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AN ELEMENTARY PROOF OF MARCELLINI SBORDONE SEMICONTINUITY THEOREM

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The weak lower semicontinuity of the functional

$$F(u) = \int_\Omega f(x, u, \nabla u)\,\mathrm{d}x$$

is a classical topic that was studied thoroughly. It was shown that if the function f is continuous and convex in the last variable, the functional is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega)$. However, the known proofs use advanced instruments of real and functional analysis. Our aim here is to present a proof understandable even for students familiar only with the elementary measure theory.

Keywords: convexity, sequential semicontinuity, calculus of variation, minimizer

Classification: 49J45, 49J20, 46E35

1. INTRODUCTION AND THE MAIN RESULT

Many optimization problems in physics may be formulated in terms of minimization of the functional in the form of

$$F(v) = \int_\Omega f(x, v, \nabla v)\,\mathrm{d}x, \tag{M}$$

over some set $\mathcal{W}$ of admissible functions. This problem is very classical, and its roots can be traced to the 17th century. The systematic study was launched later by Lagrange, du Boys–Reymond, and others (see, for instance, [8, 23]). They formulated the Euler-Lagrange equations as necessary conditions to hold for the minimizer. However, the minimizer's existence must be proved first (otherwise, even solving the Euler-Lagrange equations does not guarantee finding the solution). As many examples show, the existence of the minimizer cannot be expected in general. Therefore, we have to add some additional assumptions on the function f in (M) to assure the existence of a minimizer. The existence problem, however, is typically approached with advanced methods, including the knowledge of certain parts of the functional analysis, the measure theory, and the function spaces. We present an alternative proof of the existence of a minimizer

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under the convexity condition of f with a given boundary data. Unlike the classical ones, our proof uses only elementary methods of the measure theory and the real analysis, and its direction is easy to follow. Our calculation is done in detail, and we intentionally avoid shortening it with advanced tools.

Now, we specify the problem. Let us consider a domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary and let $f \in \mathcal{C}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. Let $u_0 \in W^{1,p}(\Omega, \mathbb{R}^d)$, we denote the set of all functions in given Sobolev space with the same boundary data as u_0 by

$$\mathcal{W} := u_0 + W_0^{1,p}(\Omega, \mathbb{R}^d). \tag{1}$$

Let $p \in (1, \infty)$, by *p-growth condition* of f we mean, if there exists $1 \leq q < p$ such that

$$(\exists c_0 > 0)(\exists c_1 \in \mathbb{R}, c_2 \in L^1(\Omega))(\forall (x, u, \xi) \in \overline{\Omega} \times \mathbb{R}^d \times \mathbb{R}^z) \\ f(x, u, \xi) \geq c_0 |\xi|^p + c_1 |u|^q + c_2(x). \tag{p-G}$$

As some authors do, the proof can be slightly simplified by considering c_1 and c_2 positive constants. Also, d is the dimension of the image of functions in (1), which can be $d = 1$ but also higher for vector-valued functions, we leave d general in statements, but we focus on the case $d = 1$. Note that the case of $p = 1$ follows immediately from Lemma 3.1. However, since the Sobolev space $W^{1,1}$ is not reflexive, even the sequential weak lower semicontinuity does not guarantee the existence of the minimizer. The case $p = \infty$ is also covered in some books (for instance, see [7]), however, this would require a different technique. We consider the convexity property of f in the last variable

$$\xi \mapsto f(x, u, \xi) \quad \text{is convex for all} \quad (x, u) \in \Omega \times \mathbb{R}^d. \tag{conv}$$

In the following text we consider only functional F and set $\mathcal{W}$ satisfying

$$(\exists u \in \mathcal{W}) \quad F(u) < \infty \quad \wedge \quad (\forall u \in \mathcal{W}) \quad F(u) > -\infty. \tag{fin}$$

Our problem is to prove that the minimizer of (M) on $\mathcal{W}$ exists under conditions (conv), (fin), and (p-G). Note that the existence follows immediately from Theorems 1.1 and 1.2. Our contribution is the new student-friendly proof of Theorem 1.1. The reader is encouraged to compare our proof with the classical proofs in [1, 7, 26] or [16]; in these proofs, useful tools are both approximations by the functions with controlled growth and regularity or replacing and extending the functions on the irregular parts of the domain, also the maximal operator is used. Later, Kałamajska wrote short proof [19] with the help of the theory of Young measures. For the sake of completeness and convenience of the reader, we include proof of Lemma 1.2 although this one is similar to the known ones.

