n_0 \in \mathbb{N}$ such that $p_a(x) < t_n$ for all $n \ge n_0$. It implies that $\|x\|_p(t_n) < a$ for all $n \ge n_0$. By the arbitrariness of a, we have $\|x\|_p(t) = \lim_{s\to t-} \|x\|_p(s)$.

Finally, it needs to show that $\|\cdot\|_p$ satisfies the conditions (FG1),(FG2),(FG4) since the other conditions hold trivially.

(FG1). For all $t \in \mathbb{R}$,

$$\|e\|_p(t) = \bigwedge\{a : p_a(e) < t\} = \begin{cases} \wedge\emptyset, & t \le 0 \\ 0, & t > 0 \end{cases}$$

$$= \begin{cases} 1, & t \le 0 \\ 0, & t > 0 \end{cases} = \widetilde{0}(t).$$

(FG2). Let $x \in G$ and $x \ne e$, specially, $p_1(x) \ne 0$. Then there exists $t_1 > 0$ such that $p_1(x) > t_1 > 0$. It implies that $1 \ge \|x\|_p(t_1) \ge 1$, hence $\|x\|_p(t_1) = 1$, I. e., $\|x\|_p \in \mathbb{R}^+(I)$.

(FG4). Let $\|x\|_p(t) = \beta$ and $\|y\|_p(s) = \gamma$. For all $\varepsilon > 0$, there exists $\alpha_1, \alpha_2 \in (0,1)$ such that $\alpha_1 < \beta + \varepsilon$ and $\alpha_2 < \gamma + \varepsilon$ with $p_{\alpha_1}(x) < t$ and $p_{\alpha_2}(y) < s$. Thus $p_{\beta+\varepsilon}(x) \le p_{\alpha_1}(x) < t$ and $p_{\gamma+\varepsilon}(y) \le p_{\alpha_2}(y) < s$. Then $p_{S(\beta+\varepsilon,\gamma+\varepsilon)}(x \oplus y) \le p_{\beta+\varepsilon}(x) + p_{\gamma+\varepsilon}(y) < t + s$. It implies that $\|x \oplus y\|_p(t+s) \le S(\beta+\varepsilon, \gamma+\varepsilon)$. By the arbitrariness of ε, hence $\|x \oplus y\|_p(t+s) \le S(\|x\|_p(t), \|y\|_p(s))$.

Therefore, $(G, \|\cdot\|_p, S)$ is a fuzzy normed gyrogroup. □

Proposition 3.4. Let $\{p_a(\cdot)\}_{a\in(0,1]}$ be a family of left continuous and nonascending gyronorms on G and $\|\cdot\|_p$ be a fuzzy gyronorm determined by $\{p_a(\cdot)\}_{a\in(0,1]}$ as Theorem 3.3. Suppose $\{q_a(\cdot)\}_{a\in(0,1]}$ is a family of gyronorms determined by Theorm 3.2. Then for each $a \in (0,1]$, $q_a(x) = p_a(x)$ for all $x \in G$.

Proof. For each $a \in (0,1]$, $x \in G$, $t > 0$, from the conclusion in Theorem 3.3, $p_a(x) < t$ if and only if $\|x\|_p(t) < a$. From the conclusion in Theorem 3.2, $\|x\|_p(t) < a$ if and only if $q_a(x) < t$. Hence $q_a(x) = p_a(x)$. $\square$

Proposition 3.5. Let $\|\cdot\|$ be a fuzzy gyronorm on G and $\{p_a(\cdot)\}_{a\in(0,1]}$ be a family of left continuous and nonascending gyronorms determined by $\|\cdot\|$ as Theorem 3.2. Suppose $\|\cdot\|_p$ is a fuzzy gyronorm determined by Theorem 3.3. Then $\|x\| = \|x\|_p$ for all $x \in G$.

Proof. For each $x \in G$ and $t > 0$, from the conclusion in Theorem 3.2, $\|x\|(t) < a$ if and only if $p_a(x) < t$. From the conclusion in Theorem 3.3, $p_a(x) < t$ if and only if $\|x\|_p(t) < a$. Hence $\|x\|(t) = \|x\|_p(t)$. $\square$

4. THE RELATIONS BETWEEN FUZZY METRICS AND FUZZY GYRONORMS ON GYROGROUPS

In this section, we study the relations between fuzzy metrics and fuzzy gyronorms on gyrogroups.

Proposition 4.1. Let $(G, \|\cdot\|, \bigvee)$ be a fuzzy normed gyrogroup. Define $d(x,y) = \|\ominus x \oplus y\|$ for all $x, y \in G$. Then $d_{\|\|}$ is a fuzzy metric on G.

Proof. (FM1). For each $x \in G$, $d_{\|\|}(x,x) = \|\ominus x \oplus x\| = \|e\| = \widetilde{0}$.

(FM2). For all $x, y \in G$,

$d_{\|\|}(x,y) = \|\ominus x \oplus y\|$

$= \|\ominus(\ominus x \oplus y)\|$ (by Proposition 2.21 (1))

$= \|gyr[\ominus x, y](\ominus y \oplus x)\|$ (by Proposition 2.21 (6))

$= \|\ominus y \oplus x\|$ (by Definition 3.1 (FG5))

$= d_{\|\|}(y,x)$.

