LIMIT STATE ANALYSIS ON THE UN-REPEATED MULTIPLE SELECTION BOUNDED CONFIDENCE MODEL

Jiangbo Zhang and Yiyi Zhao

In this paper, we study the opinion evolution over social networks with a bounded confidence rule. Node initial opinions are independently and identically distributed. At each time step, each node reviews the average opinions of several different randomly selected agents and updates its opinion only when the difference between its opinion and the average is below a threshold. First of all, we provide probability bounds of the opinion convergence and the opinion consensus, are both nontrivial events by analyzing the probability distribution of order statistics. Next, similar analyzing methods are used to provide probability bounds when the selection cover all agents. Finally, we simulate all these bounds and find that opinion fluctuations may take place. These results increase to the understanding of the role of bounded confidence in social opinion dynamics, and the possibility of fluctuation reveals that our model has fundamentally changed the behavior of general DeGroot opinion dynamical processes.

Keywords: bounded confidence, probability bounds, order statistics

Classification: 62G30, 91D30

1. INTRODUCTION

1.1. Background

Today, understanding how opinions of social peers evolve is of increasing importance to our society, for which challenges lie in the intrinsic nonlinearity, unpredictability, and randomness that are ubiquitous for complex systems like our society [4]. Social media has undeniably changed the ways of people interacting with each other at a fundamental level [13]. We now rely on social networks as a major resource of information from which we make decisions on matters that range from restaurant and movie choices to online sellers. Specially, companies and political election preferences also largely rely on social network platforms to spread ideas and opinions to others with the hope of generating influences.

In fact, the study of social interactions began long before this era of social media. In the 1970s the classical DeGroot model [6] was introduced to describe how trustful interactions among social members can lead to agreement in the asymptotic sense, where the opinions were defined as real-valued variables, and the underlying social network

DOI: 10.14736/kyb-2023-2-0254

was modeled as a fixed graph so that any social member only talks with a given set of neighbors. This model was then generalized to the so-called Hegselmann–Krause model, where nodes holds bounded confidence in the sense that nodes tend to interact only with neighbors that hold opinions within a given confidence neighborhood [9]. It is known that under such bounded confidence, social opinions continue to tend to converge in time, but the limits form different clusters [2, 15]. The Deffuant-Weisbuch model [5] stood along the same line of consideration for bounded confidence, but was restricted to pairwise interactions in a gossiping fashion [3].

1.2. Related work

The proposed social network model with bounded confidence is a generalization of the bounded confidence type of social interactions [5, 9, 12, 14]. In Deffuant-Weisbuch models, peers only meet randomly in pairs, while $\mathcal{N}_i(t)$ is assumed to be a cluster with m nodes. The boundedness of social confidence in Deffuant-Weisbuch models is inherited in our model here, where the cluster opinions describe peer pressure in nearby social groups. When m is reduced to two and selections is limited to mutual selection, the model essentially recovers the Deffuant-Weisbuch model with a homogeneous confidence bound. Bounded confidence in social interactions is also captured in the Hegselmann–Krause model, where nodes average their states among a deterministic neighborhood determined by the nodes sharing the states within a given bound. The confidence bound yields a state-dependent underlying communication graph, compared to standard DeGroot type of opinion dynamics holding a static or time-dependent communication graph among the peers [6].

For DeGroot opinion dynamics, a number of results have been demonstrated regarding the convergence and convergence rate of the network opinions into a consensus state [8, 10]. Since DeGroot opinion dynamics impose non-expansiveness of the convex hull of the node opinions over time, such convergence to consensus has been proven to be true for a number of deterministically switching network structures [1, 11]. The bounded confidence Deffuant-Weisbuch and Hegselmann–Krause models preserve this non-expansive property of the network states, and thus convergence of individual states is expected. The presence of bounded confidence, however, may forbid the node states from converging to a consensus in general. There has been a large number of literatures, where thorough numerical studies and excellent analytical results establish that social opinions in Deffuant-Weisbuch and Hegselmann–Krause models often converge to clusters [2], except to the case where external noises or different stubborn agents exist [14].

1.3. Contributions

In this paper, we propose and study opinion dynamics over a social network with bounded confidence and un-repeated multiple selection.

Firstly, probability bounds of opinion consensus and multiple limits are estimated according to the probability distributions of order statistics and special initial value setting. Secondly, probability bounds for the global selection are estimated with a similar method. Thirdly, we provide some simulations on the probability bounds of opinion limit states, including opinion fluctuation.

2. OUR MODEL

We generalize the Long-range model [14] to a new model that there are no stubborn agents and the selection rule is un-repeated. Here both rules are exists among the decentralized large-scale social network structures, such as the Internet, the social network chat tools and the social interaction platform, e.t.c. [14]. In fact, the no stubborn agents rule denotes there are no decision makers or leaders, while the un-repeated selection often happens on a large-scale network. The bounded confidence and the hyperlocality of social interactions are remained where neighbors could sample the entire network so that neighborhoods can be formed at a global scale. Bounded confidence is put in place during opinion evolution where reference opinion becomes the centroid of the opinions from the random neighbor set.

2.1. Neighbor selection process

Consider a network of n nodes indexed in the set $\mathrm{V} = \{1, 2, \ldots, n\}$. Time is slotted at $t = 0, 1, \ldots$. At each time t, each node $i \in \mathrm{V}$ randomly selects m $(1 \le m \le n)$ different nodes as its neighbor $\mathcal{N}_i(t) = \{i_1, i_2, \ldots, i_m\}$ from the network node set V, independent with other nodes' selections. This results in a set of neighbors $\mathcal{N}_i(t)$ for $i \in \mathrm{V}$ and $t = 0, 1, \ldots$. Let $\{\mathrm{V}_1, \ldots, \mathrm{V}_z\}$ be the set containing all subsets of V with m different elements, where $z = C_n^m$. For the random neighbor set $\mathcal{N}_i(t)$, we impose the following assumption.

Assumption 1. The $\{\mathcal{N}_i(t)\}$ are independent and identically distributed for $t = 0, 1, \ldots$, and

$$\mathbf{P}\{\mathcal{N}_i(t) = \mathrm{V}_k\} > 0$$

for all $i = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, z$ at any given time t.

2.2. Opinion dynamics

Each node i holds an opinion $x_i(t) \in \mathbb{R}$ at time t. After interacting with the neighbors in the set $\mathcal{N}_i(t)$, each node i computes a local average opinion

$$\mathbf{y}_i(t) = \frac{1}{m} \sum_{j \in \mathcal{N}_i(t)} x_j(t).$$

Then nodes update their opinions at time $t+1$ according to

$$x_i(t+1) = \begin{cases} (1-\delta)x_i(t) + \delta \mathbf{y}_i(t), & \text{if } |x_i(t) - \mathbf{y}_i(t)| \le \eta \\ x_i(t), & \text{otherwise} \end{cases} \tag{1}$$

for all $i \in \mathrm{V}$. Here $0 < \delta < 1$ is the mixing parameter and $\eta > 0$ is the confidence level, which are assumed to be two constants. Regarding the initial node opinions $x_1(0), \ldots, x_n(0)$. We impose the following assumption.