Note that the condition of continuity of f can be relaxed, and one may consider the Caratheodory function instead (see [7]), however, this is done easily using the Scorza-Dragoni theorem 4.1, thus we restrict ourselves to continuous functions.

Theorem 1.1. (Weak sequential lower semicontinuity) Let $\Omega \subset \mathbb{R}^n$ be a domain with a Lipchitz boundary and let $u_0 \in W^{1,p}(\Omega, \mathbb{R}^d)$. Let $f \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}^d \times \mathbb{R}^{n \times d})$ be a function such that (conv) holds. Then, the functional F given by (M) is sequentially weakly lower semicontinuous.

Lemma 1.2. (Coercivity) Let $\Omega \subset \mathbb{R}^n$ be a domain with a Lipchitz boundary and let $u_0 \in W^{1,p}(\Omega, \mathbb{R}^d)$. Let $f \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}^d \times \mathbb{R}^{n\times d})$ be a function satisfying (p-G). Let $(u_k)_k \subset \mathcal{W}$ given by (1) be a minimizing sequence of functional F given by (M). Moreover, let condition (fin) hold. Then u_k is bounded in $W^{1,p}(\Omega, \mathbb{R}^d)$.

Theorem 1.3. (Existence of minimizer) Let $\Omega \subset \mathbb{R}^n$ be a domain with a Lipchitz boundary. Let $f \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}^d \times \mathbb{R}^{n\times d})$ be a function satisfying (conv) and (p-G). Let F be the functional given by (M) satisfying (fin). Then F has a minimizer in $\mathcal{W}$ defined by (1).

The paper is structured into Section 2, where we prepare some observations and lemmata and prove Lemma 1.2, Section 3, where we prove the main result, and Appendix 4, where we present known proofs of Lemma 1.2 and Theorem 1.3 and recall some well-known theorems for the convenience of less experienced readers.

1.1. Development and connection to literature

As for the development in the literature, the result we reprove is the predecessor of the celebrated result of Acerbi and Fusco in 1984 [1]. Their methods include maximal operators, an extension of functions, and a lower semicontinuous envelope. The paper also contains the proof even for quasiconvex vector valued f, which we do not include. The result succeeded the previous results by Tonelli [34], where the Theorem 1.1 was proved with much stricter conditions. Later, the conditions were relaxed in the regularity of f; for development of this, see [12, 32] or [27]. We decided to name our paper after Marcellini and Sbordone, the authors of [27], as their assumptions are the most similar to ours. For vector-valued u, the assumption of convexity is no longer necessary and can be relaxed to various weaker properties. Initially, the result was given by Morrey [30] and generalized by Meyers [28] under stronger regularity requiring only quasi-convexity. More results were published by Ball, Liu, Dacorogna, Marcellini, and others with finer properties under more relaxed assumptions, such as polyconvexity or rank-one convexity. As the definitions of particular generalizations of convexity differ, we are led to study differences and inclusions between these classes. The topic was addressed by Šverák [36] and Alibert and Dacorogna [2] or recent results by Sil [33] and Grabovsky [17]. The ongoing trend is to ensure an improvement of convexity type by studying additional properties [18]. Note that even though (p-G) conditions can not be left out, they can be relaxed into finer scales than powers [35]. An essential tool is controlling f by the convex envelope, see [3] for studying its properties and dependence on growth conditions. Let us emphasise that the variation problems without convexity are also approached, see [10, 11].

Let us briefly overview some milestones and interesting papers in the field surrendering or following the Acerbi-Fusco theorem [1]. In 1982, a significant result about the regularity of the minimizer was done by Giaquinta and Giusti [15]. We recommend the paper by Mingione [29] for a broad survey on this topic. Approximation methods by Marcellini [26] improve the Acerbi-Fusco theorem in 1985. Significant relaxation result was presented in 1997 by Fonseca and Malý [13]. A version of convexity based on curl called A-convexity initially considered by Dacorogna [6] is shown to be the optimal variant of definition for the lower semicontinuity property by Fonseca and Müller in 1999

[14]. The Young measure theory is used by Kałamajska in 1997 to shorten the proof of the Acerbi-Fusco result. Also, by the theory of Young measure, Kristensen in 1999 provided new approximation results in [20], and in 2015 he redefined growth conditions for gradient Young functions and characterise the lower semicontinuity in this setting [21]. Kristensen and Rindler studied the case of the lower semicontinuity among $W^{1,1}$ and BV functions in this setting in 2010 [22]. Recent result by Prinari covers the lower semicontinuity and approximation properties for L^∞ functionals [31]. The splendid result by Bourdin, Francfort, and Gilles put the variational approach to address Griffith fracture models [5] in 2007.

