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MATRIX REPRESENTATION OF FINITE EFFECT ALGEBRAS

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In this paper we present representation of finite effect algebras by matrices. For each nontrivial finite effect algebra E we construct set of matrices $M(E)$ in such a way that effect algebras E_1 and E_2 are isomorphic if and only if $M(E_1) = M(E_2)$. The paper also contains the full list of matrices representing all nontrivial finite effect algebras of cardinality at most 8.

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1. INTRODUCTION

Effect algebras have been introduced by Foulis and Bennet in 1994 (see [4]) for the study of foundations of quantum mechanics (see [3]). Independently, Chovanec and Kôpka introduced an essentially equivalent structure called *D-poset* (see [8]). Another equivalent structure was introduced by Giuntini and Greuling in [5].

The most important example of an effect algebra is $(E(H), 0, I, \oplus)$, where H is a Hilbert space and $E(H)$ consists of all self-adjoint operators A on H such that $0 \leq A \leq I$. For $A, B \in E(H)$, $A \oplus B$ is defined if and only if $A + B \leq I$ and then $A \oplus B = A + B$. Elements of $E(H)$ are called *effects* and they play an important role in the theory of quantum measurements ([1, 2]).

A quantum effect may be treated as two-valued (it means 0 or 1) quantum measurement that may be unsharp (fuzzy). If there exist some pairs of effects a, b which posses an orthosum $a \oplus b$ then this orthosum correspond to a parallel measurement of two effects.

In this paper to each finite effect algebra we assign (see 3.1) a set of matrices $M(E)$ in such a way that effect algebras E_1 and E_2 are isomorphic if and only if $M(E_1) = M(E_2)$. We also present the list of matrices representing all nontrivial finite effect algebras of cardinality at most 8. Using this list it is easy to check that every effect algebras of cardinality ≤ 8 has a state. So the 9-element Riečanová's example of effect algebra without a state is smallest.

Let us start with the following definition of an effect algebra.

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Definition 1.1. In [4] an *effect algebra* is defined to be an algebraic system $(E, 0, 1, \oplus)$ consisting of a set E, two special elements $0, 1 \in E$ called the *zero* and the *unit*, and a partially defined binary operation $\oplus$ on E that satisfies the following conditions for all $p, q, r \in E$:

1. [Commutative Law] If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.

2. [Associative Law] If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ is defined, $(p \oplus q) \oplus r$ is defined, and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

3. [Orthosupplementation Law] For every $p \in E$ there exists a unique $q \in E$ such that $p \oplus q$ is defined and $p \oplus q = 1$.

4. [Zero-unit Law] If $1 \oplus p$ is defined, then $p = 0$.

For simplicity, we often refer to E, rather than to $(E, 0, 1, \oplus)$, as being an effect algebra. If $p, q \in E$, we say that p and q are orthogonal and write $p \perp q$ iff $p \oplus q$ is defined in E. If $p, q \in E$ and $p \oplus q = 1$, we call q the *orthosupplement* of p and write $p' = q$.

Definition 1.2. For effect algebras E_1, E_2 a mapping $\phi \colon E_1 \to E_2$ is said to be an *isomorphism* if ϕ is a bijection, $a \perp b \iff \phi(a) \perp \phi(b)$, $\phi(1) = 1$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$.

It is shown in [4] that the relation $\leq$ defined for $p, q \in E$ by $p \leq q$ iff $\exists r \in E$ with $p \oplus r = q$ is a partial order on E and $0 \leq p \leq 1$ holds for all $p \in E$. It is also shown that the mapping $p \mapsto p'$ is an order-reversing involution and that $q \perp p$ iff $q \leq p'$. Furtheremore, E satisfies the following *cancellation law*: If $p \oplus q \leq r \oplus q$, then $p \leq r$.

For $n \in \mathbb{N}$ and $x \in E$ let $nx = x \oplus x \oplus \cdots \oplus x$ (n-times). We write $\operatorname{ord}(x) = n \in \mathbb{N}$ if n is the greatest integer such that nx exists in E, if no such n exists, then $\operatorname{ord}(x) = \infty$.

An *atom* of an effect algebra E is a minimal element of $E \setminus \{0\}$. An effect algebra E is *atomic* if for every non-zero element $x \in E$ there exists an atom $a \in E$ such that $a \leq x$. An effect algebra E is *non-trivial* if $|E| > 1$.

We say that a finite system $F = (x_k)_{k=1}^n$ of not necessarily different elements of an effect algebra E is *orthogonal* if $x_1 \oplus x_2 \oplus \cdots \oplus x_n$ (written $\bigoplus_{k=1}^n x_k$, $\bigoplus\{x_k | k \in \{1, 2, \ldots, n\}\}$ or $\bigoplus F$) exists in E. Here we define $x_1 \oplus x_2 \oplus \cdots \oplus x_n = (x_1 \oplus x_2 \oplus \cdots \oplus x_{n-1}) \oplus x_n$ supposing that $\bigoplus_{k=1}^{n-1} x_k$ is defined and $\bigoplus_{k=1}^{n-1} x_k \leq x_n'$. We also define $\bigoplus \emptyset = 0$. An arbitrary system $G = (x_k)_{k \in H}$ of not necessarily different elements of E is called *orthogonal* if $\bigoplus K$ exists for every finite $K \subseteq G$. We say that for an orthogonal system $G = (x_k)_{k \in H}$ the element $\bigoplus G$ exists if and only if $\vee\{\bigoplus K | K \subseteq G, K \text{ finite }\}$ exists in E and then we put $\bigoplus G = \vee\{\bigoplus K | K \subseteq G, K \text{ finite }\}$. We call an effect algebra E *orthocomplete* if every orthogonal system $G = (s_k)_{k \in H}$ of elements of E has the sum $\bigoplus G$.

