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# FURTHER RESULTS ON LAWS OF LARGE NUMBERS FOR UNCERTAIN RANDOM VARIABLES

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The uncertainty theory was founded by Baoding Liu to characterize uncertainty information represented by humans. Basing on uncertainty theory, Yuhan Liu created chance theory to describe the complex phenomenon, in which human uncertainty and random phenomenon coexist. In this paper, our aim is to derive some laws of large numbers (LLNs) for uncertain random variables. The first theorem proved the Etemadi type LLN for uncertain random variables being functions of pairwise independent and identically distributed random variables and uncertain variables without satisfying the conditions of regular, independent and identically distributed (IID). Two kinds of Marcinkiewicz–Zygmund type LLNs for uncertain random variables were established in the case of $p \in (0,1)$ by the second theorem, and in the case of $p > 1$ by the third theorem, respectively. For better illustrating of LLNs for uncertain random variables, some examples were stated and explained. Compared with the existed theorems of LLNs for uncertain random variables, our theorems are the generalised results.

*Keywords:* law of large numbers, uncertain random variable, Etemadi type theorem, Marcinkiewicz–Zygmund type theorem

*Classification:* 46A45, 60F15

## 1. INTRODUCTION

In classical probability theory, Cardano first proposed a limit theorem in the sixteenth century, which later became known as the "law of large numbers" (LLN for short). Subsequently, LLN has been studied by a lot of mathematicians, including Bernoulli, Poisson, Chebyshev, Markov, Borel, Cantelli, Kolmogorov and Khinchine. After a long period of research and development, LLN has formed quite perfect theoretical system and been widely used in real life. In the 21st century, one important development in this field is LLN for random walks in random environments, which has attracted the attention of many researchers. Interested readers may refer to Komorowski and Krupa [21], Comets and Zeitouni [5], Avena et al. [2], Hollander et al. [6].

Fuzzy phenomenon is an extremely important feature of the real world. In order to model fuzzy phenomena, a concept of fuzzy set was proposed by Zadeh [48] in 1965 and possibility theory related to the theory of fuzzy set was founded by Zadeh [49]. After that, fuzzy measure, Choquet integral and Sugeno integral, were further studied in [4,

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42, 43, 44]. Valášková and Struk [43, 44] classified the fuzzy measures into eight classes: submeasure, supermeasure, submodular, supermodular, belief, plausibility, possibility and necessity, which are closed under the operations of distortion functions. The LLN of fuzzy sets was first proposed by Fullér [8] and further generalized by Hong and Ro [14]. Kwakernaak [23, 24] proposed a concept of fuzzy random variable to model complex phenomena with both randomness and fuzziness. Some LLNs for fuzzy random variables were proposed by Kruse [22] and Miyakoshi and Shimbo [38]. The readers can refer to [16, 17, 20] for more results on LLNs for fuzzy random variables.

A lot of surveys showed that human uncertainty does not behave like fuzziness. The debate focus is that the measure of union of events is not necessarily the maximum of measures of individual events (Baoding Liu [30]). In order to overcome this disadvantage, uncertainty theory was created by Baoding Liu [25] in 2007 and developed by Baoding Liu [27] in 2009, which is based on an uncertain measure satisfying normality, duality, subadditivity and product axioms. Within uncertainty theory, the uncertain measure of an event indicates the belief degree that the event will occur. Moreover, this theory provides the notion of uncertain variable to model the quantities under uncertain status. Nowadays, uncertainty theory has been successfully applied in various fields, such as uncertain differential equation (see, e. g., Baoding Liu [26], Yao and Chen [46]), uncertain programming (see, e. g., Baoding Liu [28], Liu and Chen [32]), uncertain calculus (see, e. g., Baoding Liu [27]) , uncertain control (see, e. g., Baoding Liu [29], Gao [9]), etc.

Sometimes, human uncertainties and random factors exist simultaneously in complex systems. In order to study this type of complex systems, chance theory was proposed by Yuhan Liu in [33], which introduced the concept of chance measure, integrated both probability measure and uncertain measure, and proposed the concept of uncertain random variable to model the quantities under uncertain and random conditions. These concepts are mathematical descriptions for uncertain random phenomena (i. e., mixtures of human uncertainties and randomness) and are defined on the basis of probability theory and uncertainty theory. In 2013,Yuhan Liu [34] proposed uncertain random programming as a spectrum of mathematical programming involving uncertain random variables. Then Zhou et al. [50], Qin [40], and Ke et al. [18, 19] developed uncertain random multi-objective programming, uncertain random goal programming and uncertain random multi-level programming, respectively. Besides, chance theory has successfully been applied to risk analysis (see, e. g., Liu and Ralescu [35]), graph and network (see, e. g., Baoding Liu [31]), propositional logic (see, e. g., Liu and Yao [36]) as well as uncertain random process (see, e. g., Gao and Yao [10]).

Meanwhile, LLNs have had some good research results in chance space. In Yao and Gao [47], LLN under chance measure was first proved, which shows the average of uncertain random variables being functions converges in distribution to an uncertain variable under the assumptions for independent, identically distributed (IID for short) random variables and IID regular uncertain variables. After that, LLNs under chance measures were developed by Gao and Sheng [12], Gao and Ralescu [13], Sheng et al. [41]. Recently, Nowak and Hryniewicz [39] generalized LLN under chance measure in [47], by weakening the case of independence of random variables to that of pairwise independence of random variables, but the conditions of uncertain variables that are also IID and regular.

In this paper, our aim is to derive some LLNs for uncertain random variables. Three kinds of LLNs under chance measures were obtained under some conditions of uncertain random variables. The first theorem proved the Etemadi type LLN for uncertain random variables being functions of pairwise independent and identically distributed random variables and uncertain variables without satisfying the conditions of regular and IID. Two kinds of Marcinkiewicz–Zygmund type LLNs for uncertain random variables that satisfied the conditions of the first theorem were established in the case of $p \in (0,1)$ by the second theorem, and in the case of $p > 1$ by the third theorem, respectively. The proofs of these theorems in this paper are modifications of the corresponding proofs in Nowak and Hryniewicz [39]. For better illustrating of LLNs for uncertain random variables, some examples were stated and explained. Our LLNs for uncertain random variables generalise the related results, such as Yao and Gao [47], Nowak and Hryniewicz [39].

The rest of this paper is organized as follows. In Section 2, we give some basic definitions concerning uncertainty and chance theory, including the notions of uncertain variable and uncertain random variable. In Section 3, we give our mean results: three kinds of LLNs for uncertain random variables under chance measures. In Section 4, two examples are stated and explained. Finally, a short conclusion is presented in Section 5.

## 2. PRELIMINARIES

In this section, we give some basic definitions and propositions concerning uncertainty and chance theory.

### 2.1. Uncertainty Space and Uncertain Variable

Let $\mathbb{R}$ be the set of real numbers and let $\mathcal{B}(\mathbb{R})$ be the $\sigma$- algebra of subsets of $\mathbb{R}$.

**Definition 2.1.** (Liu [25]) Let $\mathcal{L}$ be a $\sigma$-algebra on a non-empty set $\Gamma$. A set function $\mathcal{M}$ is called an uncertain measure if it satisfies the following axioms:

Axioms 1 (Normality Axiom): $\mathcal{M}\{\Gamma\} = 1$ for the universal set $\Gamma$;

Axioms 2 (Duality Axiom): $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^{c}\} = 1$ for any $\Lambda \in \mathcal{L}$;

Axioms 3 (Subadditivity Axiom): For every countable sequence of $\{\Lambda_j\} \subset \mathcal{L}$, we have

$$\mathcal{M}\left\{\bigcup_{j=1}^{\infty} \Lambda_j\right\} \le \sum_{j=1}^{\infty} \mathcal{M}\left\{\Lambda_j\right\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space, and each element $\Lambda$ in $\mathcal{L}$ is called an event. In order to obtain an uncertain measure of compound event, a product uncertain measure is defined by Liu [27] as follows:

Axioms 4 (Product Axiom): Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\left\{\Lambda_k\right\},$$

where $\Lambda_k$ are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

**Definition 2.2.** (Liu [25]) An uncertain variable $\tau$ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i. e., for any $B \in \mathcal{B}(\mathbb{R})$, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

**Definition 2.3.** (Liu [25]) The uncertainty distribution $\Psi$ of an uncertain variable $\tau$ is defined by

$$\Psi(x) = \mathcal{M}\{\tau \le x\}, \forall x \in \mathbb{R}.$$

**Definition 2.4.** (Liu [25]) Uncertain variables are said to be identically distributed if they have the same uncertainty distribution.

**Definition 2.5.** A sequence of uncertain variables $\{\tau_i\}_{i=1}^{\infty}$ on an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{\infty}\{\tau_i \in B_i\}\right\} = \bigwedge_{i=1}^{\infty} \mathcal{M}\{\tau_i \in B_i\}$$

for arbitrary $B_i \in \mathcal{B}(\mathbb{R}), i = 1, 2, \ldots, n, \ldots$.

**Remark 2.6.** Suppose that $\{\tau_i\}_{i=1}^{\infty}$ is a sequence of independent uncertain variables. Then for any $n \in \mathbb{N}$, $\tau_{k_1}, \tau_{k_2}, \ldots, \tau_{k_n}$ are independent, where $k_1 < k_2 < \ldots < k_n$.

