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DUALITY FOR A FRACTIONAL VARIATIONAL FORMULATION USING η-APPROXIMATED METHOD

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The present article explores the way η-approximated method is applied to substantiate duality results for the fractional variational problems under invexity. η-approximated dual pair is engineered and a careful study of the original dual pair has been done to establish the duality results for original problems. Moreover, an appropriate example is constructed based on which we can validate the established dual statements. The paper includes several recent results as special cases.

Keywords: duality, variational problem, optimal solution

Classification: 49J40, 49N15, 90C46, 90C32

1. INTRODUCTION

In optimization problems, we often identify the leading solutions among all possible feasible solutions. Such formulations sometimes may include more than one objective which we say vector optimization problems and we aim to extract nondominated solutions for such problems. Mechanical problems are often characterized by multiobjective variational problems along with some conditions in the form of constraints entangled with them. The same can be seen in various other phenomena like economic programming, production, inventory, and control problems.

A new chapter started with the introduction of symmetric duality by Dorn [9] in the year 1960. Later on, Mond and Hanson [18] diversified the duality results to cope with the variational problems. Hanson [11] coined invexity as a generalization of differentiable convex functions and derived Kuhn–Tucker conditions for such problems. Several new dimensions have been developed to model the complex phenomenon occurring in various engineering applications. Consequently, Mond et al. [17] expanded the results of [18] and discussed the problems of variational control under invexity. Mond and Husain [20] derived the sufficiency results to trace out optimality and duality relations for variational programs with the help of generalized invexity. Bector and Husain [7] established the duality theorems for multiobjective variational formulations. Subsequently, Nahak and Nanda [21] introduced the duality relations for the multiobjective variational problems using invexity. Later, Zhian and Qingkai [24] worked on the dual formulations for

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multiobjective problems under invexity. For solving optimization problems, Lagrange multipliers and saddle points play a prominent role in optimization theory and several researchers including Ghosh and Shaiju [10] and Li et al. [16] devised these tools for finding the optimum solutions.

Antczak [1] proposed a new approach to solve vector optimization problems by forming an equivalent problem with a modified objective function. Antczak [2] established a similarity between constrained nonlinear problem and its corresponding η-approximated problems using invexity and derived some of the duality theorems for both original as well as corresponding dual. In 2005, Antczak [3] used r-invexity to study the interrelation between optimal points of the original problems and also η-approximated problems. After a gap of two years, Antczak [4] applied η-approximation method to study constrained nonlinear problems equipped with invex functions. Husain and Ahmed [12] worked on nonlinear variational programming problems and derived several duality results using pseudo-invexity. Khazafi et al. [15] configured conditions to ensure optimality criteria and duality for multiobjective formulations with generalized of (β, ρ)-type I functions. Nahak and Behera [22] derived the duality theorems and optimality conditions for problems with variational settings under generalized $\rho - (\eta, \theta)$-B-type-I functions.

In 2014 Antczak [5] applied (ϕ, ρ)-invexity to nonconvex multiobjective problems to derive suitable duality results for mixed dual problems. Also, Antczak and Michalak [6] worked for nonconvex multiobjective problems using η-approximation method. Recently, Jayswal et al. [13] established the interrelation between a variational and modified variational problem using η-approximation method. Jha et al. [14] introduced exponential type duality for η-approximated variational problems, which we have extended to the fractional analog in the present paper especially implementing η-approximation method. We consider the associated fractional variational programming problem using the η-approximation method under invexity. In this, we constructed the η-approximated problem for the original variational problem and associated Mond–Weir type dual formulation both. We have proved several duality results for original and modified variational problems. Also, we have established an example of a fractional variational problem.

The development of the present article will move on as follows. Section 2, recalls some preliminaries and definitions which we used in the remaining part of the paper. In Section 3, we constructed Mond–Weir dual model and formulated an η-approximated variational problem by modifying both the objective function and constraints in the original problems and its dual. Moreover, we focused on relevant duality results for the considered original and modified problems. Finally, Section 4 throws light on the accomplished work in the form of conclusions.

2. PRELIMINARIES

Consider the interval $\Im = [\tau_1, \tau_2]$ and functions $\zeta_1 : \Im \times \Re^n \times \Re^n \to \Re$, $\zeta_2 : \Im \times \Re^n \times \Re^n \to \Re$ and $\aleph : \Im \times \Re^n \times \Re^n \to \Re^m$, which is continuously differentiable. For $\pi : \Im \to \Re^n$, we use $\dot{\pi}(t)$ to denote the derivative of π with regard to t.

$$\zeta_{1_\pi} = \left(\frac{\partial \zeta_1}{\partial \pi_1}, \quad \frac{\partial \zeta_1}{\partial \pi_2}, \dots, \frac{\partial \zeta_1}{\partial \pi_n} \right)^T, \zeta_{1_{\dot{\pi}}} = \left(\frac{\partial \zeta_1}{\partial \dot{\pi}_1}, \frac{\partial \zeta_1}{\partial \dot{\pi}_2}, \dots, \frac{\partial \zeta_1}{\partial \dot{\pi}_n} \right)^T,$$

where the transpose is represented by the superscript T. Moreover, $\aleph_\pi$ and $\aleph_{\dot{\pi}}$ are used to specify $m \times n$ Jacobian matrices of $\aleph$ with regard to π and $\dot{\pi}$, respectively. Assume X represents the space of continuously differentiable mappings $\pi : \Im \to \Re^n$ where we define norm by $\|\pi\| = \|\pi\|_\infty + \|D\pi\|_\infty$. The operator D can be described by

$$\sigma = D\pi \Leftrightarrow \pi(t) = \pi(\tau_1) + \int_{\tau_1}^{t} \sigma(s)\,\mathrm{d}s,$$

where the boundary value is $\pi(\tau_1)$. Therefore, $D \equiv \frac{\mathrm{d}}{\mathrm{d}t}$ excluding points where the functions are not continuous.

We work out the following fractional variational programming problem in the present paper:

$$\text{(P)} \qquad \text{minimize} \qquad \phi(\pi) = \frac{\int_{\tau_1}^{\tau_2} \zeta_1(t, \pi(t), \dot{\pi}(t))\,\mathrm{d}t}{\int_{\tau_1}^{\tau_2} \zeta_2(t, \pi(t), \dot{\pi}(t))\,\mathrm{d}t}$$

s.t.

$$\aleph(t, \pi(t), \dot{\pi}(t)) \leq 0, \ t \in \Im,$$

$$\pi(\tau_1) = \alpha, \quad \pi(\tau_2) = \beta,$$

where $\int_{\tau_1}^{\tau_2} \zeta_1(t, \pi(t), \dot{\pi}(t))\,\mathrm{d}t \geq 0$ and $\int_{\tau_1}^{\tau_2} \zeta_2(t, \pi(t), \dot{\pi}(t))\,\mathrm{d}t > 0$. The region where the constraints are satisfied (feasible region) is given by $\mathbb{F} = \{\pi \in X : \pi(\tau_1) = \alpha,\ \pi(\tau_2) = \beta \text{ and } \aleph(t, \pi, \dot{\pi}) \leq 0,\ t \in \Im\}$.

Special cases

(i) If $\int_{\tau_1}^{\tau_2} \zeta_2(t, \pi(t), \dot{\pi}(t))\,\mathrm{d}t = 1$, then the problem (P) reduces to the problem discussed in Jha et al. [14].

(ii) In addition to (i), if we consider the static case, then we get the problem discussed in Antczak [4].

Definition 2.1. A feasible point $\tilde{\pi}$ is known as the optimal solution to (P) if we have

$$\frac{\int_{\tau_1}^{\tau_2} \zeta_1(t, \pi(t), \dot{\pi}(t))\,\mathrm{d}t}{\int_{\tau_1}^{\tau_2} \zeta_2(t, \pi(t), \dot{\pi}(t))\,\mathrm{d}t} \geq \frac{\int_{\tau_1}^{\tau_2} \zeta_1(t, \tilde{\pi}(t), \dot{\tilde{\pi}}(t))\,\mathrm{d}t}{\int_{\tau_1}^{\tau_2} \zeta_2(t, \tilde{\pi}(t), \dot{\tilde{\pi}}(t))\,\mathrm{d}t}; \ \forall \pi \in \mathbb{F}.$$

Hereafter, we indicate $\zeta_1(t, \pi(t), \dot{\pi}(t))$ shortly by $\zeta_1(t, \pi, \dot{\pi})$. Suppose $\eta : \Im \times X \times X \to \Re^n$ is a differentiable multi-valued function with $\eta(t, \pi, \pi) = 0$, for all $\pi(t) \in X$. Now, we define the concept of invexity for variational problems.