Proposition 1.3. (Wei Ji [11, Proposition 3.1]) Let E be an orthocomplete atomic effect algebra. Then for every $x \in E$, there is a set $\{a_i | i \in I\}$ of mutually different atoms in E and a set $\{k_i | i \in I\}$ of positive integers such that $x = \bigoplus\{k_i a_i | i \in I\}$.

Let E be a finite effect algebra. Then A is orthocomplete and atomic. If $|E| > 1$ and E has atoms $a_1, \dots a_m$ then by Proposition 1.3 for every $x \in E$ there exist non-negative integers $k_1, \dots, k_m$ such that $x = \bigoplus_{i=1}^{m} k_i a_i$.

2. E-TEST SPACES

In [7] Gudder introduced (algebraic) E-test spaces:

Definition 2.1. Let X be a nonempty set and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $f, g \in \mathbb{N}_0^X$. We define

- $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$,
- $f + g \in \mathbb{N}_0^X$ by $(f + g)(x) = f(x) + g(x)$,
- $g - f \in \mathbb{N}_0^X$ by $(g - f)(x) = g(x) - f(x)$ if $f \leq g$,
- $\mathbf{0} \in \mathbb{N}_0^X$ by $\mathbf{0}(x) = 0$ for all $x \in X$.

A pair $(X, \mathcal{T})$ is an *E-test space* if and only if $\mathcal{T} \subseteq \mathbb{N}_0^X$ and the following conditions hold:

1. For any $x \in X$ there exists a $t \in \mathcal{T}$ such that $t(x) \neq 0$.
2. If $s, t \in \mathcal{T}$ with $s \leq t$, then $s = t$.

The elements of $\mathcal{T}$ are called *tests* of $(X, \mathcal{T})$.

Definition 2.2. Let $(X, \mathcal{T})$ be an E-test space. Let $\mathcal{E}(X, \mathcal{T}) = \{f \in \mathbb{N}_0^X : f \leq t$ for some $t \in \mathcal{T}\}$. The elements of $\mathcal{E}(X, \mathcal{T})$ are called *events* of $(X, \mathcal{T})$. Let $f, g, h \in \mathcal{E}(X, \mathcal{T})$ then we say that f, g are

1. *orthogonal* ($f \perp g$) if $f + g \in \mathcal{E}(X, \mathcal{T})$,
2. *local complements of each other* (f loc g) if $f + g \in \mathcal{T}$,
3. *perspective with axis h* ($f \approx_h g$) if $f + h \in \mathcal{T}$ and $g + h \in \mathcal{T}$,
4. *perspective* ($f \approx g$) if there exists $h \in \mathcal{E}(X, \mathcal{T})$ such that $f \approx_h g$.

We say that $(X, \mathcal{T})$ is *algebraic* if for $f, g, h \in \mathcal{E}(X, \mathcal{T})$, $f \approx g$ and $h \perp f$ imply that $h \perp g$.

Lemma 2.3. (Gudder [7, Lemma 3.1])

(a) An E-test space $(X, \mathcal{T})$ is algebraic if and only if for $f, g, h \in \mathcal{E}(X, \mathcal{T})$, $f \approx g$ and h loc f imply h loc g.

(b) If $(X, \mathcal{T})$ is algebraic, then $\approx$ is an equivalence relation on $\mathcal{E}(X, \mathcal{T})$.

Let $(X,\mathcal{T})$ be an algebraic E-test space. By Lemma 2.3 perspectivity in an algebraic E-test space is transitive, hence it is an equivalence.

If $f \in \mathcal{E}(X,\mathcal{T})$, we define $\pi(f) = \{g \in \mathcal{E}(X,\mathcal{T})\colon g \approx f\}$. The equivalence class $\pi(f)$ is called *the perspectivity class of* f. Let

$$\Pi = \Pi(X) = \{\pi(f)\colon f \in \mathcal{E}(X,\mathcal{T})\}.$$

Theorem 2.4. (Gudder [7, Theorem 3.2]) If $(X,\mathcal{T})$ is an algebraic E-test space, then $\Pi(X)$ can be organized into an effect algebra.

We explain how the $\Pi(X)$ is organized into an effect algebra:

1. We define $0, 1 \in \Pi$ by $0 = \pi(\mathbf{0}) = \{\mathbf{0}\}$ and $1 = \pi(t)$ for any $t \in \mathcal{T}$.

2. For every $f \in \mathcal{E}(X,\mathcal{T})$ we define $\pi(f)' = \pi(g)$ if g loc f (such g exists since there exists $t \in \mathcal{T}$ such that $f \leq t$ and then $t - f \in \mathcal{E}(X,\mathcal{T})$ and $(t-f)$ loc f).

3. If $f, g \in \mathcal{E}(X,\mathcal{T})$ we define $\pi(f) \oplus \pi(g) = \pi(f+g)$ when $f \perp g$.

The following Lemma shows how algebraicity of an E-test space can be checked using only tests.

Lemma 2.5. Let $(X,\mathcal{T})$ be an E-test space. Then $(X,\mathcal{T})$ is algebraic if and only if for every tests $t_1, t_2, t_3 \in \mathcal{T}$ if $t_1 + t_2 \geq t_3$ then $t_1 + t_2 - t_3 \in \mathcal{T}$.

P r o o f. Let $(X,\mathcal{T})$ be an E-test space.

Assume that for every tests $t_1, t_2, t_3 \in \mathcal{T}$ if $t_1 + t_2 \geq t_3$ then $t_1 + t_2 - t_3 \in \mathcal{T}$.