Proof. Without loss of generality, we only prove that the first $n$ uncertain variables $\tau_1, \tau_2, \ldots, \tau_n$ are independent. Let $B_{i+1} = (-\infty, \infty), i \ge n$. By the independence of $\{\tau_i\}_{i=1}^{\infty}$, we get

$$\begin{aligned}
&\mathcal{M}\left\{\bigcap_{i=1}^{\infty}\{\tau_i \in B_i\}\right\} \\
&= \mathcal{M}\left\{\bigcap_{i=1}^{n}\{\tau_i \in B_i\} \bigcap \{\tau_{n+1} \in (-\infty, \infty)\} \bigcap \{\tau_{n+2} \in (-\infty, \infty)\} \bigcap \ldots\right\} \\
&= \left\{\bigwedge_{i=1}^{n} \mathcal{M}\{\tau_i \in B_i\}\right\} \bigwedge 1 = \bigwedge_{i=1}^{n} \mathcal{M}\{\tau_i \in B_i\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathcal{M}\left\{\bigcap_{i=1}^{n}\{\tau_i \in B_i\} \bigcap \{\tau_{n+1} \in (-\infty, \infty)\} \bigcap \{\tau_{n+2} \in (-\infty, \infty)\} \bigcap \ldots\right\} \\
&= \mathcal{M}\left\{\bigcap_{i=1}^{n}\{\tau_i \in B_i\}\right\}.
\end{aligned}$$

Thus,

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}\{\tau_i \in B_i\}\right\} = \bigwedge_{i=1}^{n}\mathcal{M}\{\tau_i \in B_i\}.$$

□

**Proposition 2.7.** $\{\tau_i\}_{i=1}^{\infty}$ is a sequence of independent uncertain variables if and only if

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\{\tau_i \in B_i\}\right\} = \bigvee_{i=1}^{\infty}\mathcal{M}\{\tau_i \in B_i\}$$

for arbitrary $B_i \in \mathcal{B}(\mathbb{R})$, $i = 1, 2, \ldots, n, \ldots$.

Proof. It follows from the duality of uncertain measure that $\tau_1, \tau_2, \ldots$ are independent if and only if

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\{\tau_i \in B_i\}\right\} = 1 - \mathcal{M}\left\{\bigcap_{i=1}^{\infty}\{\tau_i \in {B_i}^c\}\right\}$$
$$= 1 - \bigwedge_{i=1}^{\infty}\mathcal{M}\{\tau_i \in {B_i}^c\} = \bigvee_{i=1}^{\infty}\mathcal{M}\{\tau_i \in B_i\}.$$

Thus, the proof is completed. □

### 2.2. Chance Space and Uncertain Random Variable

**Definition 2.8.** (Liu [33]) Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Then the product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathbb{P})$ is called a chance space. Essentially, it is another triplet,

$$(\Gamma \times \Omega, \mathcal{L} \times \mathcal{A}, \mathcal{M} \times \mathbb{P})$$

where $\Gamma \times \Omega$ is the universal set, $\mathcal{L} \times \mathcal{A}$ is the producet $\sigma-$ algebra, and $\mathcal{M} \times \mathbb{P}$ is the product measure.

**Definition 2.9.** (Liu [33]) Let $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathbb{P})$ be a chance space, and let $\Theta \in \mathcal{L} \times \mathcal{A}$ be an uncertain random event. Then the chance measure of $\Theta$ is defined as

$$Ch\{\Theta\} = \int_0^1 \mathbb{P}\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \geq r\}\,\mathrm{d}r. \tag{1}$$

Liu [33] proved that a chance measure satisfies normality, duality, and monotonicity properties, that is

(i) $Ch\{\Gamma \times \Omega\} = 1$ for the universal set $\Gamma \times \Omega$;

(ii) $Ch\{\Theta\} + Ch\{\Theta^c\} = 1$ for any event $\Theta$;

(iii) $Ch\{\Theta_1\} \leq Ch\{\Theta_2\}$ for any events $\Theta_1$ and $\Theta_2$ with $\Theta_1 \subseteq \Theta_2$.

Moreover, Hou [15] proved the subadditivity of chance measure, that is
$Ch\left\{\bigcup_{i=1}^{n}\Theta_i\right\} \leq \sum_{i=1}^{n} Ch\left\{\Theta_i\right\}$, where $\Theta_i \in \mathcal{L} \times \mathcal{A}$, $i = 1, \ldots, n$, and $n \in \mathbb{N}$.

**Definition 2.10.** (Liu [33]) An uncertain random variable is a measurable function $\xi$ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathbb{P})$ to $\mathbb{R}$, i. e., for any $B \in \mathcal{B}(\mathbb{R})$, the set

$$\{\xi \in B\} = \{(\gamma, \omega) | \xi(\gamma, \omega) \in B\}$$

is an event.

**Definition 2.11.** (Liu [33]) Let $\xi$ be an uncertain random variable. Then its chance distribution is defined by

$$\Xi(x) = Ch\{\xi \le x\}, x \in \mathbb{R}.$$

**Definition 2.12.** (Ahmadzade et al. [1]) An uncertain random sequence $\{\xi_n\}_{n=1}^{\infty}$ is said to be convergent in measure to an uncertain random variable $\xi$ if

$$\lim_{n\to\infty} Ch\{(\gamma, \omega) || \xi_n(\gamma, \omega) - \xi(\gamma, \omega)| \ge \varepsilon\} = 0$$

for any $\varepsilon > 0$.

**Definition 2.13.** (Gao and Ahmadzade [11]) Let $\Xi, \Xi_1, \Xi_2, \dots,$ be the chance distributions of uncertain random varoables $\xi, \xi_1, \xi_2, \dots,$ respectively. Then uncertain random sequence $\{\xi_n\}_{n=1}^{\infty}$ is said to be convergent in distribution to the uncertain random variable $\xi$, if

$$\lim_{n\to\infty} \Xi_n(x) = \Xi(x),$$

for all $x \in \mathbb{R}$ at which $\Xi(x)$ is continuous.

Here and in the sequel, uncertain random variables are based on the chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathbb{P})$.

## 3. MAIN RESULTS

In this section, under the chance space, we give three kinds of LLNs for uncertain random variables. In Section 3.1, we get the convergence of the sequence $\frac{S_n}{n}$ as $n \to \infty$ in the first theorem. In Section 3.2, we consider the problem of convergence of the sequence $\frac{S_n - nc}{n^{1/q}}$ as $n \to \infty$. We will give two kinds of Marcinkiewicz–Zygmund Type LLNs for uncertain random variables in the case of $q \in (0, 1), c = 0$ by the second theorem, and in the case of $q > 1, c \in \mathbb{R}$ by the third theorem.

Before we give these LLNs, we first need some basic notations and assumptions.

Let $\{\eta_i\}_{i=1}^{\infty}$ be a sequence of pairwise independent and identically distributed random variables and $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of uncertain variables. We denote by $C^I$ the class of continuous functions $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ such that $f(x, y)$ is increasing (not necessary strictly increasing) with respect to $y$ for each $x \in \mathbb{R}$, by $C^D$ the class of continuous functions $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ such that $f(x, y)$ is decreasing (not necessary strictly decreasing) with respect to $y$ for each $x \in \mathbb{R}$, by $\bar{C}^I$ the class of continuous function $g : \mathbb{R} \mapsto \mathbb{R}$ increasing (not necessary strictly increasing), and by $\bar{C}^D$ the class of continuous function $g : \mathbb{R} \mapsto \mathbb{R}$ decreasing (not necessary strictly decreasing).

Let

$$S_n = \sum_{i=1}^{n} f(\eta_i, \tau_i), n \in \mathbb{N}, f \in C^I \cup C^D,$$

$$S_n(y) = \sum_{i=1}^{n} f(\eta_i, y), y \in \mathbb{R}, n \in \mathbb{N}, f \in C^I \cup C^D,$$

$$\bar{S}_n = \sum_{i=1}^{n} g(\tau_i), n \in \mathbb{N}, g \in \bar{C}^I \cup \bar{C}^D,$$

and

$$h(y) = \mathbb{E}[f(\eta_1, y)], y \in \mathbb{R}, f \in C^I \cup C^D,$$

where we use the symbol $\mathbb{E}$ to denote the expected value with respect to probability measure $\mathbb{P}$.

**Remark 3.1.** Let $\eta_1$ be a random variable with a distribution function $F$. Assume that $f \in C^I$ and $\int_{-\infty}^{\infty} |f(x,y)|\mathrm{d}F(x) < \infty$ for any $y \in \mathbb{R}$, then $h(y) = \mathbb{E}[f(\eta_1, y)] = \int_{-\infty}^{\infty} f(x,y)\mathrm{d}F(x)$ is continuous and increasing with respect to $y$. Similarly, assume that $f \in C^D$ and$\int_{-\infty}^{\infty} |f(x,y)|\mathrm{d}F(x) < \infty$ for any $y \in \mathbb{R}$, then $h(y) = \mathbb{E}[f(\eta_1, y)] = \int_{-\infty}^{\infty} f(x,y)\mathrm{d}F(x)$ is continuous and decreasing with respect to $y$.

**Definition 3.2.** (Liu [7]) a sequence of random variables $\{\eta_i\}_{i=1}^{\infty}$ is said to be pairwise independent if for each $i, j \in \mathbb{N}$, $i \neq j$, the random variables $\eta_i$ and $\eta_j$ are independent.

**Remark 3.3.** Clearly, independence of random variables implies their pairwise independence.

The following lemma is Etemadi strong law of large numbers (SLLN for short), which is very useful in this paper and can be seen in [7, 37].

