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A CHARACTERIZATION OF UNINORMS ON BOUNDED LATTICES VIA CLOSURE AND INTERIOR OPERATORS

Gül Deniz Çayli

Uninorms on bounded lattices have been recently a remarkable field of inquiry. In the present study, we introduce two novel construction approaches for uninorms on bounded lattices with a neutral element, where some necessary and sufficient conditions are required. These constructions exploit a t-norm and a closure operator, or a t-conorm and an interior operator on a bounded lattice. Some illustrative examples are also included to help comprehend the newly added classes of uninorms.

Keywords: bounded lattice, closure operator, uninorm, interior operator, T-norm, T-conorm

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1. INTRODUCTION

Triangular norms (t-norms, for short) and triangular conorms (t-conorms, for short) were introduced by Menger [41] in 1942 and Schweizer and Sklar [46] in 1961, respectively, in the framework of probabilistic metric spaces. T-norms and t-conorms perform as natural extensions of logical connectives, namely conjunction, and disjunction, respectively, in fuzzy set theory and fuzzy logic. Therefore, these operators have been extensively used in many various branches of science, such as fuzzy set theory, fuzzy logic, fuzzy systems modeling, decision-making, probabilistic metric spaces, approximate reasoning, and information aggregation [3, 25, 26, 33, 37, 38, 39, 47].

Uninorms on the unit interval $[0, 1]$, as aggregation functions simultaneously generalizing t-norms and t-conorms, were introduced by Yager and Rybalov [50] in 1996 and studied comprehensively by Fodor et al. [30] in 1997. Since then, they have been widely involved in several research areas, such as neural networks [4], fuzzy system modeling [48, 49, 51], decision-making [52], fuzzy mathematical morphology, image processing [31], fuzzy logic, and in general [42]. Uninorms allow their neutral element to lie anywhere in the unit interval instead of point 1 (which is the case of t-norms) or point 0 (which is the case of t-conorms). There are abundant investigations concerning uninorms (e. g., [18, 19, 20, 23, 24]).

Since bounded lattices are more general structures than the unit interval, the generalization of binary aggregation operators from the real unit interval to bounded lattices

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becomes a rather hot topic. The definition of uninorms from the real unit interval to bounded lattices was straightforwardly generalized by Karaçal and Mesiar [36] in 2015. They also identified the smallest and largest uninorms on bounded lattices. Hitherto, these operators on bounded lattices have caught intensive attention, notably several construction approaches have been presented in the literature. Bodjanova and Kalina [6, 7] described the structure of uninorms derived from both t-norms and t-conorms on bounded lattices. Subsequently, Çaylı et al. [15] introduced two methods for obtaining internal and locally internal uninorms on bounded lattices based on only one of the t-norm and the t-conorm. Moreover, Çaylı [10] examined the structure of idempotent uninorms on bounded lattices with a neutral element. Dan et al. [16], and Dan and Hu [17] proposed further characterizations of uninorms on bounded lattices. We can also find some other related constructions of uninorms on bounded lattices (e. g., [2, 8, 9, 12, 28, 32, 34, 44, 53]).

In a general topology, letting the set $K \neq \emptyset$ and $\wp(K)$ be the set of all subsets of K, if a map $int : \wp(K) \to \wp(K)$ (resp. $cl : \wp(K) \to \wp(K)$) is idempotent, isotone and contractive (resp. expansive), then it is said to be an interior (resp. closure) operator on $\wp(K)$. Both these maps can be applied for generating topologies on K [27]. In especial, from the set of all interior (closure) operators on $\wp(K)$ to one of all topologies on K, a one-to-one correspondence exists. That is to say that the interior (closure) operator on $\wp(K)$ can be generated by any topology on K. Notably, interior (closure) operators on a lattice $(\wp(K), \subseteq)$ can be described when the set intersection and union are meet and join, respectively. Thence, the interior (resp. closure) operator on $\wp(K)$ to a lattice $\mathbb{L}$ was generalized by Everett [29], where the condition $int(K) = K$ (resp. $cl(\emptyset) = \emptyset$) is removed.

By using closure and interior operators on bounded lattices, the generation approaches of uninorms were improved by Ouyang and Zhang [43]. In particular, their constructions encompass, as a special case, those introduced in [36]. In this case, one can consider whether new classes of uninorms on bounded lattices with a neutral element are constructed by interior and closure operators. Motivated by this consideration, in the present study, we characterize two new classes of uninorms on bounded lattices via closure and interior operators. Characterization examinations are important working areas since they present the necessary structures for uninorms on bounded lattices. More precisely, we primarily introduce a new method for yielding uninorms on a bounded lattice $\mathbb{L}$ with the neutral element $e \in \mathbb{L}\backslash\{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$ utilizing a t-norm on $[0_{\mathbb{L}}, e]^2$ and a closure operator $\mathbb{L}$. Next, based on a t-conorm on $[e, 1_{\mathbb{L}}]^2$ and an interior operator $\mathbb{L}$, we propose a dual construction of uninorms on $\mathbb{L}$. Moreover, we investigate the relationship between our methods and the ones described in [9, 14, 53]. We also demonstrate that the tools in the present paper are different from the approaches in [9, 14, 43, 53]. Accordingly, it is worth noting that the characterization of uninorms on bounded lattices via closure and interior operators contributes to enriching and analyzing the classes of uninorms on bounded lattices.

The remainder of this paper is organized as follows: In Section 2, we provide some basic definitions and properties related to uninorms on bounded lattices. In Section 3, we develop two methods for yielding uninorms on a bounded lattice $\mathbb{L}$ with a neutral element $e \in \mathbb{L}\backslash\{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$, where some necessary and sufficient conditions are required.

These constructions exploit an interior operator on $\mathbb{L}$ and a t-conorm on $[e, 1_{\mathbb{L}}]^2$, or a closure operator on $\mathbb{L}$ and a t-norm on $[0_{\mathbb{L}}, e]^2$. Furthermore, we present some illustrative examples in order to emphasize the differences between our methods and the existing ones. In the final section, some conclusions of our discussion are listed.

2. PRELIMINARIES

In this section, we recall some basic concepts and results related to bounded lattices (for more information, see, e. g., [5]) and uninorms on them.

A poset $(\mathbb{L}, \leqslant)$ is a nonempty set $\mathbb{L}$ equipped with an order relation $\leqslant$ (i. e., a reflexive, antisymmetric and transitive binary relation). For $a, b \in \mathbb{L}$, the notation $a < b$ means that $a \leqslant b$ and $a \neq b$. The notation $a \parallel b$ implies that a and b are incomparable, i. e., neither $a \leqslant b$ nor $b < a$. $\mathbb{I}_a$ denotes the set of all elements incomparable with a, i. e., $\mathbb{I}_a = \{u \in \mathbb{L} : u \| a\}$. An element a of a subset $\mathbb{P}$ of $\mathbb{L}$ is called a smallest (resp. greatest) element of $\mathbb{P}$ if $x \geqslant a$ (resp. $x \leqslant a$) for all $x \in \mathbb{P}$. $\mathbb{L}$ is called bounded if it has a greatest (also known as top) element and a smallest (also known as bottom) element.

An element a of a poset $(\mathbb{L}, \leqslant)$ with the bottom element $0_{\mathbb{L}}$ is an atom if $0_{\mathbb{L}} < a$ and there is no element u in $\mathbb{L}$ such that $0 < u < a$ (i. e., a is a minimal element in $\mathbb{L}$ obtained by excluding $0_{\mathbb{L}}$). The concept of coatom is defined dually.

A lattice $(\mathbb{L}, \leqslant)$ is a poset such that any two elements a and b have a greatest lower bound (called meet or infimum), denoted by $a \wedge b$, as well as a smallest upper bound (called join or supremum), denoted by $a \vee b$. In this paper, unless otherwise stated, $\mathbb{L}$ denotes a bounded lattice $(\mathbb{L}, \leqslant, \wedge, \vee)$ with a top element $1_{\mathbb{L}}$ and a bottom element $0_{\mathbb{L}}$.

For $a, b \in \mathbb{L}$ with $a \leqslant b$, the subinterval $[a, b]$ of $\mathbb{L}$ is defined such that

$$[a, b] = \{u \in \mathbb{L} : a \leqslant u \leqslant b\}.$$

The subintervals $[a, b[$, $]a, b]$, and $]a, b[$ of $\mathbb{L}$ can be defined similarly. $([a, b], \leqslant, \wedge, \vee)$ is a bounded lattice with the top element b and the bottom element a.

Definition 2.1. (Çaylı et al. [15], Karaçal and Mesiar [36]) A function $U : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ is said to be a uninorm if, for any $a, b, c \in \mathbb{L}$, the following conditions are fulfilled:

(i) $U(b, a) = U(a, b)$ (commutativity);

(ii) If $b \leqslant a$, then $U(b, c) \leqslant U(a, c)$ (increasingness);

(iii) $U(b, U(a, c)) = U(U(b, a), c)$ (associativity);

(iv) There is an element $e \in \mathbb{L}$, called a neutral element, such that $U(b, e) = b$ (neutral element).

In particular, a uninorm U is a t-norm T (resp. t-conorm S) if $e = 1_{\mathbb{L}}$ (resp. $e = 0_{\mathbb{L}}$) (for more information about t-norms and t-conorms, see, e. g., [1, 11, 13, 35, 40, 45]).

Example 2.2. (i) The largest t-norm is $T^{\wedge}$ on $[a, b]^2$ defined such that $T^{\wedge}(x, y) = x \wedge y$ for all $x, y \in [a, b]$, while the smallest one T^W on $[a, b]^2$ takes the value of $x \wedge y$ if $b \in \{x, y\}$ and a otherwise. Thus, we obtain that $T^W \leqslant T \leqslant T^{\wedge}$ for any t-norm T on $[a, b]^2$.