Assumption 2. The $x_i(0), i \in \mathrm{V}$ are independent and identically distributed by the uniform distribution in $[0,1]$.

We use $\mathbb{P}$ to denote the probability of the total randomness generated by both the neighbor selection process and the nodes' initial values.

3. PRELIMINARY NOTATIONS AND LEMMAS

We first define the order statistics of opinion states as follows:

Definition 3.1. Given a moment $t = 0, 1, \dots,$

$$\mathbf{x}_{[1]}(t), \mathbf{x}_{[2]}(t), \dots, \mathbf{x}_{[n]}(t)$$

are the *order statistics* of opinion states $x_i(t), i \in \mathrm{V}$ satisfying that $\mathbf{x}_{[1]}(t) \leq \mathbf{x}_{[2]}(t) \leq \dots \leq \mathbf{x}_{[n]}(t)$. Specially, $D_{[i,j]}(t) \doteq \mathbf{x}_{[j]}(t) - \mathbf{x}_{[i]}(t)$ denotes the opinion range from $\mathbf{x}_{[i]}(t)$ to $\mathbf{x}_{[j]}(t)$, $t \in \mathbb{N}$.

Similarly, we denote

$$\mathbf{y}_{[1]}(t), \mathbf{y}_{[2]}(t), \dots, \mathbf{y}_{[z]}(t)$$

as the ordered average opinions selected from $\{\mathrm{V}_i, 1 \leq i \leq z\}$ satisfying $\mathbf{y}_{[i]}(t) \leq \mathbf{y}_{[i+1]}(t)$, $i = 1, 2, \dots, z-1$ where $z = C_n^m$.

We remark that $D_{[1,n]}(t)$ is the opinion range at time t. When $D_{[1,n]}(t) \to 0$ as $t \to \infty$, all opinions will reach a same limit, which is called *opinion consensus*.

Secondly, some graph definitions are provided as follows:

Definition 3.2. (i) A directed graph sequence $\{\mathcal{D}_t, t = 0, 1, \dots\}$ of the model (1) is defined as: $\mathcal{D}_t = (\mathrm{V}, \mathcal{E}_t)$ where $[i,j] \in \mathcal{E}_t$ if and only if $i \in \mathcal{N}_j(t)$ and $|\mathbf{y}_j(t)| \leq \eta$.

(ii) An undirected graph sequence $\{\bar{\mathcal{D}}_t, t = 0, 1, \dots\}$ of the model (1) is defined as: $\bar{\mathcal{D}}_t = (\bar{\mathrm{V}}, \bar{\mathcal{E}}_t)$ where $\bar{\mathrm{V}} = \{1, 2, \dots, z = C_n^m\}$ and $(i,j) \in \bar{\mathcal{E}}_t$ if and only if $|\bar{x}_i(t) - \bar{x}_j(t)| \leq \eta$, $\bar{x}_i(t) = \frac{1}{m}\sum_{j \in \mathrm{V}_i} x_j(t)$ is an average value at time t with $i \in \bar{\mathrm{V}}$.

Remark 3.3.

(1) $[i,j] \in \mathcal{E}_t$ but $[j,i]$ would not belong to $\mathcal{E}_t$. For example, if $\mathbf{x}(0) = (0.3, 0.4, 0.8)$, $m = 2, \delta = 0.5$, $\eta = 0.1$ and $\mathcal{N}_3(0) = \{1,2\}$, then $[2,3] \in \mathcal{E}_0$ but $[3,2] \notin \mathcal{E}_0$.

(2) Note that directed graph sequence $\{\mathcal{D}_t\}$ is stochastic because $\mathcal{N}_j(t)$ is randomly selected from $\{\mathrm{V}_i, 1 \leq i \leq z\}$, $1 \leq j \leq n$, $t = 0, 1, \dots$.

(3) Similarly, the undirected graph sequence $\bar{\mathcal{D}}_t$ is also stochastic and depend on the initial opinions and the previous $t-1$ selection averages, $t \geq 0$.

Next, we give some definitions on opinion evolutions.

Definition 3.4. (i) An event sequence $\{A_t, t = 0, 1, \dots\}$ is said to happen infinitely often (i.o.) if

$$\mathbf{P}\{\limsup_{t\to\infty} A_t\} = \mathbf{P}\{\cap_{m=1}^{\infty} \cup_{t=m}^{\infty} A_t\} = 1.$$

(ii) An event sequence $\{A_t, t = 0, 1, \dots\}$ is said to happen almost surely (a.s.) if

$$\mathbf{P}\{\liminf_{t\to\infty} A_t\} = \mathbf{P}\{\cup_{m=1}^{\infty} \cap_{t=m}^{\infty} A_t\} = 1.$$

(iii) For the opinion dynamics (1), opinions will reach *convergence* a.s. if there exist n r.v.s $B_1, \dots, B_n$ such that $\lim_{t\to\infty} x_i(t) = B_i, i \in \mathrm{V}$ a.s.. While opinions will *fluctuate* a.s. if there exists at least a node $i \in \mathrm{V}$ such that $\liminf_{t\to\infty} x_i(t) < \limsup_{t\to\infty} x_i(t), i \in \mathrm{V}$ a.s.

Remark 3.5. We remark that $\{A_t, t = 0, 1, \dots\}$ happens i.o. is equivalent to that there exists a subsequence $\{A_{t_k}, k = 1, 2, \dots\}$ of $\{A_t, t = 0, 1, \dots\}$ such that $\{A_{t_k}, k = 1, 2, \dots\}$ happens a.s. for $\{t_k, k = 1, 2, \dots\} \subset \{t, t = 0, 1, \dots\}$.

Finally, we introduce some lemmas.

Lemma 3.6. Under the Assumption 2.2, the r.v. $D_{[1,n]}(t) \in [0, 1]$ for any $t = 0, 1, \dots$ must decrease as t increases.

Lemma 3.7. (Durrett [7]) If $\mathbf{Y} = L\mathbf{X}$ where $L \in \mathbb{R}^{K\times K}$ is positive, $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{1\times K}$, and the probability densities of $\mathbf{X}$, $\mathbf{Y}$ exist, then we have

$$f_Y(y) = \frac{1}{|L|} f_X(L^{-1}y)$$

where $y = (y_1, y_2, \dots, y_K)'$.