**Lemma 3.4.** (Etemadi SLLN) If a sequence of random variables $\{\eta_i\}_{i=1}^{\infty}$ is pairwise independent, identically distributed and $\mathbb{E}[|\eta_1|] < \infty$, then

$$\mathbb{P}\left\{\omega : \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \eta_i}{n} = \mathbb{E}[\eta_1]\right\} = 1.$$

**Remark 3.5.** From Lemma 3.4, it is easy to prove that: If a sequence of random variables $\{\eta_i\}_{i=1}^{\infty}$ is pairwise independent, identically distributed and $\mathbb{E}[|\eta_1|] < \infty$, then for any $q \in (0, 1)$,

$$\mathbb{P}\left\{\omega : \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \eta_i}{n^{1/q}} = 0\right\} = 1.$$

### 3.1. The Etemadi Type LLN for Uncertain Random Variables

We first give the Etemadi type LLN for uncertain random variables being functions of pairwise independent, identically distributed random variables and uncertain variables.

**Theorem 3.6.** (The Etemadi Type LLN for Uncertain Random Variables) Let $\{\eta_i\}_{i=1}^{\infty}$ be a sequence of pairwise independent random variables with a cumulative distribution function $F$ and $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of uncertain variables. Suppose $f \in C^I \cup C^D$, and $\int_{-\infty}^{\infty} |f(x,y)|\mathrm{d}F(x) < \infty$ for any $y \in \mathbb{R}$. Set $h(\tau_k) = \int_{-\infty}^{\infty} f(x,\tau_k)\mathrm{d}F(x)$, $k \geq 1$. Denote the uncertainty distributions of uncertain variables $\sup_{k\geq 1} h(\tau_k)$ and $\inf_{k\geq 1} h(\tau_k)$ by $\Phi$ and $\Psi$, respectively. Then for any $z \in (\inf\{h(y) \mid y \in \mathbb{R}\}, \sup h(y) \mid y \in \mathbb{R}\})$ at which $\Phi(z)$ and $\Psi(z)$ are both continuous,

$$\Phi(z) \leq \liminf_{n\to\infty} Ch\left\{\frac{S_n}{n} \leq z\right\} \leq \limsup_{n\to\infty} Ch\left\{\frac{S_n}{n} \leq z\right\} \leq \Psi(z). \tag{2}$$

Furthermore, if $\sup_{k\geq 1} h(\tau_k)$ and $\inf_{k\geq 1} h(\tau_k)$ have the same uncertainty distribution on $(\inf\{h(y) \mid y \in \mathbb{R}\}, \sup h(y) \mid y \in \mathbb{R}\})$, then

$$\lim_{n\to\infty} \frac{S_n}{n} = \inf_{k\geq 1} h(\tau_k) = \inf_{k\geq 1} \left(\int_{-\infty}^{\infty} f(x,\tau_k)\mathrm{d}F(x)\right) \tag{3}$$

in the sense of convergence in distribution, i. e., for any $z \in (\inf\{h(y)|y \in \mathbb{R}\}, \sup h(y)|y \in \mathbb{R}\})$ at which $\Psi(z)$ is continuous,

$$\lim_{n\to\infty} Ch\left\{\frac{S_n}{n} \leq z\right\} = \Psi(z).$$

Proof. The proof can be completed by considering the cases $f \in C^I$ and $f \in C^D$, respectively. For any $u \in (\inf\{h(y)|y \in \mathbb{R}\},\ \sup\{h(y)|y \in \mathbb{R}\})$, define $y_0(u) = \max\{y|h(y) = u\}$.

Case 1: Assume that $f \in C^I$.

For any fixed $z \in (\inf\{h(y)|y \in \mathbb{R}\},\ \sup\{h(y)|y \in \mathbb{R}\})$ at which $\Phi(z)$ is continuous, and $\varepsilon > 0$ satisfying that $z - \varepsilon \in (\inf\{h(y)|y \in \mathbb{R}\},\ \sup\{h(y)|y \in \mathbb{R}\})$. By Lemma 3.4, there exists $N_1 \in \mathbb{N}$ such that for each $n \geq N_1$,

$$\mathbb{P}\left\{\frac{S_n(y_0(z-\varepsilon))}{n} \leq z\right\} \geq 1 - \varepsilon. \tag{4}$$

If $n \geq N_1$,

$$\begin{aligned}
Ch\left\{\frac{S_n}{n} \leq z\right\} &= \int_0^1 \mathbb{P}\left\{\mathcal{M}\left\{\frac{S_n}{n} \leq z\right\} \geq r\right\}\mathrm{d}r \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_0(z-\varepsilon))}{n} \leq z\right\} \bigcap \left\{\mathcal{M}\left\{\frac{S_n}{n} \leq z\right\} \geq r\right\}\right\}\mathrm{d}r \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_0(z-\varepsilon))}{n} \leq z\right\} \bigcap \left\{\mathcal{M}\left\{\frac{S_n}{n} \leq \frac{S_n(y_0(z-\varepsilon))}{n}\right\} \geq r\right\}\right\}\mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(z-\varepsilon))}{n} \le z\Big\} \\
&\bigcap \Big\{\mathcal{M}\Big\{\sum_{k=1}^n f(\eta_k,\tau_k) \le \sum_{k=1}^n f(\eta_k, y_0(z-\varepsilon))\Big\} \ge r\Big\}\Big\}\,\mathrm{d}r \\
&\ge \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(z-\varepsilon))}{n} \le z\Big\} \\
&\bigcap \Big\{\mathcal{M}\Big\{\bigcap_{k=1}^n \Big\{f(\eta_k,\tau_k) \le f(\eta_k, y_0(z-\varepsilon))\Big\}\Big\} \ge r\Big\}\Big\}\,\mathrm{d}r.
\end{aligned}
$$

Note that $f(x, y)$ is a increasing function of $y$ for each $x$. By (4) and $y_0(z-\varepsilon) = \max\{y|h(y) = z-\varepsilon\}$, it follows that

$$
\begin{aligned}
&Ch\Big\{\frac{S_n}{n} \le z\Big\} \ge \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(z-\varepsilon))}{n} \le z\Big\} \\
&\bigcap \Big\{\mathcal{M}\Big\{\bigcap_{k=1}^n \Big\{\tau_k \le y_0(z-\varepsilon)\Big\}\Big\} \ge r\Big\}\Big\}\,\mathrm{d}r \\
&= \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(z-\varepsilon))}{n} \le z\Big\} \bigcap \Big\{\mathcal{M}\Big\{\bigcap_{k=1}^n \Big\{h(\tau_k) \le z-\varepsilon\Big\}\Big\} \ge r\Big\}\Big\}\,\mathrm{d}r \\
&\ge \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(z-\varepsilon))}{n} \le z\Big\} \bigcap \Big\{\mathcal{M}\Big\{\bigcap_{k=1}^\infty \Big\{h(\tau_k) \le z-\varepsilon\Big\}\Big\} \ge r\Big\}\Big\}\,\mathrm{d}r \\
&= \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(z-\varepsilon))}{n} \le z\Big\} \bigcap \Big\{\mathcal{M}\Big\{\sup_{k\ge 1} h(\tau_k) \le z-\varepsilon\Big\} \ge r\Big\}\Big\}\,\mathrm{d}r \\
&\ge (1-\varepsilon)\Phi(z-\varepsilon). \qquad (5)
\end{aligned}
$$

Since $\Phi$ is continuous at $z$, by the above arguments and letting $\varepsilon \to 0$, we get

$$
\liminf_{n\to\infty} Ch\Big\{\frac{S_n}{n} \le z\Big\} \ge \Phi(z). \qquad (6)
$$

On the other hand, fix $z \in (\inf\{h(y)|y \in \mathbb{R}\}, \sup\{h(y)|y \in \mathbb{R}\})$ at which $\Psi(z)$ is continuous, and let $\varepsilon > 0$ satisfying that $z+\varepsilon \in (\inf\{h(y)|y \in \mathbb{R}\}, \sup\{h(y)|y \in \mathbb{R}\})$. By Lemma 3.4, there exists $N_2 \in \mathbb{N}$ such that for each $n \ge N_2$,

$$
\mathbb{P}\Big\{\frac{S_n(y_0(z+\varepsilon))}{n} > z\Big\} \ge 1-\varepsilon. \qquad (7)
$$

If $n \ge N_2$,

$$
\begin{aligned}
&Ch\Big\{\frac{S_n}{n} > z\Big\} = \int_0^1 \mathbb{P}\Big\{\mathcal{M}\Big\{\frac{S_n}{n} > z\Big\} \ge r\Big\}\,\mathrm{d}r \\
&\ge \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(z+\varepsilon))}{n} > z\Big\} \bigcap \Big\{\mathcal{M}\Big\{\frac{S_n}{n} > z\Big\} \ge r\Big\}\Big\}\,\mathrm{d}r
\end{aligned}
$$

$$\geq \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(z+\varepsilon))}{n} > z\Big\} \bigcap \Big\{\mathcal{M}\Big\{\frac{S_n}{n} > \frac{S_n(y_0(z+\varepsilon))}{n}\Big\} \geq r\Big\}\Big\} \mathrm{d}r$$
$$= \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(z+\varepsilon))}{n} > z\Big\}$$
$$\bigcap \Big\{\mathcal{M}\Big\{\sum_{k=1}^n f(\eta_k, \tau_k) > \sum_{k=1}^n f(\eta_k, y_0(z+\varepsilon))\Big\} \geq r\Big\}\Big\} \mathrm{d}r$$
$$\geq \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(z+\varepsilon))}{n} > z\Big\}$$
$$\bigcap \Big\{\mathcal{M}\Big\{\bigcap_{k=1}^n \Big\{f(\eta_k, \tau_k) > f(\eta_k, y_0(z+\varepsilon))\Big\}\Big\} \geq r\Big\}\Big\} \mathrm{d}r.$$