(ii) The smallest t-conorm is $S^{\vee}$ on $[a,b]^2$ defined such that $S^{\vee}(x,y) = x \vee y$ for all $x, y \in [a,b]$, while the largest one S^{W} on $[a,b]^2$ takes the value of $x \vee y$ if $a \in \{x,y\}$ and b otherwise. Thus, we obtain that $S^{\vee} \leqslant S \leqslant S^{W}$ for any t-conorm S on $[a,b]^2$.

Proposition 2.3. (Karaçal and Mesiar [36]) Let U be a uninorm on $\mathbb{L}$ with a neutral element $e \in \mathbb{L} \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$. Then, the following statements hold:

(i) $U \mid [0_{\mathbb{L}}, e]^2 : [0_{\mathbb{L}}, e]^2 \to [0_{\mathbb{L}}, e]$ is a t-norm.

(ii) $U \mid [e, 1_{\mathbb{L}}]^2 : [e, 1_{\mathbb{L}}]^2 \to [e, 1_{\mathbb{L}}]$ is a t-conorm.

Definition 2.4. (Drossos [21], Drossos and Navara [22], Everett [29]) A function $cl : \mathbb{L} \to \mathbb{L}$ is said to be a closure operator if, for any $a, b \in \mathbb{L}$, the following conditions are fulfilled:

(i) Expansion: $b \leqslant cl(b)$.

(ii) Preservation of join: $cl(a \vee b) = cl(a) \vee cl(b)$.

(iii) Idempotence: $cl\,(cl(b)) = cl(b)$.

By (i), the case (iii) is equivalent to $cl\,(cl(b)) \leqslant cl(b)$. Additionally, (ii) implies to (ii)$'$: $cl(a) \leqslant cl(b)$ if $a \leqslant b$. Observe that Birkhoff [5] defines a closure operator by (i), (ii)$'$ and (iii).

Definition 2.5. (Drossos [21], Drossos and Navara [22], Everett [29]) A function $int : \mathbb{L} \to \mathbb{L}$ is said to be an interior operator if, for any $a, b \in \mathbb{L}$, the following conditions are fulfilled:

(i) Contraction: $int(b) \leqslant b$.

(ii) Preservation of meet: $int(a \wedge b) = int(a) \wedge int(b)$.

(iii) Idempotence: $int\,(int(b)) = int(b)$.

By (i), the case (iii) is equivalent to $int(b) \leqslant int\,(int(b))$. Additionally, (ii) implies to (ii)$'$: $int(a) \leqslant int(b)$ if $a \leqslant b$. Observe that Birkhoff [5] defines an interior operator by (i), (ii)$'$ and (iii).

In the following, we recall the construction methods for uninorms on bounded lattices introduced by [9, 14, 53].

Theorem 2.6. (Çaylı [9], Theorem 8) Let $e \in \mathbb{L} \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$, $T : [0_{\mathbb{L}}, e]^2 \to [0_{\mathbb{L}}, e]$ be a t-norm, and $S : [e, 1_{\mathbb{L}}]^2 \to [e, 1_{\mathbb{L}}]$ be a t-conorm. Then the function $U : \mathbb{L}^2 \to \mathbb{L}$, given by the formula (1), is a uninorm on $\mathbb{L}$ with a neutral element e iff $x < y$ for all $x < e$ and $y \in \mathbb{I}_e$.

$$U(a,b) = \begin{cases} T(a,b) & \text{if } (a,b) \in [0_{\mathbb{L}}, e]^2, \\ a \wedge b & \begin{array}{l}\text{if } (a,b) \in [0_{\mathbb{L}}, e[\times \mathbb{I}_e \cup \mathbb{I}_e \times [0_{\mathbb{L}}, e[\\ \qquad \cup [0_{\mathbb{L}}, e[\times [e, 1_{\mathbb{L}}] \cup [e, 1_{\mathbb{L}}] \times [0_{\mathbb{L}}, e[\,,\end{array} \\ b & \text{if } (a,b) \in \{e\} \times \mathbb{I}_e, \\ a & \text{if } (a,b) \in \mathbb{I}_e \times \{e\}\,, \\ S(a \vee e, b \vee e) & \text{otherwise.} \end{cases} \tag{1}$$

Theorem 2.7. (Zhao and Wu [53], Proposition 3.5) Let $e \in \mathbb{L} \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$, $T : [0_{\mathbb{L}}, e]^2 \to [0_{\mathbb{L}}, e]$ be a t-norm, and $cl : \mathbb{L} \to \mathbb{L}$ be a closure operator. If $cl(p) \vee cl(q) \in \mathbb{I}_e$ for all $p, q \in \mathbb{I}_e$ or $cl(p) \vee cl(q) \in]e, 1_{\mathbb{L}}]$ for all $p, q \in \mathbb{I}_e$, then the function $U : \mathbb{L}^2 \to \mathbb{L}$, given by the formula (2), is a uninorm on $\mathbb{L}$ with a neutral element e iff $x < y$ for all $x < e$ and $y \in \mathbb{I}_e$.

$$U(a,b) = \begin{cases} T(a,b) & \text{if } (a,b) \in [0_{\mathbb{L}}, e]^2, \\ a \wedge b & \text{if } (a,b) \in [0_{\mathbb{L}}, e[\times \mathbb{I}_e \cup \mathbb{I}_e \times [0_{\mathbb{L}}, e[\\ & \quad \cup [0_{\mathbb{L}}, e[\times [e, 1_{\mathbb{L}}] \cup [e, 1_{\mathbb{L}}] \times [0_{\mathbb{L}}, e[, \\ b & \text{if } (a,b) \in \{e\} \times (\mathbb{I}_e \cup]e, 1_{\mathbb{L}}]), \\ a & \text{if } (a,b) \in (\mathbb{I}_e \cup]e, 1_{\mathbb{L}}]) \times \{e\}, \\ cl(a) \vee cl(b) & \text{if } (a,b) \in \mathbb{I}_e \times \mathbb{I}_e, \\ 1_{\mathbb{L}} & \text{otherwise.} \end{cases} \tag{2}$$

Theorem 2.8. (Zhao and Wu [53], Proposition 3.6) Let $e \in \mathbb{L} \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$, $T : [0_{\mathbb{L}}, e]^2 \to [0_{\mathbb{L}}, e]$ be a t-norm, $S : [e, 1_{\mathbb{L}}]^2 \to [e, 1_{\mathbb{L}}]$ be a t-conorm, and $cl : \mathbb{L} \to \mathbb{L}$ be a closure operator. If $p \parallel q$ for all $p \in \mathbb{I}_e$ and $q \in [e, 1_{\mathbb{L}}[$, then the function $U : \mathbb{L}^2 \to \mathbb{L}$, given by the formula (3), is a uninorm on $\mathbb{L}$ with a neutral element e iff $x < y$ for all $x < e$ and $y \in \mathbb{I}_e$.

$$U(a,b) = \begin{cases} T(a,b) & \text{if } (a,b) \in [0_{\mathbb{L}}, e]^2, \\ S(a,b) & \text{if } (a,b) \in [e, 1_{\mathbb{L}}]^2, \\ 1_{\mathbb{L}} & \text{if } (a,b) \in]e, 1_{\mathbb{L}}] \times \mathbb{I}_e \cup \mathbb{I}_e \times]e, 1_{\mathbb{L}}], \\ b & \text{if } (a,b) \in \{e\} \times \mathbb{I}_e, \\ a & \text{if } (a,b) \in \mathbb{I}_e \times \{e\}, \\ cl(a) \vee cl(b) & \text{if } (a,b) \in \mathbb{I}_e \times \mathbb{I}_e, \\ a \wedge b & \text{otherwise.} \end{cases} \tag{3}$$

Theorem 2.9. (Çaylı [14], Theorem 3.1) Let $e \in \mathbb{L} \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$, $T : [0_{\mathbb{L}}, e]^2 \to [0_{\mathbb{L}}, e]$ be a t-norm, and $cl : \mathbb{L} \to \mathbb{L}$ be a closure operator. Then the function $U : \mathbb{L}^2 \to \mathbb{L}$, given by the formula (4), is a uninorm on $\mathbb{L}$ with a neutral element e iff $x < y$ for all $x < e$ and $y \in \mathbb{I}_e$.

$$U(a,b) = \begin{cases} T(a,b) & \text{if } (a,b) \in [0_{\mathbb{L}}, e]^2, \\ a \wedge b & \text{if } (a,b) \in [0_{\mathbb{L}}, e[\times \mathbb{I}_e \cup \mathbb{I}_e \times [0_{\mathbb{L}}, e[\\ & \quad \cup [0_{\mathbb{L}}, e[\times [e, 1_{\mathbb{L}}] \cup [e, 1_{\mathbb{L}}] \times [0_{\mathbb{L}}, e[, \\ b & \text{if } (a,b) \in \{e\} \times (\mathbb{I}_e \cup [e, 1_{\mathbb{L}}]), \\ a & \text{if } (a,b) \in (\mathbb{I}_e \cup [e, 1_{\mathbb{L}}]) \times \{e\}, \\ cl(a) \vee cl(b) & \text{otherwise.} \end{cases} \tag{4}$$

Theorem 2.10. (Çaylı [14], Theorem 3.4) Let $e \in \mathbb{L} \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$, $T : [0_{\mathbb{L}}, e]^2 \to [0_{\mathbb{L}}, e]$ be a t-norm, $S : [e, 1_{\mathbb{L}}]^2 \to [e, 1_{\mathbb{L}}]$ be a t-conorm, and $cl : \mathbb{L} \to \mathbb{L}$ be a closure operator. If $p < q$ for all $p \in \mathbb{I}_e$, $q \in]e, 1_{\mathbb{L}}]$, and $cl(p) \vee cl(q) \in \mathbb{I}_e$ for all $p, q \in \mathbb{I}_e$, then the function $U : \mathbb{L}^2 \to \mathbb{L}$, given by the formula (5), is a uninorm on $\mathbb{L}$ with a neutral element e iff

$x < y$ for all $x < e$ and $y \in \mathbb{I}_e$.