We show that $(X,\mathcal{T})$ is algebraic using Lemma 2.3. Let $f, g, h \in \mathcal{E}(X,\mathcal{T})$, $f \approx g$ and h loc f. Then there exists $d \in \mathcal{E}(X,\mathcal{T})$ such that $f+d, g+d \in \mathcal{T}$ and $h+f \in \mathcal{T}$. Hence $(g+d)+(h+f) \geq f+d$, which implies

$$(g+d)+(h+f)-(f+d) = h+g \in \mathcal{T}.$$

It follows that h loc g so $(X,\mathcal{T})$ is algebraic.

Now assume that $(X,\mathcal{T})$ is algebraic. Let $t_1, t_2, t_3 \in \mathcal{T}$ and $t_1 + t_2 \geq t_3$. Let $f \in \mathbb{N}_0^X$ be a function such that $f(x) = \min(t_1(x), t_3(x))$ for all $x \in X$. Then $f \in \mathcal{E}(X,\mathcal{T})$ since $f \leq t_1$. Let $g = f + t_2 - t_3$. We know that $t_3 - t_2 \leq t_1$ and $t_3 - t_2 \leq t_3$ so $t_3 - t_2 \leq \min(t_1, t_3) = f$ hence $g = f + t_2 - t_3 \geq 0$. Moreover $g = f + t_2 - t_3 \leq t_2$ since $f \leq t_3$. Therefore $g \in \mathcal{E}(X,\mathcal{T})$. Let $h = t_1 - f$. Then $h \in \mathcal{E}(X,\mathcal{T})$ since $h \geq 0$ and $h \leq t_1$.

Let us observe that $f \approx g$ since $f + (t_3 - f) = t_3 \in \mathcal{T}$ and $g + (t_3 - f) = f + t_2 - t_3 + t_3 - f = t_2 \in \mathcal{T}$. Moreover $h + f = t_1 - f + f = t_1 \in \mathcal{T}$ so h loc f. By Lemma 2.3 we have h loc g thus $h + g = (t_1 - f) + (f + t_2 - t_3) = t_1 + t_2 - t_3 \in \mathcal{T}$. □

Complexity of checking if an E-test space is algebraic using Lemma 2.5 is smaller than using the Definition 2.2 since there is more events than tests.

3. MAIN THEOREM

By $M_{n\times m}(\mathbb{N}_0)$ we denote the set of all $n \times m$ matrices whose entries are elements of $\mathbb{N}_0$, i. e., natural numbers including 0.

Definition 3.1. Let E be a non-trivial finite effect algebra with atoms $a_1, \dots, a_m$ ($a_i \neq a_j$ for $i \neq j$) and $a = (a_1, \dots, a_m)$. Then define $Seq_a(E) = \{(x_1, \dots, x_m) \in \mathbb{N}_0^m \colon \bigoplus_{t=1}^{m} x_t a_t = 1\}$. Let $n = card(Seq_a(E))$ and

$$\begin{aligned} M(E) \;=\; & \Big\{[y_{ij}] \in M_{n\times m}(\mathbb{N}_0) \colon \underset{1\le i<j\le n}{\forall} \;\; \underset{1\le t\le m}{\exists} y_{it} \neq y_{jt}, \\ & \{(y_{i1}, \dots, y_{im}) \in \mathbb{N}_0^m \colon 1 \le i \le n\} = Seq_{(a_{\sigma(1)},\dots,a_{\sigma(m)})}(E) \text{ where} \\ & \sigma \colon \{1, \dots, m\} \to \{1, \dots, m\} \text{ is some permutation}\Big\}. \end{aligned}$$

If E is a non-trivial finite effect algebra and $A \in M(E)$ we say that A represents E.

It turns out that rows in $A \in M(E)$ are all elements of $Seq_{(a_{\sigma(1)},\dots,a_{\sigma(m)})}(E)$ where $\sigma \colon \{1, \dots, m\} \to \{1, \dots, m\}$ is some permutation. Moreover if $A, B \in M(E)$ then B can be obtained from A by permuting its rows and columns. By e_k we understand row $[0, \dots, 0, 1, 0, \dots, 0]$ where 1 is only at kth position.

Definition 3.2. Let $n, m \in \mathbb{N}$. Let $\mathcal{B}_{nm}$ be the set of matrices $A \in M_{n\times m}(\mathbb{N}_0)$ such that

(1) All rows and columns in A are non-zero.

(2) If r_1 is ith row in A, r_2 is jth row in A and $r_2 \ge r_1 - e_k \ge \mathbf{0}$ for some $1 \le k \le m$ then $i = j$.

(3) If r_1, r_2, r_3 are rows in A and $r_1 + r_2 \ge r_3$, then $r_1 + r_2 - r_3$ is a row in A.

The condition (2) in the above Definition follows that distinct rows are incomparable.

Definition 3.3. Let $A = [y_{ij}] \in M_{n\times m}(\mathbb{N}_0)$ and $X = \{1, 2, \dots, m\}$. Define $T(A) = \{t_1, \dots, t_n\}$ where $t_1, t_2, \dots, t_n \colon X \to \mathbb{N}_0$ are functions such that $t_i(j) = y_{ij}$.

Lemma 3.4. If $A \in \mathcal{B}_{nm}$ and $X = \{1, 2, \dots, m\}$, then $(X, T(A))$ is an algebraic E-test space.

Proof. Let $A = [y_{ij}] \in \mathcal{B}_{nm}$.