Lemma 3.8. Under the Assumption 2.1 and the Assumption 2.2, the probability density of

$$X_* = (\mathbf{x}_{[i_1]}(0), \mathbf{x}_{[i_2]}(0), \dots, \mathbf{x}_{[i_k]}(0))$$

satisfies:

$$f_{X_*}(\mathbf{x}) = \begin{cases} \dfrac{n!}{(i_1-1)!(n-i_k)!\prod_{s=1}^{k-1}(i_{s+1}-i_s-1)!} x_1^{i_1-1}(1-x_n)^{n-i_k}\prod_{s=1}^{k-1}(x_{s+1}-x_s)^{i_{s+1}-i_s-1}, \\ \qquad\qquad\qquad\qquad 0 \le x_1 \le \dots \le x_k \le 1, \\ 0, \qquad\qquad\qquad\qquad \text{otherwise,} \end{cases} \tag{2}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_k)$.

Lemma 3.9. (Durrett [7]) For the *Beta* function $B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,\mathrm{d}x$, $p > 1, q > 1$, we have $B(n, m) = \frac{(n-1)!(m-1)!}{(n+m-1)!}$ for $n, m \in \mathbb{Z}$.

4. PROBABILITY BOUNDS FOR OPINION CONVERGENCE

In this section, we will estimate the probability bounds for opinion convergence of the model (1).

4.1. The case of opinion clustering

Theorem 4.1. Let $n < 2m-1$ and $\eta < \frac{1}{m}$. Then the opinion dynamics (1) leads to network opinion clustering with a strictly positive probability. To be precise, there holds

$$\begin{aligned} &\mathbb{P}\Big\{\exists n \text{ r.v.s } B_i \text{ such that } \lim_{t\to\infty} x_i(t) = B_i, i \in \mathrm{V}\Big\} \\ \geq\ & \frac{1}{2}\sum_{s=n-m+1}^{m-1}(1-\max\{\frac{m}{m-s},\frac{m}{m+s-n}\}\eta)^n. \end{aligned} \quad (3)$$

This theorem indicates that public opinions reach stable more easily if we have a smaller confidence bound. The upper probability bound estimation of opinion clustering, is relatively difficult, because this problem is equivalent to estimating the lower bound of opinion fluctuation. The lower probability bound of the opinion fluctuation is still a hard work.

P r o o f. We first construct an initial condition that ensures the opinion convergence, and then provide a lower probability bound of the probability for this initial condition.

Step 1: An initial condition that ensures the opinion convergence.

In fact, if $\mathbf{y}_{[1]}(0) > \mathbf{x}_{[s]}(0)+\eta, \mathbf{y}_{[z]}(0) < \mathbf{x}_{[s+1]}(0)-\eta$ for some $s = 2,3,\dots,n-1$ and $z = C_n^m$, then $\min_{1\leq j\leq n, 1\leq s\leq z}|\mathbf{y}_s(0)-\mathbf{x}_j(0)| > \eta$ by Remark 3.3. Note that $\mathbf{y}_j > \eta$ for any $j = 1,2,\dots,n$ and any selected average opinions. Thus $x_i(1) = x_i(0)$, $i \in \mathrm{V}$. Recursively, $x_i(t+1) = x_i(t)$ for any $t = 1,2,\dots$ and $i \in \mathrm{V}$. Thus, opinions will keep unchanged based on the opinion dynamics (1).

Denote $A_s^{(1)} = \{\mathbf{y}_{[1]}(0) > \mathbf{x}_{[s]}(0)+\eta\}$, $A_s^{(2)} = \{\mathbf{y}_{[z]}(0) < \mathbf{x}_{[s+1]}(0)-\eta\}$, $A_s = A_s^{(1)} \cap A_s^{(2)}$ and $A = \cup_{s=2}^{n-1} A_s$.

Step 2: We will prove that $\mathbb{P}\{A\} = \mathbb{P}\{\cup_{s=n-m+1}^{m-1} A_s\}$ if $n < 2m-1$ and $\mathbb{P}\{A\} = 0$ if $n \geq 2m-1$.

(i) If $s \notin \{1,2,\dots,m\} \cup \{n-m+1,\dots,n\}$, then $\mathbb{P}\{A_s\} = 0$. In fact, if $s \notin \{1,2,\dots,m\}$, then $\mathbf{x}_{[s]}(0) \geq \frac{1}{m}\sum_{j=1}^m \mathbf{x}_{[j]}(0) = \mathbf{y}_{[1]}(0)$. Therefore, $\mathbb{P}\{A_s\} \leq \mathbb{P}\{A_s^{(1)}\} = 0$. Similarly, if $s \notin \{n-m+1,\dots\}$, then $\mathbf{x}_{[s+1]}(0) \leq \mathbf{y}_{[C_n^m]}(0)$, thus $\mathbb{P}\{A_s\} \leq \mathbb{P}\{A_s^{(2)}\} = 0$.

(ii) If $n \geq 2m-1$, then $\mathbb{P}\{A_s\} = 0$ for any $s \in \{1,\dots,m\} \cup \{n-m+1,\dots,n\}$. This can be deduced by the following fact: If $s \in \{1,2,\dots,m\}$, by $n-m+1 \geq m$, $\mathbf{x}_{[s+1]}(0) \leq \mathbf{x}_{[m+1]}(0) \leq \mathbf{x}_{[n-m+1]}(0) \leq \frac{1}{m}\sum_{j=n-m+1}^n \mathbf{x}_{[j]}(0) = \mathbf{y}_{[C_n^m]}(0)$, then $\mathbb{P}\{A_s\} \leq \mathbb{P}\{A_s^{(2)}\} = 0$; Similarly, if $s \in \{n-m+1, n-m+2,\dots,n\}$, then $\mathbf{x}_{[s]}(0) > \mathbf{y}_{[1]}(0)$, and $\mathbb{P}\{A_s\} \leq \mathbb{P}\{A_s^{(1)}\} = 0$.

(iii) Consider the condition of $n < 2m-1$ or $n-m+1 < m$. If $s \in \{1,2,\dots,n-m\}$, then $\mathbb{P}\{A_s\} \leq \mathbb{P}\{A_s^{(1)}\} = 0$ because $\mathbf{x}_{[s]}(0) \leq \mathbf{y}_{[1]}(0)$; Similarly, $\mathbb{P}\{A_s\} \leq \mathbb{P}\{A_s^{(2)}\} = 0$ if $s \in \{m, m+1,\dots,n\}$.

In a sum, $\mathbb{P}\{A\} = \mathbb{P}\{\cup_{s=n-m+1}^{m-1} A_s\}$.

Step 3: Two traits of $\{A_s\}$ are provided.

(1) $A^{(1)}_{n-m+1} \supset A^{(1)}_{n-m+2} \supset \cdots \supset A^{(1)}_{m-1}$ and $A^{(2)}_{n-m+1} \subset A^{(2)}_{n-m+2} \subset \cdots \subset A^{(2)}_{m-1}$. This can be deduced by $\{\mathbf{y}_{[1]}(0) > \mathbf{x}_{[s]}(0) + \eta\} \supset \{\mathbf{y}_{[1]}(0) > \mathbf{x}_{[s+1]}(0) + \eta\}$ for $s = n-m+1, \ldots, m-1$ and $\{\mathbf{y}_{[C_n^m]}(0) < \mathbf{x}_{[s]}(0) - \eta\} \subset \{\mathbf{y}_{[C_n^m]}(0) < \mathbf{x}_{[s+1]}(0) - \eta\}$ for $s = n-m+1, \ldots, m-1$.