Definition 2.2. The functional $\int_{\tau_1}^{\tau_2} \zeta_1(t,\pi,\dot{\pi})\,\mathrm{d}t$ is known as invex (strictly invex) at $\tilde{\pi} \in X$ with regard to η provided

$$\int_{\tau_1}^{\tau_2} \zeta_1(t,\pi,\dot{\pi})\,\mathrm{d}t - \int_{\tau_1}^{\tau_2} \zeta_1(t,\tilde{\pi},\dot{\tilde{\pi}})\,\mathrm{d}t$$

$$\geq (>) \int_{\tau_1}^{\tau_2} \left\{ \eta(t,\pi,\tilde{\pi})^T \zeta_{1_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}}) + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\pi})\right)^T \zeta_{1_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}}) \right\} \mathrm{d}t;\ \forall\, \pi \in X.$$

Now, we give the criterion of optimality for the problem (P) which is the subcase of Theorem 3.3 given in Zalmai [23].

Theorem 2.3. Assume $\tilde{\pi} \in \mathbb{F}$ represents an optimal solution to the variational problem (P) and Slater's constraint qualification (see, Theorem 2.1 of Chandra et al. [8]) be satisfied at $\pi \in \mathbb{F}$. Then, a smooth piecewise function $\varsigma : \Im \to \Re^m$, $(\varsigma(t))^T \geq 0$ exists so that

$$\Psi(\tilde{\pi})\zeta_{1_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}}) + (\varsigma(t))^T \aleph_\pi(t,\tilde{\pi},\dot{\tilde{\pi}})$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}[\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}}) - \Phi\zeta_{2_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}}) + (\varsigma(t))^T \aleph_{\dot{\pi}}(t,\tilde{\pi},\dot{\tilde{\pi}})], \tag{1}$$

$$(\varsigma(t))^T \aleph(t,\tilde{\pi},\dot{\tilde{\pi}}) = 0, \tag{2}$$

where $\Phi(\tilde{\pi})$ is same as that of the numerator of $\phi(\tilde{\pi})$ whereas $\Psi(\tilde{\pi})$ is same as that of the denominator.

3. DUAL FORMULATION

In this section, we establish duality results for Mond–Weir type dual [19] for the original variational problem (P) with the help of a modified Mond–Weir dual problem, consisting of the modified objective function.

The following Mond–Weir type dual model is taken into consideration for the problem (P):

$$\text{(D)} \qquad \text{maximize} \qquad \phi(\sigma) = \frac{\int_{\tau_1}^{\tau_2} \zeta_1(t,\sigma,\dot{\sigma})\,\mathrm{d}t}{\int_{\tau_1}^{\tau_2} \zeta_2(t,\sigma,\dot{\sigma})\,\mathrm{d}t}$$

s.t.

$$\sigma(\tau_1) = \alpha, \quad \sigma(\tau_2) = \beta, \tag{3}$$

$$\Psi(\sigma)\zeta_{1_\pi}(t,\sigma,\dot{\sigma}) - \Phi(\sigma)\zeta_{2_\pi}(t,\sigma,\dot{\sigma}) + (\varsigma(t))^T \aleph_\pi(t,\sigma,\dot{\sigma})$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}[\Psi(\sigma)\zeta_{1_{\dot{\pi}}}(t,\sigma,\dot{\sigma}) - \Phi(\sigma)\zeta_{2_{\dot{\pi}}}(t,\sigma,\dot{\sigma}) + (\varsigma(t))^T \aleph_{\dot{\pi}}(t,\sigma,\dot{\sigma})], t \in \Im, \tag{4}$$

$$\int_{\tau_1}^{\tau_2} \varsigma(t)^T \aleph(t,\sigma,\dot{\sigma})\ \mathrm{d}t \geq 0,\ \ t \in \Im, \tag{5}$$

$$\varsigma(t)^T \geq 0,\ \ t \in \Im, \tag{6}$$

where $\Phi(\sigma)$ and $\Psi(\sigma)$ are as depicted in Theorem 2.3 whereas $\varsigma(t) : \Im \to \Re^m$ is a smooth piecewise function. The set containing all feasible solutions to the modified problem (D) is denoted by $\mathbb{W}$. For $(\tilde{\sigma}, \varsigma(t)) \in \mathbb{W}$, we construct $(\mathrm{P}_\eta(\tilde{\sigma}))$ and $(\mathrm{D}_\eta(\tilde{\sigma}))$ as follows:

$$(\mathrm{P}_\eta(\tilde{\sigma})) \qquad \text{minimize} \int_{\tau_1}^{\tau_2} \left\{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\} \mathrm{d}t$$

$$+ \int_{\tau_1}^{\tau_2} \Bigg\{\eta(t,\pi,\tilde{\sigma}) \left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}$$

$$+ \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\sigma})\right) \left\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\} \Bigg\}\mathrm{d}t$$

s.t.

$$\pi(\tau_1) = \alpha, \quad \pi(\tau_2) = \beta, \tag{7}$$

$$\int_{\tau_1}^{\tau_2} \varsigma(t)^T \aleph(t,\tilde{\sigma},\dot{\tilde{\sigma}})\,\mathrm{d}t + \int_{\tau_1}^{\tau_2} \Bigg\{\eta(t,\pi,\tilde{\sigma})\varsigma(t)^T \aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})$$

$$+ \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\sigma})\right) \varsigma(t)^T \aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\Bigg\}\mathrm{d}t \leq 0,\ \ t \in \Im, \tag{8}$$

$$(\mathrm{D}_\eta(\tilde{\sigma})) \qquad \text{maximize} \int_{\tau_1}^{\tau_2} \left\{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\} \mathrm{d}t$$

$$+ \int_{\tau_1}^{\tau_2} \Bigg\{\eta(t,\sigma,\tilde{\sigma}) \left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}$$

$$+ \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right) \left\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\} \Bigg\}\mathrm{d}t$$

s.t.

$$\sigma(\tau_1) = \alpha, \quad \sigma(\tau_2) = \beta, \tag{9}$$

$$\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \varsigma(t)^T\aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}[\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \varsigma(t)^T\aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})],\ \ t \in \Im, \tag{10}$$

$$\int_{\tau_1}^{\tau_2} \varsigma(t)^T \aleph(t,\tilde{\sigma},\dot{\tilde{\sigma}})\,\mathrm{d}t + \int_{\tau_1}^{\tau_2} \Bigg\{\eta(t,\sigma,\tilde{\sigma})\varsigma(t)^T \aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})$$

$$+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)\varsigma(t)^T\aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\Bigg\}\mathrm{d}t \geq 0,\ t\in\Im, \tag{11}$$

$$\varsigma(t)^T \geq 0,\ t\in\Im, \tag{12}$$

where $\varsigma(t):\Im\to\Re^m$ is a piecewise smooth function. $\mathbb{F}(\tilde{\sigma})$ and $\mathbb{W}(\tilde{\sigma})$ denote all feasible solutions of problems $(\mathrm{P}_\eta(\tilde{\sigma}))$ and $(\mathrm{D}_\eta(\tilde{\sigma}))$, respectively.

Now, we demonstrate weak duality between modified variational problem $(\mathrm{P}_\eta(\tilde{\sigma}))$ and modified Mond–Weir dual problem $(\mathrm{D}_\eta(\tilde{\sigma}))$.

Proposition 3.1. (Weak duality for the modified problems) Let π and $(\sigma,\varsigma(t))$ be feasible solutions of problems $(\mathrm{P}_\eta(\tilde{\sigma}))$ and $(\mathrm{D}_\eta(\tilde{\sigma}))$, respectively. Then

$$\int_{\tau_1}^{\tau_2}\left\{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t+\int_{\tau_1}^{\tau_2}\Bigg\{\eta(t,\pi,\tilde{\sigma})\left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right.$$

$$\left.-\Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\sigma})\right)\left\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\Bigg\}\mathrm{d}t$$

$$\geq\int_{\tau_1}^{\tau_2}\left\{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t+\int_{\tau_1}^{\tau_2}\Bigg\{\eta(t,\sigma,\tilde{\sigma})\left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right.$$

$$\left.-\Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)\left\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\Bigg\}\mathrm{d}t.$$

Proof. Since π and $(\sigma,\varsigma(t))$ are feasible solutions of problems $(\mathrm{P}_\eta(\tilde{\sigma}))$ and $(\mathrm{D}_\eta(\tilde{\sigma}))$, respectively, we have

$$\int_{\tau_1}^{\tau_2}\varsigma(t)^T\aleph(t,\tilde{\sigma},\dot{\tilde{\sigma}})\,\mathrm{d}t+\int_{\tau_1}^{\tau_2}\Bigg\{\eta(t,\pi,\tilde{\sigma})\varsigma(t)^T\aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})$$

$$+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\sigma})\right)\varsigma(t)^T\aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\Bigg\}\mathrm{d}t\leq 0 \tag{13}$$

and

$$\int_{\tau_1}^{\tau_2}\varsigma(t)^T\aleph(t,\tilde{\sigma},\dot{\tilde{\sigma}})\,\mathrm{d}t+\int_{\tau_1}^{\tau_2}\Bigg\{\eta(t,\sigma,\tilde{\sigma})\varsigma(t)^T\aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})$$

$$+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)\varsigma(t)^T\aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\Bigg\}\mathrm{d}t\geq 0. \tag{14}$$