The proper list of references would be overwhelming. Thus, we had to omit a lot. We strongly recommend the recent survey paper by Benešová and Kružík [4] that offers an overview and directions for further study of development and questions still open.

1.2. Question of convexity

Let us emphasise that replacing the convexity with the quasiconvecxity changes the problem's difficulty. By the quasiconvexity, several different properties may be considered; for example, continuous $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasiconvex if for every $\xi \in \mathbb{R}^{m\times n}$, for every open $\Omega \subset \mathbb{R}^n$, and every function $u \in C_0^1(\Omega, \mathbb{R}^m)$ we have

$$|\Omega| f(\xi) \leq \int_\Omega f(\xi + Du(x))\ \mathrm{d}x.$$

This restriction allows us to simplify the proof a lot, as much harder tools would be required, as the class is much wider than the class of convex functions. As we aim for student-friendly proof, we cover only the convex case.

The proof can also be simplified by either omitting the second variable in (M), the functions itself or by assuming convexity even in this variable. We do not use this simplification.

2. SOME AUXILIARY RESULTS AND OBSERVATIONS

In this and the following text, we use measure theory. However, even though the formulation of lemmata and observations are more general, we apply them only for the Lebesgue measure in spaces $\mathbb{R}$ and $\mathbb{R}^n$.

Observation 2.1. Let (Ω, μ) be a finite measure space and let $f \in L^0(\Omega)$ and $g \in L^1(\Omega)$ be functions for which $g(x) \leq f(x)$ holds for a.e. $x \in \Omega$. Let $\Omega_k \subset \Omega$ be sets such that $|\Omega \setminus \Omega_k| \to 0$. Then we get

$$\lim_k \left(\int_{\Omega_k} f \,\mathrm{d}\mu \right) = \int_\Omega f \,\mathrm{d}\mu.$$

Note, that *partition of a set* M means the pair-wise disjoint family M_m such that

$$\bigcup_m M_m = M.$$

Given a partition $\mathcal{P}$ we denote the *norm of the partition* by

$$\nu(\mathcal{P}) := \max_{j \in m} \{\operatorname{diam}(P_j)\}.$$

Lemma 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $u \in L^1(\Omega)$ and let $\mathcal{P} = (P_j)_{j=1}^m$ be a partition of Ω. Define

$$u_{\mathcal{P}} := \sum_{j=1}^{m} \chi_{P_j} \fint_{P_j} u.$$

Then we have

$$\lim_{\nu(\mathcal{P}) \to 0} |\{|u_{\mathcal{P}} - u| > \varepsilon\}| = 0 \quad (\forall \varepsilon > 0).$$

Proof. Let us first show that the statement is valid for continuous functions. Pick $\varepsilon, \gamma > 0$. Given a positive η let us denote

$$\Omega_\eta := \{x \in \Omega : \operatorname{dist}(x, \Omega^c) > \eta\}.$$

We choose η so small that

$$|\Omega \setminus \Omega_{2\eta}| < \gamma.$$

Since u is uniformly continuous on Ω_η, there exists $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |u(x) - u(y)| < \varepsilon \quad \forall x, y \in \Omega_\eta.$$

Now choose a partition $\mathcal{P}$ with

$$\nu(\mathcal{P}) < \min\{\eta, \delta\}. \tag{2}$$

For all $x \in \Omega_{2\eta}$ we have that

$$|u(x) - u_{\mathcal{P}}(x)| < \varepsilon.$$

Hence

$$|\{|u - u_{\mathcal{P}}| > \varepsilon\}| < \gamma$$

as soon as (2) holds. The convergence in measure for the continuous function is proved.