(2) $A_s \cap A_j = \emptyset$, $s \neq j$. In fact, assume that $s < j$, $\{\mathbf{y}_{[C_n^m]}(0) < \mathbf{x}_{[s+1]}(0) - \eta\} \cap \{\mathbf{y}_{[1]}(0) > \mathbf{x}_{[j]}(0) + \eta\} = \emptyset$, thus $A_s \cap A_j \subset A_s^{(2)} \cap A_j^{(1)} = \emptyset$. Similarly, $A_s \cap A_j = \emptyset$ if $s > j$. Therefore, $\mathbb{P}\{A\} = \mathbb{P}\{\cup_{s=n-m+1}^{m-1} A_s\} = \sum_{s=n-m+1}^{m-1} \mathbb{P}\{A_s\}$.

Step 4: The lower probability bound of $\mathbb{P}\{A\}$ is estimated by the order statistics method.

Along to Definition 3.1, we get that

$$\{\mathbf{y}_{[1]}(0) > \mathbf{x}_{[s]}(0) + \eta\} \supset \{\frac{m-s}{m}\mathbf{x}_{[s+1]}(0) + \frac{s-1}{m}\mathbf{x}_{[1]}(0) - \frac{m-1}{m}\mathbf{x}_{[s]}(0) > \eta\}$$

which can be obtained by

$$\mathbf{y}_{[1]}(0) = \sum_{j=1}^{m} \mathbf{x}_{[j]}(0) \geq \frac{1}{m}\left((s-1)\mathbf{x}_{[1]}(0) + \mathbf{x}_{[s]}(0) + (m-s)\mathbf{x}_{[s+1]}(0)\right).$$

Similarly,

$$\begin{aligned} & \{\mathbf{y}_{[C_n^m]}(0) < \mathbf{x}_{[s+1]}(0) - \eta\} \\ \supset \; & \left\{\frac{m-1}{m}\mathbf{x}_{[s+1]}(0) - \frac{m+s-n}{m}\mathbf{x}_{[s]}(0) - \frac{n-s-1}{m}\mathbf{x}_{[n]}(0) > \eta\right\} \end{aligned}$$

which can be obtained by

$$\mathbf{y}_{[C_n^m]}(0) = \sum_{j=n-m+1}^{n} \mathbf{x}_{[j]}(0) \leq \frac{1}{m}\left((s-n+m)\mathbf{x}_{[s]}(0) + \mathbf{x}_{[s+1]}(0) + (n-s-1)\mathbf{x}_{[n]}(0)\right).$$

By Lemma 3.8, the probability density of $\mathbf{X}_s = (\mathbf{x}_{[1]}(0), \mathbf{x}_{[s]}(0), \mathbf{x}_{[s+1]}(0), \mathbf{x}_{[n]}(0)) \in \mathbb{R}^4$ is:

$$f_{\mathbf{X}_s}(\mathbf{x}_s) = \begin{cases} \frac{n!}{(s-2)!(n-s-2)!}(x_2 - x_1)^{s-2}(x_4 - x_3)^{n-s-2}, & 0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathbf{x}_s = (x_1, x_2, x_3, x_4)$.

Therefore, for $s = n-m+1, n-m+2, \dots, m-1$, by Lemma 3.9,

$$\begin{aligned}
\mathbb{P}\{A_s\} &\overset{(a)}{\geq} \int_{\frac{m-s}{m}x_3+\frac{s-1}{m}x_1+\frac{m-1}{m}x_2>\eta,\, \frac{m-1}{m}x_3-\frac{m+s-n}{m}x_2-\frac{n-s-1}{m}x_4>\eta} f_{\mathbf{X}_s}(\mathbf{x}_s)\,\mathrm{d}\mathbf{x}_s \\
&\overset{(b)}{\geq} \int_{\frac{m-s}{m}(x_3-x_2)>\eta,\, \frac{m+s-n}{m}(x_3-x_2)>\eta} f_{\mathbf{X}_s}(\mathbf{x}_s)\,\mathrm{d}\mathbf{x}_s \\
&= \frac{n!}{(s-2)!(n-s-2)!}\int_0^{1-\eta^*} d_{x_2}\int_0^{x_2}(x_2-x_1)^{s-2}\,\mathrm{d}x_1\int_{x_2+\eta^*}^1 \mathrm{d}x_3\int_{x_3}^1 (x_4-x_3)^{n-s-2}\,\mathrm{d}x_4 \\
&\geq \frac{n!}{(s-2)!(n-s-2)!}\int_0^{1-\eta^*} \mathrm{d}_{x_2}\int_{x_2+\eta^*}^1 x_2^{s-1}(1-x_3)^{n-s-1}\,\mathrm{d}x_3 \\
&= \frac{n!}{(s-2)!(n-s-1)!}\int_0^{1-\eta^*} x_2^{s-1}(1-\eta^*-x_2)^{n-s}\,\mathrm{d}x_2 \\
&= \frac{n!}{(s-2)!(n-s-1)!}(1-\eta^*)^n B(s, n-s+1) = (1-\eta^*)^n
\end{aligned}$$

where (a) holds because
$\{A_s\} \supseteq \{\frac{m-s}{m}x_3 + \frac{s-1}{m}x_1 + \frac{m-1}{m}x_2 > \eta, \frac{m-1}{m}x_3 - \frac{m+s-n}{m}x_2 - \frac{n-s-1}{m}x_4 > \eta\}$,
(b) comes from $\{\frac{m-s}{m}x_3 + \frac{s-1}{m}x_1 + \frac{m-1}{m}x_2 > \eta, \frac{m-1}{m}x_3 - \frac{m+s-n}{m}x_2 - \frac{n-s-1}{m}x_4 > \eta\} \supseteq \{\frac{m-s}{m}(x_3-x_2) > \eta, \frac{m+s-n}{m}(x_3-x_2) > \eta\}$ and $\eta^* = \max\{\frac{m}{m-s}, \frac{m}{m+s-n}\}\eta$.

In a sum, $\mathbb{P}\{A\} \geq \frac{1}{2}\sum_{s=n-m+1}^{m-1}\Big(1-\max\{\frac{m}{m-s}, \frac{m}{m+s-n}\}\eta\Big)^n$ where $n < 2m-1$ and $\eta < \frac{1}{m}$.