From (13) and (14), we get

$$\int_{\tau_1}^{\tau_2}\Bigg\{\eta(t,\pi,\tilde{\sigma})\varsigma(t)^T\aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\sigma})\right)\varsigma(t)^T\aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\Bigg\}\mathrm{d}t$$

$$\leq \int_{\tau_1}^{\tau_2} \left\{\eta(t,\sigma,\tilde{\sigma})\varsigma(t)^T \aleph_{\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)\varsigma(t)^T \aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t. \tag{15}$$

On the other hand, from equation (10), we have

$$\int_{\tau_1}^{\tau_2} \eta(t,\sigma,\tilde{\sigma}) \left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \varsigma(t)^T\aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t$$

$$= \int_{\tau_1}^{\tau_2} \eta(t,\sigma,\tilde{\sigma})\left\{\frac{\mathrm{d}}{\mathrm{d}t}[\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \varsigma(t)^T\aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})]\right\}\mathrm{d}t$$

which using integration by parts gives

$$\int_{\tau_1}^{\tau_2} \eta(t,\sigma,\tilde{\sigma}) \left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \varsigma(t)^T\aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t$$

$$= |\eta(t,\sigma,\tilde{\sigma})[\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \varsigma(t)^T\aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})]|_{\tau_1}^{\tau_2}$$
$$- \int_{\tau_1}^{\tau_2} \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)[\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \varsigma(t)^T\aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})]\mathrm{d}t. \tag{16}$$

Equation (16) together with (9) and $\eta(t,\tilde{\sigma},\tilde{\sigma}) = 0$ yields

$$\int_{\tau_1}^{\tau_2} \eta(t,\sigma,\tilde{\sigma}) \left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \varsigma(t)^T\aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t$$

$$= -\int_{\tau_1}^{\tau_2} \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)[\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \varsigma(t)^T\aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})]\mathrm{d}t.$$

That is,

$$\int_{\tau_1}^{\tau_2} \left\{\eta(t,\sigma,\tilde{\sigma}) \left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\right.$$
$$\left. + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)[\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})]\right\}\mathrm{d}t$$
$$= -\int_{\tau_1}^{\tau_2} \left\{\eta(t,\sigma,\tilde{\sigma})\varsigma(t)^T\aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)\varsigma(t)^T\aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t. \tag{17}$$

In the same way as in above for the feasible point $(\pi, \varsigma(t))$ in $(\mathrm{D}_\eta(\tilde{\sigma}))$, we have

$$\int_{\tau_1}^{\tau_2} \Big\{ \eta(t,\pi,\tilde{\sigma}) \left\{ \Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \right\}$$

$$+ \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\sigma}) \right) [\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})] \Big\} \mathrm{d}t$$

$$= -\int_{\tau_1}^{\tau_2} \left\{ \eta(t,\pi,\tilde{\sigma})\varsigma(t)^T \aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\sigma}) \right) \varsigma(t)^T \aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \right\} \mathrm{d}t. \tag{18}$$

On combining (15), (17) and (18), we obtain

$$\int_{\tau_1}^{\tau_2} \Big\{ \eta(t,\pi,\tilde{\sigma}) \left\{ \Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \right\}$$

$$+ \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\sigma}) \right) [\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})] \Big\} \mathrm{d}t$$

$$\geq \int_{\tau_1}^{\tau_2} \Big\{ \eta(t,\sigma,\tilde{\sigma}) \left\{ \Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \right\}$$

$$+ \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma}) \right) [\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})] \Big\} \mathrm{d}t,$$

which upon adding $\int_{\tau_1}^{\tau_2} \left\{ \Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \right\} \mathrm{d}t$ on both sides of the above inequality, we attain

$$\int_{\tau_1}^{\tau_2} \left\{ \Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \right\} \mathrm{d}t + \int_{\tau_1}^{\tau_2} \Big\{ \eta(t,\pi,\tilde{\sigma}) \left\{ \Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \right.$$

$$\left. - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \right\} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\sigma}) \right) [\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})] \Big\} \mathrm{d}t$$

$$\geq \int_{\tau_1}^{\tau_2} \left\{ \Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \right\} \mathrm{d}t + \int_{\tau_1}^{\tau_2} \Big\{ \eta(t,\sigma,\tilde{\sigma}) \left\{ \Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \right.$$

$$\left. - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \right\} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma}) \right) [\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})] \Big\} \mathrm{d}t.$$

Hence the result. □

Theorem 3.2. (Weak duality for the original problems) Let $\tilde{\pi}$ and $(\tilde{\sigma}, \tilde{\varsigma}(t))$ are feasible solutions of (P) and (D), respectively. Suppose that the functions $\int_{\tau_1}^{\tau_2} \{\Psi(\pi)\zeta_1(t,\pi,\dot{\pi}) - \Phi(\pi)\zeta_2(t,\, \pi,\, \dot{\pi})\}\mathrm{d}t$ and $\int_{\tau_1}^{\tau_2} \tilde{\varsigma}(t)^T \aleph(t,\pi,\dot{\pi})\,\mathrm{d}t$ are invex at $\tilde{\sigma}$ on X with regard to η. Then

$$\frac{\int_{\tau_1}^{\tau_2} \zeta_1(t,\tilde{\pi},\dot{\tilde{\pi}})\,\mathrm{d}t}{\int_{\tau_1}^{\tau_2} \zeta_2(t,\tilde{\pi},\dot{\tilde{\pi}})\,\mathrm{d}t} \geq \frac{\int_{\tau_1}^{\tau_2} \zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}})\,\mathrm{d}t}{\int_{\tau_1}^{\tau_2} \zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\,\mathrm{d}t}.$$

Proof. As a first move, we will show $\tilde{\pi}$ and $(\tilde{\sigma}, \tilde{\varsigma}(t))$ are feasible to $(\mathrm{P}_\eta(\tilde{\sigma}))$ and $(\mathrm{D}_\eta(\tilde{\sigma}))$, respectively. Since $\tilde{\pi}$ is feasible to (P), we have

$$\aleph(t,\tilde{\pi},\dot{\tilde{\pi}}) \leq 0.$$

Using the fact $\tilde{\varsigma}(t)^T \in \Re^m$ and $\tilde{\varsigma}(t)^T \geq 0$ in the above inequality, we obtain

$$\int_{\tau_1}^{\tau_2} \tilde{\varsigma}(t)^T \aleph(t,\tilde{\pi},\dot{\tilde{\pi}})\ \mathrm{d}t \leq 0. \tag{19}$$

Due to invexity of $\int_{\tau_1}^{\tau_2} \tilde{\varsigma}(t)^T \aleph(t,\pi,\dot{\pi})\,\mathrm{d}t$ at $\tilde{\sigma}$ on X with regard to η, we get

$$\int_{\tau_1}^{\tau_2} \tilde{\varsigma}(t)^T \aleph(t,\tilde{\pi},\dot{\tilde{\pi}})\,\mathrm{d}t \geq \int_{\tau_1}^{\tau_2} \tilde{\varsigma}(t)^T \aleph(t,\tilde{\sigma},\dot{\tilde{\sigma}})\,\mathrm{d}t$$

$$+ \int_{\tau_1}^{\tau_2} \left\{\eta(t,\tilde{\pi},\tilde{\sigma})\tilde{\varsigma}(t)^T \aleph_{\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\pi},\tilde{\sigma})\right)\tilde{\varsigma}(t)^T \aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t. \tag{20}$$

In view of (19), inequality (20) yields

$$\int_{\tau_1}^{\tau_2} \tilde{\varsigma}(t)^T \aleph(t,\tilde{\sigma},\dot{\tilde{\sigma}})\,\mathrm{d}t + \int_{\tau_1}^{\tau_2} \left\{\eta(t,\tilde{\pi},\tilde{\sigma})\tilde{\varsigma}(t)^T \aleph_{\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right.$$

$$\left. + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\pi},\tilde{\sigma})\right)\tilde{\varsigma}(t)^T \aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t \leq 0,$$

which validates that feasibility of $\tilde{\pi}$ to $(\mathrm{P}_\eta(\tilde{\sigma}))$. On the flip side, due to feasibility of $(\tilde{\sigma}, \tilde{\varsigma}(t))$ to (D) and inequality (5), we can have

$$\int_{\tau_1}^{\tau_2} \tilde{\varsigma}(t)^T \aleph(t,\tilde{\sigma},\dot{\tilde{\sigma}})\ \mathrm{d}t \geq 0.$$

Since $\eta(t,\tilde{\sigma},\tilde{\sigma})=0$, we can interpret easily that

$$\int_{\tau_1}^{\tau_2} \tilde{\varsigma}(t)^T \aleph(t,\tilde{\sigma},\dot{\tilde{\sigma}})\,\mathrm{d}t + \int_{\tau_1}^{\tau_2} \Big\{ \eta(t,\tilde{\sigma},\tilde{\sigma})\tilde{\varsigma}(t)^T \aleph_{\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\sigma},\tilde{\sigma})\right) \tilde{\varsigma}(t)^T \aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \Big\}\mathrm{d}t \geq 0,$$

which at the same time depicts the feasibility of $(\tilde{\sigma},\tilde{\varsigma}(t))$ to $(\mathrm{D}_\eta(\tilde{\sigma}))$, and therefore, using Proposition 3.1, we get