Given a $\alpha, \varepsilon > 0$ and $u \in L^1(\Omega)$ we first choose v continuous such that

$$\|u - v\|_{L^1} \leq \varepsilon\alpha,$$

then, by the triangle inequality

$$|\{|u - u_{\mathcal{P}}| > 3\varepsilon\}| \leq |\{|u - v| > \varepsilon\}| + |\{|v - v_{\mathcal{P}}| > \varepsilon\}| + |\{|u_{\mathcal{P}} - v_{\mathcal{P}}| > \varepsilon\}| =: I + II + III.$$

Note that $u \mapsto u_{\mathcal{P}}$ is non-expansive mapping in L^1, so we estimate

$$\|u_{\mathcal{P}} - v_{\mathcal{P}}\|_{L^1} \leq \varepsilon\alpha.$$

Hence, by Chebyschev inequality 4.8, we have $I + III < 2\alpha$. Since v is continuous, it is enough to choose the norm of the partition small to obtain $II < \alpha$. □

3. PROOF OF THEOREM 1.1 AND THEOREM 1.3

Let $u \in L^0(\Omega, \mathbb{R}^d)$ and $v \in L^p(\Omega, \mathbb{R}^z)$. Given such functions let us consider

$$J_\Omega(u,v) := \int_\Omega f(x, u(x), v(x))\,\mathrm{d}x. \tag{3}$$

Let us denote the norm topology in L^p by $\mathcal{L}^p$ and the topology of weak convergence by $\mathcal{L}^p_w$. We also denote the convergence in the Lebesgue measure by $\stackrel{\mathrm{d}x}{\to}$. Let us define the topology

$$\tau = \mathcal{L}^0 \times \mathcal{L}^p_w$$

for $(u_k, v_k) \in L^0(\Omega, \mathbb{R}^d) \times L^p(\Omega, \mathbb{R}^z)$ by

$$(u_k, v_k) \stackrel{\tau}{\to} (u,v) \quad \text{if} \ \left(u_k \stackrel{\mathrm{d}x}{\to} u \wedge v_k \stackrel{L^p}{\rightharpoonup} v\right).$$

The following lemma is somewhat similar to the Eisen Lemma (see [9]).

Lemma 3.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p \in [1, \infty)$, $f \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}^d \times \mathbb{R}^z)$ be function satisfying (conv) and let $b \in L^1(\Omega)$ be a function such that

$$f(x,u,v) \geq b(x) \quad \forall (x,u,v) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^z. \tag{4}$$

Then the functional J_Ω defined in (3) is sequentially lower semicontinuous with respect to topology τ.

Lemma 3.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p \in (1, \infty)$ and let $f \in \mathcal{C}(\overline{\Omega} \times \mathbb{R}^d \times \mathbb{R}^z)$ be a function satisfying (p-G) and (conv). Then J_Ω defined in (3) is sequentially lower semicontinuous with respect to topology $\mathcal{L}^q \times \mathcal{L}^p_w$.

P r o o f. (Proof of Lemma 3.1) Choose $\varepsilon > 0$. Let (u_k, v_k) be a sequence convergent to (u,v) in the topology τ. Since u_k converges to u in measure, by the Riesz thm. 4.4 one may assume that $u_k(x) \to u(x)$ for a.e. $x \in \Omega$. Hence, by the Yegorov thm. 4.6 and the Luzin thm. 4.7 for every l natural there exists a compact set $\Omega_l \subset \Omega$ such that

$$\left(|\Omega \setminus \Omega_l| \leq \frac{1}{l}\right) \wedge (u_k \rightrightarrows u \quad \text{on } \Omega_l) \wedge (u \in \mathcal{C}(\Omega_l)).$$

Define

$$S := \sup_k \sup_{x \in \Omega_l} |u_k(x)|.$$

We may assume that $S < \infty$. Note that the function $(x,u) \mapsto f(x,u,\xi)$ is uniformly continuous on $\Omega_l \times [-S,S] \times \overline{B(0,2j)}$ for arbitrary j natural. Now, let us denote by η_j the strictly positive number such that for all $x, \overline{x} \in \Omega_l, u, \overline{u} \in [-S,S]$

$$|(\overline{x}, \overline{u}) - (x,u)| < 6\eta_j \tag{5}$$

implies

$$|f(\overline{x}, \overline{u}, \xi) - f(x,u,\xi)| < \varepsilon \quad (\forall |\xi| \leq 2j).$$