We remark that $\mathbb{P}\{A\} \geq (2m-n-1)(1-m\eta)^n$ for simplicity. □

4.2. The case of opinion consensus

Theorem 4.2. Let $\eta > 0$. Then along the opinion dynamics (1) the network will reach a consensus with some positive probability. Precisely there holds

$$\mathbb{P}\Big\{\exists a \text{ r.v. } B_* \text{ such that } \lim_{t\to\infty} x_i(t) = B_*, i \in \mathrm{V}\Big\} \geq n!\eta^{n-1}(1-\eta). \tag{4}$$

Theorem 4.3. If $\eta \leq \frac{1}{m+1}$, then

$$\mathbb{P}\Big\{\exists a \text{ r.v. } B_* \text{ such that } \lim_{t\to\infty} x_i(t) = B_*, i \in \mathrm{V}\Big\} \leq 1 - \frac{n(n-1)}{2}\Big(\frac{\eta}{m}\Big)^{n-2}(1-(m+1)\eta)^2. \tag{5}$$

According to these theorems, the event that agent opinions of the opinion dynamics (1) reach consensus is non-singular.

P r o o f of Theorem 4.2. We prove this theorem with the following two steps.

(Step 1): If $\eta \geq D_{[1,n]}(0)$, then $D_{[1,n]}(t) \to 0$ as $t \to \infty$;

In fact, at time 0, for any agent selected average opinion $\mathbf{y}_{[j]}(0)$ of agent $[i]$, $i \in \mathrm{V}$, $j \in \bar{\mathrm{V}}$, we have

$$|\mathbf{y}_{[i]}(0)| = |\mathbf{y}_{[j]}(0) - \mathbf{x}_{[i]}(0)| \leq \frac{1}{m}|\sum_{j=1}^{m} D_{[1,n]}(0)| \leq \eta.$$

At time 1, by Lemma 3.6, we get that $D_{[1,n]}(1) \leq D_{[1,n]}(0)$. Similarly, for any agent selected average opinion $\mathbf{y}_{[j]}(0)$ of agent $[i]$, $i \in \mathrm{V}$, $j \in \bar{\mathrm{V}}$,

$$|\mathbf{y}_{[i]}(1)| = |\mathbf{y}_{[j]}(1) - \mathbf{x}_{i}(1)| \leq \frac{1}{m}|\sum_{j=1}^{m} D_{[1,n]}(1)| \leq D_{[1,n]}(0) \leq \eta.$$

Recursively, we get that for any $i \in \mathrm{V}$, $j \in \bar{\mathrm{V}}$ and $t = 0, 1, \ldots$,

$$|\mathbf{x}_{[i]}(t)| \leq \eta. \tag{6}$$

At any time $t \in \mathbb{N}$, by Assumption 1 and Lemma 3.6, $D_{[1,n]}(t) \to \mathbf{x}^*$ as $t \to \infty$.

If $\mathbf{x}^* > 0$, then for any small real number $0 < \epsilon < \frac{\delta \mathbf{x}^*}{m-\delta}$, there exists a random variable $T \in \mathbb{N}$, when $t > T$, $\mathbf{x}^* \leq D_{[1,n]}(t) < \mathbf{x}^* + \epsilon$. Note that

$$\mathbf{y}_{[C_n^m]}(t) \geq \frac{m-1}{m}\mathbf{x}_{[1]}(t) + \frac{1}{m}\mathbf{x}_{[n]}(t) = \mathbf{x}_{[1]}(t) + \frac{D_{[1,n]}(t)}{m},$$

and agent [1] at time t has a positive probability to select the average opinion $\mathbf{y}_{[C_n^m]}(t)$, thus

$$D_{[1,n]}(t+1) \leq D_{[1,n]}(t) - \delta\frac{D_{[1,n]}(t)}{m}$$

if agent [1] at time t selects the average opinion $\mathbf{y}_{[C_n^m]}(t)$. By $\epsilon < \frac{\delta \mathbf{x}^*}{m-\delta}$,

$$D_{[1,n]}(t+1) \leq (1 - \frac{\delta}{m})(\mathbf{x}^* + \epsilon) < \mathbf{x}^*.$$

Note that agent [1] has a strictly positive probability to select the average opinion $\mathbf{y}_{[C_n^m]}(t)$, by (6) and Borel-Cantelli lemma, it provides a contradiction. Thus, $\lim_{t\to\infty} D_{[1,n]}(t) = 0$ a.s.

(Step 2): The lower bound of the probability $\mathbb{P}\{\eta \geq D_{[1,n]}(0)\}$ can be estimated.

Set $\mathbf{X}_c = (\mathbf{x}_{[1]}(0), \mathbf{x}_{[n]}(0))'$, then by Lemma 3.8,

$$f_{\mathbf{X}_c}(x) = \begin{cases} \frac{n!}{(n-2)!}(x_2 - x_1)^{n-2}, & 0 \leq x_1 \leq x_2 \leq 1 \\ 0, & \text{others.} \end{cases}$$

Therefore,

$$\begin{aligned} \mathbb{P}\{\lim_{t\to\infty} D_{[1,n]}(t) = 0\} &\geq \mathbb{P}\{D_{[1,n]}(0) \leq \eta\} \\ &= \int_{x_2 - x_1 \leq \eta, 0 \leq x_1 \leq x_2 \leq 1} \frac{n!}{(n-2)!}(x_2 - x_1)^{n-2}\, \mathrm{d}x_1 \mathrm{d}x_2 \\ &= \frac{n!}{(n-2)!}\int_0^{1-\eta} \mathrm{d}x_1 \int_{x_1}^{x_1+\eta} (x_2 - x_1)^{n-2}\, \mathrm{d}x_2 = n\eta^{n-1}(1-\eta). \end{aligned}$$

□

P r o o f . of Theorem 4.3. We prove this theorem with the following two steps:

(Step 1): If $D_{[1,2]}(0) > \frac{m^2+m-1}{m}\eta$, $D_{[2,n]}(0) < \frac{1}{m}\eta$, then opinions will not reach consensus.

We will prove that $\mathbf{x}_{[1]}(t)$ will keep unchanged, $t = 0, 1, \ldots$ and $x_j(t) \in [\mathbf{x}_{[2]}(0), \mathbf{x}_{[n]}(0)]$ for any $j \neq [1]$.

(i) At time 0, Note that

$$|\mathbf{y}_{[1]}(0)| = \frac{1}{m}|\sum_{[j_k]\in\mathcal{N}_{[1]}(0), 1\leq k\leq m} D_{[1,j_k]}(0)| \geq \frac{1}{m}|D_{[1,1]}(0) + \sum_{[j_k]\in\mathcal{N}_{[1]}(0), j_k\neq 1} D_{[1,j_k]}(0)|$$
$$\geq \frac{m-1}{m}\min_{k\geq 2} D_{[1,k]}(0) > \frac{m-1}{m}(m + \frac{m-1}{m})\eta > \eta,$$

$$|\mathbf{y}_{[i]}(0)| = \frac{1}{m}|\sum_{[j_s]\in\mathcal{N}_{[i]}(0), 1\leq s\leq m} D_{[i,j_s]}(0)| > \frac{1}{m}\left((m + \frac{m-1}{m})\eta - \max_{j\geq 2} D_{[i,j]}(0)\right) \geq \eta$$

for $[1] \in \mathcal{N}_{[i]}(0)$ and $|\mathbf{y}_{[i]}(0)| \leq \eta$ for $[1] \notin \mathcal{N}_{[i]}(0)$. Thus, $\mathbf{x}_{[1]}(1) = \mathbf{x}_{[1]}(0)$ and $\mathbf{x}_{[i]}(1) \in [\mathbf{x}_{[2]}(0), \mathbf{x}_{[n]}(0)]$ for $i \geq 2$. Besides, the minimum order is preserved. Furthermore, $D_{[1,2]}(1) \geq D_{[1,2]}(0) > \frac{m^2+m-1}{m}\eta$, $D_{[2,n]}(1) \leq D_{[2,n]}(0) < \frac{\eta}{m}$.