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} \left\{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t + \int_{\tau_1}^{\tau_2} \Big\{ \eta(t,\tilde{\pi},\tilde{\sigma}) \left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right.\\
&\quad \left. -\Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\pi},\tilde{\sigma})\right)[\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})] \Big\}\mathrm{d}t\\
&\geq \int_{\tau_1}^{\tau_2} \left\{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t + \int_{\tau_1}^{\tau_2} \Big\{ \eta(t,\tilde{\sigma},\tilde{\sigma}) \left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right.\\
&\quad \left. -\Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\sigma},\tilde{\sigma})\right)[\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})] \Big\}\mathrm{d}t.
\end{aligned}$$

Using the fact that $\eta(t,\tilde{\sigma},\tilde{\sigma})=0$, the above inequality implies that

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} \left\{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t + \int_{\tau_1}^{\tau_2} \Big\{ \eta(t,\tilde{\pi},\tilde{\sigma}) \left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right.\\
&\quad \left. -\Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\pi},\tilde{\sigma})\right)[\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})] \Big\}\mathrm{d}t\\
&\geq \int_{\tau_1}^{\tau_2} \left\{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t. \qquad (21)
\end{aligned}$$

Due to invexity of $\int_{\tau_1}^{\tau_2} \{\Psi(\pi)\zeta_1(t,\pi,\dot{\pi}) - \Phi(\pi)\zeta_2(t,\pi,\dot{\pi})\}\,\mathrm{d}t$ at $\tilde{\sigma}$ on X with regard to η, we obtain

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} \left\{\Psi(\tilde{\pi})\zeta_1(t,\tilde{\pi},\dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_2(t,\tilde{\pi},\dot{\tilde{\pi}})\right\}\mathrm{d}t\\
&\geq \int_{\tau_1}^{\tau_2} \left\{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t + \int_{\tau_1}^{\tau_2} \Big\{ \eta(t,\tilde{\pi},\tilde{\sigma}) \left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right.
\end{aligned}$$

$$-\Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\pi},\tilde{\sigma})\right)[\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})]\Big\}\mathrm{d}t. \quad (22)$$

By (21) and (22), we have

$$\int_{\tau_1}^{\tau_2} \left\{\Psi(\tilde{\pi})\zeta_1(t,\tilde{\pi},\dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_2(t,\tilde{\pi},\dot{\tilde{\pi}})\right\} \mathrm{d}t \geq \int_{\tau_1}^{\tau_2} \left\{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\} \mathrm{d}t,$$

That is,

$$\frac{\int_{\tau_1}^{\tau_2} \zeta_1(t,\tilde{\pi},\dot{\tilde{\pi}})\,\mathrm{d}t}{\int_{\tau_1}^{\tau_2} \zeta_2(t,\tilde{\pi},\dot{\tilde{\pi}})\,\mathrm{d}t} \geq \frac{\int_{\tau_1}^{\tau_2} \zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}})\,\mathrm{d}t}{\int_{\tau_1}^{\tau_2} \zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\mathrm{d}t}.$$

Hence the result. □

Example 3.3. Consider X and $\pi : \Im \to \Im,\ \Im = [0,1]$ denote the sets of continuously differentiable functions. We take the following primal and dual pair in our problem.

$$\text{(P1)} \qquad \text{minimize} \quad \phi(\pi) = \frac{\int_0^1 \Big(\pi(t)\ \arctan\pi(t) + t\pi^2(t) + \pi(t) + 1\Big)\,\mathrm{d}t}{\int_0^1 \Big(\sin\pi(t) + \pi(t) + 1\Big)\,\mathrm{d}t}$$

s.t.

$$\aleph(t,\pi,\dot{\pi}) = \pi(t) - 1 \leq 0,$$
$$\pi(0) = 0, \quad \pi(1) = 1.$$

Take $\Omega = \{\pi \in X : \pi(0) = 0,\ \pi(1) = 1 \text{ and } \pi(t) - 1 \leq 0, \text{ where } t \in \Im\}$ as the set containing all feasible solutions and define $\eta : \Im \times X \times X \to \Re$ by $\eta(t,\pi,\tilde{\pi}) = -\pi^2(t) + \tilde{\pi}^2(t)$. Also, let $\varsigma(t)^T = 1$ and $\tilde{\pi}(t) = 0$.

$$\text{(D1)} \qquad \text{maximize} \quad \phi(\sigma) = \frac{\int_0^1 \Big(\sigma(t)\ \arctan\sigma(t) + t\sigma^2(t) + \sigma(t) + 1\Big)\,\mathrm{d}t}{\int_0^1 \Big(\sin\sigma(t) + \sigma(t) + 1\Big)\,\mathrm{d}t}$$

s.t.

$$\sigma(0) = 0, \quad \sigma(1) = 1,$$
$$(\sin\sigma(t) + \sigma(t) + 1)\left(\arctan\sigma(t) + \frac{\sigma(t)}{1+\sigma^2(t)} + 2t\sigma(t) + 1\right)$$
$$-(\sigma(t)\ \arctan\sigma(t) + t\sigma^2(t) + \sigma(t) + 1)(\cos\sigma(t) + 1) + \varsigma(t)^T = 0,$$
$$\int_0^1 \varsigma(t)^T(\sigma(t) - 1) \geq 0,$$

$$\varsigma(t)^T \geq 0,\ t \in \Im.$$

The modified problem for the dual pair (P1) and (D1) for the feasible point $(\tilde{\sigma}(t), \varsigma(t)^T) = (0, 1)$ can be constructed as follows.

$$(\text{P1}_{(\eta)}(\tilde{\sigma})) \qquad \text{minimize} \quad \int_0^1 \pi^2(t)\mathrm{d}t$$

s.t.

$$\pi(0) = 0,\ \pi(1) = 1,$$

$$\int_0^1 (-\pi^2(t) - 1)\varsigma(t)^T \mathrm{d}t \leq 0.$$

$$(\text{D1}_{(\eta)}(\tilde{\sigma})) \qquad \text{maximize} \quad \int_0^1 \sigma^2(t)\mathrm{d}t$$

s.t.

$$\sigma(0) = 0,\ \sigma(1) = 1,$$

$$1 - \varsigma(t)^T = 0,$$

$$\int_0^1 (-\sigma^2(t) - 1)\varsigma(t)^T \mathrm{d}t \geq 0,$$

$$\varsigma(t)^T \geq 0.$$

Clearly, $\tilde{\pi}(t) = 0$ and $(\tilde{\sigma}(t), \varsigma(t)^T) = (0, 1)$ constitute feasible solution to $(\text{P1}_{(\eta)}(\tilde{\sigma}))$ and $(\text{D1}_{(\eta)}(\tilde{\sigma}))$, respectively. Moreover, weak duality for modified problems can also be verified for the chosen feasible points.

Note that η-approximated method transforms the dual pair (P1) and (D1) to a simple (non-fractional) variational dual problem $(\text{P1}_{(\eta)}(\tilde{\sigma}))$ and $(\text{D1}_{(\eta)}(\tilde{\sigma}))$ for which duality theorems can be derived easily. Moreover, the method converts the nonlinear dual problems to linear dual problems as witnessed in [13]. Further, [1] deal with an example where a nonconvex variational dual problem is transformed into a convex variational dual problem using η-approximation method. This suggests that the complexity of the problems can be reduced for certain classes of problems using η-approximation method and the interrelation between the two problems can give valuable information.

Theorem 3.4. Any normal optimal solution to (P) will be an optimal solution to $(\text{P}_\eta(\tilde{\pi}))$.

P r o o f. Due to optimality of normal solution $\tilde{\pi}$ of (P), we can say there exist $\varsigma(t) : \Im \to \Re^m_+$ such that

$$\Psi(\tilde{\pi})\zeta_{1_\pi}(t, \tilde{\pi}, \dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_\pi}(t, \tilde{\pi}, \dot{\tilde{\pi}}) + (\varsigma(t))^T \aleph_\pi(t, \tilde{\pi}, \dot{\tilde{\pi}})$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}[\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t, \tilde{\pi}, \dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t, \tilde{\pi}, \dot{\tilde{\pi}}) + (\varsigma(t))^T \aleph_{\dot{\pi}}(t, \tilde{\pi}, \dot{\tilde{\pi}})], \tag{23}$$

$$\varsigma(t)^T\aleph(t,\tilde{\pi},\dot{\tilde{\pi}}) = 0, \tag{24}$$

$$\varsigma(t)^T \geq 0. \tag{25}$$

If possible, suppose $\tilde{\pi}$ does not represent the optimal solution to $(\mathrm{P}_\eta(\tilde{\pi}))$. Then there exists $\bar{\varsigma} \in \Omega(\tilde{\pi})$ satisfying