$$U(a,b) = \begin{cases} T(a,b) & \text{if } (a,b) \in [0_{\mathbb{L}}, e]^2, \\ S(a,b) & \text{if } (a,b) \in [e, 1_{\mathbb{L}}]^2, \\ a \wedge b & \begin{array}{l}\text{if } (a,b) \in [0_{\mathbb{L}}, e[\times \mathbb{I}_e \cup \mathbb{I}_e \times [0_{\mathbb{L}}, e[\\ \qquad \cup [0_{\mathbb{L}}, e[\times [e, 1_{\mathbb{L}}] \cup [e, 1_{\mathbb{L}}] \times [0_{\mathbb{L}}, e[\,, \end{array} \\ b & \text{if } (a,b) \in \{e\} \times \mathbb{I}_e, \\ a & \text{if } (a,b) \in \mathbb{I}_e \times \{e\}\,, \\ cl(a) \vee cl(b) & \text{if } (a,b) \in \mathbb{I}_e \times \mathbb{I}_e, \\ a \vee b & \text{otherwise.} \end{cases} \tag{5}$$

3. CONSTRUCTION APPROACHES FOR UNINORMS

In this section, we introduce in Theorem 3.1 a novel method for getting the family of uninorm $U_{(T,cl)}$ on a bounded lattice $\mathbb{L}$ with a neutral element e. The uninorm $U_{(T,cl)}$ is derived from a t-norm T on $[0_{\mathbb{L}}, e]^2$ and a closure operator cl on $\mathbb{L}$. In addition, we propose in Theorem 3.11 a different method to obtain the family of uninorm $U_{(S,int)}$ on $\mathbb{L}$ with a neutral element e. This construction is based on the existence of a t-conorm S on $[e, 1_{\mathbb{L}}]^2$ and an interior operator int on $\mathbb{L}$.

Theorem 3.1. Let $e \in \mathbb{L} \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$ and $T : [0_{\mathbb{L}}, e]^2 \to [0_{\mathbb{L}}, e]$ be a t-norm. The function $U_{(T,cl)} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$, given by the formula (6), is a uninorm on $\mathbb{L}$ with a neutral element e for every closure operator $cl : \mathbb{L} \to \mathbb{L}$ iff $f > g$ and $d \vee f \in \mathbb{I}_e \cup \{1_{\mathbb{L}}\}$ for all $d, f \in \mathbb{I}_e$ and $g \in [0_{\mathbb{L}}, e[$.

$$U_{(T,cl)}(a,b) = \begin{cases} T(a,b) & \text{if } (a,b) \in [0_{\mathbb{L}}, e]^2, \\ 1_{\mathbb{L}} & \text{if } (a,b) \in \,]e, 1_{\mathbb{L}}]^2\,, \\ cl(a) \vee cl(b) & \text{if } (a,b) \in \,]e, 1_{\mathbb{L}}] \times \mathbb{I}_e \cup \mathbb{I}_e \times \,]e, 1_{\mathbb{L}}]\,, \\ a \vee b & \text{if } (a,b) \in \mathbb{I}_e \times \mathbb{I}_e, \\ a & \text{if } (a,b) \in (\mathbb{I}_e \cup [e, 1_{\mathbb{L}}]) \times \{e\}\,, \\ b & \text{if } (a,b) \in \{e\} \times (\mathbb{I}_e \cup [e, 1_{\mathbb{L}}])\,, \\ a \wedge b & \text{otherwise.} \end{cases} \tag{6}$$

Proof. *Necessity.* Let the function $U_{(T,cl)}$ be a uninorm on $\mathbb{L}$ with a neutral element e. We first demonstrate that $f > g$ for all $f \in \mathbb{I}_e$, $g \in [0_{\mathbb{L}}, e[$. Suppose that there exist the elements $f \in \mathbb{I}_e$, $g \in \,]0_{\mathbb{L}}, e[$ such that $f \parallel g$. Then we have that

$$U_{(T,cl)}(g, U_{(T,cl)}(f, 1_{\mathbb{L}})) = U_{(T,cl)}(g, cl(f) \vee cl(1_{\mathbb{L}})) = U_{(T,cl)}(g, 1_{\mathbb{L}}) = g \wedge 1_{\mathbb{L}} = g,$$

and

$$U_{(T,cl)}(U_{(T,cl)}(g, f), 1_{\mathbb{L}}) = U_{(T,cl)}(g \wedge f, 1_{\mathbb{L}}) = g \wedge f \wedge 1_{\mathbb{L}} = g \wedge f,$$

which contradicts the associativity property of $U_{(T,cl)}$. Consequently, $f > g$ for all $f \in \mathbb{I}_e$, $g \in [0_{\mathbb{L}}, e[\,$.

Now, we verify that $d \vee f \in \mathbb{I}_e \cup \{1_{\mathbb{L}}\}$ for all $d, f \in \mathbb{I}_e$. Suppose that there exist the elements $d, f \in \mathbb{I}_e$ such that $e < d \vee f < 1_{\mathbb{L}}$. Then, for the closure operator $cl : \mathbb{L} \to \mathbb{L}$ given by $cl\,(x) = x$ for all $x \in \mathbb{L}$, we get that

$$U_{(T,cl)}(d \vee f, U_{(T,cl)}(d, f)) = U_{(T,cl)}(d \vee f, d \vee f) = 1_{\mathbb{L}},$$

and

$$\begin{aligned} U_{(T,cl)}(U_{(T,cl)}(d \vee f, d), f) &= U_{(T,cl)}(cl\,(d \vee f) \vee cl\,(d)\,, f) = U_{(T,cl)}(cl\,(d \vee f)\,, f) \\ &= U_{(T,cl)}(d \vee f, f) = cl\,(d \vee f) \vee cl\,(f) = cl\,(d \vee f) = d \vee f, \end{aligned}$$

which contradicts the associativity property of $U_{(T,cl)}$. Consequently, $d \vee f \in \mathbb{I}_e \cup \{1_{\mathbb{L}}\}$ for all $d, f \in \mathbb{I}_e$.

Sufficiency. Let $f > g$ and $d \vee f \in \mathbb{I}_e \cup \{1_{\mathbb{L}}\}$ for all $d, f \in \mathbb{I}_e$ and $g \in [0_{\mathbb{L}}, e[$. We verify that $U_{(T,cl)}$ is a uninorm on $\mathbb{L}$ with a neutral element e. Clearly, $U_{(T,cl)}$ is commutative and e is a neutral element of $U_{(T,cl)}$. Therefore, it remains to verify the increasingness and associativity of $U_{(T,cl)}$. Increasingness: We prove that, for all $a, b, c \in \mathbb{L}$, $U_{(T,cl)}(a, c) \leqslant U_{(T,cl)}(b, c)$ if $a \leqslant b$. If $c = e$, then $U_{(T,cl)}(a, c) = U_{(T,cl)}(a, e) = a \leqslant b = U_{(T,cl)}(b, e) = U_{(T,cl)}(b, c)$. If $(a, b) \in [0_{\mathbb{L}}, e[^2 \cup \{e\}^2 \cup]e, 1_{\mathbb{L}}]^2 \cup \mathbb{I}_e^2$, the increasingness holds. So, we consider all remaining possible cases.

1. Let $a \in [0_{\mathbb{L}}, e[$.

1.1. $b = e$ and $c \in [0_{\mathbb{L}}, e[$,

$$U_{(T,cl)}(a, c) = T(a, c) \leqslant c = U_{(T,cl)}(e, c) = U_{(T,cl)}(b, c).$$

1.2. $b = e$ and $c \in]e, 1_{\mathbb{L}}] \cup \mathbb{I}_e$,

$$U_{(T,cl)}(a, c) = a \wedge c \leqslant c = U_{(T,cl)}(e, c) = U_{(T,cl)}(b, c).$$

1.3. $b \in]e, 1_{\mathbb{L}}] \cup \mathbb{I}_e$ and $c \in [0_{\mathbb{L}}, e[$,

$$U_{(T,cl)}(a, c) = T(a, c) \leqslant b \wedge c = U_{(T,cl)}(b, c).$$

1.4. $b, c \in]e, 1_{\mathbb{L}}]$,

$$U_{(T,cl)}(a, c) = a \wedge c \leqslant 1_{\mathbb{L}} = U_{(T,cl)}(b, c).$$

1.5. ($b \in \mathbb{I}_e$ and $c \in]e, 1_{\mathbb{L}}]$) or ($b \in]e, 1_{\mathbb{L}}]$ and $c \in \mathbb{I}_e$),

$$U_{(T,cl)}(a, c) = a \wedge c \leqslant cl\,(b) \vee cl\,(c) = U_{(T,cl)}(b, c).$$

1.6. $b, c \in \mathbb{I}_e$,

$$U_{(T,cl)}(a, c) = a \wedge c \leqslant b \vee c = U_{(T,cl)}(b, c).$$

2. Let $a = e$ and $b \in]e, 1_{\mathbb{L}}]$.

2.1. $c \in [0_{\mathbb{L}}, e[$,

$$U_{(T,cl)}(a, c) = U_{(T,cl)}(e, c) = c = b \wedge c = U_{(T,cl)}(b, c).$$

2.2. $c \in]e, 1_{\mathbb{L}}]$,

$$U_{(T,cl)}(a,c) = U_{(T,cl)}(e,c) = c \leqslant 1_{\mathbb{L}} = U_{(T,cl)}(b,c).$$

2.3. $c \in \mathbb{I}_e$,

$$U_{(T,cl)}(a,c) = U_{(T,cl)}(e,c) = c \leqslant cl\,(b) \vee cl\,(c) = U_{(T,cl)}(b,c).$$

3. Let $a \in \mathbb{I}_e$ and $b \in]e, 1_{\mathbb{L}}]$.

3.1. $c \in [0_{\mathbb{L}}, e[$,

$$U_{(T,cl)}(a,c) = a \wedge c \leqslant b \wedge c = U_{(T,cl)}(b,c).$$

3.2. $c \in]e, 1_{\mathbb{L}}]$,

$$U_{(T,cl)}(a,c) = cl\,(a) \vee cl\,(c) \leqslant 1_{\mathbb{L}} = U_{(T,cl)}(b,c).$$

3.3. $c \in \mathbb{I}_e$,

$$U_{(T,cl)}(a,c) = a \vee c \leqslant cl\,(b) \vee cl\,(c) = U_{(T,cl)}(b,c).$$

Associativity: We prove that $U_{(T,cl)}(a, U_{(T,cl)}(b,c)) = U_{(T,cl)}(U_{(T,cl)}(a,b),c)$ for all $a,b,c \in \mathbb{L}$. The associativity holds if $e \in \{a,b,c\}$. So, we consider all remaining possible cases.