Let $X = \{1, 2, \dots, m\}$ and $T(A) = \{t_1, \dots, t_n\}$. Then $(X, T(A))$ is an E-test space by (1) and (2) in the Definition 3.2. By Lemma 2.5 and (3) $(X, T(A))$ is algebraic. □

The matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \notin B_{23}$ since the condition (2) in the Definition 3.2 is not satisfied: $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \ge \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} - e_1 \ge \mathbf{0}$ but $(X, T(A))$ is an algebraic E-test space since rows in A are incomparable and algebraicity follows from the Lemma 2.5. So $(X, T(A))$ can be an algebraic E-test space for $A \notin B_{nm}$. Equivalent matrix representation is $A' = \begin{bmatrix} 1 & 1 \end{bmatrix}$ (from cancellativity it follows that first and third atom must be the same).

Definition 3.5. Let $A \in \mathcal{B}_{nm}$, $X = \{1, 2, \ldots, m\}$ and $\mathcal{T} = T(A) = \{t_1, \ldots, t_n\}$. Then $(X, \mathcal{T})$ is an algebraic E-test space by Lemma 3.4. By Theorem 2.4 $\Pi(A) = \Pi(X)$ can be organized into an effect algebra.

We describe when for a matrix A with m columns there exists a non-trivial effect algebra E with m atoms such that $A \in M(E)$:

Theorem 3.6. Let $A \in M_{n\times m}(\mathbb{N}_0)$ for $n, m \in \mathbb{N}$. Then there exists a non-trivial finite effect algebra E with atoms $a_1, \ldots, a_m$ ($a_i \neq a_j$ for $i \neq j$) such that $A \in M(E)$ if and only if $A \in \mathcal{B}_{nm}$.

Proof.

- $\Rightarrow$ Let $A = [y_{ij}] \in M_{n\times m}(\mathbb{N}_0)$ and E be a non-trivial finite effect algebra with atoms $a_1, \ldots, a_m$ ($a_i \neq a_j$ for $i \neq j$) and $a = (a_1, \ldots, a_m)$ such that $A \in M(E)$. Then
  (1) Each row in A is non-zero: if $(x_1, \ldots, x_m)$ is a row in A and $x_1 = x_2 = \ldots = x_m = 0$ then $0 = \bigoplus_{t=1}^{m} x_t a_t = 1$ and we get a contradiction since E is non-trivial.
  Each column in A is non-zero: let $1 \leq j \leq m$ then there exist non-negative integers $k_1, \ldots, k_m$ such that $a_j' = \bigoplus_{i=1}^{m} k_i a_i$ so
  $$1 = a_j \oplus a_j' = \bigoplus_{i=1}^{j-1} k_i a_i \oplus (k_j + 1) a_j \oplus \bigoplus_{i=j+1}^{m} k_i a_i$$
  and $(k_1, \ldots, k_{j-1}, k_j + 1, k_{j+1}, \ldots, k_m) \in Seq_a(E)$ is a row in A with a non-zero jth coordinate. It follows that the jth column is also non-zero.
  (2) If $r_1 = (y_{i1}, \ldots, y_{im})$ and $r_2 = (y_{j1}, \ldots, y_{jm})$ and $r_2 \geq r_1 - e_k \geq \mathbf{0}$ for some $1 \leq k \leq m$ then $y_{jt} \geq y_{it}$ for $t \neq k$ and $y_{jk} \geq y_{ik} - 1$. Therefore
  $$\bigoplus_{t=1}^{m} (y_{it}) a_t = \bigoplus_{t=1}^{m} (y_{jt}) a_t = 1$$
  so by cancellation law we have
  $$a_k = \bigoplus_{t=1}^{k-1} (y_{jt} - y_{it}) a_t \oplus (y_{jk} - y_{ik} + 1) a_k \oplus \bigoplus_{t=1}^{m} (y_{jt} - y_{it}) a_t.$$
  But a_k is an atom, so $(y_{i1}, \ldots, y_{im}) = (y_{j1}, \ldots, y_{jm})$. Suppose that $i \neq j$ then we obtain a contradiction since for $A \in M(E)$ we have $\forall_{1\leq i<j\leq n} \ \exists_{1\leq t\leq m} \ y_{it} \neq y_{jt}$. Hence $i = j$.
  (3) Let r_1, r_2, r_3 be rows in A, $r_1 + r_2 \geq r_3$ and $r_1 = (y_{i1}, \ldots, y_{im})$, $r_2 = (y_{j1}, \ldots, y_{jm})$, $r_3 = (y_{k1}, \ldots, y_{km})$ then $\bigoplus_{t=1}^{m} y_{it} a_t = \bigoplus_{t=1}^{m} y_{jt} a_t = \bigoplus_{t=1}^{m} y_{kt} a_t = 1$

and $y_{it} + y_{jt} \geq y_{kt}$ for $1 \leq t \leq m$. Let $x = (x_1, \dots, x_m)$ where $x_t = \min(y_{it}, y_{kt})$ for $1 \leq t \leq m$. By cancellation law

$$\bigoplus_{t=1}^{m} (y_{it} - x_t) a_t = \bigoplus_{t=1}^{m} (y_{kt} - x_t) a_t. \tag{1}$$

We have $r_3 - r_2 \leq r_1$ and $r_3 - r_2 \leq r_3$ so $r_3 - r_2 \leq \min(r_1, r_3)$ and $y_{kt} - y_{jt} \leq x_t$ for $1 \leq t \leq m$. Hence $x_t - y_{kt} + y_{jt} \geq 0$ for $1 \leq t \leq m$ and by (1), we obtain

$$\bigoplus_{t=1}^{m} (y_{it} - x_t + x_t - y_{kt} + y_{jt}) a_t = \bigoplus_{t=1}^{m} (y_{kt} - x_t + x_t - y_{kt} + y_{jt}) a_t,$$

so

$$\bigoplus_{t=1}^{m} (y_{it} - y_{kt} + y_{jt}) a_t = \bigoplus_{t=1}^{m} (y_{jt}) a_t = 1$$

and $(y_{i1} + y_{j1} - y_{k1}, \dots, y_{it} + y_{jt} - y_{kt}, \dots, y_{im} + y_{jm} - y_{km}) \in Seq_a(E)$ so $r_1 + r_2 - r_3$ is a row in A.