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\big\{\Psi(\tilde{\pi})\zeta_1(t,\bar{\pi},\dot{\bar{\pi}}) - \Phi(\tilde{\pi})\zeta_2(t,\bar{\pi},\dot{\bar{\pi}})\big\}\mathrm{d}t + \int_{\tau_1}^{\tau_2}\bigg\{\eta(t,\bar{\varsigma},\tilde{\pi})\big\{\Psi(\tilde{\pi})\zeta_{1_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})\\
&\quad -\Phi(\tilde{\pi})\zeta_{2_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})\big\} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\bar{\varsigma},\tilde{\pi})\right)\big\{\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}})\big\}\bigg\}\mathrm{d}t\\
&\quad < \int_{\tau_1}^{\tau_2}\big\{\Psi(\tilde{\pi})\zeta_1(t,\tilde{\pi},\dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_2(t,\tilde{\pi},\dot{\tilde{\pi}})\big\}\mathrm{d}t + \int_{\tau_1}^{\tau_2}\bigg\{\eta(t,\tilde{\pi},\tilde{\pi})\big\{\Psi(\tilde{\pi})\zeta_{1_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})\\
&\quad - \Phi(\tilde{\pi})\zeta_{2_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})\big\} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\pi},\tilde{\pi})\right)\big\{\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}})\big\}\bigg\}\mathrm{d}t.
\end{aligned} \tag{26}$$

Using the condition that $\eta(t,\tilde{\pi},\tilde{\pi}) = 0$, we obtain

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\bigg\{\eta(t,\bar{\varsigma},\tilde{\pi})\big\{\Psi(\tilde{\pi})\zeta_{1_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})\big\}\\
&\quad + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\bar{\varsigma},\tilde{\pi})\right)\big\{\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}})\big\}\bigg\}\mathrm{d}t < 0.
\end{aligned} \tag{27}$$

Since $\bar{\varsigma}$ is a feasible solution of $(\mathrm{P}_\eta(\tilde{\pi}))$, we use inequality (8) to get

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\varsigma(t)^T\aleph(t,\bar{\pi},\dot{\bar{\pi}})\,\mathrm{d}t + \int_{\tau_1}^{\tau_2}\bigg\{\eta(t,\bar{\varsigma},\tilde{\pi})\varsigma(t)^T\aleph_\pi(t,\tilde{\pi},\dot{\tilde{\pi}})\\
&\quad +\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\bar{\varsigma},\tilde{\pi})\right)\varsigma(t)^T\aleph_{\dot{\pi}}(t,\tilde{\pi},\dot{\tilde{\pi}})\bigg\}\mathrm{d}t \leq 0.
\end{aligned}$$

Equation (24) together with above inequality yields

$$\int_{\tau_1}^{\tau_2}\bigg\{\eta(t,\bar{\varsigma},\tilde{\pi})\varsigma(t)^T\aleph_\pi(t,\tilde{\pi},\dot{\tilde{\pi}}) + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\bar{\varsigma},\tilde{\pi})\right)\varsigma(t)^T\aleph_{\dot{\pi}}(t,\tilde{\pi},\dot{\tilde{\pi}})\bigg\}\mathrm{d}t \leq 0. \tag{28}$$

On the other hand, multiplying both sides of equation (23) by $\eta(t,\bar{\varsigma},\tilde{\pi})$ and then integrating the output from τ_1 to τ_2, we get

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\bigg\{\eta(t,\bar{\varsigma},\tilde{\pi})\big\{\Psi(\tilde{\pi})\zeta_{1_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}}) + (\varsigma(t))^T\aleph_\pi(t,\tilde{\pi},\dot{\tilde{\pi}})\big\}\bigg\}\mathrm{d}t\\
&\quad = \int_{\tau_1}^{\tau_2}\bigg\{\eta(t,\bar{\varsigma},\tilde{\pi})\frac{\mathrm{d}}{\mathrm{d}t}\big\{\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}}) + (\varsigma(t))^T\aleph_{\dot{\pi}}(t,\tilde{\pi},\dot{\tilde{\pi}})\big\}\bigg\}\mathrm{d}t.
\end{aligned}$$

Now, integrating by parts the right-hand side of the above equation and utilizing the fact that $\eta(t,\tilde{\pi},\tilde{\pi})=0$ gives rise to

$$\int_{\tau_1}^{\tau_2}\Big\{\eta(t,\bar{\varsigma},\tilde{\pi})\big\{\Psi(\tilde{\pi})\zeta_{1_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})-\Phi(\tilde{\pi})\zeta_{2_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})+(\varsigma(t))^T\aleph_\pi(t,\tilde{\pi},\dot{\tilde{\pi}})\big\}\Big\}\mathrm{d}t$$

$$=-\int_{\tau_1}^{\tau_2}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\bar{\varsigma},\tilde{\pi})\Big)\big\{\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}})-\Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}})+(\varsigma(t))^T\aleph_{\dot{\pi}}(t,\tilde{\pi},\dot{\tilde{\pi}})\big\}\mathrm{d}t.$$

Rewriting the above equation, we get

$$\int_{\tau_1}^{\tau_2}\Big\{\eta(t,\bar{\varsigma},\tilde{\pi})\big\{\Psi(\tilde{\pi})\zeta_{1_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})-\Phi(\tilde{\pi})\zeta_{2_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})\big\}$$

$$+\Big(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\bar{\varsigma},\tilde{\pi})\Big)\big\{\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}})-\Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}})\big\}\Big\}\mathrm{d}t$$

$$=-\int_{\tau_1}^{\tau_2}\Big\{\eta(t,\bar{\varsigma},\tilde{\pi})(\varsigma(t))^T\aleph_\pi(t,\tilde{\pi},\dot{\tilde{\pi}})+\Big(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\bar{\varsigma},\tilde{\pi})\Big)(\varsigma(t))^T\aleph_{\dot{\pi}}(t,\tilde{\pi},\dot{\tilde{\pi}})\Big\}\mathrm{d}t,$$

which on using equation (28) gives

$$\int_{\tau_1}^{\tau_2}\Big\{\eta(t,\bar{\varsigma},\tilde{\pi})\big\{\Psi(\tilde{\pi})\zeta_{1_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})-\Phi(\tilde{\pi})\zeta_{2_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})\big\}$$

$$+\Big(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\bar{\varsigma},\tilde{\pi})\Big)\big\{\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}})-\Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}})\big\}\Big\}\mathrm{d}t\geq 0, \tag{29}$$

which contradicts equation (27). Therefore $\tilde{\pi}$ is an optimal solution of $(\mathrm{P}_\eta(\tilde{\pi}))$. Hence the proof is complete. □

Theorem 3.5. (Strong duality for the modified problem) If $\tilde{\pi}$ be the normal optimal solution to $(\mathrm{P}_\eta(\tilde{\pi}))$. Then there exists a piecewise smooth function $\tilde{\varsigma}(t):\ \Im\to\Re_+^m$ so that $(\tilde{\pi},\tilde{\varsigma})$ will also be an optimal solution to $(D_\eta(\tilde{\pi}))$.

P r o o f. If $\tilde{\pi}$ represent both normal as well as optimal solution to $(\mathrm{P}_\eta(\tilde{\pi}))$, we may get a piecewise smooth function $\tilde{\varsigma}(t):\Im\to\Re_+^m$ satisfying (23)–(25). With the help of equations (23) and (24) along with $\eta(t,\tilde{\pi},\tilde{\pi})=0$, we obtain

$$\Psi(\tilde{\pi})\zeta_{1_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})-\Phi(\tilde{\pi})\zeta_{2_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})+(\tilde{\varsigma}(t))^T\aleph_\pi(t,\tilde{\pi},\dot{\tilde{\pi}})$$

$$=\frac{\mathrm{d}}{\mathrm{d}t}[\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}})-\Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}})+(\tilde{\varsigma}(t))^T\aleph_{\dot{\pi}}(t,\tilde{\pi},\dot{\tilde{\pi}})].$$

$$\int_{\tau_1}^{\tau_2}\tilde{\varsigma}(t)^T\aleph(t,\tilde{\pi},\dot{\tilde{\pi}})\,\mathrm{d}t+\int_{\tau_1}^{\tau_2}\Big\{\eta(t,\tilde{\pi},\tilde{\pi})\tilde{\varsigma}(t)^T\aleph_\pi(t,\tilde{\pi},\dot{\tilde{\pi}})$$

$$+\Big(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\pi},\tilde{\pi})\Big)\tilde{\varsigma}(t)^T\aleph_{\dot{\pi}}(t,\tilde{\pi},\dot{\tilde{\pi}})\Big\}\mathrm{d}t=0,$$

which confirms feasibility of $(\tilde{\pi}, \tilde{\varsigma})$ to $(D_\eta(\tilde{\pi}))$. Also, if $(\tilde{\pi}, \tilde{\varsigma})$ is not the optimal solution of $(D_\eta(\tilde{\pi}))$, there exist $(\tilde{\sigma}, \bar{\xi})$ in $S(\tilde{\pi})$ in such a way that

$$\int_{\tau_1}^{\tau_2} \{\Psi(\tilde{\pi})\zeta_1(t, \tilde{\pi}, \dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_2(t, \tilde{\pi}, \dot{\tilde{\pi}})\}\mathrm{d}t + \int_{\tau_1}^{\tau_2} \Big\{\eta(t, \tilde{\sigma}, \tilde{\pi})\{\Psi(\tilde{\pi})\zeta_{1_\pi}(t, \tilde{\pi}, \dot{\tilde{\pi}})$$

$$-\Phi(\tilde{\pi})\zeta_{2_\pi}(t, \tilde{\pi}, \dot{\tilde{\pi}})\} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t, \tilde{\sigma}, \tilde{\pi})\right)\{\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t, \tilde{\pi}, \dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t, \tilde{\pi}, \dot{\tilde{\pi}})\}\Big\}\mathrm{d}t$$

$$> \int_{\tau_1}^{\tau_2} \{\Psi(\tilde{\pi})\zeta_1(t, \tilde{\pi}, \dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_2(t, \tilde{\pi}, \dot{\tilde{\pi}})\}\mathrm{d}t + \int_{\tau_1}^{\tau_2} \Big\{\eta(t, \tilde{\pi}, \tilde{\pi})\{\Psi(\tilde{\pi})\zeta_{1_\pi}(t, \tilde{\pi}, \dot{\tilde{\pi}})$$

$$- \Phi(\tilde{\pi})\zeta_{2_\pi}(t, \tilde{\pi}, \dot{\tilde{\pi}})\} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t, \tilde{\pi}, \tilde{\pi})\right)\{\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t, \tilde{\pi}, \dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t, \tilde{\pi}, \dot{\tilde{\pi}})\}\Big\}\mathrm{d}t. \quad (30)$$