(FM3). For all $x, y, z \in G$, $t, r \in \mathbb{R}$,

$d_{\|\|}(x,z)(t) = \|\ominus x \oplus z\|(t)$

$= \|(\ominus x \oplus y) \oplus gyr[\ominus x, y](\ominus y \oplus z)\|(t)$ (by Proposition 2.21 (7))

$\leq \bigwedge_{r\in\mathbb{R}} \|\ominus x \oplus y\|(r) \bigvee \|\ominus y \oplus z\|(t-r)$ (by Definition 3.1 (FG4))

$= (\|\ominus x \oplus y\| + \|\ominus y \oplus z\|)(t)$

$= (d_{\|\|}(x,y) + d_{\|\|}(y,z))(t)$.

(FM4). For all $x, y \in G$ and $x \neq y$, then $\ominus x \oplus y \neq e$, hence $d_{\|\|}(x, y) = \| \ominus x \oplus y\| \in \mathbb{R}^+(I)$. □

The fuzzy metric induced by a fuzzy gyronorm in Proposition 4.1 is called a fuzzy gyronorm metric on G.

Proposition 4.2. Let $(G, \|\cdot\|, \bigvee)$ be a fuzzy normed gyrogroup. Then the fuzzy gyronorm metric $d_{\|\|}$ with respect to $(\|\cdot\|, \bigvee)$ is invariant under the left gyrotranslation: $d_{\|\|}(a \oplus x, a \oplus y) = d_{\|\|}(x, y)$ for all $x, y, a \in G$.

P r o o f. Let $x, y, a \in G$. Then

$$\begin{aligned}
&d_{\|\|}(a \oplus x, a \oplus y)\\
&\quad = \| \ominus (a \oplus x) \oplus (a \oplus y)\|\\
&\quad = \|gyr[a, x](\ominus x \ominus a) \oplus (a \oplus y)\| \quad \text{(by Proposition 2.21 (6))}\\
&\quad = \|gyr[x, a]gyr[a, x](\ominus x \ominus a) \oplus gyr[x, a](a \oplus y)\|\\
&\quad = \|(\ominus x \ominus a) \oplus gyr[x, a](a \oplus y)\| \quad \text{(by Proposition 2.21 (9))}\\
&\quad = \| \ominus x \oplus y\| = d_{\|\|}(x, y). \quad \text{(by Proposition 2.21 (7))}
\end{aligned}$$

□

Theorem 4.3. Let G be a gyrogroup with a fuzzy metric d and S be a continuous s-norm. Suppose d is invariant under the left gyrotranslation, I. e., $d(a\oplus x, a\oplus y) = d(x, y)$ for all $x, y, a \in G$. Define $\|x\|_d = d(e, x)$, then $(\|\cdot\|_d, S)$ is a fuzzy gyronorm on G.

P r o o f. It is clear that $\|x\|_d \in \mathbb{R}^*(I)$ for all $x \in G$. Then we show $\|x\|_d$ satisfies the conditions in Definition 3.1.

(FG1). $\|e\|_d = d(e, e) = \widetilde{0}$.

(FG2). For all $x \in G$ and $x \neq e$, thus $\|x\|_d = d(e, x) \in \mathbb{R}^+(I)$.

(FG3). For all $x \in G$, $\|x\|_d = d(e, x) = d(x, e) = d(\ominus x \oplus x, \ominus x) = d(e, \ominus x) = \| \ominus x\|_d$.

(FG4). For all $x, y \in G$, $t, r \in \mathbb{R}$,

$$\begin{aligned}
&\|x \oplus y\|_d(t) = d(e, x \oplus y)(t) = d(\ominus x, y)(t)\\
&\leq (d(\ominus x, e) + d(e, y))(t)\\
&= (d(e, x) + d(e, y))(t)\\
&= (\|x\|_d + \|y\|_d)(t)\\
&\leq S(\|x\|_d(r), \|y\|_d(t - r)).
\end{aligned}$$

(FG5). For all $x, y, a, b \in G$,

$$\begin{aligned}
&\|gyr[a, b](x)\|_d = d(e, \ominus(a \oplus b) \oplus (a \oplus (b \oplus x)))\\
&= d(a \oplus b, a \oplus (b \oplus x))\\
&= d(b, b \oplus x)\\
&= d(e, x)\\
&= \|x\|_d.
\end{aligned}$$

□

Proposition 4.4. Let $(\|\cdot\|, \bigvee)$ be a fuzzy gyronorm on G and $d_{\|\|}$ be a fuzzy metric determined by $\|\cdot\|$ as Proposition 4.1. Suppose $\|\cdot\|_{d_{\|\|}}$ is a fuzzy gyronorm determined by Theorem 4.3. Then $\|x\| = \|x\|_{d_{\|\|}}$ for all $x \in G$.

Proof. According to Proposition 4.1 and Proposition 4.2, we have $\|\ominus x \oplus y\|_{d_{\|\|}} = d_{\|\|}(e, \ominus x \oplus y) = d_{\|\|}(x, y) = \|\ominus x \oplus y\|$. □

5. FUZZY TOPOLOGICAL GYROGROUP INDUCED BY FUZZY GYRONORMS

Many researchers are interested in the study of topological groups and fuzzy topological groups. In 2017, Atiponrat [2] introduced the notion of topological gyrogroups, which as being a generalization of a topological group. In this section, we intend to construct a stratified fuzzy topology with the help of level topologies. Therefore, this fuzzy topology on gyrogroup will form a fuzzy topological gyrogroup.