Without loss of generality one may suppose that η_j is decreasing and

$$|u(x) - u_k(x)| \le \eta_k \quad (\forall x \in \overline{\Omega_l}). \tag{6}$$

Moreover, since the function u is uniformly continuous, by the triangle inequality and double usage of (6), for each j, there exists positive $\delta_j < \eta_j$ (δ_j decreasing) such that for all $x, y \in \overline{\Omega}$, one has

$$|x - y| < \delta_j \quad \text{implies} \quad |(x, u_k(x)) - (y, u_k(y))| < 3\eta_j \quad (\forall k \ge j).$$

Use the previous estimates to observe

$$|x - y| < \delta_j \quad \text{implies} \quad |(x, u_k(x)) - (y, u(y))| < 6\eta_j \quad (\forall k \ge j). \tag{7}$$

For j, k naturals denote

$$E_{k,j} := \{|v_k| \le j\} \cap \Omega_l, \quad G_j := \{|v| \le j\} \cap \Omega_l. \tag{8}$$

Note that, using the Chebychev inequality thm. 4.8 and boundedness of v_k in L^p we obtain that there exists $C > 0$ independent of j, k such that

$$|\Omega_l \setminus G_j| \le C/j^p \quad \text{and} \quad |\Omega_l \setminus E_{k,j}| \le C/j^p. \tag{9}$$

By uniform continuity of $(x, u, \xi) \mapsto f(x, u, \xi)$ on $\Omega_l \times [-S, S] \times \overline{B(0, 2j)}$, for every natural j there exists γ_j such that

$$|\xi - \overline{\xi}| < \gamma_j \Rightarrow |f(x, u, \xi) - f(x, u, \overline{\xi})| < \varepsilon \quad (\forall (x, u) \in \overline{\Omega_l} \times [-S, S] \quad \text{and} \quad \forall \xi, \overline{\xi} \in \overline{B(0, 2j)} \tag{10}$$

By Lemma 2.2 we obtain that for each j natural there exists a finite partition of G_j, consider such a partition $\mathcal{P} = (K^j_m)_{m=1}^{M_j}$ satisfying

(i) $\nu(\mathcal{P}) < \delta_j$.

(ii) $\sum_m \left|\{x \in K^j_m : |v - ⨍_{K^j_m} v| > \gamma_j\}\right| < 1/j$.

By the weak convergence of v_k to v in L^p one has

$$\lim_k ⨍_{K^j_m} (v - v_k) = 0.$$

Therefore, one may assume without the loss of generality,

$$\left| ⨍_{K^j_m} (v - v_k) \right| \le \gamma_k \quad (\forall k \ge j) \tag{11}$$

(if this doesn't hold, one can pass to the appropriate sub-sequence). Set

$$Q^j_m := \left\{ x \in K^j_m : |v(x) - ⨍_{K^j_m} v| \le \gamma_j \right\} \tag{12}$$

and denote

$$Q_j := \bigcup_m Q^j_m. \tag{13}$$

Note that by (8) and the property (ii) of the parts K^j_m one has $|G_j \setminus Q_j| \le \frac{1}{j}$. Pick $a_j > j$ such that

$$\left| \fint_{K^j_m} v_j \chi_{E^c_{j,a_j}} \,\mathrm{d}x \right| < \gamma_j. \tag{14}$$

Moreover, let $Q^j_{m,t}$ be a partition of Q^j_m such that

$$\operatorname{diam}(Q^j_{m,t}) < \delta_{a_j},$$

and let us choose arbitrary $x^j_{m,t} \in Q^j_{m,t}$, creating $2\delta_{a_j}$ net. For the following estimate, denote

$$\begin{aligned}
\xi_{j,m} &:= \fint_{K^j_m} v \,\mathrm{d}x, & \xi^{\spadesuit}_{j,m} &:= \fint_{K^j_m} v_j \,\mathrm{d}x,\\
\xi^{\heartsuit}_{j,m} &:= \fint_{K^j_m} v_j \chi_{E_{j,a_j}} \,\mathrm{d}x, & \tilde{v}_j &:= v_j \chi_{E_{j,a_j}},\\
T &:= \max_{(x,u)\in\overline{\Omega}_l\times[-S,S]} |f(x,u,0)|.
\end{aligned} \tag{15}$$