Using the fact that $\eta(t, \tilde{\pi}, \tilde{\pi}) = 0$, we obtain

$$\int_{\tau_1}^{\tau_2} \Big\{\eta(t, \tilde{\sigma}, \tilde{\pi})\{\Psi(\tilde{\pi})\zeta_{1_\pi}(t, \tilde{\pi}, \dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_\pi}(t, \tilde{\pi}, \dot{\tilde{\pi}})\}$$

$$+ \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t, \tilde{\sigma}, \tilde{\pi})\right)\{\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t, \tilde{\pi}, \dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t, \tilde{\pi}, \dot{\tilde{\pi}})\}\Big\}\mathrm{d}t > 0. \quad (31)$$

Since $(\tilde{\sigma}, \bar{\xi})$ is feasible to $(D_\eta(\tilde{\pi}))$, we use equation (10) to get

$$\Psi(\tilde{\pi})\zeta_{1_\pi}(t, \tilde{\pi}, \dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_\pi}(t, \tilde{\pi}, \dot{\tilde{\pi}}) + \bar{\xi}(t)\aleph_\pi(t, \tilde{\pi}, \dot{\tilde{\pi}})$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}[\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t, \tilde{\pi}, \dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t, \tilde{\pi}, \dot{\tilde{\pi}}) + \bar{\xi}(t)\aleph_{\dot{\pi}}(t, \tilde{\pi}, \dot{\tilde{\pi}})],$$

which upon multiplying both sides by $\eta(t, \tilde{\sigma}, \tilde{\pi})$ and then integrating from τ_1 to τ_2, produces

$$\int_{\tau_1}^{\tau_2} \Big\{\eta(t, \tilde{\sigma}, \tilde{\pi})\{\Psi(\tilde{\pi})\zeta_{1_\pi}(t, \tilde{\pi}, \dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_\pi}(t, \tilde{\pi}, \dot{\tilde{\pi}}) + \bar{\xi}(t)\aleph_\pi(t, \tilde{\pi}, \dot{\tilde{\pi}})\}\Big\}\mathrm{d}t$$

$$= \int_{\tau_1}^{\tau_2} \Big\{\eta(t, \tilde{\sigma}, \tilde{\pi})\frac{\mathrm{d}}{\mathrm{d}t}\{\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t, \tilde{\pi}, \dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t, \tilde{\pi}, \dot{\tilde{\pi}}) + \bar{\xi}(t)\aleph_{\dot{\pi}}(t, \tilde{\pi}, \dot{\tilde{\pi}})\}\Big\}\mathrm{d}t.$$

Now, integrating the right side of above equation and making use of $\eta(t, \tilde{\pi}, \tilde{\pi}) = 0$, we obtain

$$\int_{\tau_1}^{\tau_2} \Big\{\eta(t, \tilde{\sigma}, \tilde{\pi})\{\Psi(\tilde{\pi})\zeta_{1_\pi}(t, \tilde{\pi}, \dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_\pi}(t, \tilde{\pi}, \dot{\tilde{\pi}}) + \bar{\xi}(t)\aleph_\pi(t, \tilde{\pi}, \dot{\tilde{\pi}})\}\Big\}\mathrm{d}t$$

$$= -\int_{\tau_1}^{\tau_2} \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t, \tilde{\sigma}, \tilde{\pi})\right)\{\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t, \tilde{\pi}, \dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t, \tilde{\pi}, \dot{\tilde{\pi}}) + \bar{\xi}(t)\aleph_{\dot{\pi}}(t, \tilde{\pi}, \dot{\tilde{\pi}})\}\mathrm{d}t.$$

Rewriting the above equation, we get

$$\int_{\tau_1}^{\tau_2} \Big\{\eta(t, \tilde{\sigma}, \tilde{\pi})\{\Psi(\tilde{\pi})\zeta_{1_\pi}(t, \tilde{\pi}, \dot{\tilde{\pi}}) - \Phi(\tilde{\pi})\zeta_{2_\pi}(t, \tilde{\pi}, \dot{\tilde{\pi}})\}$$

$$+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\sigma},\tilde{\pi})\right)\left\{\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}})-\Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}})\right\}\Bigg\}\mathrm{d}t$$

$$=-\int_{\tau_1}^{\tau_2}\left\{\eta(t,\tilde{\sigma},\tilde{\pi})\bar{\xi}(t)\aleph_{\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\sigma},\tilde{\pi})\right)\bar{\xi}(t)\aleph_{\dot{\pi}}(t,\tilde{\pi},\dot{\tilde{\pi}})\right\}\mathrm{d}t,$$

which on using equation (31) gives

$$\int_{\tau_1}^{\tau_2}\left\{\eta(t,\tilde{\sigma},\tilde{\pi})\bar{\xi}(t)\aleph_{\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\sigma},\tilde{\pi})\right)\bar{\xi}(t)\aleph_{\dot{\pi}}(t,\tilde{\pi},\dot{\tilde{\pi}})\right\}\mathrm{d}t<0. \tag{32}$$

Since $(\tilde{\sigma},\bar{\xi})$ is feasible to $(D_\eta(\tilde{\pi}))$, we use inequality (11) to get

$$\int_{\tau_1}^{\tau_2}\bar{\xi}(t)\aleph(t,\tilde{\pi},\dot{\tilde{\pi}})\,\mathrm{d}t+\int_{\tau_1}^{\tau_2}\Bigg\{\eta(t,\tilde{\sigma},\tilde{\pi})\bar{\xi}(t)\aleph_{\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})$$

$$+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\sigma},\tilde{\pi})\right)\bar{\xi}(t)\aleph_{\dot{\pi}}(t,\tilde{\pi},\dot{\tilde{\pi}})\Bigg\}\mathrm{d}t\geq 0.$$

Using equation (24), the above inequality yields

$$\int_{\tau_1}^{\tau_2}\left\{\eta(t,\tilde{\sigma},\tilde{\pi})\bar{\xi}(t)\aleph_{\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\sigma},\tilde{\pi})\right)\bar{\xi}(t)\aleph_{\dot{\pi}}(t,\tilde{\pi},\dot{\tilde{\pi}})\right\}\mathrm{d}t\geq 0,$$

which contradicts equation (32). Hence, the solution $(\tilde{\sigma},\tilde{\varsigma})$ is optimal to $(D_\eta(\tilde{\pi}))$. □

Theorem 3.6. (Strong duality for original problem) If $\tilde{\pi}$ is a normal optimal solution to (P) and if all the assumptions specified in Theorem 3.2 are satisfied, then $(\tilde{\pi},\tilde{\varsigma})$ will become an optimal solution to (D) and have equal optimal values.

P r o o f. As the solution $\tilde{\pi}$ is a normal optimal solution to the problem (P), $\tilde{\pi}$ is also optimality of $(\mathrm{P}_\eta(\tilde{\pi}))$. With the help of Theorem 3.5, we infer that $(\tilde{\pi},\tilde{\varsigma})$ is also an optimality solution of $(\mathrm{D}_\eta(\tilde{\pi}))$. Hence,

$$\Psi(\tilde{\pi})\zeta_{1_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})-\Phi(\tilde{\pi})\zeta_{2_\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})+(\tilde{\varsigma}(t))^T\aleph_{\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})$$

$$=\frac{\mathrm{d}}{\mathrm{d}t}[\Psi(\tilde{\pi})\zeta_{1_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}})-\Phi(\tilde{\pi})\zeta_{2_{\dot{\pi}}}(t,\tilde{\pi},\dot{\tilde{\pi}})+(\tilde{\varsigma}(t))^T\aleph_{\dot{\pi}}(t,\tilde{\pi},\dot{\tilde{\pi}})],$$

$$\int_{\tau_1}^{\tau_2}\tilde{\varsigma}(t)^T\aleph(t,\tilde{\pi},\dot{\tilde{\pi}})\,\mathrm{d}t+\int_{\tau_1}^{\tau_2}\Bigg\{\eta(t,\tilde{\pi},\tilde{\pi})\tilde{\varsigma}(t)^T\aleph_{\pi}(t,\tilde{\pi},\dot{\tilde{\pi}})$$

$$+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\pi},\tilde{\pi})\right)\tilde{\varsigma}(t)^T\aleph_{\dot{\pi}}(t,\tilde{\pi},\dot{\tilde{\pi}})\Bigg\}\mathrm{d}t\geq 0,$$

$$\tilde{\varsigma}(t)(t)^T\geq 0, t\in\Im,$$

which asserts that $(\tilde{\pi},\tilde{\varsigma})$ is also feasible to (D). As all the conditions mentioned in the weak duality theorem are fulfilled at $(\tilde{\pi},\tilde{\varsigma})$, so $(\tilde{\pi},\tilde{\varsigma})$ turns out to be the optimal solution to (D) giving the same extremal value as that of (P). □

Theorem 3.7. (Converse dual statement for the modified problem) Let $(\tilde{\sigma}, \tilde{\varsigma})$ be an optimal solution of $(D_\eta(\tilde{\sigma}))$. Then $\tilde{\sigma}$ is an optimal solution of $(\mathrm{P}_\eta(\tilde{\sigma}))$.