Some sufficient conditions which make gyrogroups with some topologies become topological gyrogroups are found.

Proposition 5.1. Let $(G, \|\cdot\|, \bigvee)$ be a fuzzy normed gyrogroup and $d_{\|\|}$ be a fuzzy gyronorm metric with respect to $(\|\cdot\|, \bigvee)$. Then the following conditions are equivalent:

(1) Right-gyrotranslation inequality: $d_{\|\|}(x \oplus a, y \oplus a) \leq d_{\|\|}(x, y) + \widetilde{0}$, I. e.,

$$\|\ominus (x \oplus a) \oplus (y \oplus a)\| \leq \|\ominus x \oplus y\| + \widetilde{0} \text{ for all } x, y, a \in G;$$

(2) $d_{\|\|}(x \oplus y, a \oplus b) \leq d_{\|\|}(x, a) + d_{\|\|}(y, b) + \widetilde{0}$, I. e.,

$$\|\ominus (x \oplus y) \oplus (a \oplus b)\| \leq \|\ominus x \oplus a\| + \|\ominus y \oplus b\| + \widetilde{0} \text{ for all } x, y, a, b \in G.$$

Proof. (1) ⇒ (2). We have

$$\begin{aligned} d_{\|\|}(x \oplus y, a \oplus b) &\leq d_{\|\|}(x \oplus y, x \oplus b) + d_{\|\|}(x \oplus b, a \oplus b) \\ &= d_{\|\|}(y, b) + d_{\|\|}(x \oplus b, a \oplus b) \leq d_{\|\|}(y, b) + d_{\|\|}(x, a) + \widetilde{0}. \end{aligned}$$

(2) ⇒ (1). We have

$$d_{\|\|}(x \oplus a, y \oplus a) \quad \leq d_{\|\|}(x, y) + d_{\|\|}(a, a) + \widetilde{0} = d_{\|\|}(x, y) + \widetilde{0}.$$

□

Theorem 5.2. Let $(G, \|\cdot\|, \bigvee)$ be a fuzzy normed gyrogroup and $\alpha \in [0, 1)$. Suppose the fuzzy gyronorms satisfying right-gyrotranslation inequality. Then the family

$$\mathcal{B}_\alpha = \{B(x, r, t) : x \in G, r \in (0, 1 - \alpha), t > 0\}$$

is a base for a topology τ_α on G, where $B(x, r, t) = \{y : \|\ominus x \oplus y\|(t) < r\}$. Also, G is a topological gyrogroup endowed with the topology τ_α.

Proof. For all $x, y \in G$. Since $\| \ominus x \oplus y\| = \| \ominus y \oplus x\|$, then we have $y \in B(x, r, t) \Leftrightarrow x \in B(y, r, t)$. Let $z \in B(x, r_1, t) \cap B(y, r_2, s)$ where $r_1, r_2 \in (0, 1 - \alpha)$, $t, s > 0$. Thus $\| \ominus x \oplus z\|(t) < r_1$ and $\| \ominus y \oplus z\|(s) < r_2$. Then there exist $t_0 < t$ and $s_0 < s$ such that $\| \ominus x \oplus z\|(t_0) < r_1$ and $\| \ominus y \oplus z\|(s_0) < r_2$. Let $r = r_1 \wedge r_2$ a nd $p = t - t_0 \wedge s - s_0$. It needs to show $B(z, r, p) \subseteq B(x, r_1, t) \cap B(y, r_2, s)$. Let $u \in B(z, r, p)$, then $\|\ominus z \oplus u\| < r$. Thus

$$\begin{aligned}
&\| \ominus x \oplus u\|(t) = \| \ominus x \oplus (z \oplus (\ominus z \oplus u))\|(t) \quad \text{(by Proposition 2.21 (2))}\\
&= \|(\ominus x \oplus z) \oplus gyr[\ominus x, z](\ominus z \oplus u)\|(t) \quad \text{(by Proposition 2.21 (5))}\\
&\leq \| \ominus x \oplus z\|(t_0) \vee \| \ominus z \oplus u\|(t - t_0)\\
&< r_1 \vee r\\
&= r_1.
\end{aligned}$$

Then $u \in B(x, r_1, t)$. Similarly, $\| \ominus y \oplus u\|(s) \leq \| \ominus y \oplus z\|(s_0) \bigvee \| \ominus z \oplus u\|(s - s_0) < r_2 \bigvee r = r_2$. Thus $u \in B(y, r_2, s)$. Hence $B(z, r, p) \subseteq B(x, r_1, t) \cap B(y, r_2, s)$. Then (G, τ_α) is a topology space.