Note that $f(x,y)$ is a increasing function of $y$ for each $x$. By (7) and $y_0(z+\varepsilon) = \max\{y|h(y) = z+\varepsilon\}$, it follows that

$$Ch\Big\{\frac{S_n}{n} > z\Big\} \geq \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(z+\varepsilon))}{n} > z\Big\}$$
$$\bigcap \Big\{\mathcal{M}\Big\{\bigcap_{k=1}^n \Big\{\tau_k > y_0(z+\varepsilon)\Big\}\Big\} \geq r\Big\}\Big\} \mathrm{d}r$$
$$= \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(z+\varepsilon))}{n} > z\Big\} \bigcap \Big\{\mathcal{M}\Big\{\bigcap_{k=1}^n \Big\{h(\tau_k) > z+\varepsilon\Big\}\Big\} \geq r\Big\}\Big\} \mathrm{d}r$$
$$\geq \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(z+\varepsilon))}{n} > z\Big\} \bigcap \Big\{\mathcal{M}\Big\{\bigcap_{k=1}^\infty \Big\{h(\tau_k) > z+\varepsilon\Big\}\Big\} \geq r\Big\}\Big\} \mathrm{d}r$$
$$\geq \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(z+\varepsilon))}{n} > z\Big\} \bigcap \Big\{\mathcal{M}\Big\{\inf_{k\geq 1} h(\tau_k) > z+\varepsilon\Big\} \geq r\Big\}\Big\} \mathrm{d}r$$
$$= \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(z+\varepsilon))}{n} > z\Big\} \bigcap \Big\{1 - \mathcal{M}\Big\{\inf_{k\geq 1} h(\tau_k) \leq z+\varepsilon\Big\} \geq r\Big\}\Big\} \mathrm{d}r$$
$$\geq (1-\varepsilon)(1-\Psi(z+\varepsilon)). \tag{8}$$

Applying the duality property of $Ch$, it yields

$$Ch\Big\{\frac{S_n}{n} \leq z\Big\} \leq 1 - (1-\varepsilon)(1-\Psi(z+\varepsilon)).$$

Since $\Psi$ is continuous at $z$, letting $\varepsilon \to 0$, we get

$$\limsup_{n\to\infty} Ch\Big\{\frac{S_n}{n} \leq z\Big\} \leq \Psi(z). \tag{9}$$

Then for any $z \in (\inf\{h(y)|y \in \mathbb{R}\}, \sup\{h(y)|y \in \mathbb{R}\})$ at which $\Phi(z)$ and $\Psi(z)$ are both continuous, (2) follows, combining (6) and (9).

Case 2: Assume that $f \in C^D$, then $-f \in C^I$. By considering $-f$ and $-z$ instead of $f$ and $z$ in (2), we get

$$\mathcal{M}\left\{\sup_{k\geq 1}(-h(\tau_k)) < -z\right\} \leq \liminf_{n\to\infty} Ch\left\{\frac{-S_n}{n} < -z\right\}$$
$$\leq \limsup_{n\to\infty} Ch\left\{\frac{-S_n}{n} < -z\right\} \leq \mathcal{M}\left\{\inf_{k\geq 1}(-h(\tau_k)) < -z\right\}.$$

Then we obtain

$$\mathcal{M}\left\{\sup_{k\geq 1} h(\tau_k) \leq z\right\} \leq \liminf_{n\to\infty} Ch\left\{\frac{S_n}{n} \leq z\right\}$$
$$\leq \limsup_{n\to\infty} Ch\left\{\frac{S_n}{n} \leq z\right\} \leq \mathcal{M}\left\{\inf_{k\geq 1} h(\tau_k) \leq z\right\},$$

by the duality properties of $\mathcal{M}$ and $Ch$. Then for any $z \in (\inf\{h(y)|y \in \mathbb{R}\}, \sup\{h(y)|y \in \mathbb{R}\})$ at which $\Phi(z)$ and $\Psi(z)$ are both continuous, (2) follows.

Furthermore, if $\sup_{k\geq 1} h(\tau_k)$ and $\inf_{k\geq 1} h(\tau_k)$ have the same uncertainty distribution on $(\inf\{h(y) \mid y \in \mathbb{R}\}, \sup h(y) \mid y \in \mathbb{R}\})$, (3) holds by (2). Thus, the proof of Theorem 3.6 is completed. □

**Remark 3.7.** Let $F$ be a distribution function and $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of IID uncertain variables with continuous uncertainty distributions. Suppose $f \in C^I \cup C^D$, and $\int_{-\infty}^{\infty} |f(x,y)| \mathrm{d}F(x) < \infty$ for any $y \in \mathbb{R}$. Set $h(\tau_k) = \int_{-\infty}^{\infty} f(x,\tau_k) \mathrm{d}F(x)$, $k \geq 1$. Then the uncertainty distributions of $\sup_{k\geq 1} h(\tau_k)$ and $\inf_{k\geq 1} h(\tau_k)$ are as the same as that of $h(\tau_1)$ on $(\inf\{h(y) \mid y \in \mathbb{R}\}, \sup h(y) \mid y \in \mathbb{R}\})$.

P r o o f. First, we show that $\{h(\tau_k)\}_{k=1}^{\infty}$ is a sequence of uncertain variables with the same distribution on $(\inf\{h(y)|\ y \in \mathbb{R}\}, \sup h(y)|\ y \in \mathbb{R}\})$. The proof can be completed by considering the cases $f \in C^I$ and $f \in C^D$, respectively. For any $u \in (\inf\{h(y)|y \in \mathbb{R}\},\ \sup\{h(y)|y \in \mathbb{R}\})$, we define $y_0(u) = \max\{y|h(y) = u\}$ and $y_1(u) = \min\{y|h(y) = u\}$. Assume that $f \in C^I$, then $h(y)$ is increasing. For any $z \in (\inf\{h(y)|y \in \mathbb{R}\}, \sup\{h(y)|y$
$\in \mathbb{R}\})$, it follows from Definition 2.4 that

$$\mathcal{M}\{\tau_k \leq y_0(z)\} = \mathcal{M}\{\tau_1 \leq y_0(z)\}, \quad k \geq 1.$$

Hence, $\mathcal{M}\{h(\tau_k) \leq z\} = \mathcal{M}\{h(\tau_1) \leq z\}, \quad k \geq 1.$

Assume that $f \in C^D$, then $h(y)$ is decreasing. Note that $\{\tau_k\}_{k=1}^{\infty}$ is a sequence of identically distributed uncertain variables with continuous uncertainty distributions. For any $z \in (\inf\{h(y)|y \in \mathbb{R}\}, \sup\{h(y)|y \in \mathbb{R}\})$, we have

$$\mathcal{M}\{\tau_k \leq y_1(z)\} = \mathcal{M}\{\tau_1 \leq y_1(z)\}, \quad k \geq 1.$$

By the continuity of the uncertainty distributions of $\{\tau_k\}_{k=1}^{\infty}$ , we get

$$\mathcal{M}\{\tau_k < y_1(z)\} = \mathcal{M}\{\tau_1 < y_1(z)\}, \quad k \geq 1,$$

i. e.,

$$\mathcal{M}\{\tau_k \geq y_1(z)\} = \mathcal{M}\{\tau_1 \geq y_1(z)\}, \quad k \geq 1.$$

It follows that $\mathcal{M}\left\{h(\tau_k) \leq z\right\} = \mathcal{M}\left\{h(\tau_1) \leq z\right\}, \quad k \geq 1$. Thus, $\{h(\tau_k)\}_{k=1}^{\infty}$ is a sequence of uncertain variables with the same distribution on $(\inf\{h(y) \mid y \in \mathbb{R}\}, \sup h(y) \mid y \in \mathbb{R}\})$.

From Theorem 1.7 (b) in [45] and the fact in Remark 3.1that $h\,(y)$ is a continuous function, we know that $h\,(\tau_i)$ is a measurable function, i. e., for any $B_i \in \mathcal{B}(\mathbb{R})$, the set $\{h\,(\tau_i) \in B_i\} = \{\gamma \in \Gamma \mid h\,(\tau_i(\gamma)) \in B_i\} \in \mathcal{L}$. Next, applying Proposition 2.7, we can get

$$\begin{aligned}\mathcal{M}&\left\{\bigcap_{i=1}^{\infty}\{h\,(\tau_i) \in B_i\}\right\} = \mathcal{M}\left\{\bigcap_{i=1}^{\infty}\left\{\tau_i \in h^{-1}\,(B_i)\right\}\right\}\\ &= \bigwedge_{i=1}^{\infty}\mathcal{M}\left\{\tau_i \in h^{-1}\,(B_i)\right\} = \bigwedge_{i=1}^{\infty}\mathcal{M}\left\{h\,(\tau_i) \in B_i\right\},\end{aligned} \tag{10}$$

where $h^{-1}\,(B_i) =: \{x \in \mathbb{R} : h(x) \in B_i\} \in \mathcal{B}(\mathbb{R})$. Hence, $\{h(\tau_k)\}_{k=1}^{\infty}$ is a sequence of independent uncertain variables with the same distribution on $(\inf\{h(y) \mid y \in \mathbb{R}\}, \sup h(y) \mid y \in \mathbb{R}\})$.