Recursively, $D_{[1,2]}(t) \geq D_{[1,2]}(t-1) \geq \cdots \geq D_{[1,2]}(0) > \frac{m^2+m-1}{m}\eta$. Thus, opinions will not reach consensus in this condition.

(Step 2): We estimate the lower probability bound of $\mathbb{P}\{D_{[1,2]}(0) > \frac{m^2+m-1}{m}\eta, D_{[2,n]}(0) < \frac{1}{m}\eta\}$.

Denote $\mathbf{X}_c = (\mathbf{x}_{[1]}(0), D_{[1,2]}(0), D_{[2,n]}(0))'$ and $\mathbf{X}^{(3)} = (\mathbf{x}_{[1]}(0), \mathbf{x}_{[2]}(0), \mathbf{x}_{[n]}(0))'$. Then $\mathbf{X}_c = L_2\mathbf{X}^{(3)}$ where

$$L_2 = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

By Lemma 3.7, we get that $f_{\mathbf{X}_c}(\mathbf{y}) = \frac{1}{|L_2|} f_{\mathbf{X}^{(3)}}(L_2^{-1}\mathbf{y}) = f_{\mathbf{X}^{(3)}}(L_2^{-1}\mathbf{y})$, where $\mathbf{y} = (y_1, y_2, y_3)$, $0 \leq y_1, y_2, y_3 \leq 1$, $y_1 + y_2 + y_3 \leq 1$.

By Lemma 3.8,

$$f_{\mathbf{X}^{(3)}}(\mathbf{x}) = \begin{cases} \frac{n!}{(n-3)!}(x_3 - x_2)^{n-3}, & 0 \leq x_1 \leq x_2 \leq x_3 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

where $\mathbf{x} = (x_1, x_2, x_3)$. Thus,

$$f_{\mathbf{X}_c}(\mathbf{y}) = \begin{cases} \frac{n!}{(n-3)!} y_3^{n-3}, & 0 \leq y_j \leq 1, \sum_{i=1}^{3} y_i \leq 1, j = 1, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

In a sum,

$$
\begin{aligned}
&\mathbb{P}\{D_{[1,2]}(0) > \frac{m^2+m-1}{m}\eta, D_{[2,n]}(0) < \frac{1}{m}\eta\} \\
=&\int\int\int_{0\le y_1+y_2+y_3\le 1, y_2\ge \frac{m^2+m-1}{m}\eta, y_3\le \frac{\eta}{m}} \frac{n!}{(n-3)!} y_3^{n-3}\,\mathrm{d}y_1\,\mathrm{d}y_2\mathrm{d}y_3 \\
\ge&\frac{n!}{(n-3)!}\int_0^{1-(m+1)\eta}\mathrm{d}y_1\int_{\frac{m^2+m-1}{m}\eta}^{1-\frac{\eta}{m}-y_1}\mathrm{d}y_2\int_0^{\frac{\eta}{m}} y_3^{n-3}\,\mathrm{d}y_3 \\
=&n(n-1)\left(\frac{\eta}{m}\right)^{n-2}\int_0^{1-(m+1)\eta}(1-(m+1)\eta-y_1)\,\mathrm{d}y_1 \\
=&\frac{n(n-1)}{2}\left(\frac{\eta}{m}\right)^{n-2}(1-(m+1)\eta)^2
\end{aligned}
$$

if $\eta < \frac{1}{m+1}$. □

4.3. The case of global selection

In this subsection, we consider the case that $m = n$. Then the network $\mathcal{D}_t = (\mathrm{V}, \mathcal{E}_t)$ satisfies $\mathcal{N}_i(t) = V$ for any $i \in V$ and $t \in \mathbb{N}$. Thus, for any $i \in V$, $x_i(t+1) = x_i(t) + \delta(\mathbf{y}_1(t) - x_i(t))$ if $|\mathbf{y}_1(t) - x_i(t)| \le \eta$ and $x_i(t+1) = x_i(t)$ otherwise. Here $\mathbf{y}_1(t) = \sum_{j\in V}\frac{x_j(t)}{n}$. Based the previous subsections, the probability bounds of opinion limit states can be estimated as follows:

Theorem 4.4. Suppose $m = n$. For the opinion dynamics (1), the distribution of limit states $B_1, B_2, \dots, B_n$ satisfies

(P1.) $\mathbb{P}\{B_1 = B_2 = \cdots = B_n\}$
$\ge \left(\frac{\eta}{n}\right)^{n-1}\frac{n-\eta(n-1)}{n}\sum_{s=2}^{\lfloor n/2\rfloor}\frac{n!}{(s-2)!(n-s-2)!}\left(1-\frac{n(n-s)+s^2}{(n-s)n}\eta\right)\frac{s^{2(n-s-1)}}{(n-s)^{n-2s}}$ and
$\mathbb{P}\{B_1 = B_2 = \cdots = B_n\} \le (2\eta)^{n-1}(n - 2\eta(n-1))$ for $\eta \le \frac{1}{2}$;

(P2.) $\mathbb{P}\{|B_i - B_j| > \eta, \forall i \ne j, i, j \in \mathrm{V}\} \ge \sum_{s=1}^{n-1}\frac{n!(1-K_0\eta-n\eta)^{n-1}}{(s-1)!(n-s-1)!}\left(\frac{s}{n-s}\right)^{n-2s}\frac{1}{n-1}$ and
$\mathbb{P}\{|B_i - B_j| > \eta, \forall i \ne j, i, j \in \mathrm{V}\} < (1-\eta)^n$ for $\eta \le \frac{1}{n^2}$
where $K_0 = \max\{\frac{(n-s)^2}{s}, \frac{s(s+1)}{n-s}\}$.

Theorem 4.4 shows that both the opinion consensus event and $\{|B_i - B_j| > \eta, \forall i \ne j\}$ are non-trivial. Due to $\{D_{[1,n]}(0) < 2\eta\}$ is a small probability event, opinion consensus is a comparative 'rare' event.

P r o o f. We divide the proof into two parts.

Part 1. The probability bound estimation for the consensus event.