Proof. If feasible solution $\tilde{\sigma}$ fails to be optimal to $(\mathrm{P}_\eta(\tilde{\sigma}))$, there exists $\pi \in \Omega(\tilde{\sigma})$ such that

$$\int_{\tau_1}^{\tau_2} \big\{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\big\}\mathrm{d}t + \int_{\tau_1}^{\tau_2} \bigg\{\eta(t,\pi,\tilde{\sigma})\big\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\big\} + \bigg(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\sigma})\bigg)\big\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\big\}\bigg\}\mathrm{d}t$$
$$< \int_{\tau_1}^{\tau_2} \big\{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\big\}\mathrm{d}t + \int_{\tau_1}^{\tau_2} \bigg\{\eta(t,\tilde{\sigma},\tilde{\sigma})\big\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\big\} + \bigg(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\sigma},\tilde{\sigma})\bigg)\big\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\big\}\bigg\}\mathrm{d}t.$$

Making use of $\eta(t,\tilde{\sigma},\tilde{\sigma}) = 0$, the relation stated above can be written as

$$\int_{\tau_1}^{\tau_2} \bigg\{\eta(t,\pi,\tilde{\sigma})\big\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\big\} + \bigg(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\sigma})\bigg)\big\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\big\}\bigg\}\mathrm{d}t < 0. \tag{33}$$

Since $(\tilde{\sigma}, \tilde{\varsigma})$ is a feasible solution of $(D_\eta(\tilde{\sigma}))$, we use inequality (11) to get

$$\int_{\tau_1}^{\tau_2} \tilde{\varsigma}(t)^T \aleph(t,\tilde{\sigma},\dot{\tilde{\sigma}})\,\mathrm{d}t + \int_{\tau_1}^{\tau_2} \bigg\{\eta(t,\tilde{\sigma},\tilde{\sigma})\tilde{\varsigma}(t)^T \aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \bigg(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\sigma},\tilde{\sigma})\bigg)\tilde{\varsigma}(t)^T \aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\bigg\}\mathrm{d}t \geq 0.$$

Now, using $\eta(t,\tilde{\sigma},\tilde{\sigma}) = 0$, the above inequality reduces to

$$\int_{\tau_1}^{\tau_2} \tilde{\varsigma}(t)^T \aleph(t,\tilde{\sigma},\dot{\tilde{\sigma}})\,\mathrm{d}t \;\geq\; 0. \tag{34}$$

Using feasibility of $\tilde{\sigma}$ to $(\mathrm{P}_\eta(\tilde{\sigma}))$, one can have

$$\int_{\tau_1}^{\tau_2} \tilde{\varsigma}(t)^T \aleph(t,\tilde{\sigma},\dot{\tilde{\sigma}})\,\mathrm{d}t + \int_{\tau_1}^{\tau_2} \bigg\{\eta(t,\pi,\tilde{\sigma})\tilde{\varsigma}(t)^T \aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \bigg(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\sigma})\bigg)\tilde{\varsigma}(t)^T \aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\bigg\}\mathrm{d}t \leq 0,$$

which by using (34), reduces to

$$\int_{\tau_1}^{\tau_2} \bigg\{\eta(t,\pi,\tilde{\sigma})\tilde{\varsigma}(t)^T \aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \bigg(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\sigma})\bigg)\tilde{\varsigma}(t)^T \aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\bigg\}\mathrm{d}t \leq 0. \tag{35}$$

Combining inequalities (33) and (35), we get

$$\int_{\tau_1}^{\tau_2}\Big\{\eta(t,\pi,\tilde{\sigma})\big\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})+\tilde{\varsigma}(t)^T\aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})\big\}$$

$$+\Big(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\sigma})\Big)\big\{\Psi(\tilde{\sigma})\zeta_{1_{\dot\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_{\dot\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})+\tilde{\varsigma}(t)^T\aleph_{\dot\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\big\}\Big\}\mathrm{d}t<0. \quad (36)$$

Due to the fact that $(\tilde{\sigma},\tilde{\varsigma})$ is feasible to $(D_\eta(\tilde{\sigma}))$, we obtain

$$\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})+(\tilde{\varsigma}(t))^T\aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})$$

$$=\frac{\mathrm{d}}{\mathrm{d}t}[\Psi(\tilde{\sigma})\zeta_{1_{\dot\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_{\dot\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})+(\tilde{\varsigma}(t))^T\aleph_{\dot\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})].$$

Multiplying $\eta(t,\pi,\tilde{\sigma})$ to either side of the equation and integrating the resultant from τ_1 to τ_2, we obtain

$$\int_{\tau_1}^{\tau_2}\Big\{\eta(t,\pi,\tilde{\sigma})\big\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})+(\tilde{\varsigma}(t))^T\aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})\big\}\Big\}\mathrm{d}t$$

$$=\int_{\tau_1}^{\tau_2}\Big\{\eta(t,\pi,\tilde{\sigma})\frac{\mathrm{d}}{\mathrm{d}t}\big\{\Psi(\tilde{\sigma})\zeta_{1_{\dot\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_{\dot\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})+(\tilde{\varsigma}(t))^T\aleph_{\dot\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\big\}\Big\}\mathrm{d}t.$$

Applying integrating by parts along with $\eta(t,\tilde{\sigma},\tilde{\sigma})=0$, one can get

$$\int_{\tau_1}^{\tau_2}\Big\{\eta(t,\pi,\tilde{\sigma})\big\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})+(\tilde{\varsigma}(t))^T\aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})\big\}\Big\}\mathrm{d}t$$

$$=-\int_{\tau_1}^{\tau_2}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\sigma})\Big)\big\{\Psi(\tilde{\sigma})\zeta_{1_{\dot\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_{\dot\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})+(\tilde{\varsigma}(t))^T\aleph_{\dot\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\big\}\mathrm{d}t.$$

Rearranging the above equation, we get

$$\int_{\tau_1}^{\tau_2}\Big\{\eta(t,\pi,\tilde{\sigma})\big\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})+(\tilde{\varsigma}(t))^T\aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})\big\}\Big\}\mathrm{d}t$$

$$+\int_{\tau_1}^{\tau_2}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\pi,\tilde{\sigma})\Big)\big\{\Psi(\tilde{\sigma})\zeta_{1_{\dot\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_{\dot\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})+(\tilde{\varsigma}(t))^T\aleph_{\dot\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\big\}\mathrm{d}t=0,$$

which violates (36) and hence $\tilde{\sigma}$ becomes maximizer of $(\mathrm{P}_\eta(\tilde{\sigma}))$. □

Theorem 3.8. (Converse duality for the original problems) If $(\tilde{\sigma},\tilde{\varsigma})$ denotes an optimal solution of the original dual (D) satisfying $\aleph(t,\tilde{\sigma},\dot{\tilde{\sigma}})=0$. Under the assumption that $\int_{\tau_1}^{\tau_2}\{\Psi(\pi)\zeta_1(t,\pi,\dot\pi)-\Phi(\pi)\zeta_2(t,\pi,\dot\pi)\}\mathrm{d}t$ and $\int_{\tau_1}^{\tau_2}\tilde{\varsigma}(t)\aleph(t,\pi,\dot\pi)\,\mathrm{d}t$ are invex at $\tilde{\sigma}$ on X with regard to η, $\tilde{\sigma}$ becomes an optimal solution to (P).