Hence $A \in \mathcal{B}_{nm}$.

$\Leftarrow$ Let $A = [y_{ij}] \in \mathcal{B}_{nm}$.

Let $X = \{1, 2, \dots, m\}$ and $\mathcal{T} = T(A) = \{t_1, \dots, t_n\}$. Then $(X, \mathcal{T})$ is an algebraic E-test space by Lemma 3.4. By Theorem 2.4 $\Pi(X)$ can be organized into an effect algebra.

We show that $A \in M(\Pi(X))$. It is enough to show that $Seq(\Pi(X)) = \{t_1, \dots, t_n\}$ since the condition $\forall_{1 \leq i < j \leq n} \; \exists_{1 \leq t \leq m} \; y_{it} \neq y_{jt}$ follows from (2).

First we describe atoms in $\Pi(X)$. Let $e_i \colon X \to \mathbb{N}_0$ be a function such that $e_i(x) = \begin{cases} 0 & x \neq i \\ 1 & x = i \end{cases}$ for $1 \leq i \leq m$. Then $e_i \in \mathcal{E}(X, \mathcal{T})$ for $1 \leq i \leq m$.

We show that $\{\pi(e_1), \dots, \pi(e_m)\}$ is the set of mutually different atoms in $\Pi(X)$. Let $f \in \mathcal{E}(X, \mathcal{T})$ and let $\pi(f)$ be an atom in $\Pi(X)$. Then $\mathbf{0} < f$ so there exists $i \in \{1, \dots, m\}$ such that $f(i) > 0$ so $f(i) \geq 1$ and $f \geq e_i$ so $\pi(f) \geq \pi(e_i) > 0$ then $\pi(f) = \pi(e_i)$.

Let $k \in \{1, \dots, m\}$. We show that $\pi(e_k)$ is an atom in $\Pi(X)$. Let $f \in \mathcal{E}(X, \mathcal{T})$ and $\pi(e_k) \geq \pi(f) > 0$. Then there exists $g \in \mathcal{E}(X, \mathcal{T})$ such that $f \perp g$ and $\pi(e_k) = \pi(f) \oplus \pi(g) = \pi(f + g)$ so $e_k \approx_h f + g$ for some $h \in \mathcal{E}(X, \mathcal{T})$. Then $e_k + h \in \mathcal{T}$ and $f + g + h \in \mathcal{T}$, so $r_1 = e_k + h$ and $r_2 = f + g + h$ are rows in A. Moreover, $r_2 \geq r_1 - e_k = h \geq \mathbf{0}$, by (2) we have $r_1 = r_2$ so $e_k + h = f + g + h$ thus $e_k = f + g$ therefore $f = e_k$ and $g = \mathbf{0}$ since $\pi(f) > 0$. Hence $\pi(f) = \pi(e_k)$ so $\pi(e_k)$ is an atom.

Now we show that $\pi(e_i) \neq \pi(e_j)$ for $1 \leq i < j \leq m$. Assume that $\pi(e_i) = \pi(e_j)$ for $i, j \in \{1, \dots, m\}$. Then there exists $f \in \mathcal{E}(X, \mathcal{T})$ such that $e_i \approx_f e_j$ thus $e_i + f, e_j + f \in \mathcal{T}$ and $r_1 = e_i + f$, $r_2 = e_j + f$ are rows in A. Then $r_2 \geq f =$

$r_1 - e_i \geq \mathbf{0}$ so $r_1 = r_2$ by (2), hence $e_i = e_j$ so $i = j$. This ends the proof that $\{\pi(e_1), \ldots, \pi(e_m)\}$ is the set of mutually different atoms in $\Pi(X)$.

Now we prove that $Seq_{(\pi(e_1),\ldots,\pi(e_m))}(\Pi(X)) = \{t_1, \ldots, t_n\}$.

Let $i \in \{1, \ldots, n\}$. Then $t_i = (y_{i1}, \ldots, y_{im}) \in Seq_{(\pi(e_1),\ldots,\pi(e_m))}(\Pi(X))$ is equivalent to

$$\bigoplus_{t=1}^{m} y_{it}\pi(e_t) = 1.$$

We know that $t_i = y_{i1}e_1 + \cdots + y_{im}e_m \in \mathcal{T}$ thus

$$1_{EA} = \pi(t_i) = \pi(y_{i1}e_1 + \cdots + y_{im}e_m) = \bigoplus_{t=1}^{m} y_{it}\pi(e_t)$$

so $t_i \in Seq_{(\pi(e_1),\ldots,\pi(e_m))}(\Pi(X))$ and $\{t_1, \ldots, t_n\} \subseteq Seq_{(\pi(e_1),\ldots,\pi(e_m))}(\Pi(X))$.

Let $t = (x_1, \ldots, x_m) \in Seq_{(\pi(e_1),\ldots,\pi(e_m))}(\Pi(X))$ then

$$1_{EA} = \bigoplus_{t=1}^{m} x_t\pi(e_t) = \pi(x_1e_1 + \cdots + x_me_m) = \pi(t)$$

hence $t \in \mathcal{T} = \{t_1, \ldots, t_n\}$ so $Seq_{(\pi(e_1),\ldots,\pi(e_m))}(\Pi(X) \subseteq \{t_1, \ldots, t_n\}$ thus

$$Seq_{(\pi(e_1),\ldots,\pi(e_m))}(\Pi(X)) = \{t_1, \ldots, t_n\}$$

and $A \in M(\Pi(X))$.