Define

$$\mathbb{B}_s \triangleq \{\frac{n-s}{n}D_{[s,s+1]}(0) \ge \frac{s}{n}D_{[1,s]}(0), \frac{s}{n}D_{[s,s+1]}(0) + \frac{n-s}{n}D_{[1,n]}(0) \le \eta\}$$

and

$$\mathbb{U}_s \triangleq \{\frac{s}{n}D_{[1,n]}(0) + \frac{n-s}{n}D_{[s+1,n]}(0) \le \frac{n}{n-s}\eta, \frac{s}{n}D_{[s,s+1]}(0) \ge \frac{n-s}{n}D_{[s+1,n]}(0)\}.$$

Further, we define $\mathbb{B}'_s = \{y_j \in (0,\eta), y_k \in (-\frac{n}{n-s}\eta, 0), j \ne k, 1 \le j \le s, s+1 \le k \le n\}$ for $s = 1, 2, \ldots, n-1$ and $\mathbb{U}'_s = \{y_j \in (-\eta, 0), y_k \in (0, \frac{n}{s-1}\eta), j \ne k, s \le j \le n, 1 \le k < s\}$ for $s = 2, 3, \ldots, n$ where $y_k = \bar{x}_{[1]}(0) - x_{[k]}(0)$. It shows that

$$\{B_1 = B_2 = \cdots = B_n\} \overset{a)}{=} \cup_{s=1}^{n-1}\{0 < y_k < \eta, 0 < -y_j < \eta + \frac{\sum_{r=1}^{s} y_r}{n-s}, 1 \le k \le s\} \cup$$
$$\cup_{s=2}^{n}\{0 < -y_j < \eta, s \le j \le n, 0 < y_k < \eta - \frac{\sum_{r=s}^{n} y_j}{s-1}, 1 \le k < s\} \overset{b)}{\supseteqq} \bigcup_{s=1}^{n-1} \mathbb{B}_s \cup \bigcup_{s=2}^{n} \mathbb{U}_s \tag{7}$$

where $a)$ is based on the model (1) and $b)$ holds by the definition of $\mathbb{B}_s$ and $\mathbb{U}_s$. Furthermore, by the definition of $\mathbb{B}_s$ and $\mathbb{U}_s$, $\mathbb{B}_s \cup \mathbb{U}_{s+1}, s = 1, 2, \ldots, n-1$ are mutually exclusive events.

Then by the inequality (7),

$$\mathbb{P}\{B_1 = B_2 = \cdots = B_n\} \ge \sum_{s=1}^{n-1} \mathbb{P}\{\mathbb{B}'_s \cup \mathbb{U}'_{s+1}\}.$$

By the definition of the event $\mathbb{B}'_s$,

$$\bar{x}_{[1]}(0) \in [\frac{1}{n}\left(sx_{[1]}(0) + (n-s)x_{[s+1]}(0)\right), \frac{1}{n}\left(sx_{[s]}(0) + (n-s)x_{[n]}(0)\right)].$$

Then

$$y_k \in [\frac{n-s}{n}D_{[s,s+1]}(0) - \frac{s}{n}D_{[1,s]}(0), \frac{s}{n}D_{[1,s]}(0) + \frac{n-s}{n}D_{[1,n]}(0)] \text{ if } 1 \le k \le s,$$

$$y_k \in \left[-\frac{s}{n}D_{[1,n]}(0) - \frac{n-s}{n}D_{[s+1,n]}(0), -\frac{s}{n}D_{[s,s+1]}(0) + \frac{n-s}{n}D_{[s+1,n]}(0)\right]$$
if $s+1 \le k \le n$.

Obviously,

$$\mathbb{B}'_s \supseteqq \mathbb{B}_s \triangleq \{\frac{n-s}{n}D_{[s,s+1]}(0) \ge \frac{s}{n}D_{[1,s]}(0), \frac{s}{n}D_{[s,s+1]}(0) + \frac{n-s}{n}D_{[1,n]}(0) \le \eta\}$$

$1 \le k \le s$ and

$$\mathbb{U}'_s \supseteqq \mathbb{U}_s \triangleq \{\frac{s}{n}D_{[1,n]}(0) + \frac{n-s}{n}D_{[s+1,n]}(0) \le \frac{n}{n-s}\eta, \frac{s}{n}D_{[s,s+1]}(0) \ge \frac{n-s}{n}D_{[s+1,n]}(0)\}$$

for $s+1 \le k \le n$.

Define

$$\mathcal{Z}_s = (z_1, z_2, z_3, z_4) \triangleq (x_{[1]}(0), D_{[1,s]}(0), D_{[s,s+1]}(0), D_{[s+1,n]}(0))$$

and

$$X_s = (x_{[1]}(0), x_{[s]}(0), x_{[s+1]}(0), x_{[n]}(0)).$$

Then $\mathcal{Z}_s = LX_s$ where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

By Lemma 3.8 and $z_3 \geq \max\{\frac{s}{n-s}z_2, \frac{n-s}{s}z_4\}$ in the event $\mathbb{B}_s \cup \mathbb{U}_{s+1}$, we have

$$\begin{aligned}
&\mathbb{P}\{\mathbb{B}_s \cup \mathbb{U}_{s+1}\} \\
&\quad \overset{a)}{\geq} \int_{z_2 \leq \frac{n-s}{s}z_3, z_4 \leq \frac{s}{n-s}z_3, z_4 + \frac{s}{n}(z_2+z_3) \leq \frac{n\eta}{n-s}, z_2 + \frac{n-s}{n}(z_3+z_4) \leq \eta} \frac{n! z_2^{s-2} z_4^{n-s-2}}{(s-2)!(n-s-2)!} \,\mathrm{d}\mathcal{Z}_s \\
&\quad \overset{b)}{\geq} \frac{n!}{(s-2)!(n-s-2)!} \int_{sz_2 + (n-s)z_4 \leq s\eta} z_2^{s-2} z_4^{n-s-2} \,\mathrm{d}\mathcal{Z}_s
\end{aligned}$$

where $a)$ follows from the inequality (7) and $b)$ holds because $\{\frac{n}{n-s}z_2 + z_3 + z_4 \leq \frac{n}{n-s}\eta\} \subset \{z_4 + \frac{s}{n}(z_2 + z_3) \leq \frac{n}{n-s}\eta\}$, $z_3 \leq \frac{s}{n}\eta$ by $\frac{n}{n-s}z_2 + z_3 + z_4 \leq \frac{n}{n-s}\eta$, $z_2 \leq \frac{n-s}{s}z_3$ and $z_4 \leq \frac{s}{n-s}z_3$. We further get that