At last, we show that the distributions of $\sup_{k\geq 1} h(\tau_k)$ and $\inf_{k\geq 1} h(\tau_k)$ are as the same as that of $h(\tau_1)$ on $(\inf\{h(y) \mid y \in \mathbb{R}\}, \sup h(y) \mid y \in \mathbb{R}\})$. From the fact that $\{h(\tau_k)\}_{k=1}^{\infty}$ is a sequence of IID uncertain variables with continuous uncertainty distributions on $(\inf\{h(y) \mid y \in \mathbb{R}\}, \sup h(y) \mid y \in \mathbb{R}\})$, we have: for any $z \in (\inf\{h(y)|y \in \mathbb{R}\}, \sup\{h(y)|y \in \mathbb{R}\})$,

$$\begin{aligned}\mathcal{M}&\left\{\sup_{k\geq 1} h(\tau_k) \leq z\right\} = \mathcal{M}\left\{\bigcap_{k=1}^{\infty}\{h(\tau_k) \leq z\}\right\}\\ &= \bigwedge_{k=1}^{\infty}\mathcal{M}\left\{h(\tau_k) \leq z\right\} = \mathcal{M}\left\{h(\tau_1) \leq z\right\},\end{aligned} \tag{11}$$

and

$$\begin{aligned}\mathcal{M}&\left\{\inf_{k\geq 1} h(\tau_k) < z\right\} = \mathcal{M}\left\{\bigcup_{k=1}^{\infty}\{h(\tau_k) < z\}\right\}\\ &= \bigvee_{k=1}^{\infty}\mathcal{M}\left\{h(\tau_k) < z\right\} = \mathcal{M}\left\{h(\tau_1) < z\right\}.\end{aligned} \tag{12}$$

Thus, the proof is completed. □

Applying Theorem 3.6 and Remark 3.7, we can immediately obtain:

**Corollary 3.8.** Let $\{\eta_i\}_{i=1}^{\infty}$ be a sequence of pairwise independent random variables with a cumulative distribution function $F$ and $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of IID uncertain variables with continuous uncertainty distributions. Suppose $f \in C^I \cup C^D$, and $\int_{-\infty}^{\infty} |f(x,y)|\mathrm{d}F(x) < \infty$ for any $y \in \mathbb{R}$. Set $h(\tau_k) = \int_{-\infty}^{\infty} f(x,\tau_k)\mathrm{d}F(x)$, $k \geq 1$. Then

$$\lim_{n\to\infty}\frac{S_n}{n} = h(\tau_1) = \int_{-\infty}^{\infty} f(x,\tau_1)\mathrm{d}F(x) \tag{13}$$

in the sense of convergence in distribution, i. e., for any $z \in (\inf\{h(y)|y \in \mathbb{R}\}, \sup h(y)|y \in \mathbb{R}\})$ at which $\mathcal{M}\{h(\tau_1) \le z\}$ is continuous,

$$\lim_{n\to\infty} Ch\left\{\frac{S_n}{n} \le z\right\} = \mathcal{M}\{h(\tau_1) \le z\}.$$

The following Corollary 3.9 is a direct consequence of Theorem 3.6 in a degenerate situation for IID uncertain variables with $f(x,y) = g(y)$, $x, y \in \mathbb{R}$.

**Corollary 3.9.** Let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of IID uncertain variables with continuous uncertainty distributions. Suppose $g \in \bar{C}^I \cup \bar{C}^D$. Then

$$\lim_{n\to\infty} \frac{\bar{S}_n}{n} = g(\tau_1) \tag{14}$$

in the sense of convergence in distribution, i. e., for any $z \in (\inf\{g(y)|y \in \mathbb{R}\}, \sup g(y)|y \in \mathbb{R}\})$ at which $\mathcal{M}\{g(\tau_1) \le z\}$ is continuous,

$$\lim_{n\to\infty} Ch\left\{\frac{\bar{S}_n}{n} \le z\right\} = \mathcal{M}\{g(\tau_1) \le z\}.$$

### 3.2. The Marcinkiewicz–Zygmund type LLNs for uncertain random variables

In this subsection, we give two types of Marcinkiewicz–Zygmund LLNs for uncertain random variables being functions of pairwise independent, identically distributed random variables and uncertain variables.

**Theorem 3.10.** (The Type I of Marcinkiewicz–Zygmund LLN for Uncertain Random Variables) Let $\{\eta_i\}_{i=1}^{\infty}$ be a sequence of pairwise independent random variables with a cumulative distribution function $F$ and $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of uncertain variables. Suppose $f \in C^I \cup C^D$, and $\int_{-\infty}^{\infty} |f(x,y)| \mathrm{d}F(x) < \infty$ for any $y \in \mathbb{R}$. Set $h(\tau_k) = \int_{-\infty}^{\infty} f(x,\tau_k) \mathrm{d}F(x)$, $k \ge 1$. And suppose that the uncertainty distributions of uncertain variables $\sup_{k\ge1} h(\tau_k)$ and $\inf_{k\ge1} h(\tau_k)$, denoted by $\Phi$ and $\Psi$, respectively satisfy

$$\lim_{y\to\infty} \Phi(y) = 1, \quad \lim_{y\to-\infty} \Psi(y) = 0. \tag{15}$$

Then for any $q \in (0,1)$, $\lim\limits_{n\to\infty} \frac{S_n}{n^{1/q}} = 0$ in the sense of convergence in measure, i. e., for any $\varepsilon > 0$,

$$\lim_{n\to\infty} Ch\left\{\left|\frac{S_n}{n^{1/q}}\right| \ge \varepsilon\right\} = 0. \tag{16}$$

P r o o f. For any $u \in (\inf\{h(y)|y \in \mathbb{R}\},\ \sup\{h(y)|y \in \mathbb{R}\})$, define $y_0(u) = \max\{y|h(y) = u\}$. Fix $\varepsilon > 0$. Let $0 < \varepsilon_1, \varepsilon_2 < 1$. By Remark 3.5, there exist $N_3, N_4 \in \mathbb{N}$, such that

$$\mathbb{P}\left\{\frac{S_n(y_0(u))}{n^{1/q}} < \varepsilon\right\} > 1 - \varepsilon_1, \quad n \ge N_3, \tag{17}$$

and

$$\mathbb{P}\left\{\frac{S_n(y_0(u))}{n^{1/q}} > -\varepsilon\right\} > 1-\varepsilon_2, \quad n \geq N_4. \tag{18}$$

If $f \in C^I$, letting $n \geq N_3$, we have

$$\begin{aligned}
&Ch\left\{\frac{S_n}{n^{1/q}} < \varepsilon\right\} = \int_0^1 \mathbb{P}\left\{\mathcal{M}\left\{\frac{S_n}{n^{1/q}} < \varepsilon\right\} \geq r\right\} \mathrm{d}r \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_0(u))}{n^{1/q}} < \varepsilon\right\} \bigcap \left\{\mathcal{M}\left\{\frac{S_n}{n^{1/q}} < \varepsilon\right\} \geq r\right\}\right\} \mathrm{d}r \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_0(u))}{n^{1/q}} < \varepsilon\right\} \bigcap \left\{\mathcal{M}\left\{\frac{S_n}{n^{1/q}} \leq \frac{S_n(y_0(u))}{n^{1/q}}\right\} \geq r\right\}\right\} \mathrm{d}r \\
&= \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_0(u))}{n^{1/q}} < \varepsilon\right\} \bigcap \left\{\mathcal{M}\left\{\sum_{k=1}^n f(\eta_k, \tau_k) \leq \sum_{k=1}^n f(\eta_k, y_0(u))\right\} \geq r\right\}\right\} \mathrm{d}r \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_0(u))}{n^{1/q}} < \varepsilon\right\} \bigcap \left\{\mathcal{M}\left\{\bigcap_{k=1}^n \left\{f(\eta_k, \tau_k) \leq f(\eta_k, y_0(u))\right\}\right\} \geq r\right\}\right\} \mathrm{d}r.
\end{aligned}$$

Since $f(x, y)$ is an increasing function of $y$ for each $x$, by (17), we get

$$\begin{aligned}
&Ch\left\{\frac{S_n}{n^{1/q}} < \varepsilon\right\} \geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_0(u))}{n^{1/q}} < \varepsilon\right\} \bigcap \left\{\mathcal{M}\left\{\bigcap_{k=1}^n \left\{\tau_k \leq y_0(u)\right\}\right\} \geq r\right\}\right\} \mathrm{d}r \\
&\geq \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_0(u))}{n^{1/q}} < \varepsilon\right\} \bigcap \left\{\mathcal{M}\left\{\bigcap_{k=1}^\infty \left\{h(\tau_k) \leq u\right\}\right\} \geq r\right\}\right\} \mathrm{d}r \\
&= \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_0(u))}{n^{1/q}} < \varepsilon\right\} \bigcap \left\{\mathcal{M}\left\{\sup_{k\geq 1} h(\tau_k) \leq u\right\} \geq r\right\}\right\} \mathrm{d}r \\
&= \int_0^1 \mathbb{P}\left\{\left\{\frac{S_n(y_0(u))}{n^{1/q}} < \varepsilon\right\} \bigcap \left\{\Phi(u) \geq r\right\}\right\} \mathrm{d}r \\
&> (1-\varepsilon_1)\Phi(u).
\end{aligned} \tag{19}$$