1. Let $a \in [0_{\mathbb{L}}, e[$.

1.1. $b, c \in [0_{\mathbb{L}}, e[$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}\,(a, T(b,c)) = T\,(a, T(b,c)) \\ &= T(T(a,b),c) = U_{(T,cl)}(T(a,b),c) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b),c). \end{aligned}$$

1.2. $b \in [0_{\mathbb{L}}, e[$ and $c \in]e, 1_{\mathbb{L}}] \cup \mathbb{I}_e$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}\,(a, b \wedge c) = U_{(T,cl)}\,(a,b) = T(a,b) \\ &= T(a,b) \wedge c = U_{(T,cl)}(T(a,b),c) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b),c). \end{aligned}$$

1.3. $b \in]e, 1_{\mathbb{L}}] \cup \mathbb{I}_e$ and $c \in [0_{\mathbb{L}}, e[$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}\,(a, b \wedge c) = U_{(T,cl)}\,(a,c) = T(a,c) \\ &= U_{(T,cl)}\,(a,c) = U_{(T,cl)}\,(a \wedge b, c) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b),c). \end{aligned}$$

1.4. $b, c \in]e, 1_{\mathbb{L}}]$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}\,(a, 1_{\mathbb{L}}) = a \wedge 1_{\mathbb{L}} = a = a \wedge c \\ &= U_{(T,cl)}\,(a,c) = U_{(T,cl)}\,(a \wedge b, c) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b),c). \end{aligned}$$

1.5. ($b \in \mathbb{I}_e$ and $c \in]e, 1_{\mathbb{L}}]$) or ($b \in]e, 1_{\mathbb{L}}]$ and $c \in \mathbb{I}_e$),

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}\,(a, cl\,(b) \vee cl\,(c)) = a \wedge (cl\,(b) \vee cl\,(c)) \\ &= a = a \wedge c = U_{(T,cl)}\,(a,c) = U_{(T,cl)}\,(a \wedge b, c) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b),c). \end{aligned}$$

1.6. $b, c \in \mathbb{I}_e$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}\left(a, b \vee c\right) = a \wedge (b \vee c) = a \\ &= a \wedge c = U_{(T,cl)}\left(a, c\right) = U_{(T,cl)}\left(a \wedge b, c\right) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b), c). \end{aligned}$$

2. Let $a \in]e, 1_{\mathbb{L}}] \cup \mathbb{I}_e$.

2.1. $b, c \in [0_{\mathbb{L}}, e[$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}(a, T\left(b,c\right)) = a \wedge T\left(b,c\right) \\ &= T\left(b,c\right) = U_{(T,cl)}\left(b, c\right) = U_{(T,cl)}\left(a \wedge b, c\right) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b), c). \end{aligned}$$

2.2. $b \in [0_{\mathbb{L}}, e[$ and $c \in]e, 1_{\mathbb{L}}] \cup \mathbb{I}_e$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}\left(a, b \wedge c\right) = U_{(T,cl)}\left(a, b\right) = a \wedge b \\ &= b = b \wedge c = U_{(T,cl)}\left(b, c\right) = U_{(T,cl)}\left(a \wedge b, c\right) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b), c). \end{aligned}$$

3. Let $a \in]e, 1_{\mathbb{L}}]$.

3.1. $b \in]e, 1_{\mathbb{L}}]$ and $c \in [0_{\mathbb{L}}, e[$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}\left(a, b \wedge c\right) = U_{(T,cl)}\left(a, c\right) = a \wedge c \\ &= c = 1_{\mathbb{L}} \wedge c = U_{(T,cl)}\left(1_{\mathbb{L}}, c\right) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b), c). \end{aligned}$$

3.2. $b \in \mathbb{I}_e$ and $c \in [0_{\mathbb{L}}, e[$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}\left(a, b \wedge c\right) = U_{(T,cl)}\left(a, c\right) = a \wedge c = c \\ &= \left(cl\left(a\right) \vee cl\left(b\right)\right) \wedge c = U_{(T,cl)}\left(cl\left(a\right) \vee cl\left(b\right), c\right) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b), c). \end{aligned}$$

3.3. $b, c \in]e, 1_{\mathbb{L}}]$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}\left(a, 1_{\mathbb{L}}\right) = 1_{\mathbb{L}} \\ &= U_{(T,cl)}(1_{\mathbb{L}}, c) = U_{(T,cl)}(U_{(T,cl)}(a,b), c). \end{aligned}$$

3.4. $b \in]e, 1_{\mathbb{L}}]$ and $c \in \mathbb{I}_e$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}\left(a, cl\left(b\right) \vee cl\left(c\right)\right) = 1_{\mathbb{L}} \\ &= cl\left(1_{\mathbb{L}}\right) \vee cl\left(c\right) = U_{(T,cl)}(1_{\mathbb{L}}, c) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b), c). \end{aligned}$$

3.5. $b \in \mathbb{I}_e$ and $c \in]e, 1_{\mathbb{L}}]$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}\left(a, cl\left(b\right) \vee cl\left(c\right)\right) = 1_{\mathbb{L}} \\ &= U_{(T,cl)}(cl\left(a\right) \vee cl\left(b\right), c) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b), c). \end{aligned}$$

3.6. $b, c \in \mathbb{I}_e$,

$$\begin{aligned} U_{(T,cl)}\left(a, U_{(T,cl)}(b,c)\right) &= U_{(T,cl)}(a, b \vee c) \\ &= \begin{cases} U_{(T,cl)}(a, 1_{\mathbb{L}}) & \text{if } b \vee c = 1_{\mathbb{L}}, \\ cl(a) \vee cl(b \vee c) & \text{if } b \vee c \in \mathbb{I}_e, \end{cases} \\ &= \begin{cases} 1_{\mathbb{L}} & \text{if } b \vee c = 1_{\mathbb{L}}, \\ cl(a \vee b \vee c) & \text{if } b \vee c \in \mathbb{I}_e, \end{cases} \\ &= cl(a \vee b \vee c) \\ &= U_{(T,cl)}(cl(a) \vee cl(b), c) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b), c). \end{aligned}$$

4. Let $a \in \mathbb{I}_e$.
4.1. $b \in \left]e, 1_{\mathbb{L}}\right]$ and $c \in \left[0_{\mathbb{L}}, e\right[$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}(a, b \wedge c) = U_{(T,cl)}(a, c) = a \wedge c = c \\ &= (cl(a) \vee cl(b)) \wedge c = U_{(T,cl)}(cl(a) \vee cl(b), c) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b), c). \end{aligned}$$

4.2. $b, c \in \left]e, 1_{\mathbb{L}}\right]$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}(a, 1_{\mathbb{L}}) = cl(a) \vee cl(1_{\mathbb{L}}) \\ &= 1_{\mathbb{L}} = U_{(T,cl)}(cl(a) \vee cl(b), c) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b), c). \end{aligned}$$

4.3. $b \in \left]e, 1_{\mathbb{L}}\right]$ and $c \in \mathbb{I}_e$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}(a, cl(b) \vee cl(c)) = cl(a) \vee cl(b) \vee cl(c) \\ &= U_{(T,cl)}(cl(a) \vee cl(b), c) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b), c). \end{aligned}$$

5. Let $a, b \in \mathbb{I}_e$.
5.1. $c \in \left[0_{\mathbb{L}}, e\right[$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}(a, b \wedge c) = U_{(T,cl)}(a, c) = a \wedge c \\ &= c = (a \vee b) \wedge c = U_{(T,cl)}(a \vee b, c) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b), c). \end{aligned}$$

5.2. $c \in \left]e, 1_{\mathbb{L}}\right]$,

$$\begin{aligned} U_{(T,cl)}(a, U_{(T,cl)}(b,c)) &= U_{(T,cl)}(a, cl(b) \vee cl(c)) \\ &= cl(a \vee b \vee c) \\ &= \begin{cases} 1_{\mathbb{L}} & \text{if } a \vee b = 1_{\mathbb{L}}, \\ cl(a \vee b \vee c) & \text{if } a \vee b \in \mathbb{I}_e, \end{cases} \\ &= \begin{cases} U_{(T,cl)}(1_{\mathbb{L}}, c) & \text{if } a \vee b = 1_{\mathbb{L}}, \\ cl(a \vee b) \vee cl(c) & \text{if } a \vee b \in \mathbb{I}_e, \end{cases} \\ &= U_{(T,cl)}(a \vee b, c) \\ &= U_{(T,cl)}(U_{(T,cl)}(a,b), c). \end{aligned}$$

5.3. $c \in \mathbb{I}_e$,

$$\begin{aligned} U_{(T,cl)}\left(a, U_{(T,cl)}(b,c)\right) &= U_{(T,cl)}(a, b \vee c) \\ &= \begin{cases} U_{(T,cl)}(a, 1_{\mathbb{L}}) & \text{if } b \vee c = 1_{\mathbb{L}}, \\ a \vee b \vee c & \text{if } b \vee c \in \mathbb{I}_e, \end{cases} \\ &= \begin{cases} cl(a) \vee cl(1_{\mathbb{L}}) & \text{if } b \vee c = 1_{\mathbb{L}}, \\ a \vee b \vee c & \text{if } b \vee c \in \mathbb{I}_e, \end{cases} \\ &= \begin{cases} 1_{\mathbb{L}} & \text{if } b \vee c = 1_{\mathbb{L}}, \\ a \vee b \vee c & \text{if } b \vee c \in \mathbb{I}_e, \end{cases} \end{aligned}$$

and

$$\begin{aligned} U_{(T,cl)}(U_{(T,cl)}(a,b),c) &= U_{(T,cl)}(a \vee b, c) \\ &= \begin{cases} U_{(T,cl)}(1_{\mathbb{L}}, c) & \text{if } a \vee b = 1_{\mathbb{L}}, \\ a \vee b \vee c & \text{if } a \vee b \in \mathbb{I}_e, \end{cases} \\ &= \begin{cases} cl(1_{\mathbb{L}}) \vee cl(c) & \text{if } a \vee b = 1_{\mathbb{L}}, \\ a \vee b \vee c & \text{if } a \vee b \in \mathbb{I}_e, \end{cases} \\ &= \begin{cases} 1_{\mathbb{L}} & \text{if } a \vee b = 1_{\mathbb{L}}, \\ a \vee b \vee c & \text{if } a \vee b \in \mathbb{I}_e, \end{cases} \end{aligned}$$

implying that $U_{(T,cl)}\left(a, U_{(T,cl)}(b,c)\right) = U_{(T,cl)}(U_{(T,cl)}(a,b),c)$.