Then it needs to show the binary operation $\oplus : \tau_\alpha \times \tau_\alpha \to \tau_\alpha$ is continuous. For all $U \in \tau_\alpha$, given $x \oplus y \in U$, there exist $\lambda \in (0, 1 - \alpha)$, $s > 0$ such that $B(x \oplus y, \lambda, s) \subseteq U$. It is shown that $B(x, \lambda, \frac{s}{2}) \oplus B(y, \lambda, \frac{s}{2}) \subseteq B(x \oplus y, \lambda, s)$. Let $a \in B(x, \lambda, \frac{s}{2})$ and $b \in B(y, \lambda, \frac{s}{2})$. Then we have $\| \ominus x \oplus a\|(\frac{s}{2}) < \lambda$ and $\| \ominus y \oplus b\|(\frac{s}{2}) < \lambda$. According the Right-gyrotranslation inequality and $\widetilde{0}(s) = 0$ with $s > 0$. Thus

$$\begin{aligned}
&(\| \ominus (x \oplus y) \oplus (a \oplus b)\|)(s) \leq (\| \ominus x \oplus a\| + \| \ominus y \oplus b\| + \widetilde{0})(s)\\
&\leq \| \ominus x \oplus a\|(\tfrac{s}{2}) \vee \| \ominus y \oplus b\|(\tfrac{s}{2}) \vee 0 < \lambda.
\end{aligned}$$

It follows that $a \oplus b \in B(x \oplus y, \lambda, s)$. Hence $B(x, \lambda, \frac{s}{2}) \oplus B(y, \lambda, \frac{s}{2}) \subseteq B(x \oplus y, \lambda, s) \subseteq U$ with $B(x, \lambda, \frac{s}{2}) \in \tau_\alpha$ and $B(y, \lambda, \frac{s}{2}) \in \tau_\alpha$.

Finally it is shown $\ominus : \tau_\alpha \to \tau_\alpha$ is continuous. For all $V \in \tau_\alpha$, it is clear that $(\ominus)^{-1}(V) = V \in \tau_\alpha$. Then the operation of taking the inverse is continuous. Therefore, G is a topological gyrogroup endowed with the topology τ_α. $\square$

Remark 5.3. Let $(G, \| \cdot \|, \bigvee)$ be a fuzzy normed gyrogroup and $\alpha \in [0, 1)$. The family $\{\tau_\alpha : \alpha \in [0, 1)\}$ is a decreasing family of topologies, since $B_\alpha \subseteq B_\beta$ whenever $\beta < \alpha$. By the definition of τ_α, we have $\tau_0 = \tau^{\|\cdot\|}$

Let $A \subseteq G$. Then $A \in \tau_\alpha$ if and only if for each $x \in A$, there exist $t > 0$ and $r \in (0, 1 - \alpha)$ such that $B(x, r, t) \subseteq A$. It is equivalent to $A = \bigcup_{\substack{0<r<1-\alpha, t>0\\ B(x,r,t)\subseteq A}} B(x, r, t)$. Besides, the family $\{\tau_\alpha : \alpha \in [0, 1)\}$ satisfies LT-property. For all $A \in \tau_\alpha$, let $A_\beta = \bigcup_{\substack{0<r<1-\beta, t>0\\ B(x,r,t)\subseteq A}} B(x, r, t)$ with $\beta \in (\alpha, 1)$. It is clear that $(A_\beta)_{\beta\in(\alpha,1)} \in \tau_\beta$. Thus

$$A = \bigcup_{\substack{0<r<1-\alpha, t>0\\ B(x,r,t)\subseteq A}} B(x, r, t) = \bigcup_{\beta>\alpha} \bigcup_{\substack{0<r<1-\beta, t>0\\ B(x,r,t)\subseteq A}} B(x, r, t) = \bigcup_{\beta>\alpha} A_\beta.$$

According to Proposition 2.13, we can construct a fuzzy topology, which is the following Proposition 5.4.

Proposition 5.4. Let $(G, \|\cdot\|, \bigvee)$ be a fuzzy normed gyrogroup and $\{\tau_\alpha : \alpha \in [0,1)\}$ be the family of topologies induced by this fuzzy gyronorms. Then

$$\mathscr{T}^{\|\cdot\|} := \{\lambda : [\lambda]^\alpha \in \tau_\alpha, \forall \alpha \in [0,1)\}.$$

is the finest stratified fuzzy topology satisfying $\iota_\alpha(\mathscr{T}^{\|\cdot\|}) = \tau_\alpha$. As a consequence, we have $\iota(\mathscr{T}^{\|\cdot\|}) = \tau^{\|\cdot\|}$.

Let $(G, \|\cdot\|, \bigvee)$ be a fuzzy normed gyrogroup, we introduce an open ball in G.

Definition 5.5. Let $(G, \|\cdot\|, \bigvee)$ be a fuzzy normed gyrogroup, $x, y \in G$, $r \in (0,1)$, $t > 0$ and $\beta \in (0,1)$. Then the fuzzy set $\beta\mu_r(x,t)$ is called β open ball with the center x and radius r, where

$$\beta\mu_r(x,t)(y) = \begin{cases} \beta, & y \in B(x,r,t), \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 5.6. Let $(G, \|\cdot\|, \bigvee)$ be a fuzzy normed gyrogroup. Then the family

$$\mathcal{B}_1 = \{\beta\mu_r(x,t) : x \in G, r, \beta \in (0,1), t > 0\}$$

is a base for $\omega(\tau^{\|\cdot\|})$.