Let $\varepsilon_1 \to 0$ and $u \to \infty$, then $\lim_{n\to\infty} Ch\left\{\frac{S_n}{n^{1/q}} < \varepsilon\right\} = 1$. Hence, by the duality property of $Ch$,

$$\lim_{n\to\infty} Ch\left\{\frac{S_n}{n^{1/q}} \geq \varepsilon\right\} = 0. \tag{20}$$

On the other hand, for $n \geq N_4$,

$$Ch\Big\{\frac{S_n}{n^{1/q}} > -\varepsilon\Big\} = \int_0^1 \mathbb{P}\Big\{\mathcal{M}\Big\{\frac{S_n}{n^{1/q}} > -\varepsilon\Big\} \geq r\Big\}\,\mathrm{d}r$$
$$\geq \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(u))}{n^{1/q}} > -\varepsilon\Big\}\bigcap\Big\{\mathcal{M}\Big\{\frac{S_n}{n^{1/q}} > -\varepsilon\Big\} \geq r\Big\}\Big\}\,\mathrm{d}r$$
$$\geq \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(u))}{n^{1/q}} > -\varepsilon\Big\}\bigcap\Big\{\mathcal{M}\Big\{\frac{S_n}{n^{1/q}} > \frac{S_n(y_0(u))}{n^{1/q}}\Big\} \geq r\Big\}\Big\}\,\mathrm{d}r$$
$$= \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(u))}{n^{1/q}} > -\varepsilon\Big\}\bigcap\Big\{\mathcal{M}\Big\{\sum_{k=1}^n f(\eta_k,\tau_k) > \sum_{k=1}^n f(\eta_k, y_0(u))\Big\} \geq r\Big\}\Big\}\,\mathrm{d}r$$
$$\geq \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(u))}{n^{1/q}} > -\varepsilon\Big\}\bigcap\Big\{\mathcal{M}\Big\{\bigcap_{k=1}^n \Big\{f(\eta_k,\tau_k) > f(\eta_k, y_0(u))\Big\}\Big\} \geq r\Big\}\Big\}\,\mathrm{d}r.$$

Since $f(x,y)$ is a increasing function of $y$ for each $x$, by (18), we get

$$Ch\Big\{\frac{S_n}{n^{1/q}} > -\varepsilon\Big\} \geq \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(u))}{n^{1/q}} > -\varepsilon\Big\}\bigcap\Big\{\mathcal{M}\Big\{\bigcap_{k=1}^n \Big\{\tau_k > y_0(u)\Big\}\Big\} \geq r\Big\}\Big\}\,\mathrm{d}r$$
$$\geq \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(u))}{n^{1/q}} > -\varepsilon\Big\}\bigcap\Big\{\mathcal{M}\Big\{\bigcap_{k=1}^\infty \Big\{h(\tau_k) > u\Big\}\Big\} \geq r\Big\}\Big\}\,\mathrm{d}r$$
$$\geq \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(u))}{n^{1/q}} > -\varepsilon\Big\}\bigcap\Big\{\mathcal{M}\Big\{\inf_{k\geq 1} h(\tau_k) > u\Big\} \geq r\Big\}\Big\}\,\mathrm{d}r$$
$$= \int_0^1 \mathbb{P}\Big\{\Big\{\frac{S_n(y_0(u))}{n^{1/q}} > -\varepsilon\Big\}\bigcap\Big\{(1-\Psi(u)) \geq r\Big\}\Big\}\,\mathrm{d}r$$
$$> (1-\varepsilon_2)(1-\Psi(u)). \tag{21}$$

Letting $\varepsilon_2 \to 0$ and $u \to -\infty$, we have $\lim_{n\to\infty} Ch\{\frac{S_n}{n^{1/q}} > -\varepsilon\} = 1$. Hence, by the duality property of $Ch$,

$$\lim_{n\to\infty} Ch\Big\{\frac{S_n}{n^{1/q}} \leq -\varepsilon\Big\} = 0. \tag{22}$$

From (20), (22) and the subadditive property of $Ch$, it follows that

$$\lim_{n\to\infty} Ch\Big\{\Big|\frac{S_n}{n^{1/q}}\Big| \geq \varepsilon\Big\} = \lim_{n\to\infty} Ch\Big\{\Big\{\frac{S_n}{n^{1/q}} \leq -\varepsilon\Big\}\bigcup\Big\{\frac{S_n}{n^{1/q}} \geq \varepsilon\Big\}\Big\}$$
$$\leq \lim_{n\to\infty} Ch\Big\{\frac{S_n}{n^{1/q}} \leq -\varepsilon\Big\} + \lim_{n\to\infty} Ch\Big\{\frac{S_n}{n^{1/q}} \geq \varepsilon\Big\} = 0. \tag{23}$$

If $f \in C^D$, then $-f \in C^I$. By considering $-f$ instead of $f$ in (23), we get

$$\lim_{n\to\infty} Ch\Big\{\Big|\frac{\sum_{k=1}^n(-f(\eta_k,\tau_k))}{n^{1/q}}\Big| \geq \varepsilon\Big\} = \lim_{n\to\infty} Ch\Big\{\Big|\frac{\sum_{k=1}^n f(\eta_k,\tau_k)}{n^{1/q}}\Big| \geq \varepsilon\Big\} = 0. \tag{24}$$

The proof of Theorem 3.10 is completed. □

Applying Theorem 3.10 and Remark 3.7, we can immediately obtain:

**Corollary 3.11.** Let $\{\eta_i\}_{i=1}^{\infty}$ be a sequence of pairwise independent random variables with a cumulative distribution function $F$ and $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of IID uncertain variables with continuous uncertainty distributions, which satisfy

$$\lim_{y\to\infty} \mathcal{M}\{\tau_1 \le y\} = 1, \quad \lim_{y\to-\infty} \mathcal{M}\{\tau_1 \le y\} = 0. \tag{25}$$

Suppose $f \in C^I \cup C^D$, and $\int_{-\infty}^{\infty} |f(x,y)| \mathrm{d}F(x) < \infty$ for any $y \in \mathbb{R}$. Then for any $q \in (0,1)$, $\lim\limits_{n\to\infty} \frac{S_n}{n^{1/q}} = 0$ in the sense of convergence in measure, i.e., for any $\varepsilon > 0$,

$$\lim_{n\to\infty} Ch\left\{\left|\frac{S_n}{n^{1/q}}\right| \ge \varepsilon\right\} = 0.$$

The following Corollary 3.12 is a direct consequence of Theorem 3.10 in a degenerate situation for IID uncertain variables with $f(x,y) = g(y), x, y \in \mathbb{R}$.

**Corollary 3.12.** Let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of IID uncertain variables with continuous uncertainty distributions, satisfying (25). Suppose $g \in \bar{C}^I \cup \bar{C}^D$. Then for any $q \in (0,1)$, $\lim\limits_{n\to\infty} \frac{\bar{S}_n}{n^{1/q}} = 0$ in the sense of convergence in measure, i.e., for any $\varepsilon > 0$,

$$\lim_{n\to\infty} Ch\left\{\left|\frac{\bar{S}_n}{n^{1/q}}\right| \ge \varepsilon\right\} = \lim_{n\to\infty} \mathcal{M}\left\{\left|\frac{\bar{S}_n}{n^{1/q}}\right| \ge \varepsilon\right\} = 0.$$

**Theorem 3.13.** (The Type II of Marcinkiewicz–Zygmund LLN for Uncertain Random Variables) Let $\{\eta_i\}_{i=1}^{\infty}$ be a sequence of pairwise independent random variables with a cumulative distribution function $F$ and $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of uncertain variables. Suppose $f \in C^I \cup C^D$, and $\int_{-\infty}^{\infty} |f(x,y)| \mathrm{d}F(x) < \infty$ for any $y \in \mathbb{R}$. Set $h(\tau_k) = \int_{-\infty}^{\infty} f(x,\tau_k) \mathrm{d}F(x)$, $k \ge 1$. Denote the uncertainty distributions of uncertain variables $\sup_{k\ge1} h(\tau_k)$ and $\inf_{k\ge1} h(\tau_k)$ by $\Phi$ and $\Psi$, respectively. Then for any $q > 1$, $c \in (\inf\{h(y)|y \in \mathbb{R}\}, \sup\{h(y)|y \in \mathbb{R}\})$ at which $\Phi(c)$ and $\Psi(c)$ are both continuous, and $z \in \mathbb{R}$,

$$\Phi(c) \le \liminf_{n\to\infty} Ch\left\{\frac{S_n - nc}{n^{1/q}} \le z\right\} \le \limsup_{n\to\infty} Ch\left\{\frac{S_n - nc}{n^{1/q}} \le z\right\} \le \Psi(c). \tag{26}$$

Furthermore, if $\sup_{k\ge1} h(\tau_k)$ and $\inf_{k\ge1} h(\tau_k)$ have the same uncertainty distribution on $(\inf\{h(y) \mid y \in \mathbb{R}\}, \sup h(y) \mid y \in \mathbb{R}\})$, then for any $q > 1$, $c \in (\inf\{h(y)|y \in \mathbb{R}\}, \sup\{h(y)|y \in \mathbb{R}\})$ at which $\Psi(c)$ is continuous, $\lim\limits_{n\to\infty} \frac{S_n - nc}{n^{1/q}} = m_{f,c}$ in the sense of convergence in distribution, where $m_{f,c}$ is an uncertain variable with constant uncertainty distribution equal to $\Psi(c)$, i.e., for any $z \in \mathbb{R}$,

$$\lim_{n\to\infty} Ch\left\{\frac{S_n - nc}{n^{1/q}} \le z\right\} = \Psi(c). \tag{27}$$

Proof. Fix $z \in \mathbb{R}$ and $c \in (\inf\{h(y|y \in \mathbb{R}\}, \sup\{h(y)|y \in \mathbb{R}\})$ at which $\Phi(c)$ and $\Psi(c)$ are both continuous. For each small enough $\delta > 0$, there exists $N \in \mathbb{N}$ such that $\frac{|z|}{n^{1-1/q}} < \delta,\ n \ge N,\ q > 1$ and $c + \delta, c - \delta \in (\inf\{h(y)|y \in \mathbb{R}\}, \sup\{h(y)|y \in \mathbb{R}\})$.