Therefore, $U_{(T,cl)}$ is an associative, commutative, and increasing function on $\mathbb{L}$ with a neutral element e. Accordingly, $U_{(T,cl)}$ is a uninorm on $\mathbb{L}$. □

Remark 3.2. Notice that the uninorm $U_{(T,cl)} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ in Theorem 3.1 can be also defined such that

$$U_{(T,cl)}(a,b) = \begin{cases} T(a,b) & \text{if } (a,b) \in [0_{\mathbb{L}}, e]^2, \\ 1_{\mathbb{L}} & \text{if } (a,b) \in]e, 1_{\mathbb{L}}]^2, \\ a & \text{if } (a,b) \in [0_{\mathbb{L}}, e[\times \mathbb{I}_e \cup [0_{\mathbb{L}}, e[\times [e, 1_{\mathbb{L}}] \\ & \qquad \cup (\mathbb{I}_e \cup [e, 1_{\mathbb{L}}]) \times \{e\}, \\ b & \text{if } (a,b) \in \mathbb{I}_e \times [0_{\mathbb{L}}, e[\cup [e, 1_{\mathbb{L}}] \times [0_{\mathbb{L}}, e[\\ & \qquad \cup \{e\} \times (\mathbb{I}_e \cup [e, 1_{\mathbb{L}}]), \\ a \vee b & \text{if } (a,b) \in \mathbb{I}_e \times \mathbb{I}_e, \\ cl(a) \vee cl(b) & \text{if } (a,b) \in \mathbb{I}_e \times]e, 1_{\mathbb{L}}] \cup]e, 1_{\mathbb{L}}] \times \mathbb{I}_e. \end{cases}$$

Remark 3.3. From Remark 3.2, the structure of the uninorm $U_{(T,cl)} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ is illustrated in Figure 1.

If we take in Theorem 3.1 the t-norm $T : [0_{\mathbb{L}}, e]^2 \to [0_{\mathbb{L}}, e]$ stated by $T = T^{\wedge}$, we define the corresponding uninorm as the following structure:

Corollary 3.4. Let $e \in \mathbb{L} \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$. The function $U_{(cl)} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$, given by the formula (7), is a uninorm on $\mathbb{L}$ with a neutral element e for every closure operator $cl : \mathbb{L} \to \mathbb{L}$ iff $f > g$ and $d \vee f \in \mathbb{I}_e \cup \{1_{\mathbb{L}}\}$ for all $d, f \in \mathbb{I}_e$ and $g \in [0_{\mathbb{L}}, e[$.

$$U_{(cl)}(a,b) = \begin{cases} 1_{\mathbb{L}} & \text{if } (a,b) \in]e, 1_{\mathbb{L}}]^2, \\ cl(a) \vee cl(b) & \text{if } (a,b) \in]e, 1_{\mathbb{L}}] \times \mathbb{I}_e \cup \mathbb{I}_e \times]e, 1_{\mathbb{L}}], \\ a \vee b & \text{if } (a,b) \in \mathbb{I}_e \times \mathbb{I}_e, \\ a & \text{if } (a,b) \in (\mathbb{I}_e \cup [e, 1_{\mathbb{L}}]) \times \{e\}, \\ b & \text{if } (a,b) \in \{e\} \times (\mathbb{I}_e \cup [e, 1_{\mathbb{L}}]), \\ a \wedge b & \text{otherwise}. \end{cases} \tag{7}$$

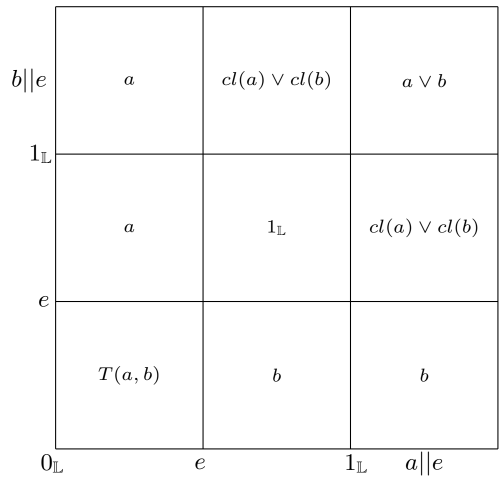

Fig. 1. Uninorm $U_{(T,cl)} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ in Theorem 3.1.

If we allow in Theorem 3.1 to be an atom of the element $e \in \mathbb{L} \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$, we define the corresponding uninorm as the following structure:

Corollary 3.5. Let $e \in \mathbb{L} \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$ be an atom. The function $U_{(e,cl)} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$, given by the formula (8), is a uninorm on $\mathbb{L}$ with a neutral element e for every closure operator $cl : \mathbb{L} \to \mathbb{L}$ iff $d \vee f \in \mathbb{I}_e \cup \{1_{\mathbb{L}}\}$ for all $d, f \in \mathbb{I}_e$.

$$U_{(e,cl)}(a,b) = \begin{cases} 1_{\mathbb{L}} & \text{if } (a,b) \in \left]e, 1_{\mathbb{L}}\right]^2, \\ cl(a) \vee cl(b) & \text{if } (a,b) \in \left]e, 1_{\mathbb{L}}\right] \times \mathbb{I}_e \cup \mathbb{I}_e \times \left]e, 1_{\mathbb{L}}\right], \\ a \vee b & \text{if } (a,b) \in \mathbb{I}_e \times \mathbb{I}_e, \\ a & \text{if } (a,b) \in (\mathbb{I}_e \cup [e, 1_{\mathbb{L}}]) \times \{e\}, \\ b & \text{if } (a,b) \in \{e\} \times (\mathbb{I}_e \cup [e, 1_{\mathbb{L}}]), \\ 0_{\mathbb{L}} & \text{otherwise.} \end{cases} \tag{8}$$

Remark 3.6. Let $e \in \mathbb{L} \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$, $S : [e, 1_{\mathbb{L}}]^2 \to [e, 1_{\mathbb{L}}]$ be a t-conorm and $cl : \mathbb{L} \to \mathbb{L}$ be a closure operator. We introduce in Theorem 3.1 a new construction approach for uninorms on bounded lattices. To be more precise,

(i) If $(a,b) \in \left]e, 1_{\mathbb{L}}\right]^2 \cup \left]e, 1_{\mathbb{L}}\right] \times \mathbb{I}_e \cup \mathbb{I}_e \times \left]e, 1_{\mathbb{L}}\right] \cup \mathbb{I}_e^2$, the method in [9, Theorem 8] puts for $U(a,b)$ the value of $S(a \vee e, b \vee e)$. On the other hand, when $(a,b) \in \left]e, 1_{\mathbb{L}}\right]^2$ (resp. $(a,b) \in \mathbb{I}_e^2$) our construction puts for $U_{(T,cl)}(a,b)$ the value of $1_{\mathbb{L}}$ (resp. $a \vee b$). Moreover, in our construction $U_{(T,cl)}(a,b) = cl(a) \vee cl(b)$ for $(a,b) \in \left]e, 1_{\mathbb{L}}\right] \times \mathbb{I}_e \cup \mathbb{I}_e \times \left]e, 1_{\mathbb{L}}\right]$. However, both constructions coincide in the remaining domains;

(ii) If $(a,b) \in \left]e, 1_{\mathbb{L}}\right]^2 \cup \mathbb{I}_e^2$, the method in [14, Theorem 3.1] puts for $U(a,b)$ the value of $cl(a) \vee cl(b)$. On the other hand, when $(a,b) \in \left]e, 1_{\mathbb{L}}\right]^2$ (resp. $(a,b) \in \mathbb{I}_e^2$) our construction puts for $U_{(T,cl)}(a,b)$ the value of $1_{\mathbb{L}}$ (resp. $a \vee b$). However, both constructions coincide in the remaining domains;

(iii) If $(a,b) \in \,]e,1_{\mathbb{L}}]^2$ (resp. $(a,b) \in \,]e,1_{\mathbb{L}}] \times \mathbb{I}_e \cup \mathbb{I}_e \times \,]e,1_{\mathbb{L}}]$), the method in [14, Theorem 3.4] puts for $U(a,b)$ the value of $S(a,b)$ (resp. $a \vee b$). Furthermore, in [14, Theorem 3.4] $U(a,b) = cl(a) \vee cl(b)$ for $(a,b) \in \mathbb{I}_e^2$. On the other hand, when $(a,b) \in \,]e,1_{\mathbb{L}}]^2$ (resp. $(a,b) \in \,]e,1_{\mathbb{L}}] \times \mathbb{I}_e \cup \mathbb{I}_e \times \,]e,1_{\mathbb{L}}]$) our construction puts for $U_{(T,cl)}(a,b)$ the value of $1_{\mathbb{L}}$ (resp. $cl(a) \vee cl(b)$). Moreover, in our construction $U_{(T,cl)}(a,b) = a \vee b$ for $(a,b) \in \mathbb{I}_e^2$. However, both constructions coincide in the remaining domains;

(iv) If $(a,b) \in \mathbb{I}_e^2$ (resp. $(a,b) \in \,]e,1_{\mathbb{L}}]^2 \cup \,]e,1_{\mathbb{L}}] \times \mathbb{I}_e \cup \mathbb{I}_e \times \,]e,1_{\mathbb{L}}]$), the method in [53, Proposition 3.5] puts for $U(a,b)$ the value of $cl(a) \vee cl(b)$ (resp. $1_{\mathbb{L}}$). On the other hand, when $(a,b) \in \mathbb{I}_e^2$ (resp. $(a,b) \in \,]e,1_{\mathbb{L}}] \times \mathbb{I}_e \cup \mathbb{I}_e \times \,]e,1_{\mathbb{L}}]$) our construction puts for $U_{(T,cl)}(a,b)$ the value of $a \vee b$ (resp. $cl(a) \vee cl(b)$). Moreover, in our construction $U_{(T,cl)}(a,b) = 1_{\mathbb{L}}$ for $(a,b) \in \,]e,1_{\mathbb{L}}]^2$. However, both constructions coincide in the remaining domains.