Proof. It firstly needs to show $\beta\mu_r(x,t)$ with $x \in G, r, \beta \in (0,1), t > 0$ is lower semicontinuous. For all $\varepsilon > 0$, given $x_0 \in G$, we need to find $U \subseteq G$ with $x_0 \in U$ such that $\beta\mu_r(x,t)(y) < \beta\mu_r(x,t)(x_0) + \varepsilon$ for all $y \in U$. We only consider $\beta\mu_r(x,t)(y) = \beta$ since if $\beta\mu_r(x,t)(y) = 0$, then this inequality is clearly true. Suppose $\beta\mu_r(x,t)(y) = \beta$, thus $y \in B(x,r,t)$. Then there exists $s < t$ such that $y \in B(x,r,s)$. Let $U = B(x_0, r, t-s)$, then we have $\|\ominus x_0 \oplus x\|(t) \leq \|\ominus x_0 \oplus y\|(t-s) \vee \|\ominus y \oplus x\|(s) < r$. It implies that $x_0 \in B(x,r,t)$, hence $\beta\mu_r(x,t)(x_0) = \beta$. Then the inequality $\beta\mu_r(x,t)(y) < \beta\mu_r(x,t)(x_0) + \varepsilon$ is true. Finally, let $\lambda \in \omega(\tau^{\|\cdot\|})$ and $\lambda(x) > 0$. For all $\delta \in (0,1)$ satisfying $\lambda(x) > \delta$ there exists $r \in (0,1), t > 0$ such that $\lambda(y) > \lambda(x) - \delta$ for all $y \in B(x,r,t)$. Let $\beta = \lambda(x) - \delta$, then $\lambda(y) \geq \beta = \beta\mu_r(x,t)(y)$. Therefore, $\mathcal{B}_1$ is a base for $\omega(\tau^{\|\cdot\|})$. $\square$

Proposition 5.7. Let $(G, \|\cdot\|, \bigvee)$ be a fuzzy normed gyrogroup. Then the family

$$\mathcal{B}_2 = \{\beta\mu_r(x,t) : x \in G, r \in (0,1), t > 0, \beta \in (0, 1-r)\}$$

is a base for $\mathscr{T}^{\|\cdot\|}$.

Proof. Firstly, we show that $\mathcal{B}_2 \subseteq \mathscr{T}^{\|\cdot\|}$. For each $\alpha, \beta \in (0, 1-r)$, if $\alpha < \beta$, then $[\beta\mu_r(x,t)]^\alpha = B(x,r,t)$ with $x \in G, r \in (0,1), t > 0$. Since $1 - r > \beta$, then $1 - r > \alpha$. It follows that $B(x,r,t) \in B_\alpha \subseteq \tau_\alpha$. Hence $\beta\mu_r(x,t) \in \mathscr{T}^{\|\cdot\|}$. Let $\lambda \in \mathscr{T}^{\|\cdot\|}$ and $\lambda(x) > 0$. Then $[\lambda]^\alpha \in \tau_\alpha$, according to the definition of τ_α, there exists $r_1 < 1 - \alpha$ satisfying $B(x, r_1, t) \subseteq [\lambda]^\alpha$. That is, for every $y \in B(x, r_1, t)$, we have $\lambda(y) > \alpha$. Let $\alpha = \alpha\mu_{r_1}(x,t)(y)$, then $\lambda > \alpha\mu_{r_1}(x,t)$. Therefore, $\mathcal{B}_2$ is a base for $\mathscr{T}^{\|\cdot\|}$. $\square$

Theorem 5.8. Let $(G, \|\cdot\|, \bigvee)$ be a fuzzy normed gyrogroup and $d_{\|\|}$ be a fuzzy metric determined by $\|\cdot\|$ as Proposition 4.1. Then the fuzzy topology associated with $d_{\|\|}$ as Definition 2.16 is coarser than $\mathscr{T}^{d_{\|\|}}$.

P r o o f. It is clear that $\mathscr{T}^{d_{\|\|}} = \mathscr{T}^{\|\cdot\|}$ since $d_{\|\|}(x,y) = \|\ominus x \oplus y\|$ for all $x, y \in G$. First, we show that $\beta\mu_{1-r}(x,t) \leq B(x;\widetilde{t})$ with $r \in (0,1), t > 0, \beta \in (0,r)$. For $y \in G$, if $B(x;\widetilde{t})(y) = 1 - d_{\|\|}(x,y)(t) > r$, then we have $y \in B(x, 1-r, t)$. Thus $\beta\mu_{1-r}(x,t)(y) = \beta < r < B(x;\widetilde{t})(y)$. On the other hand, if $B(x;\widetilde{t})(y) = 1 - d_{\|\|}(x,y)(t) \leq r$, then we have $y \notin B(x, 1-r, t)$, which implies that $\beta\mu_{1-r}(x,t)(y) = 0 \leq r$. Hence that $[\beta\mu_{1-r}(x,t)]^\alpha \subseteq [B(x;\widetilde{t})]^\alpha$ for all $\alpha \in [0,1)$. Since $\beta\mu_{1-r}(x,t) \in \mathscr{T}^{d_{\|\|}}$, then $[\beta\mu_{1-r}(x,t)]^\alpha \in \tau_\alpha$. Thus $[B(x;\widetilde{t})]^\alpha \in \tau_\alpha$. It follows that $B(x;\widetilde{t}) \in \mathscr{T}^{d_{\|\|}}$. □

Definition 5.9. Let $(X, \mathscr{T}_X), (Y, \mathscr{T}_Y)$ be two fuzzy topological spaces. We say that the mapping $f : (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ is continuous if $f^{-1}(U) \in \mathscr{T}_X$ for all $U \in \mathscr{T}_Y$.