Note that (14) estimate the difference of functions above as

$$\left| \xi^{\spadesuit}_{j,m} - \xi^{\heartsuit}_{j,m} \right| = \left| \fint_{K^j_m} v_j \chi_{E^c_{j,a_j}} \,\mathrm{d}x \right| < \gamma_j. \tag{16}$$

Let us estimate

$$\begin{aligned}
\int_{Q_j} f(x,u,v)\,\mathrm{d}x &\overset{(7),(5)}{\le} \sum_m \int_{Q^j_m} f(x^j_m, u_j(x^j_m), v)\,\mathrm{d}x + |\Omega_l|\varepsilon\\
&\overset{(12),(10)}{\le} \sum_m \int_{Q^j_m} f(x^j_m, u_j(x^j_m), \xi_{j,m})\,\mathrm{d}x + 2|\Omega_l|\varepsilon\\
&\overset{(11),(10)}{\le} \sum_m \int_{Q^j_m} f(x^j_m, u_j(x^j_m), \xi^{\spadesuit}_{j,m})\,\mathrm{d}x + 3|\Omega_l|\varepsilon\\
&\overset{(7),(5)}{\le} \sum_m \sum_t \int_{Q^j_{m,t}} f(x^j_{m,t}, u_j(x^j_{m,t}), \xi^{\spadesuit}_{j,m})\,\mathrm{d}x + 4|\Omega_l|\varepsilon\\
&\overset{(16),(10)}{\le} \sum_m \sum_t \int_{Q^j_{m,t}} f(x^j_{m,t}, u_j(x^j_{m,t}), \xi^{\heartsuit}_{j,m})\,\mathrm{d}x + 5|\Omega_l|\varepsilon\\
&\overset{\text{Thm 4.3}}{\le} \sum_m \sum_t \int_{Q^j_{m,t}} f(x^j_{m,t}, u_j(x^j_{m,t}), \tilde{v}_j)\,\mathrm{d}x + 5|\Omega_l|\varepsilon.
\end{aligned}$$

Note that using the Jensen inequality in the last estimate is the crucial step, where convexity (conv) comes into play. Note that we may question if more general versions, such as quasi-convexity, can be used in this step. Following the fact that $x^j_{m,t}$ is a $2\delta_{a_j}$ net and the inequalities (5),(7) and (6), we estimate

$$\begin{aligned} &\leq \sum_m \sum_t \int_{Q^j_{m,t}} f(x, u_j, \tilde{v}_j)\,\mathrm{d}x + 6|\Omega_l|\varepsilon \\ &\overset{(13),\text{part. of } Q^j_m}{=} \int_{Q_j} f(x, u_j, \tilde{v}_j)\,\mathrm{d}x + 6|\Omega_l|\varepsilon. \end{aligned}$$

To estimate we split th domain into $Q_j \cap E_{j,a_j}$ where $\tilde{v}_j = v_j$ and $Q_j \cap E^c_{j,a_j}$, where we estimate $|f| \leq T$. In the last estimate, we add to the right-hand side integral of non-negative $f - b$ over the set $\Omega_l \setminus (E_{j,a_j} \cap Q_j)$, so we estimate

$$\begin{aligned} &\overset{(15)}{\leq} \int_{E_{j,a_j} \cap Q_j} f(x, u_j, v_j)\,\mathrm{d}x + T|E^c_{j,a_j}| + 6|\Omega_l|\varepsilon \\ &\overset{(9)}{\leq} \int_{\Omega_l} f(x, u_j, v_j)\,\mathrm{d}x + \frac{TC}{a_j^p} + 6|\Omega_l|\varepsilon - \int_{\Omega_l \setminus (E_{j,a_j} \cap Q_j)} b(x)\,\mathrm{d}x. \end{aligned}$$

Passing to the limit with $\varepsilon \to 0_+$ yields

$$J_{Q_j}(u, v) \leq J_{\Omega_l}(u_j, v_j) + \frac{TC}{a_j^p} - \int_{\Omega_l \setminus (E_{j,a_j} \cap Q_j)} b(x)\,\mathrm{d}x.$$