Remark 3.7. Let $e \in \mathbb{L} \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$. If we define the closure operator $cl : \mathbb{L} \to \mathbb{L}$ such that $cl(x) = x$ for all $x \in \mathbb{L}$, then the following statements hold:

(i) the uninorm $U_{(T,cl)}$ in Theorem 3.1 coincides with the uninorm in [14, Theorem 3.1], where e is a coatom;

(ii) the uninorm $U_{(T,cl)}$ in Theorem 3.1 coincides with the uninorm in [14, Theorem 3.4], where the t-conorm $S : [e,1_{\mathbb{L}}]^2 \to [e,1_{\mathbb{L}}]$ is $S = S^W$;

(iii) the uninorm $U_{(T,cl)}$ in Theorem 3.1 coincides with the uninorm in [53, Proposition 3.5], where $b_1 \parallel b_2$ for all $b_1 \in [e,1_{\mathbb{L}}[$, $b_2 \in \mathbb{I}_e$;

(iv) the uninorm $U_{(T,cl)}$ in Theorem 3.1 coincides with the uninorm in [53, Proposition 3.6], where $b_1 \parallel b_2$ for all $b_1 \in [e,1_{\mathbb{L}}[$, $b_2 \in \mathbb{I}_e$, and the t-conorm $S : [e,1_{\mathbb{L}}]^2 \to [e,1_{\mathbb{L}}]$ is $S = S^W$.

It should be pointed out that the uninorm constructed by the method in Theorem 3.1 does not have to coincide with those introduced in [9, Theorem 8], [14, Theorems 3.1 and 3.4], and [53, Propositions 3.5 and 3.6]. In the following examples, we demonstrate this observation.

Example 3.8. Consider the lattice $\mathbb{L}_1 = \{0_{\mathbb{L}_1}, u, v, y, z, r, e, 1_{\mathbb{L}_1}\}$ characterized by Hasse diagram in Figure 2.

Define the closure operator $cl : \mathbb{L}_1 \to \mathbb{L}_1$ such that $cl(x) = x$ for all $x \in \mathbb{L}_1$. By using the construction approach in Theorem 3.1, the uninorm $U^1_{(T,cl)} : \mathbb{L}_1 \times \mathbb{L}_1 \to \mathbb{L}_1$ is given as in Table 1. Clearly, $U^1_{(T,cl)}(r,r) = 1_{\mathbb{L}_1}$, $U^1_{(T,cl)}(r,u) = r$ and $U^1_{(T,cl)}(y,z) = z$. On the other hand, the uninorms U^1 and U^2 constructed by [14, Theorem 3.1] and [53, Proposition 3.5], respectively, satisfy that $U^1(r,r) = r$ and $U^2(r,u) = 1_{\mathbb{L}_1}$. Moreover, the uninorm U^3 in [9, Theorem 8] satisfies that $U^3(y,z) = 1_{\mathbb{L}_1}$. Hence, $U^1_{(T,cl)}$ differs from the uninorms U^1, U^2 and U^3 on $\mathbb{L}_1$.

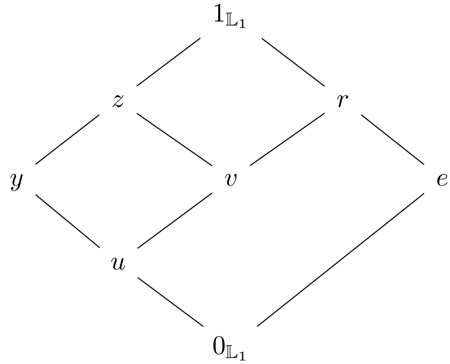

Fig. 2. The lattice $\mathbb{L}_1$.

$U^1_{(T,cl)}$	$0_{\mathbb{L}_1}$	e	u	v	y	z	r	$1_{\mathbb{L}_1}$
$0_{\mathbb{L}_1}$	$0_{\mathbb{L}_1}$	$0_{\mathbb{L}_1}$	$0_{\mathbb{L}_1}$	$0_{\mathbb{L}_1}$	$0_{\mathbb{L}_1}$	$0_{\mathbb{L}_1}$	$0_{\mathbb{L}_1}$	$0_{\mathbb{L}_1}$
e	$0_{\mathbb{L}_1}$	e	u	v	y	z	r	$1_{\mathbb{L}_1}$
u	$0_{\mathbb{L}_1}$	u	u	v	y	z	r	$1_{\mathbb{L}_1}$
v	$0_{\mathbb{L}_1}$	v	v	v	z	z	r	$1_{\mathbb{L}_1}$
y	$0_{\mathbb{L}_1}$	y	y	z	y	z	$1_{\mathbb{L}_1}$	$1_{\mathbb{L}_1}$
z	$0_{\mathbb{L}_1}$	z	z	z	z	z	$1_{\mathbb{L}_1}$	$1_{\mathbb{L}_1}$
r	$0_{\mathbb{L}_1}$	r	r	r	$1_{\mathbb{L}_1}$	$1_{\mathbb{L}_1}$	$1_{\mathbb{L}_1}$	$1_{\mathbb{L}_1}$
$1_{\mathbb{L}_1}$	$0_{\mathbb{L}_1}$	$1_{\mathbb{L}_1}$	$1_{\mathbb{L}_1}$	$1_{\mathbb{L}_1}$	$1_{\mathbb{L}_1}$	$1_{\mathbb{L}_1}$	$1_{\mathbb{L}_1}$	$1_{\mathbb{L}_1}$

Tab. 1. Uninorm $U^1_{(T,cl)}$ on $\mathbb{L}_1$.

Example 3.9. Consider the lattice $\mathbb{L}_2 = \{0_{\mathbb{L}_2}, s, m, n, e, 1_{\mathbb{L}_2}\}$ characterized by Hasse diagram in Figure 3.

Define the closure operator $cl : \mathbb{L}_2 \to \mathbb{L}_2$ such that $cl(0_{\mathbb{L}_2}) = cl(s) = s$, $cl(n) = cl(m) = m$ and $cl(e) = cl(1_{\mathbb{L}_2}) = 1_{\mathbb{L}_2}$. By virtue of the construction approach in Theorem 3.1, the uninorm $U^2_{(T,cl)} : \mathbb{L}_2 \times \mathbb{L}_2 \to \mathbb{L}_2$ is given as in Table 2 when considering the t-norm $T^{\wedge} : [0_{\mathbb{L}_2}, e]^2 \to [0_{\mathbb{L}_2}, e]$. Clearly, $U^2_{(T,cl)}(n, n) = n$. On the other hand, the uninorms U^4 and U^5 constructed by [14, Theorem 3.4] and [53, Proposition 3.6], respectively, satisfy that $U^4(n, n) = U^5(n, n) = m$. Hence, $U^2_{(T,cl)}$ differs from the uninorms U^4 and U^5 on $\mathbb{L}_2$.

Remark 3.10. Notice that the uninorm $U_{(T,cl)}$ in Theorem 3.1 coincides with the t-conorm S^W on $[e, 1_{\mathbb{L}}]^2$. However, $U_{(T,cl)}$ does not have to coincide with another t-conorm except S^W on $[e, 1_{\mathbb{L}}]^2$. To demonstrate this observation, considering the lattice $\mathbb{L}_1$ in Figure 2, we define the closure operator $cl : \mathbb{L}_1 \to \mathbb{L}_1$ such that $cl(x) = 1_{\mathbb{L}_1}$ for all $x \in \mathbb{L}_1$. Assume that the uninorm $U_{(T,cl)} \mid [e, 1_{\mathbb{L}_1}]^2$ is the t-conorm $S^{\vee} : [e, 1_{\mathbb{L}_1}]^2 \to [e, 1_{\mathbb{L}_1}]$. By

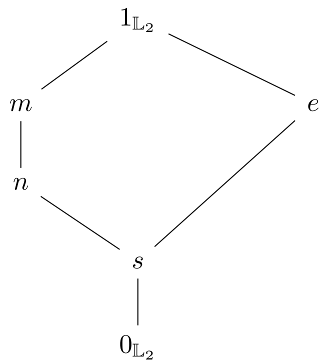

Fig. 3. The lattice $\mathbb{L}_2$.