□

In the following Lemma we describe events in $(X, T(A))$.

Lemma 3.7. Let E be a non-trivial finite effect algebra with atoms $a_1, \ldots, a_m$ ($a_i \neq a_j$ for $i \neq j$), $A \in M_{n\times m}(\mathbb{N}_0)$ and $A \in M(E)$. Let $f\colon X \to \mathbb{N}_0$ be a function, where $X = \{1, \ldots, m\}$. Then

$$f \in \mathcal{E}(X, T(A)) \iff \bigoplus_{i=1}^{m} f(i)a_i \text{ exists in } E.$$

P r o o f. By Theorem 3.6 and Lemma 3.4 $(X, T(A))$ is an algebraic E-test space.

Assume that $f \in \mathcal{E}(X, T(A))$ then there exists $t \in T(A)$ such that $\mathbf{0} \leq f \leq t$ and

$$\bigoplus_{i=1}^{m} t(i)a_i = 1$$

since $A \in M(E)$. Hence $\bigoplus_{i=1}^{m} t(i)a_i$ exists in E so $\bigoplus_{i=1}^{m} f(i)a_i$ exists in E.

Assume that $x = \bigoplus_{i=1}^{m} f(i)a_i$ exists in E. Let $x' = \bigoplus_{i=1}^{m} k_i a_i$ for $k_i \in \mathbb{N}_0$ for all $i \in X$. Then

$$1 = x \oplus x' = \bigoplus_{i=1}^{m} (f(i) + k_i)a_i.$$

Let $t\colon X \to \mathbb{N}_0$ be a function such that $t(i) = f(i) + k_i$ for all $i \in X$. Then $t \in T(A)$ since $A \in M(E)$. Moreover $\mathbf{0} \le f \le t$ so $f \in \mathcal{E}(X, T(A))$. □

Theorem 3.8. Let E be a non-trivial finite effect algebra with m atoms, $A \in M_{n\times m}(\mathbb{N}_0)$ and $A \in M(E)$. Then effect algebras E and $\Pi(A)$ are isomorphic.

Proof. Let E be a non-trivial finite effect algebra with atoms $a_1, \dots, a_m$ ($a_i \neq a_j$ for $i \neq j$), $A \in M_{n\times m}(\mathbb{N}_0)$ and $A \in M(E)$. Then $A \in \mathcal{B}_{nm}$ by Theorem 3.6. Let $X = \{1, 2, \dots, m\}$ and $T(A) = \{t_1, \dots, t_n\}$. Then $(X, T(A))$ is an algebraic E-test space by Lemma 3.4. By Theorem 2.4 $\Pi(A) = \Pi(X)$ can be organized into an effect algebra.

Let $\phi\colon E \to \Pi(X)$ be a function such that if $x = \bigoplus_{i=1}^{m} x_i a_i$ then $\phi(x) = \pi(f)$, where $f\colon X \to \mathbb{N}_0$ is a function such that $f(i) = x_i$ for all $i \in X$. By Lemma 3.7, we have $f \in \mathcal{E}(X, T(A))$.

Now we show that ϕ is well-defined. Let $x = \bigoplus_{i=1}^{m} x_i a_i = \bigoplus_{i=1}^{m} y_i a_i$ in E and $f, g\colon X \to \mathbb{N}_0$ be functions such that $f(i) = x_i$ and $g(i) = y_i$ for all $i \in X$. Let $x' = \bigoplus_{i=1}^{m} z_i a_i$ then

$$\bigoplus_{i=1}^{m} (x_i + z_i)a_i = 1, \qquad \bigoplus_{i=1}^{m} (y_i + z_i)a_i = 1.$$

Let $h\colon X \to \mathbb{N}_0$ be a function such that $h(i) = z_i$ for all $i \in X$. Then $f + h,\ g + h \in T(A)$ so $f \approx_h g$ hence $\pi(f) = \pi(g)$.

Now we show that ϕ is a bijection.

Let $x, y \in E$ and assume that $\phi(x) = \phi(y)$. Then there exists $x_1, \dots, x_m, y_1, \dots, y_m \in \mathbb{N}_0$ such that $x = \bigoplus_{i=1}^{m} x_i a_i$, $y = \bigoplus_{i=1}^{m} y_i a_i$ and if $f, g\colon X \to \mathbb{N}_0$ are functions such that $f(i) = x_i$ and $g(i) = y_i$ for all $i \in X$ and $\pi(f) = \pi(g)$. Then there exists $h \in \mathcal{E}(X, T(A))$ such that $f \approx_h g$ so $f + h \in T(A)$ and $g + h \in T(A)$ thus

$$\bigoplus_{i=1}^{m} (x_i + z_i)a_i = 1, \qquad \bigoplus_{i=1}^{m} (y_i + z_i)a_i = 1,$$

where $z_i = h(i)$ for all $i \in X$. Hence

$$x \oplus \bigoplus_{i=1}^{m} (z_i)a_i = y \oplus \bigoplus_{i=1}^{m} (z_i)a_i$$

and $x = y$ by cancellation law.

Let $f \in \mathcal{E}(X, T(A))$. By Lemma 3.7 we have $x = \bigoplus_{i=1}^{m} (f(i))a_i$ exists in E and $\phi(x) = \pi(f)$. This ends the proof that ϕ is a bijection.

By Proposition 1.3 there exist $x_1, \dots, x_m \in \mathbb{N}_0$ such that $1 = \bigoplus_{i=1}^{m} x_i a_i$ then $\phi(1) = \pi(f)$ where $f(i) = x_i$ for all $i \in X$. Moreover $f \in T(A)$ since $A \in M(E)$. Hence $\pi(f) = \phi(1) = 1$.