Using Observation 2.1 and (4), passing to limit for $j \to \infty$ we obtain

$$J_\Omega(u, v) \leq \limsup_j J_{\Omega_l}(u_j, v_j),$$

which is equivalent to sequential lower semicontinuity of J_Ω on the topological space $(L^0(\Omega) \times L^p(\Omega), \tau)$. □

P r o o f. (Proof of the Lemma 3.2) Given a function satisfying (p-G), define the auxiliary function

$$\tilde{f}(x, u, \xi) := f(x, u, \xi) - c_1|u|^q$$

and functional

$$\tilde{J}(u, v) := \int_\Omega \tilde{f}(x, u, v)\,\mathrm{d}x.$$

By the Lemma 3.1, the functional $\tilde{J}$ is sequentially lower semicontinuous with respect to topology τ. Hence it is sequentially lower semicontinuous with respect to $\mathcal{L}^q \times \mathcal{L}^p_w$ and thus

$$J(u, v) = \tilde{J}(u, v) + c_1 \int_\Omega |u|^q\,\mathrm{d}x \leq \liminf_k \tilde{J}(u_k, v_k) + c_1 \lim_k \int_\Omega |u_k|^q\,\mathrm{d}x = \liminf_k J(u_k, v_k).$$

□

Proof. (Proof of Theorem 1.1) Let $u_k \rightharpoonup u$ in $W^{1,p}$. Using the Rellich–Kondrachev theorem 4.5 without loss of generality, we may suppose that $u_k \to u$ strongly in L^p while $\nabla u_k \overset{L^p}{\rightharpoonup} \nabla u$. Hence, by the preceding Lemma 3.2, we obtain that

$$F(u) = J(u, \nabla u) \le \liminf_k J(u_k, \nabla u_k) = \liminf_k F(u_k).$$

□

4. APPENDIX

4.1. Coercivity and the existence of the minimizer

Proof. (Proof of Lemma 1.2, Coercivity) Let $\tilde{u}_k \in \mathcal{W}$ be a minimizing sequence of F. Denote

$$u_k := \tilde{u}_k - u_0. \tag{17}$$

First, we prove the sequence is bounded in $W^{1,p}(\Omega)$. Suppose the contrary that u_k is unbounded. Without loss of generality (otherwise, pass to a proper subsequence), suppose

$$0 < \|u_k\|_{W^{1,p}} \to \infty,$$

the Friedrich inequality thm. 4.2 implies

$$\|\nabla u_k\|_p \to \infty.$$

This implication does not have to be true in general, but thanks to $u_k \equiv 0$ on the boundary, it has to hold. Now, by (p-G) we estimate

$$\begin{aligned} F(\tilde{u}_k) &\ge c_0 \|\nabla \tilde{u}_k\|_p^p - |c_1| \|\tilde{u}_k\|_q^q + \|c_2\|_1 \\ &\overset{(17)}{\ge} c_0(\|\nabla u_k\|_p^p - \|\nabla u_0\|_p^p) - |c_1|(\|u_k\|_q^q + \|u_0\|_q^q) + \|c_2\|_1 \\ &\overset{\text{Thm. 4.2}}{\ge} c_0 \|\nabla u_k\|_p^p - C(\Omega) c_1 \|\nabla u_k\|_q^q + C(u_0, c_1, c_2, \Omega) \\ &\overset{\text{Hölder in.}}{\ge} c_0 \|\nabla u_k\|_p^p - C(c_1, \Omega) \|\nabla u_k\|_p^q + C(u_0, c_1, c_2, \Omega) \\ &= \|\nabla u_k\|_p^p \left(c_0 - C(c_1, \Omega) \|\nabla u_k\|^{q-p} - \|\nabla u_k\|_p^{-p} C(u_0, c_1, c_2, \Omega) \right). \end{aligned}$$

As $p > q$, the second and the third terms in the bracket tend to zero, and $c_0 > 0$, the remaining expression diverges to infinity as k tends to infinity. But this contradicts the fact that u_k is a minimizing sequence. □