$U^2_{(T,cl)}$	$0_{\mathbb{L}_2}$	s	n	m	e	$1_{\mathbb{L}_2}$
$0_{\mathbb{L}_2}$	$0_{\mathbb{L}_2}$	$0_{\mathbb{L}_2}$	$0_{\mathbb{L}_2}$	$0_{\mathbb{L}_2}$	$0_{\mathbb{L}_2}$	$0_{\mathbb{L}_2}$
s	$0_{\mathbb{L}_2}$	s	s	s	s	s
n	$0_{\mathbb{L}_2}$	s	n	m	n	$1_{\mathbb{L}_2}$
m	$0_{\mathbb{L}_2}$	s	m	m	m	$1_{\mathbb{L}_2}$
e	$0_{\mathbb{L}_2}$	s	n	m	e	$1_{\mathbb{L}_2}$
$1_{\mathbb{L}_2}$	$0_{\mathbb{L}_2}$	s	$1_{\mathbb{L}_2}$	$1_{\mathbb{L}_2}$	$1_{\mathbb{L}_2}$	$1_{\mathbb{L}_2}$

Tab. 2. Uninorm $U^2_{(T,cl)}$ on $\mathbb{L}_2$.

applying the construction approach in Theorem 3.1, we obtain

$$U_{(T,cl)}(u,r) = cl(u) \vee cl(r) = 1_{\mathbb{L}_1} > r = S^{\vee}(r,r) = U_{(T,cl)}(r,r),$$

for $u \in \mathbb{I}_e$ and $r > e$ with $u < r$. It contradicts the increasingness property of $U_{(T,cl)}$. Therefore, $U_{(T,cl)}$ does not have to coincide with any t-conorm except the t-conorm S^W on $[e, 1_{\mathbb{L}_1}]^2$.

We suggest in Theorem 3.11 a dual construction method for uninorms on bounded lattices. Namely, based on a t-conorm S on $[e, 1_{\mathbb{L}}]^2$ and an interior operator int on $\mathbb{L}$, we define the family of uninorm $U_{(S,int)}$ on $\mathbb{L}$ with a neutral element e.

Theorem 3.11. Let $e \in \mathbb{L}\backslash\{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$ and $S : [e, 1_{\mathbb{L}}]^2 \to [e, 1_{\mathbb{L}}]$ be a t-conorm. The function $U_{(S,int)} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$, given by the formula (9), is a uninorm on $\mathbb{L}$ with a neutral element e for every interior operator $int : \mathbb{L} \to \mathbb{L}$ iff $f < h$ and $d \wedge f \in \mathbb{I}_e \cup \{0_{\mathbb{L}}\}$ for all

$d, f \in \mathbb{I}_e$, $h \in]e, 1_{\mathbb{L}}]$.

$$U_{(S,int)}(a,b) = \begin{cases} S(a,b) & \text{if } (a,b) \in [e, 1_{\mathbb{L}}]^2, \\ 0_{\mathbb{L}} & \text{if } (a,b) \in [0_{\mathbb{L}}, e[^2, \\ int(a) \wedge int(b) & \text{if } (a,b) \in \mathbb{I}_e \times [0_{\mathbb{L}}, e[\cup [0_{\mathbb{L}}, e[\times \mathbb{I}_e, \\ a \wedge b & \text{if } (a,b) \in \mathbb{I}_e \times \mathbb{I}_e, \\ a & \text{if } (a,b) \in (\mathbb{I}_e \cup [0_{\mathbb{L}}, e]) \times \{e\}, \\ b & \text{if } (a,b) \in \{e\} \times (\mathbb{I}_e \cup [0_{\mathbb{L}}, e]), \\ a \vee b & \text{otherwise.} \end{cases} \tag{9}$$

Proof. It is similar to that of Theorem 3.1. □

Remark 3.12. Notice that the uninorm $U_{(S,int)} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ in Theorem 3.11 can be also defined such that

$$U_{(S,int)}(a,b) = \begin{cases} S(a,b) & \text{if } (a,b) \in [e, 1_{\mathbb{L}}]^2, \\ 0_{\mathbb{L}} & \text{if } (a,b) \in [0_{\mathbb{L}}, e[^2, \\ a & \text{if } (a,b) \in]e, 1_{\mathbb{L}}] \times \mathbb{I}_e \cup]e, 1_{\mathbb{L}}] \times [0_{\mathbb{L}}, e] \\ & \qquad \cup (\mathbb{I}_e \cup [0_{\mathbb{L}}, e]) \times \{e\}, \\ b & \text{if } (a,b) \in \mathbb{I}_e \times]e, 1_{\mathbb{L}}] \cup [0_{\mathbb{L}}, e] \times]e, 1_{\mathbb{L}}] \\ & \qquad \cup \{e\} \times (\mathbb{I}_e \cup [0_{\mathbb{L}}, e]), \\ a \wedge b & \text{if } (a,b) \in \mathbb{I}_e \times \mathbb{I}_e, \\ int(a) \wedge int(b) & \text{if } (a,b) \in \mathbb{I}_e \times [0_{\mathbb{L}}, e[\cup [0_{\mathbb{L}}, e[\times \mathbb{I}_e. \end{cases}$$

Remark 3.13. From Remark 3.12, the structure of the uninorm $U_{(S,int)} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ is illustrated in Figure 4.

If we take in Theorem 3.11 the t-conorm $S : [e, 1_{\mathbb{L}}]^2 \to [e, 1_{\mathbb{L}}]$ given by $S = S^{\vee}$, we define the corresponding uninorm as the following structure:

Corollary 3.14. Let $e \in \mathbb{L} \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$. The function $U_{(int)} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$, given by the formula (10), is a uninorm on $\mathbb{L}$ with a neutral element e for every interior operator $int : \mathbb{L} \to \mathbb{L}$ iff $f < h$ and $d \wedge f \in \mathbb{I}_e \cup \{0_{\mathbb{L}}\}$ for all $d, f \in \mathbb{I}_e$, $h \in]e, 1_{\mathbb{L}}]$.

$$U_{(int)}(a,b) = \begin{cases} 0_{\mathbb{L}} & \text{if } (a,b) \in [0_{\mathbb{L}}, e[^2, \\ int(a) \wedge int(b) & \text{if } (a,b) \in \mathbb{I}_e \times [0_{\mathbb{L}}, e[\cup [0_{\mathbb{L}}, e[\times \mathbb{I}_e, \\ a \wedge b & \text{if } (a,b) \in \mathbb{I}_e \times \mathbb{I}_e, \\ a & \text{if } (a,b) \in (\mathbb{I}_e \cup [0_{\mathbb{L}}, e]) \times \{e\}, \\ b & \text{if } (a,b) \in \{e\} \times (\mathbb{I}_e \cup [0_{\mathbb{L}}, e]), \\ a \vee b & \text{otherwise.} \end{cases} \tag{10}$$

If we allow in Theorem 3.11 to be a coatom of the element $e \in \mathbb{L} \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$, we define the corresponding uninorm as the following structure:

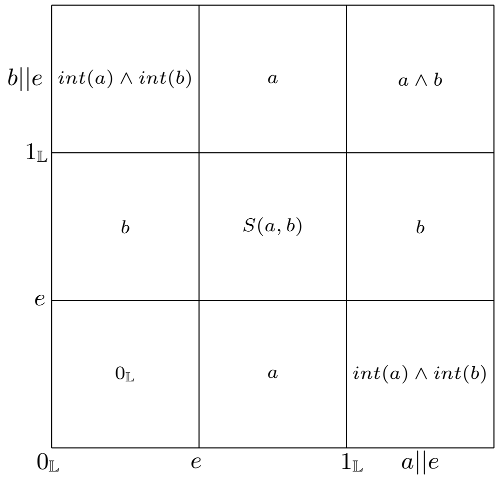

Fig. 4. Uninorm $U_{(S,int)} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ in Theorem 3.11.

Corollary 3.15. Let $e \in \mathbb{L}\backslash\{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$ be a coatom. The function $U_{(e,int)} : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$, given by the formula (11), is a uninorm on $\mathbb{L}$ with a neutral element e for every interior operator $int : \mathbb{L} \to \mathbb{L}$ iff $d \wedge f \in \mathbb{I}_e \cup \{0_{\mathbb{L}}\}$ for all $d, f \in \mathbb{I}_e$.

$$U_{(e,int)}(a,b) = \begin{cases} 0_{\mathbb{L}} & \text{if } (a,b) \in [0_{\mathbb{L}}, e[^2, \\ int(a) \wedge int(b) & \text{if } (a,b) \in \mathbb{I}_e \times [0_{\mathbb{L}}, e[\cup [0_{\mathbb{L}}, e[\times \mathbb{I}_e, \\ a \wedge b & \text{if } (a,b) \in \mathbb{I}_e \times \mathbb{I}_e, \\ a & \text{if } (a,b) \in (\mathbb{I}_e \cup [0_{\mathbb{L}}, e]) \times \{e\}, \\ b & \text{if } (a,b) \in \{e\} \times (\mathbb{I}_e \cup [0_{\mathbb{L}}, e]), \\ 1_{\mathbb{L}} & \text{otherwise.} \end{cases} \tag{11}$$

Remark 3.16. Let $e \in \mathbb{L}\backslash\{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$, $T : [0_{\mathbb{L}}, e]^2 \to [0_{\mathbb{L}}, e]$ be a t-norm and $int : \mathbb{L} \to \mathbb{L}$ be an interior operator. We suggest in Theorem 3.11 a new construction approach for uninorms on bounded lattices. To be more precise,

(i) If $(a,b) \in [0_{\mathbb{L}}, e[^2 \cup [0_{\mathbb{L}}, e[\times \mathbb{I}_e \cup \mathbb{I}_e \times [0_{\mathbb{L}}, e[\cup \mathbb{I}_e^2$, the method in [9, Theorem 11] puts for $U(a,b)$ the value of $T(a \wedge e, b \wedge e)$. On the other hand, when $(a,b) \in [0_{\mathbb{L}}, e[^2$ (resp. $(a,b) \in \mathbb{I}_e^2$) our construction puts for $U_{(S,int)}(a,b)$ the value of $0_{\mathbb{L}}$ (resp. $a \wedge b$). Moreover, in our construction $U_{(S,int)}(a,b) = int(a) \wedge int(b)$ for $(a,b) \in [0_{\mathbb{L}}, e[\times \mathbb{I}_e \cup \mathbb{I}_e \times [0_{\mathbb{L}}, e[$. However, both constructions coincide in the remaining domains;