Proof. If we presume that $(\tilde{\sigma}, \tilde{\varsigma})$ is not an optimal solution of $(D_\eta(\tilde{\sigma}))$, then we get a feasible point $(\sigma, \bar{\varsigma}) \in S(\tilde{\sigma})$ which satisfies the following conditions

$$\int_{\tau_1}^{\tau_2} \{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\}\mathrm{d}t + \int_{\tau_1}^{\tau_2} \Big\{\eta(t,\tilde{\sigma},\tilde{\sigma})\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})$$
$$-\Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\} + \Big(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\sigma},\tilde{\sigma})\Big)\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\}\Big\}\mathrm{d}t$$
$$< \int_{\tau_1}^{\tau_2} \{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\}\mathrm{d}t + \int_{\tau_1}^{\tau_2} \Big\{\eta(t,\sigma,\tilde{\sigma})\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})$$
$$-\Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\} + \Big(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\Big)\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\}\Big\}\mathrm{d}t.$$

Due to the fact that $\eta(t,\tilde{\sigma},\tilde{\sigma}) = 0$, the last expression settles down to

$$\int_{\tau_1}^{\tau_2} \Big\{\eta(t,\sigma,\tilde{\sigma})\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\}$$
$$+ \Big(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\Big)\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\}\Big\}\mathrm{d}t > 0. \quad (37)$$

Also, by the feasibility of $(\sigma, \bar{\varsigma})$ in $(D_\eta(\tilde{\sigma}))$, we have

$$\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + (\bar{\varsigma}(t))^T\aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}[\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + (\bar{\varsigma}(t))^T\aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})].$$

Multiplying both sides of the above equation by $\eta(t,\sigma,\tilde{\sigma})$ and then integrating between τ_1 and τ_2, we get

$$\int_{\tau_1}^{\tau_2} \Big\{\eta(t,\sigma,\tilde{\sigma})\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + (\bar{\varsigma}(t))^T\aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})\}\Big\}\mathrm{d}t$$
$$= \int_{\tau_1}^{\tau_2} \Big\{\eta(t,\sigma,\tilde{\sigma})\frac{\mathrm{d}}{\mathrm{d}t}\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + (\bar{\varsigma}(t))^T\aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\}\Big\}\mathrm{d}t,$$

which upon integrating and using the condition $\eta(t,\tilde{\sigma},\tilde{\sigma}) = 0$, provides

$$\int_{\tau_1}^{\tau_2} \Big\{\eta(t,\sigma,\tilde{\sigma})\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + (\bar{\varsigma}(t))^T\aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}})\}\Big\}\mathrm{d}t$$
$$= -\int_{\tau_1}^{\tau_2} \Big(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\Big)\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + (\bar{\varsigma}(t))^T\aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\}\mathrm{d}t.$$

Rearranging the above equation, we get

$$\int_{\tau_1}^{\tau_2} \Big\{\eta(t,\sigma,\tilde{\sigma})\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\}$$

$$+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)\left\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\Bigg\}\mathrm{d}t$$

$$=-\int_{\tau_1}^{\tau_2}\left\{\eta(t,\sigma,\tilde{\sigma})(\bar{\varsigma}(t))^T\aleph_{\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)(\bar{\varsigma}(t))^T\aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t,$$

which along with inequality (37) gives

$$\int_{\tau_1}^{\tau_2}\left\{\eta(t,\sigma,\tilde{\sigma})(\bar{\varsigma}(t))^T\aleph_{\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)(\bar{\varsigma}(t))^T\aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t \;<\; 0.$$

Using the condition $\aleph(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \;=\; 0$, we can rewrite the above inequality as

$$\int_{\tau_1}^{\tau_2}\left\{\bar{\varsigma}(t))^T\aleph(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t+\int_{\tau_1}^{\tau_2}\Bigg\{\eta(t,\sigma,\tilde{\sigma})(\bar{\varsigma}(t))^T\aleph_{\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})$$

$$+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)(\varsigma(t))^T\aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\Bigg\}\mathrm{d}t \;<\; 0,$$

which disagree that $(\sigma,\bar{\varsigma})$ is feasible to $(D_\eta(\tilde{\sigma}))$ and hence $(\tilde{\sigma},\tilde{\varsigma})$ is certainly an optimal solution to $(D_\eta(\tilde{\sigma}))$. By converse duality of modified problem, $\tilde{\sigma}$ is an optimal solution of $(\mathrm{P}_\eta(\tilde{\sigma}))$. It remains to show that $\tilde{\sigma}$ is an optimal solution of (P). Let us assume $\tilde{\sigma}$ does not represent a minimal solution to the problem (P). Therefore, we have a feasible point $\sigma\in\Omega$ satisfying

$$\int_{\tau_1}^{\tau_2}\left\{\Psi(\sigma)\zeta_1(t,\sigma,\dot{\sigma})-\Phi(\sigma)\zeta_2(t,\sigma,\dot{\sigma})\right\}\mathrm{d}t \;<\; \int_{\tau_1}^{\tau_2}\left\{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t. \tag{38}$$

Since $\int_{\tau_1}^{\tau_2}\left\{\Psi(\pi)\zeta_1(t,\pi,\dot{\pi})-\Phi(\pi)\zeta_2(t,\pi,\dot{\pi})\right\}\mathrm{d}t$ is invex at $\tilde{\sigma}$ with regard to η, we have

$$\int_{\tau_1}^{\tau_2}\Big\{\left\{\Psi(\sigma)\zeta_1(t,\sigma,\dot{\sigma})-\Phi(\sigma)\zeta_2(t,\sigma,\dot{\sigma})\right\}-\left\{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\Big\}\mathrm{d}t$$

$$\geq\int_{\tau_1}^{\tau_2}\Bigg\{\eta(t,\sigma,\tilde{\sigma})\left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}$$

$$+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)\left\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\Bigg\}\mathrm{d}t.$$

Using inequality (38), the above inequality reduces to

$$\int_{\tau_1}^{\tau_2}\Bigg\{\eta(t,\sigma,\tilde{\sigma})\left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}$$

$$+\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)\left\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})-\Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\Bigg\}\mathrm{d}t<0. \tag{39}$$

Also, by assumption, $\int_{\tau_1}^{\tau_2}\left\{\tilde{\varsigma}(t))^T\aleph(t,\pi,\dot{\pi})\right\}\mathrm{d}t$ is invex at $\tilde{\sigma}$ on X with regard to η, we get

$$\int_{\tau_1}^{\tau_2}\left\{\tilde{\varsigma}(t))^T\aleph(t,\sigma,\dot{\sigma})\right\}\mathrm{d}t-\int_{\tau_1}^{\tau_2}\left\{\tilde{\varsigma}(t))^T\aleph(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t$$

$$\geq \int_{\tau_1}^{\tau_2} \left\{ \eta(t,\sigma,\tilde{\sigma})(\tilde{\varsigma}(t))^T \aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}}) + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)(\tilde{\varsigma}(t))^T \aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \right\}\mathrm{d}t.$$

Since $\sigma \in \Omega$ represents a feasible solution, the above expression turns out to be

$$\int_{\tau_1}^{\tau_2} \left\{\tilde{\varsigma}(t))^T \aleph(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t + \int_{\tau_1}^{\tau_2} \left\{ \eta(t,\sigma,\tilde{\sigma})(\tilde{\varsigma}(t))^T \aleph_\pi(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \right.$$

$$\left. + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)(\tilde{\varsigma}(t))^T \aleph_{\dot{\pi}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \right\}\mathrm{d}t \leq 0,$$

indicating σ is a feasible solution to the modified problem $(\mathrm{P}_\eta(\tilde{\sigma}))$. As $\tilde{\sigma}$ is the minimal solution to $(\mathrm{P}_\eta(\tilde{\sigma}))$, therefore, we can have

$$\int_{\tau_1}^{\tau_2} \left\{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t + \int_{\tau_1}^{\tau_2} \left\{ \eta(t,\tilde{\sigma},\tilde{\sigma})\left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \right.\right.$$

$$\left. -\Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\tilde{\sigma},\tilde{\sigma})\right)\left\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\} \Big\}\mathrm{d}t$$

$$\leq \int_{\tau_1}^{\tau_2} \left\{\Psi(\tilde{\sigma})\zeta_1(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_2(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\}\mathrm{d}t + \int_{\tau_1}^{\tau_2} \left\{ \eta(t,\sigma,\tilde{\sigma})\left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) \right.\right.$$

$$\left. - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)\left\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\} \Big\}\mathrm{d}t.$$

Since $\eta(t,\tilde{\sigma},\tilde{\sigma}) = 0$, the last expression gives

$$\int_{\tau_1}^{\tau_2} \left\{ \eta(t,\sigma,\tilde{\sigma})\left\{\Psi(\tilde{\sigma})\zeta_{1_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_\pi}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\} \right.$$

$$\left. + \left(\frac{\mathrm{d}}{\mathrm{d}t}\eta(t,\sigma,\tilde{\sigma})\right)\left\{\Psi(\tilde{\sigma})\zeta_{1_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}}) - \Phi(\tilde{\sigma})\zeta_{2_{\dot{\pi}}}(t,\tilde{\sigma},\dot{\tilde{\sigma}})\right\} \right\}\mathrm{d}t \geq 0,$$

which contradicts (39). Therefore, we can conclude that $\tilde{\sigma}$ is a minimal solution of (P). □

4. CONCLUSIONS

In the excogitated paper, we analyzed fractional variational problems using η-approximated functions by modifying the objective function and constraints for the original problem. We derived duality results for the modified dual pair and the same can be used to validate the duality results of the original problem. Moreover, constructed example relates the original problem with the modified problem, and projected duality results can be easily checked with this example.

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