Proof. (Proof of Theorem 1.3, Existence of minimizer) Let $f \in \mathcal{C}(\Omega \times \mathbb{R} \times \mathbb{R}^n)$ be a function satisfying (p-G) and (conv). Let u_k be a minimizing sequence. By Lemma 1.2 we have that u_k is bounded in $W^{1,p}(\Omega)$. Alaoglu theorem implies that the closed ball in $W^{1,p}(\Omega)$ is compact, therefore we may assume that u_k is weakly convergent to $u \in W^{1,p}(\Omega)$. Let us emphasise that by the Rellich–Kondrachev theorem 4.5, we also may assume that $u_k \to u$ in $L^q(\Omega)$. We get

$$(u_k, \nabla u_k) \to (u, \nabla u) \quad \text{in the topology} \quad \mathcal{L}^q \times \mathcal{L}^p_w.$$

Using the Theorem 1.1 we obtain that

$$F(u) \le \liminf_{k\to\infty} F(u_k) = \inf\{F(u)|u \in \mathcal{W}\}.$$

Hence, u is the minimizer. □

4.2. Recalled theorems

The following results are standard knowledge. The proofs can be found in [7, 24, 25] etc.

Theorem 4.1. (Scorza-Dragoni) Let $\varepsilon > 0$ and $f : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ be measurable in the first variable and continuous in the second and third one. Then there exists compact set $K \subset \Omega$ such that $|\Omega \setminus K| < \varepsilon$ and $f|_{K\times\mathbb{R}\times\mathbb{R}^n}$ is continuous.

Theorem 4.2. (Friedrichs inequality) Let $\Omega \subset \mathbb{R}^n$ be a domain with Lipchitz boundary, $p \in \langle 1, \infty\rangle$. There exists constant C_Ω such that

$$\|u\|_p \le C_\Omega \|\nabla u\|_p$$

holds for all $u \in W_0^{1,p}(\Omega)$.

Theorem 4.3. (Jensen inequality) Let $N \subset \mathbb{R}^n$ be a convex set. Let μ be a measure with a finite total variation on N. Let $\varphi : N \to \mathbb{R}$ be a convex function. Let $E \subset N$ be a measurable set. Then, the following inequality holds

$$\varphi\left(\fint_E x \,\mathrm{d}\mu(x)\right) \le \fint_E \varphi(x)\,\mathrm{d}\mu(x).$$

Theorem 4.4. (Riesz) Let (Ω, μ) be a space with measure and let u_k be a sequence of measurable functions such that

$$u_k \xrightarrow{\mu} u \quad u_k \text{ converge in measure to } u.$$

Then, there exists an increasing sequence of indices l_k such that

$$u_{l_k}(x) \to u(x) \quad \text{for } \mu- \text{ a.e. } x \in \Omega.$$

Theorem 4.5. (Rellich–Kondrachev) Let $p \in (1, \infty)$ and let $1 \le q < \frac{np}{n-p}$ for $p < n$ or $1 \le q < \infty$ for $p \ge n$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipchitz boundary. And let $u_k \in W^{1,p}(\Omega)$ be a bounded sequence in $W^{1,p}(\Omega)$, then there exists an increasing sequence of indices l_k and $u \in L^q(\Omega)$ such that

$$\|u - u_{l_k}\|_{L^q(\Omega)} \to 0.$$

Theorem 4.6. (Yegorov) Let $\Omega \subset \mathbb{R}^n$ measurable, let u_k be a sequence of measurable functions such that

$$u_k(x) \to u(x) \quad \text{for } \mu - \text{a.e. } x \in \Omega.$$

Then, there exists the sequence of sets N_j and the strictly increasing sequence k_j such that

$$\text{(Y1)}\, N_j \subset N_{j+1}, \quad \text{(Y2)}\, \mu(\Omega \setminus N_j) \le 1/j, \quad \text{(Y3)}\, |u - u_{k_j}| < 1/j \quad \text{on } N_j.$$

Theorem 4.7. (Luzin) Let μ be a complete Radon measure on locally compact space and measurable function u finite μ-almost everywhere. Then for open K and $\varepsilon > 0$ there exist open $G \subset K$ such that u is continuous on $K \setminus G$ and $\mu(G) < \varepsilon$.

Theorem 4.8. (Chebysev inequality) Let $\Omega \subset \mathbb{R}^n$ measurable, $u \in L^0(\Omega)$, $p \in \langle 1, \infty)$ and let $t \in (0, \infty)$ then one has

$$t^p \left|\{|u| > t\}\right| \leq \int_\Omega |u(x)|^p \,\mathrm{d}x.$$

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