(ii) If $(a,b) \in [0_{\mathbb{L}}, e[^2 \cup \mathbb{I}_e^2$, the method in [14, Theorem 3.10] puts for $U(a,b)$ the value of $int(a) \wedge int(b)$. On the other hand, when $(a,b) \in [0_{\mathbb{L}}, e[^2$ (resp. $(a,b) \in \mathbb{I}_e^2$) our construction puts for $U_{(S,int)}(a,b)$ the value of $0_{\mathbb{L}}$ (resp. $a \wedge b$). However, both constructions coincide in the remaining domains;

(iii) If $(a,b) \in [0_{\mathbb{L}}, e[^2$ (resp. $(a,b) \in [0_{\mathbb{L}}, e[\times \mathbb{I}_e \cup \mathbb{I}_e \times [0_{\mathbb{L}}, e[)$, the method in [14, Theorem 3.12] puts for $U(a,b)$ the value of $T(a,b)$ (resp. $a \wedge b$). Furthermore, in [14, Theorem 3.12] $U(a,b) = int(a) \wedge int(b)$ for $(a,b) \in \mathbb{I}_e^2$. On the other hand, when $(a,b) \in [0_{\mathbb{L}}, e[^2$ (resp. $(a,b) \in [0_{\mathbb{L}}, e[\times \mathbb{I}_e \cup \mathbb{I}_e \times [0_{\mathbb{L}}, e[)$ our construction puts for $U_{(S,int)}(a,b)$ the value of $0_{\mathbb{L}}$ (resp. $int(a) \wedge int(b)$). Moreover, in our construction $U_{(S,int)}(a,b) = a \wedge b$ for $(a,b) \in \mathbb{I}_e^2$. However, both constructions coincide in the remaining domains;

(iv) If $(a,b) \in \mathbb{I}_e^2$ (resp. $(a,b) \in [0_{\mathbb{L}}, e[^2 \cup [0_{\mathbb{L}}, e[\times \mathbb{I}_e \cup \mathbb{I}_e \times [0_{\mathbb{L}}, e[)$, the method in [53, Corollary 4.2] puts for $U(a,b)$ the value of $int(a) \wedge int(b)$ (resp. $0_{\mathbb{L}}$). On the other hand, when $(a,b) \in \mathbb{I}_e^2$ (resp. $(a,b) \in [0_{\mathbb{L}}, e[\times \mathbb{I}_e \cup \mathbb{I}_e \times [0_{\mathbb{L}}, e[)$ our construction puts for $U_{(S,int)}(a,b)$ the value of $a \wedge b$ (resp. $int(a) \wedge int(b)$). Moreover, in our construction $U_{(S,int)}(a,b) = 0_{\mathbb{L}}$ for $(a,b) \in [0_{\mathbb{L}}, e[^2$. However, both constructions coincide in the remaining domains.

Remark 3.17. Let $e \in \mathbb{L} \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$. If we define the interior operator $int : \mathbb{L} \to \mathbb{L}$ such that $int(x) = x$ for all $x \in \mathbb{L}$, then the following statements hold:

(i) the uninorm $U_{(S,int)}$ in Theorem 3.11 coincides with the uninorm in [14, Theorem 3.10], where e is an atom;

(ii) the uninorm $U_{(S,int)}$ in Theorem 3.11 coincides with the uninorm in [14, Theorem 3.12], where the t-norm $T : [0_{\mathbb{L}}, e]^2 \to [0_{\mathbb{L}}, e]$ is $T = T^W$;

(iii) the uninorm $U_{(S,int)}$ in Theorem 3.11 coincides with the uninorm in [53, Corollary 4.2], where $c_1 \parallel c_2$ for all $c_1 \in]0_{\mathbb{L}}, e]$, $c_2 \in \mathbb{I}_e$;

(iv) the uninorm $U_{(S,int)}$ in Theorem 3.11 coincides with the uninorm in [53, Corollary 4.4], where $c_1 \parallel c_2$ for all $c_1 \in]0_{\mathbb{L}}, e]$, $c_2 \in \mathbb{I}_e$ and the t-norm $T : [0_{\mathbb{L}}, e]^2 \to [0_{\mathbb{L}}, e]$ is $T = T^W$

Similarly to Examples 3.8 and 3.9, we can show that the uninorm obtained via the approach in Theorem 3.11 does not have to coincide with the ones introduced by [9, Theorem 11], [14, Theorems 3.10 and 3.12], and [53, Corollaries 4.2 and 4.4].

Remark 3.18. Let $e \in \mathbb{L} \backslash \{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$, $cl : \mathbb{L} \to \mathbb{L}$ be a closure operator, and $int : \mathbb{L} \to \mathbb{L}$ be an interior operator. The uninorms obtained by the methods in Theorems 3.1 and 3.11 do not have to coincide with those introduced by [43, Theorems 4.1 and 5.1]. That is to say

(i) the uninorm $U_{(T,cl)}$ in Theorem 3.1 satisfies that $U_{(T,cl)}(0_{\mathbb{L}}, 1_{\mathbb{L}}) = 0_{\mathbb{L}}$ and $U_{(T,cl)}(1_{\mathbb{L}}, x) = 1_{\mathbb{L}}$ for any $x \in \mathbb{I}_e$;

(ii) the uninorm $U_{(S,int)}$ in Theorem 3.11 satisfies that $U_{(S,int)}(0_{\mathbb{L}}, 1_{\mathbb{L}}) = 1_{\mathbb{L}}$ and $U_{(S,int)}(0_{\mathbb{L}}, x) = 0_{\mathbb{L}}$ for any $x \in \mathbb{I}_e$;

(iii) the uninorm U in [43, Theorem 4.1] satisfies that $U(0_{\mathbb{L}}, 1_{\mathbb{L}}) = 1_{\mathbb{L}}$ and $U(0_{\mathbb{L}}, x) = x$ for any $x \in \mathbb{I}_e$;

(iv) the uninorm U in [43, Theorem 5.1] satisfies that $U(0_{\mathbb{L}}, 1_{\mathbb{L}}) = 0_{\mathbb{L}}$ and $U(1_{\mathbb{L}}, x) = x$ for any $x \in \mathbb{I}_e$.

Remark 3.19. Notice that the uninorm $U_{(S,int)}$ in Theorem 3.11 coincides with the t-norm T^W on $[0_{\mathbb{L}}, e]^2$. However, $U_{(S,int)}$ does not have to coincide with another t-norm except T^W on $[0_{\mathbb{L}}, e]^2$. To illustrate this fact, take the lattice $\mathbb{L}_3 = \{0_{\mathbb{L}_3}, k, s, n, e, 1_{\mathbb{L}_3}\}$ depicted by Hasse diagram in Figure 5. We define the interior operator $int : \mathbb{L}_3 \to \mathbb{L}_3$ such that $int(0_{\mathbb{L}_3}) = 0_{\mathbb{L}_3}$, $int(e) = int(s) = int(k) = k$, $int(n) = n$ and $int(1_{\mathbb{L}_3}) = 1_{\mathbb{L}_3}$. Assume that the uninorm $U_{(S,int)} \mid [0_{\mathbb{L}_3}, e]^2$ is the t-norm $T' : [0_{\mathbb{L}_3}, e]^2 \to [0_{\mathbb{L}_3}, e]$ given as in Table 3.

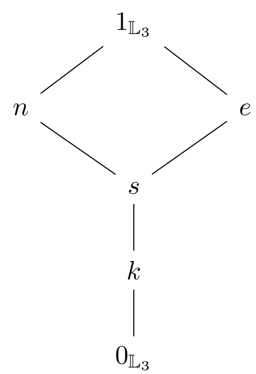

Fig. 5. The lattice $\mathbb{L}_3$.

T'	$0_{\mathbb{L}_3}$	k	s	e
$0_{\mathbb{L}_3}$	$0_{\mathbb{L}_3}$	$0_{\mathbb{L}_3}$	$0_{\mathbb{L}_3}$	$0_{\mathbb{L}_3}$
k	$0_{\mathbb{L}_3}$	$0_{\mathbb{L}_3}$	$0_{\mathbb{L}_3}$	k
s	$0_{\mathbb{L}_3}$	$0_{\mathbb{L}_3}$	s	s
e	$0_{\mathbb{L}_3}$	k	s	e

Tab. 3. T-norm T' on $[0_{\mathbb{L}_3}, e]^2$.

By applying the construction approach in Theorem 3.11, we obtain

$$U_{(S,int)}\left(U_{(S,int)}(s,s), n\right) = U_{(S,int)}(T'(s,s), n) = U_{(S,int)}(s,n) = int(s) \wedge int(n) = k,$$

and

$$U_{(S,int)}\left(s, U_{(S,int)}(s,n)\right) = U_{(S,int)}(s, int(s) \wedge int(n)) = U_{(S,int)}(s,k) = T'(s,k) = 0_{\mathbb{L}_3},$$

which contradicts the associativity property of $U_{(S,int)}$. Therefore, $U_{(S,int)}$ does not have to coincide with any t-norm except the t-norm T^W on $[0_{\mathbb{L}_3}, e]^2$.

4. CONCLUSION

This paper characterized two new families of uninorms on bounded lattices by virtue of the closure and interior operators. We introduced two novel methods to obtain uninorms on a bounded lattice $\mathbb{L}$ with a neutral element $e \in \mathbb{L}\backslash\{0_{\mathbb{L}}, 1_{\mathbb{L}}\}$, where some necessary and sufficient conditions are required. It should be noted that our methods are derived from a t-norm on $[0_{\mathbb{L}}, e]^2$ and a closure operator on $\mathbb{L}$, or a t-conorm on $[e, 1_{\mathbb{L}}]^2$ and an interior operator on $\mathbb{L}$. Subsequently, some specific examples were included to help comprehend the newly added classes of uninorms. Furthermore, we investigate how our approaches compare with some methods outlined in [9, 14, 53]. We also demonstrate that our construction approaches do not have to coincide with the known ones.

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Gül Deniz Çaylı, Department of Mathematics, Faculty of Science, Karadeniz Technical University, 61080 Trabzon. Turkey.
e-mail: guldeniz.cayli@ktu.edu.tr