Definition 5.10. A triple $(G, \mathscr{T}, \oplus)$ is called a fuzzy topological gyrogroup if the following hold:

(1) $(G, \mathscr{T})$ is a fuzzy topological space;

(2) $(G, \oplus)$ is a gyrogroup;

(3) The binary operation $\oplus : G \times G \to G$ is continuous, where $G \times G$ is endowed with the product topology;

(4) The operation of taking the inverse $\ominus(\cdot) : G \to G$, I. e. $x \to \ominus x$, is also continuous.

Theorem 5.11. Let $(G, \|\cdot\|, \bigvee)$ be a fuzzy normed gyrogroup. Suppose the fuzzy gyronorms satisfying right-gyrotranslation inequality. Then $(G, \mathscr{T}^{\|\cdot\|}, \oplus)$ is a fuzzy topological gyrogroup endowed with the finest stratified fuzzy topology $\mathscr{T}^{\|\cdot\|}$.

P r o o f. We only need to show the binary operation $\oplus : \mathscr{T}^{\|\cdot\|} \times \mathscr{T}^{\|\cdot\|} \to \mathscr{T}^{\|\cdot\|}$ and the operation of taking the inverse $\ominus(\cdot) : \mathscr{T}^{\|\cdot\|} \to \mathscr{T}^{\|\cdot\|}$ are continuous. For all $A \in \mathscr{T}^{\|\cdot\|}$. We have $[A]^\alpha \in \tau_\alpha$ for all $\alpha \in [0,1)$. Given $x \oplus y \in [A]^\alpha$, then there exist $r \in (0, 1-\beta), t > 0$ such that $B(x \oplus y, r, t) \subseteq A$ for all $B(x \oplus y, r, t) \in \bigcap_{\beta > \alpha} B_\beta \subseteq B_\alpha$. Let $\gamma = \vee\beta$, then $\gamma\mu_r(x \oplus y, t) \leq A$ since $[\gamma\mu_r(x \oplus y, t)]^\alpha = B(x \oplus y, r, t) \subseteq [A]^\alpha$. We know that $[\gamma\mu_r(x, \frac{t}{2})]^\alpha = B(x, r, \frac{t}{2}) \in \tau_\alpha$ and $[\gamma\mu_r(y, \frac{t}{2})]^\alpha = B(y, r, \frac{t}{2}) \in \tau_\alpha$ for all $\alpha \in [0,1)$. It implies that $\gamma\mu_r(x, \frac{t}{2}), \gamma\mu_r(y, \frac{t}{2}) \in \mathscr{T}^{\|\cdot\|}$. It is shown that

$$
\begin{aligned}
&(\gamma\mu_r(x, \tfrac{t}{2}) \oplus \gamma\mu_r(y, \tfrac{t}{2}))(x \oplus y) = \textstyle\bigwedge_{x_1 \oplus y_1 = x \oplus y} \gamma\mu_r(x, \tfrac{t}{2})(x_1) \vee \gamma\mu_r(y, \tfrac{t}{2})(x_2) \\
&\quad \leq \gamma\mu_r(x, \tfrac{t}{2})(x) \vee \gamma\mu_r(y, \tfrac{t}{2})(x) = \gamma = \gamma\mu_r(x \oplus y, t)(x \oplus y).
\end{aligned}
$$

It implies that $\gamma\mu_r(x, \frac{t}{2}) \oplus \gamma\mu_r(y, \frac{t}{2}) \leq \gamma\mu_r(x \oplus y, t) \leq A$. Hence $\oplus$ is continuous. For all $\alpha \in [0,1)$,

$$y \in [(\ominus)^{-1}(A)]^\alpha \Leftrightarrow (\ominus)^{-1}(A)(y) > \alpha \Leftrightarrow A(\ominus y) > \alpha \Leftrightarrow \ominus y \in A^\alpha \Leftrightarrow y \in A^\alpha.$$

It follows that $[(\ominus)^{-1}(A)]^\alpha = A^\alpha \in \tau_\alpha$. Hence $(\ominus)^{-1}(A) \in \mathscr{T}^{\|\cdot\|}$. Therefore, $\ominus$ is continuous. □

Proposition 5.12. Let $(G, \|\cdot\|, \bigvee)$ be a fuzzy normed gyrogroup and $\alpha \in [0,1)$. Suppose the fuzzy gyronorms satisfying right-gyrotranslation inequality. Then the topology τ_α associated with $\|\cdot\|$ is T_2.

Proof. Let $x \neq y \in G$, it follows that $\|\ominus x \oplus y\| \in \mathbb{R}^+(I)$, then there exists t_0 such that $\|\ominus x \oplus y\|(t_0) = 1$. For each $z \in B(x, r, \frac{t_0}{2}) \cap B(y, r, \frac{t_0}{2})$ with $r \in (0,1)$. It implies that

$$\begin{aligned}
&\|\ominus x \oplus y\|(t_0) = \|\ominus x \oplus (z \oplus (\ominus z \oplus y))\|(t_0)\\
&= \|(\ominus x \oplus z) \oplus gyr[\ominus x, z](\ominus z \oplus y)\|(t_0) \quad \text{(by Proposition 2.21 (5))}\\
&\leq (\|\ominus x \oplus z\| + \|\ominus z \oplus y\|)(t_0)\\
&\leq \|\ominus x \oplus z\|(\tfrac{t_0}{2}) \vee \|\ominus z \oplus y\|(\tfrac{t_0}{2})\\
&< r \vee r\\
&= r.
\end{aligned}$$