Let $x, y \in E$. We show that $x \perp y \iff \phi(x) \perp \phi(y)$. By Proposition 1.3 there exist $x_1, \dots, x_m, y_1, \dots, y_m \in \mathbb{N}_0$ such that $x = \bigoplus_{i=1}^{m} x_i a_i$ and $y = \bigoplus_{i=1}^{m} y_i a_i$. Let $f, g \colon X \to \mathbb{N}_0$ be functions such that $f(i) = x_i$ and $g(i) = y_i$ for all $i \in X$. Then $f, g \in \mathcal{E}(X, T(A))$ by Lemma 3.7 .

If $x \perp y$ then $x \oplus y = \bigoplus_{i=1}^{m} (x_i + y_i) a_i$ exists in E. By Lemma 3.7 we have $f + g \in \mathcal{E}(X, T(A))$ so $\pi(f + g) = \pi(f) \oplus \pi(g) = \phi(x) \oplus \phi(y)$ exists in $\Pi(A)$ so $\phi(x) \perp \phi(y)$.

If $\phi(x) \perp \phi(y)$ then $\pi(f + g) = \pi(f) \oplus \pi(g) = \phi(x) \oplus \phi(y)$ exists in $\Pi(A)$ so $f + g \in \mathcal{E}(X, T(A))$ and $\bigoplus_{i=1}^{m} (x_i + y_i) a_i = x \oplus y$ exists in E by Lemma 3.7. Hence $x \perp y$.

Now we show that $\phi(x \oplus y) = \phi(x) \oplus \phi(y)$ for all $x, y \in E$ such that $x \perp y$.

Let $x, y \in E$ and $x = \bigoplus_{i=1}^{m} f(i) a_i$, $y = \bigoplus_{i=1}^{m} g(i) a_i$ for some functions $f, g \colon X \to \mathbb{N}_0$. Then $x \oplus y = \bigoplus_{i=1}^{m} (f(i) + g(i)) a_i$ and

$$\phi(x) \oplus \phi(y) = \pi(f) \oplus \pi(g) = \pi(f + g) = \phi(x \oplus y)$$

so $\phi \colon E \to \Pi(X)$ is an isomorphism of effect algebras. □

Corollary 3.9. Let E_1, E_2 be non-trivial finite effect algebras. Then E_1 and E_2 are isomorphic if and only if $M(E_1) = M(E_2)$.

P r o o f. Let E_1, E_2 be non-trivial finite effect algebras.

- $\Leftarrow$ If $M(E_1) = M(E_2)$ and $A \in M(E_1)$ then E_1 and $\Pi(A)$ are isomorphic by Theorem 3.8. Moreover, E_2 and $\Pi(A)$ are isomorphic by Theorem 3.8. Hence E_1 and E_2 are isomorphic.

- $\Rightarrow$ Assume that E_1 and E_2 are isomorphic. Let $\phi \colon E_1 \to E_2$ be an isomorphism. If $a_1, \dots, a_m$ ($a_i \neq a_j$ for $i \neq j$) are atoms of E_1 then $\phi(a_1), \dots, \phi(a_m)$ are atoms of E_2. Let us observe that

$$(x_1, \dots, x_m) \in Seq_{(a_1, \dots, a_m)}(E_1) \iff \bigoplus_{i=1}^{m} x_i a_i = 1 \text{ in } E_1 \iff$$

$$\bigoplus_{i=1}^{m} x_i \phi(a_i) = 1 \text{ in } E_2 \iff (x_1, \dots, x_n) \in Seq_{(\phi(a_1), \dots, \phi(a_m))}(E_2)$$

since ϕ is an isomorphism. Hence $Seq_{(a_1, \dots, a_m)}(E_1) = Seq_{(\phi(a_1), \dots, \phi(a_m))}(E_2)$ and $M(E_1) = M(E_2)$. □

By Corollary 3.9, the cardinality (up to isomorphism) of finite non-trivial effect algebras with m atoms and k elements is equal to cardinality of the set $\{A \in B_{nm} : |\Pi(A)| \leq k \text{ and } n \in \mathbb{N}\}/\sim$, where $A \sim B$ if and only if B can be obtained from A by some permutation of rows or columns.

List of matrices representing nontrivial finite effect algebras of cardinatily at most 8:

- 2-elem.: $[1]$,
- 3-elem.: $[2]$,
- 4-elem.: $[3]$, $\begin{bmatrix} 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$,
- 5-elem.: $[4]$, $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$,
- 6-elem.: $[5]$, $\begin{bmatrix} 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$,
$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$
- 7-elem.: $[6]$, $\begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 1 \\ 4 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$,
$\begin{bmatrix} 0 & 0 & 2 \\ 1 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$,
$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$
- 8-elem.: $[7]$, $\begin{bmatrix} 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$, $\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix}$,
$\begin{bmatrix} 4 & 0 \\ 2 & 2 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$,
$\begin{bmatrix} 0 & 1 & 1 \\ 5 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 1 & 1 \\ 3 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$,

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2 \\ 1 & 2 & 0 \\ 4 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

So we have (up to isomorphism):

- one 2-element effect algebra,
- one 3-element effect algebra,
- three 4-element effect algebras,
- four 5-element effect algebras,
- ten 6-element effect algebras,
- fourteen 7-element effect algebras,
- forty 8-element effect algebras,

The list of all effect algebras up to 11-elements can be found at
https://www.mat.savba.sk/∼hycko/wprepea/.

Definition 3.10. A state on an effect algebra E is a mapping $s\colon E \to [0,1] \subseteq \mathbb{R}$ such that $s(1) = 1$ and if $a \oplus b$ is defined, then $s(a \oplus b) = s(a) + s(b)$.