If $z \geq 0$ and $n \geq N$, we get

$$Ch\left\{\frac{S_n}{n} \leq c\right\} \leq Ch\left\{\frac{S_n - nc}{n^{1/q}} \leq z\right\} \leq Ch\left\{\frac{S_n}{n} \leq c + \delta\right\}. \tag{28}$$

By Theorem 3.6, we have

$$\Phi(c) \leq \liminf_{n\to\infty} Ch\left\{\frac{S_n}{n} \leq c\right\}, \tag{29}$$

$$\limsup_{n\to\infty} Ch\left\{\frac{S_n}{n} \leq c + \delta\right\} \leq \Psi(c + \delta). \tag{30}$$

From (28), (29) and (30), it follows that

$$\begin{aligned}\Phi(c) &\leq \liminf_{n\to\infty} Ch\left\{\frac{S_n - nc}{n^{1/q}} \leq z\right\} \\ &\leq \limsup_{n\to\infty} Ch\left\{\frac{S_n - nc}{n^{1/q}} \leq z\right\} \leq \Psi(c + \delta).\end{aligned} \tag{31}$$

Since $\Psi$ is continuous at $c$ and $\delta > 0$ is arbitrary, letting $\delta \to 0$ in (31), we get

$$\Phi(c) \leq \liminf_{n\to\infty} Ch\left\{\frac{S_n - nc}{n^{1/q}} \leq z\right\} \leq \limsup_{n\to\infty} Ch\left\{\frac{S_n - nc}{n^{1/q}} \leq z\right\} \leq \Psi(c). \tag{32}$$

If $z < 0$ and $n \geq N$, we get

$$Ch\left\{\frac{S_n}{n} \leq c - \delta\right\} \leq Ch\left\{\frac{S_n - nc}{n^{1/q}} \leq z\right\} \leq Ch\left\{\frac{S_n}{n} \leq c\right\}. \tag{33}$$

By Theorem 3.6, we have

$$\Phi(c - \delta) \leq \liminf_{n\to\infty} Ch\left\{\frac{S_n}{n} \leq c - \delta\right\}, \tag{34}$$

$$\limsup_{n\to\infty} Ch\left\{\frac{S_n}{n} \leq c\right\} \leq \Psi(c). \tag{35}$$

From (33), (34) and (35), it follows that

$$\begin{aligned}\Phi(c - \delta) &\leq \liminf_{n\to\infty} Ch\left\{\frac{S_n - nc}{n^{1/q}} \leq z\right\} \\ &\leq \limsup_{n\to\infty} Ch\left\{\frac{S_n - nc}{n^{1/q}} \leq z\right\} \leq \Psi(c).\end{aligned} \tag{36}$$

Since $\Phi$ is continuous at $c$ and $\delta > 0$ is arbitrary, letting $\delta \to 0$ in (36), we get

$$\Phi(c) \leq \liminf_{n\to\infty} Ch\left\{\frac{S_n - nc}{n^{1/q}} \leq z\right\} \leq \limsup_{n\to\infty} Ch\left\{\frac{S_n - nc}{n^{1/q}} \leq z\right\} \leq \Psi(c). \tag{37}$$

Thus, (26) is proved.

Furthermore, if $\sup_{k\geq 1} h(\tau_k)$ and $\inf_{k\geq 1} h(\tau_k)$ have the same uncertainty distribution on $(\inf\{h(y) \mid y \in \mathbb{R}\}, \sup h(y) \mid y \in \mathbb{R}\})$, (27) holds by (26). The proof of Theorem 3.13 is completed. □

Applying Theorem 3.13 and Remark 3.7, we can immediately obtain:

**Corollary 3.14.** Let $\{\eta_i\}_{i=1}^{\infty}$ be a sequence of pairwise independent random variables with a cumulative distribution function $F$ and $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of IID uncertain variables with continuous uncertainty distributions. Suppose $f \in C^I \cup C^D$, and $\int_{-\infty}^{\infty} |f(x,y)| \mathrm{d}F(x) < \infty$ for any $y \in \mathbb{R}$. Set $h(\tau_k) = \int_{-\infty}^{\infty} f(x,\tau_k) \mathrm{d}F(x)$, $k \geq 1$. then for any $q > 1$, $c \in (\inf\{h(y)|y \in \mathbb{R}\}, \sup\{h(y)|y \in \mathbb{R}\})$ at which $\mathcal{M}\{h(\tau_1) \leq c\}$ is continuous, $\lim\limits_{n\to\infty} \frac{S_n - nc}{n^{1/q}} = m_{f,c}$ in the sense of convergence in distribution, where $m_{f,c}$ is an uncertain variable with constant uncertainty distribution equal to $\mathcal{M}\{h(\tau_1) \leq c\}$, i. e., for any $z \in \mathbb{R}$,

$$\lim_{n\to\infty} Ch\left\{\frac{S_n - nc}{n^{1/q}} \leq z\right\} = \mathcal{M}\{h(\tau_1) \leq c\}. \tag{38}$$

The following Corollary 3.15 is a direct consequence of Theorem 3.13 in a degenerate situation for IID uncertain variables with $f(x,y) = g(y), x, y \in \mathbb{R}$.

**Corollary 3.15.** Let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of IID uncertain variables with continuous uncertainty distributions. Suppose $g \in \bar{C}^I \cup \bar{C}^D$. Then for any $q > 1$, $c \in (\inf\{g(y)|y \in \mathbb{R}\}, \sup\{g(y)|y \in \mathbb{R}\})$ at which $\mathcal{M}\{g(\tau_1) \leq c\}$ is continuous, $\lim\limits_{n\to\infty} \frac{\bar{S}_n - nc}{n^{1/q}} = m_{g,c}$ in the sense of convergence in distribution, where $m_{g,c}$ is an uncertain variable with constant uncertainty distribution equal to $\mathcal{M}\{g(\tau_1) \leq c\}$, i. e., for any $z \in \mathbb{R}$,

$$\lim_{n\to\infty} Ch\left\{\frac{\bar{S}_n - nc}{n^{1/q}} \leq z\right\} = \lim_{n\to\infty} \mathcal{M}\left\{\frac{\bar{S}_n - nc}{n^{1/q}} \leq z\right\} = \mathcal{M}\{g(\tau_1) \leq c\}. \tag{39}$$

## 4. EXAMPLES

The following two examples are the applications of Theorem 3.6 and Theorem 3.13, but they are not obtained by the existed results of LLNs for uncertain random variables, such as in Yao and Gao [47], Nowak and Hryniewicz [39].

**Example 4.1.** Let $\{\eta_i\}_{i=1}^{\infty}$ be a sequence of independent random variables with a cumulative distribution

$$\mathbb{P}\{\eta_1 = x\} = p^x(1-p)^{1-x}, \quad x = 0, 1, \quad 0 < p < 1,$$

and $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of independent, normally distributed uncertain variables with uncertainty distribution

$$\psi(x) = \left(1 + exp\left(\frac{-\pi x}{\sqrt{3}}\right)\right)^{-1}, \ x \in \mathbb{R}.$$

Let $f_1(x,y) = x + g(y)$, $f_2(x,y) = x - g(y)$, $g(y) = 0\mathbb{I}_{\{y\le 0\}} + y\mathbb{I}_{\{0<y<1\}} + 1\mathbb{I}_{\{y\ge 1\}}$, for any $x, y \in \mathbb{R}$. Obviously, $f_1 \in C^I$, $f_2 \in C^D$, but $g(y)$ is not strictly increasing with respect to $y$, and $\mathbb{E}[f_1(\eta_1, y)] = p + g(y)$, $\mathbb{E}[f_2(\eta_1, y)] = p - g(y)$. Then, by Theorem 3.6 and Theorem 3.13, we have:

*(I)*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n (\eta_k + g(\tau_k))}{n} = p + g(\tau_1)$$

in the sense of convergence in distribution, i. e., for any $z \in (p, p+1)$,

$$\lim_{n\to\infty} Ch\left\{ \frac{\sum_{k=1}^n (\eta_k + g(\tau_k))}{n} \le z \right\} = \psi(z - p),$$

and

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n (\eta_k - g(\tau_k))}{n} = p - g(\tau_1)$$

in the sense of convergence in distribution, i. e., for any $z \in (p-1, p)$,

$$\lim_{n\to\infty} Ch\left\{ \frac{\sum_{k=1}^n (\eta_k - g(\tau_k))}{n} \le z \right\} = 1 - \psi(p - z).$$