This is a contradiction. Hence $B(x, r, \frac{t_0}{2}) \cap B(y, r, \frac{t_0}{2}) = \varnothing$. □

Theorem 5.13. Let $(G, \|\cdot\|, \bigvee)$ be a fuzzy normed gyrogroup. Then the topology τ_α associated with $\|\cdot\|$ is T_i if and only if the fuzzy topology $\mathscr{T}^{\|\cdot\|}$ associated with $\|\cdot\|$ is $\alpha - T_i$ for all $\alpha \in [0,1)$ and $i = 0,1,2$,

Proof. We only prove one case: $i = 2$, because the other proofs of $i = 0,1$ are similar. If (G, τ_α) is T_2, then for each $x \neq y \in G$, there exists $r_1, r_2 \in (0, 1-\alpha), t_1, t_2 > 0$ such that $B(x, r_1, t_1) \cap B(y, r_2, t_2) = \varnothing$ satisfying $x \in B(x, r_1, t_1)$ and $x \in B(y, r_2, t_2)$. Given $\beta \in (\alpha, 1 - r_1 \vee r_2)$. Let $U = \beta\mu_{r_1}(x, t_1)$ and $V = \beta\mu_{r_2}(y, t_2)$. For $z \in G$, we have $(U \cap V)(z) = U(z) \wedge V(z) = 0$, which implies U, V are disjoint. Also, $U(x) \wedge V(y) = \beta\mu_{r_1}(x, t_1)(x) \wedge \beta\mu_{r_2}(y, t_2)(y) = \beta > \alpha$. On the contrary, for each $x \neq y \in G$, there exist disjoint $\gamma\mu_{r_3}(x, t_3), \gamma\mu_{r_4}(y, t_4) \in \mathscr{T}^{\|\cdot\|}$ such that $\gamma\mu_{r_3}(x, t_3)(x) \wedge \gamma\mu_{r_4}(y, t_4)(y) > \alpha$ satisfying $r_3, r_4 \in (0, 1-\alpha), t_3, t_4 > 0$ and $\gamma \in (\alpha, 1 - r_3 \vee r_4)$. Thus $B(x, r_3, t_3) = [\gamma\mu_{r_3}(x, t_3)]^\alpha \in \tau_\alpha$ and $B(y, r_4, t_4) = [\gamma\mu_{r_4}(y, t_4)]^\alpha \in \tau_\alpha$. Hence $B(x, r_3, t_3) \cap B(y, r_4, t_4) = \varnothing$. □

6. FUZZIFYING TOPOLOGICAL GYROGROUP INDUCED BY FUZZY GYRONORMS

Zhang and Yan [20] introduced the notions of L-fuzzifying topological groups as a generalization of topological group, Here we introduce a fuzzifying topological gyrogroup induced by fuzzy gyronorms.

Definition 6.1. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be two fuzzifying topological spaces. We say that a mapping $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous if $\mathcal{T}_Y(U) \leq \mathcal{T}_X(f^{-1}(U))$ for all $U \subseteq Y$.

Definition 6.2. A triple $(G, \mathcal{T}, \oplus)$ is called a fuzzifying topological gyrogroup if the following hold:

(1) $(G, \mathcal{T})$ is a fuzzifying topological space;

(2) $(G, \oplus)$ is a gyrogroup;

(3) The binary operation $\oplus : G \times G \to G$ is continuous, where $G \times G$ is endowed with the product topology;

(4) The operation of taking the inverse $\ominus(\cdot) : G \to G$, I. e. $x \to \ominus x$, is also continuous.

According to some results in [20], we have the following similar conclusions:

Proposition 6.3. Let $(G, \oplus)$ be a gyrogroup and $\mathcal{T}$ be a fuzzifying topology on G. Then the binary operation $\oplus : G \times G \to G$ is continuous if and only if for every $x, y \in G$, $W \subseteq G$, $p_{x \oplus y}(W) \leq \bigvee_{U \oplus V \subseteq W} p_x(U) \wedge p_y(V)$, where $P = \{p_x | x \in G\}$ is a fuzzifying neighborhood structure determined by $\mathcal{T}$.

Proposition 6.4. Let $(G, \oplus)$ be a gyrogroup and $\mathcal{T}$ be a fuzzifying topology on G. Then the operation of taking the inverse $\ominus(\cdot) : G \to G$ is continuous if and only if for every $x \in G$, $W \subseteq G$, $p_{\ominus x}(W) \leq \bigvee_{\ominus U \subseteq W} p_x(U)$, where $P = \{p_x | x \in G\}$ is a fuzzifying neighborhood structure determined by $\mathcal{T}$.

Proposition 6.5. Let $(G, \oplus)$ be a gyrogroup and $\mathcal{T}$ be a fuzzifying topology on G. Then

(1) the binary operation $\oplus : G \times G \to G$ is continuous if and only if for every $x, y \in G$, $A \subseteq G$ with $x \oplus y \in A$, $\mathcal{T}(A) \leq \bigvee_{B \oplus C \subseteq A, B \in \dot{x}, C \in \dot{y}} \mathcal{T}(B) \wedge \mathcal{T}(C)$.

(2) the operation of taking the inverse $\ominus(\cdot) : G \to G$ is continuous if and only if for every $x \in G$, $A \subseteq G$ with $\ominus x \in A$, $\mathcal{T}(A) \leq \bigvee_{\ominus B \subseteq A, B \in \dot{x}} \mathcal{T}(B)$.