Definition 3.11. Let

$$A = [y_{ij}] \in M_{n\times m}(\mathbb{N}_0)$$

then denote

$$\begin{bmatrix} y_{11} & \dots & y_{1m} & 1 \\ \vdots & & \vdots & \vdots \\ y_{n1} & \dots & y_{nm} & 1 \end{bmatrix}$$

by $(A|1)$.

The following theorem gives the necessary condition enabling algebra to have a state.

Theorem 3.12. Let E be a non-trivial finite effect algebra with atoms $a_1, \dots, a_m$. Let $A \in M(E)$ and $B = (A|1)$. If E has a state then rank $A =$ rank B.

P r o o f. Let

$$A = [y_{ij}] \in M(E) \cap M_{n\times m}(\mathbb{N}_0).$$

Suppose that E has a state $h\colon E \to [0,1]$. Let $s_t = h(a_t) \geq 0$ for $1 \leq t \leq m$. We know that $\bigoplus_{t=1}^{m} y_{it}a_t = 1$ for $1 \leq i \leq n$ since $A \in M(E)$. Hence $1 = h(1) = h(\bigoplus_{t=1}^{m} y_{it}a_t) = \sum_{t=1}^{m} y_{it}h(a_t) = \sum_{t=1}^{m} y_{it}s_t$. Thus

$$A \cdot \begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

By Rouché-Capelli Theorem [12], rank $A =$ rank B, where $B = (A|1)$. □

Problem. Prove or disprove: if E is a finite effect algebra with atoms $a_1, \dots, a_m$, $A \in M(E)$ and $B = (A|1)$ then E has a state if and only if rank $A =$ rank B.

In [6] Greechie gives an example of finite effect algebra that has no states. This effect algebra E has 12 atoms $\{a_1, \dots, a_{12}\}$ such that $a_1 \oplus a_2 \oplus a_3 \oplus a_4 = 1$, $a_5 \oplus a_6 \oplus a_7 \oplus a_8 = 1$, $a_9 \oplus a_{10} \oplus a_{11} \oplus a_{12} = 1$, $a_1 \oplus a_5 \oplus a_9 = 1$, $a_2 \oplus a_6 \oplus a_{10} = 1$, $a_3 \oplus a_7 \oplus a_{11} = 1$, $a_4 \oplus a_8 \oplus a_{12} = 1$. Let

$$A = \begin{bmatrix} 1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&1&1&1 \\ 1&0&0&0&1&0&0&0&1&0&0&0 \\ 0&1&0&0&0&1&0&0&0&1&0&0 \\ 0&0&1&0&0&0&1&0&0&0&1&0 \\ 0&0&0&1&0&0&0&1&0&0&0&1 \end{bmatrix}$$

then $A \in M(E)$ and the echelon form of $(A|1)$ is (using Maxima, see [9]).

$$\begin{bmatrix} 1&0&0&0&1&0&0&0&1&0&0&0&1\\ 0&1&0&0&0&1&0&0&0&1&0&0&1\\ 0&0&1&0&0&0&1&0&0&0&1&0&1\\ 0&0&0&1&0&0&0&1&0&0&0&1&1\\ 0&0&0&0&1&1&1&1&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix}.$$

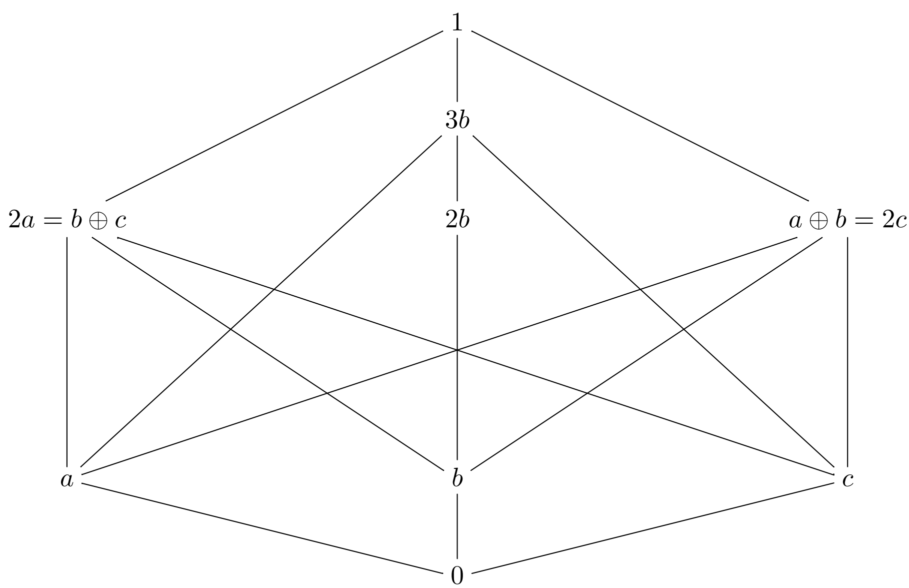
So rank $A = 6$ and rank $(A|1) = 7$. Thus E has no state by 3.12.

In [10], Riečanová found example of finite effect algebra E that has no states. This effect algebra E has 3 atoms $\{a, b, c\}$ such that $a \oplus b \oplus c = 1$, $3a = 4b = 3c = 1$. Let

$$B = \begin{bmatrix} 1&1&1\\ 3&0&0\\ 0&4&0\\ 0&0&3 \end{bmatrix}$$

then $B \in M(E)$ and rank $B = 3$ and rank $(B|1) = 4$ thus E has no state by 3.12.

This effect algebra E is represented by the following Hasse diagram:



The above effect algebra E has 9 elements. Using the list of all matrices representing effect algebras which have at most 8 elements it is easy to check that every effect algebras of cardinality ≤ 8 has a state. So the Riečanová's example is smallest one.

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