*(II)*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n g(\tau_k)}{n} = g(\tau_1)$$

in the sense of convergence in distribution, i. e., for any $z \in (0, 1)$,

$$\lim_{n\to\infty} Ch\left\{ \frac{\sum_{k=1}^n g(\tau_k)}{n} \le z \right\} = \psi(z),$$

and

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n (-g(\tau_k))}{n} = -g(\tau_1)$$

in the sense of convergence in distribution, i. e., for any $z \in (-1, 0)$,

$$\lim_{n\to\infty} Ch\left\{ \frac{\sum_{k=1}^n (-g(\tau_k))}{n} \le z \right\} = 1 - \psi(-z).$$

*(III)* For any $q > 1$, $c \in (p, p+1)$, $\lim_{n\to\infty} \frac{\sum_{k=1}^n (\eta_k + g(\tau_k)) - nc}{n^{1/q}} = m_{f_1,c}$ in the sense of convergence in distribution, where $m_{f_1,c}$ is an uncertain variable with constant uncertainty distribution equal to $\psi(c-p)$, i. e., for any $z \in \mathbb{R}$,

$$\lim_{n\to\infty} Ch\left\{ \frac{\sum_{k=1}^n (\eta_k + g(\tau_k)) - nc}{n^{1/q}} \le z \right\} = \psi(c - p),$$

and for any $q > 1$, $c \in (p-1, p)$, $\lim_{n\to\infty} \frac{\sum_{k=1}^n (\eta_k - g(\tau_k)) - nc}{n^{1/q}} = m_{f_2,c}$ in the sense of convergence in distribution, where $m_{f_2,c}$ is an uncertain variable with constant uncertainty distribution equal to $1 - \psi(p-c)$, i. e., for any $z \in \mathbb{R}$,

$$\lim_{n\to\infty} Ch\left\{ \frac{\sum_{k=1}^n (\eta_k - g(\tau_k)) - nc}{n^{1/q}} \le z \right\} = 1 - \psi(p - c).$$

$(IV)$ For any $q>1$, $c\in(0,1)$, $\lim\limits_{n\to\infty}\frac{\sum_{k=1}^{n}g(\tau_k^3)-nc}{n^{1/q}}=m_{g,c}$ in the sense of convergence in distribution, where $m_{g,c}$ is an uncertain variable with constant uncertainty distribution equal to $\psi(\sqrt[3]{c})$, i. e., for any $z\in\mathbb{R}$,

$$\lim_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}g(\tau_k^3)-nc}{n^{1/q}}\leq z\right\}=\psi(\sqrt[3]{c}),$$

and for any $q>1$, $c\in(-1,0)$, $\lim\limits_{n\to\infty}\frac{-\sum_{k=1}^{n}g(\tau_k^3)-nc}{n^{1/q}}=m_{-g,c}$ in the sense of convergence in distribution, where $m_{-g,c}$ is an uncertain variable with constant uncertainty distribution equal to $1-\psi(\sqrt[3]{-c})$, i. e., for any $z\in\mathbb{R}$,

$$\lim_{n\to\infty} Ch\left\{\frac{-\sum_{k=1}^{n}g(\tau_k^3)-nc}{n^{1/q}}\leq z\right\}=1-\psi(\sqrt[3]{-c}).$$

**Example 4.2.** We consider a sequence of independent Rademacher random variables $\{\bar{\varepsilon}_i\}_{i=-1}^{\infty}$, i. e., random variables with distribution $\mathbb{P}\{\bar{\varepsilon}_i=-1\}=\mathbb{P}\{\bar{\varepsilon}_i=1\}=\frac{1}{2}$, $i\in\mathbb{N}\cup\{-1,0\}$, and a sequence of IID indicator functions $\mathbb{I}_{A_i}$, $i\in\mathbb{N}$, such that $\mathbb{P}\{A_i\}=a$, $0<a<1$, $i\in\mathbb{N}$. We assume that both sequences are independent. Let

$$\eta_i=\mathbb{I}_{A_i}\bar{\varepsilon}_i+=\mathbb{I}_{A_i^c}\bar{\varepsilon}_{i-1}\bar{\varepsilon}_{i-2},\quad i\in\mathbb{N}.$$

Obviously, $\{\eta_i\}_{i=1}^{\infty}$ is a sequence of Rademacher distributed and pairwise independent, but dependent (see [3] for proof) random variables. Moreover, let $\{\tau_i\}_{i=1}^{\infty}$ be a sequence of independent uncertain variables. $\{\tau_{2i}\}_{i=1}^{\infty}$ has a cumulative distribution function $\psi(x)=0\mathbb{I}_{\{x\leq 0\}}+x\mathbb{I}_{\{0<x<1\}}+1\mathbb{I}_{\{x\geq 1\}}$, $x\in\mathbb{R}$. $\{\tau_{2i-1}\}_{i=1}^{\infty}$ has a cumulative distribution function $\varphi(x)=0\mathbb{I}_{\{x\leq 0\}}+x\mathbb{I}_{\{0<x<1+\delta\}}+1\mathbb{I}_{\{x\geq 1+\delta\}}$, $x\in\mathbb{R}$, where $\delta$ is a given small positive constant. Then, by Theorem 3.6 and Theorem 3.13, we have:
$(I)$ If $z\in(-\infty,0]$,

$$\lim_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}(\eta_k+\tau_k)}{n}\leq z\right\}=0,$$

if $z\in[1+\delta,\infty)$,

$$\lim_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}(\eta_k+\tau_k)}{n}\leq z\right\}=1,$$

and if $z\in(0,1+\delta)$,

$$\frac{z}{1+\delta}\leq\liminf_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}(\eta_k+\tau_k)}{n}\leq z\right\}$$
$$\leq\limsup_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}(\eta_k+\tau_k)}{n}\leq z\right\}\leq z\bigwedge 1.$$

$(II)$ If $z\in(-\infty,-1-\delta]$,

$$\lim_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}(\eta_k-\tau_k)}{n}\leq z\right\}=0,$$

if $z \in [0, \infty)$,

$$\lim_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}(\eta_k - \tau_k)}{n} \le z\right\} = 1,$$

and if $z \in (-1-\delta, 0)$,

$$(1+z)\bigvee 0 \le \liminf_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}(\eta_k - \tau_k)}{n} \le z\right\}$$
$$\le \limsup_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}(\eta_k - \tau_k)}{n} \le z\right\} \le 1 + \frac{z}{1+\delta}.$$

$(III)$ For any $q > 1$, $z \in \mathbb{R}$, if $c \in (-\infty, 0]$,

$$\lim_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}(\eta_k + \tau_k) - nc}{n^{1/q}} \le z\right\} = 0,$$

if $c \in [1+\delta, \infty)$,

$$\lim_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}(\eta_k + \tau_k) - nc}{n^{1/q}} \le z\right\} = 1,$$

and if $c \in (0, 1+\delta)$,

$$\frac{c}{1+\delta} \le \liminf_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}(\eta_k + \tau_k) - nc}{n^{1/q}} \le z\right\}$$
$$\le \limsup_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}(\eta_k + \tau_k) - nc}{n^{1/q}} \le z\right\} \le c \bigwedge 1.$$

$(IV)$ For any $q > 1$, $z \in \mathbb{R}$, if $c \in (-\infty, -1-\delta]$,

$$\lim_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}(\eta_k - \tau_k) - nc}{n^{1/q}} \le z\right\} = 0,$$

if $c \in [0, \infty)$,

$$\lim_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}(\eta_k - \tau_k) - nc}{n^{1/q}} \le z\right\} = 1,$$

and if $c \in (-1-\delta, 0)$,

$$(1+c)\bigvee 0 \le \liminf_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}(\eta_k - \tau_k) - nc}{n^{1/q}} \le z\right\}$$
$$\le \limsup_{n\to\infty} Ch\left\{\frac{\sum_{k=1}^{n}(\eta_k - \tau_k) - nc}{n^{1/q}} \le z\right\} \le 1 + \frac{c}{1+\delta}.$$

## 5. CONCLUSIONS

This paper is devoted to the development of the limit theory of uncertain random variables. We have considered uncertain random variables in the form of continuous functions of a random variable and an uncertain variable. It has been additionally assumed that the functions are monotone (not necessary strictly monotone) with respect to their second arguments. This paper's main contribution is the Etemadi type LLN and two types of Marcinkiewicz–Zygmund LLNs for such defined sequences of uncertain random variables. The first theorem proved the Etemadi type LLN for uncertain random variables being the above defined functions of pairwise independent and identically distributed random variables and uncertain variables without satisfying the conditions of regular and IID. Two kinds of Marcinkiewicz–Zygmund type LLNs for uncertain random variables that satisfied the conditions of the first theorem were established in the case of $p \in (0, 1)$ by the second theorem, and in the case of $p > 1$ by the third theorem, respectively. Obviously, compared with the existed theorems of LLNs for uncertain random variables, our theorems are the generalised results. For example, in Yao and Gao [47], Nowak and Hryniewicz [39], LLNs for uncertain random variables need the conditions that the continuous functions of a random variable and an uncertain variable are strictly monotone with respect to their second arguments and uncertain variables are regular and IID. One possible direction for further research is to study the Chow type law for delayed sums of uncertain random variables being functions of pairwise independent, identically distributed random variables and uncertain variables without satisfying the conditions of regular and IID. Although there are some difficulties to handle, we will actively explore more and better ways to solve this problem.

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