Theorem 6.6. Let $(G, \|\cdot\|, \bigvee)$ be a fuzzy normed gyrogroup and $\alpha \in [0, 1)$. Suppose the fuzzy gyronorms satisfying right-gyrotranslation inequality. Define $\mathcal{T}(A) = \bigvee\{\alpha : A \in \tau_\alpha\}$, where τ_α is a topology associated with $\|\cdot\|$. Then $(G, \mathcal{T}, \oplus)$ is a fuzzifying topological gyrogroup.

P r o o f. First, we show that $\mathcal{T}$ is a fuzzifying topology.

(FY1). It is clear $\mathcal{T}(X) = \bigvee\{\alpha : X \in \tau_\alpha\} = \bigvee[0, 1) = 1$ and $\mathcal{T}(\varnothing) = \bigvee[0, 1) = 1$.

(FY2). For all $U, V \in G$ and $a > 0$, let $\mathcal{T}(U) \wedge \mathcal{T}(V) > p$. Then there exist $\alpha, \beta > a$ such that $U \in \tau_\alpha$ and $V \in \tau_\beta$. Let $\eta = \alpha \vee \beta$, then $\eta > a$. For every $x \in U \cap V$, there exists $t > 0$ and $r \in (0, 1 - \eta)$ such that $B(x, r, t) \subseteq U \cap V$. It implies $\mathcal{T}(U \cap V) \geq \eta > a$. Hence $\mathcal{T}(U \cap V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V)$.

(FY3). For all $U_j \in G$ with $j \in J$ and $b > 0$, let $\bigwedge_{j \in J} \mathcal{T}(U_j) > b$. Then for all $j \in J$, $\mathcal{T}(U_j) > b$. That is for each $x \in U_j$, there exists $t_0 > 0, r_0 \in (0, 1 - \alpha)$ such that $B(x, r, t) \subseteq U_j \subseteq \bigcup_{j \in J} U_j$. It follows $\mathcal{T}(\bigcup_{j \in J} U_j) > b$. Hence $\mathcal{T}(\bigcup_{j \in J} U_j) \geq \bigwedge_{j \in J} \mathcal{T}(U_j)$.

Then it needs to show the binary operation $\oplus$ and the operation of taking the inverse $\ominus(\cdot)$ are continuous. That is we need to show for every $x, y \in G$, $A \subseteq G$ with $x \oplus y \in A$, $\mathcal{T}(A) \leq \bigvee_{B \oplus C \subseteq A, B \in \dot{x}, C \in \dot{y}} \mathcal{T}(B) \wedge \mathcal{T}(C)$. Let $w > 0$ and $\mathcal{T}(A) > w$. Then there exists $\alpha > w$ such that $A \in \tau_\alpha$. Thus exists $s > 0, \lambda \in (0, 1 - \alpha)$ such that $B(x \oplus y, \lambda, s) \subseteq A$. Let $B = B(x, \lambda, \frac{s}{2})$ and $C = B(y, \lambda, \frac{s}{2})$. Then $B, C \in \tau_\alpha$, which implies that $\mathcal{T}(B) > w$ and $\mathcal{T}(C) > w$. According to the proof of Theorem 5.2, we know $B \oplus C = B(x, \lambda, \frac{s}{2}) \oplus B(y, \lambda, \frac{s}{2}) \subseteq B(x \oplus y, \lambda, s) \subseteq A$. Hence $\bigvee_{B \oplus C \subseteq A, B \in \dot{x}, C \in \dot{y}} \mathcal{T}(B) \wedge \mathcal{T}(C) > w$.

On the other hand, for every $x \in G$, $A \subseteq G$ with $\ominus x \in A$, it needs to show $\mathcal{T}(A) \leq \bigvee_{\ominus B \subseteq A, B \in \dot{x}} \mathcal{T}(B)$. Let $m > 0$ and $\mathcal{T}(A) > m$. Then there exists $\alpha > m$ such that $A \in \tau_\alpha$. Thus exists $s_0 > 0, \lambda_0 \in (0, 1-\alpha)$ such that $B(x, \lambda_0, s_0) \subseteq A$. Let $B = \ominus B = B(x, \lambda_0, s_0)$. Then $B \in \tau_\alpha$,which implies that $\mathcal{T}(B) > m$. Hence $\bigvee_{\ominus B \subseteq A, B \in \dot{x}} \mathcal{T}(B) > m$. □

7. CONCLUSIONS AND FUTURE WORK

In the present paper, the notion of Morsi's fuzzy gyronorms on gyrogroups is introduced. The relationships between fuzzy metrics (in the sense of Morsi), fuzzy gyronorms, gyronorms on gyrogroups are studied. Also we have found some sufficient conditions which can make a fuzzy normed gyrogroup to be a topological gyrogroup and an fuzzy topological gyrogroup. Meanwhile, the relations between topological gyrogroups, fuzzy topological gyrogroups and stratified fuzzy topological gyrogroups are studied. At last, a fuzzifying topology determined by level topological gyrogroups is introduced and it is proved that the fuzzifying topology is compatible with gyrogroup structure.

A direction worthy of future work is to study further some properties in fuzzy normed gyrogroups. Such as, separability and fuzzy metrization in fuzzy topological gyrogroups.

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