$$\begin{aligned}
&\mathbb{P}\{\mathbb{B}_s \cup \mathbb{U}_{s+1}\} \\
&\overset{(a)}{\geq} \frac{n!}{(s-2)!(n-s-2)!}\left(1 - \frac{n^2 - sn + s^2}{(n-s)n}\eta\right) \\
&\quad \left\{\int_0^{\frac{n-s}{n}\eta} z_2^{s-2}\,\mathrm{d}z_2 \int_{(\frac{s}{n-s})^2 z_2}^{\frac{s^2}{(n-s)n}} z_4^{n-s-2}(\frac{s}{n}\eta - \frac{n-s}{s}z_4)\,\mathrm{d}z_4 \right. \\
&\quad \left. + \int_0^{\frac{s^2}{(n-s)n}\eta} z_4^{n-s-2} dz_4 \int_{(\frac{n-s}{s})^2 z_4}^{\frac{n-s}{n}\eta} z_2^{s-2}(\frac{s}{n}\eta - \frac{n}{n-s}z_2)\,\mathrm{d}z_2 \right\} \\
&\geq \frac{n!}{(s-2)!(n-s-2)!} \frac{s^{2(n-s-1)}\eta}{(n-s)^{n-2s}} \left(\frac{\eta}{n}\right)^{n-2} \left(1 - \frac{(n-s)n + s^2}{(n-s)n}\eta\right) \\
&\quad \left\{\frac{s}{n(n-1)(n-s)} + \frac{1}{s}\left(\frac{1}{n-1} - \frac{\eta}{n}\right) - \frac{s}{s-1}\left(\frac{1}{n(n-2)} - \frac{\eta}{n^2}\right)\right\} \\
&\geq \frac{n!\left(1 - \frac{(n-s)n+s^2}{(n-s)n}\eta\right)}{(s-2)!(n-s-2)!} \frac{s^{2(n-s-1)}\eta}{(n-s)^{n-2s}} \left(\frac{\eta}{n}\right)^{n-2} \left\{\frac{s}{n(n-1)(n-s)} + \left(\frac{1}{s} - \frac{1}{n}\right)\left(\frac{1}{n-1} - \frac{\eta}{n}\right)\right\} \\
&\geq \left(\frac{\eta}{n}\right)^{n-1} \frac{n - \eta(n-1)}{n} \frac{n!}{(s-2)!(n-s-2)!} \left(1 - \frac{n(n-s) + s^2}{(n-s)n}\eta\right) \frac{s^{2(n-s-1)}}{(n-s)^{n-2s}}
\end{aligned}$$

where (a) holds by the fact that $z_3 \geq \max\{\frac{s}{n-s}z_2, \frac{n-s}{s}z_4\}$ and
$\eta \leq \min_{s\in\{1,2,\dots,\lfloor\frac{n}{2}\rfloor\}}\left\{\frac{(n-s)n}{(n-s)n+s^2}\right\} = \frac{\frac{n^2}{2}}{\frac{n^2}{2}+\frac{n^2}{4}} = \frac{2}{3}$.

For the upper probability bound for the consensus event, note that $\{B_1 = B_2 = \dots = B_n\} \subset \{D_{[1,n]}(0) < 2\eta\}$, by

$$\mathbb{P}\{D_{[1,n]<2\eta}\} = n(n-1)\int_0^{2\eta} r^{n-2}(1-r)\,\mathrm{d}r = (2\eta)^{n-1}[n-2\eta(n-1)],$$

we get that

$$\mathbb{P}\{B_1 = B_2 = \dots = B_n\} \leq (2\eta)^{n-1}[n-2\eta(n-1)].$$

Part 2: The probability bound estimation for the $\{|B_i - B_j| \geq \eta, i \neq j\}$ event.

Denote $\mathcal{D} = \{|B_i - B_j| \geq \eta, i \neq j\}$.

$$\begin{aligned}\mathcal{D} \supset \{|x_{[i]}(0) - \bar{x}_{[1]}(0)| > \eta\} \cap \{D_{[i,i+1]}(0) > \eta\}\\ \cap\{0 < \bar{x}_{[1]}(0) - x_{[i_0]}(0) < \eta, x_{[i_0+1]}(0) - \bar{x}_{[1]}(0)\\ > \eta + \frac{1}{n-1}(\bar{x}_{[1]}(0) - x_{[i_0]}(0)), 1 \leq i_0 \leq n-1\}\\ \cap\{0 < x_{[j_0+1]}(0) - \bar{x}_{[1]}(0) < \eta, \bar{x}_{[1]}(0) - x_{[j_0]}(0)\\ > \eta + \frac{1}{n-1}(x_{[j_0+1]}(0) - \bar{x}_{[1]}(0)), 1 \leq i_0 \leq n-1\}.\end{aligned}$$

According to Lemma 3.8,

$$\begin{aligned}&\mathbb{P}\{\mathcal{D}\} \geq \sum_{s=1}^{n-1}\frac{n!}{(s-2)!(n-s-2)!}\\ &\int\int\cdots\int_{\frac{n-s}{n}w_3-\frac{s}{n}w_2>\frac{s(s+1)}{n}\eta,\frac{s}{n}w_3-\frac{n-s}{n}w_4>\frac{(n-s)^2}{n}\eta,\sum w_i\leq 1-n\eta,w_i\geq 0} w_2^{s-2}w_4^{n-s-2}\,\mathrm{d}w_1\,\mathrm{d}w_2\,\mathrm{d}w_3\mathrm{d}w_4\\ &\geq \sum_{s=1}^{n-1}\frac{n!}{(s-1)!(n-s-1)!}\int_{K_0\eta}^{1-n\eta}\left(\frac{n-s}{s}\right)^{s-1}\left(\frac{s}{n-s}\right)^{n-s-1}(w_3-K_0\eta)^{n-2}\,\mathrm{d}w_3\\ &= \sum_{s=1}^{n-1}\frac{n!(1-K_0\eta-n\eta)^{n-1}}{(n-1)(s-1)!(n-s-1)!}\left(\frac{n-s}{s}\right)^{s-1}\left(\frac{s}{n-s}\right)^{n-s-1},\end{aligned}$$

where $K_0 = \max\{\frac{(n-s)^2}{s}, \frac{s(s+1)}{n-s}\} \geq \bar{K} = \max\{s+1, n-s\}$ for any $s \in \mathcal{V}$.

For the upper bound, note that

$$\{x_{[s]}(0) - \bar{x}_{[1]}(0) < \eta, \bar{x}_{[1]}(0) - x_{[s-1]}(0) > \eta + \frac{x_{[s]}(0) - \bar{x}_{[1]}(0)}{n-1}\} \subset \{D_{[s-1,s]}(0) > \eta\}$$

and

$$\{\bar{x}_{[1]}(0) - x_{[s-1]}(0) < \eta, x_{[s]}(0) - \bar{x}_{[1]}(0) > \eta + \frac{\bar{x}_{[1]}(0) - x_{[s-1]}(0)}{n-1}\} \subset \{D_{[s-1,s]}(0) > \eta\},$$

thus, $\mathcal{D} \subset \{D_{[i,i+1]}(0) > \eta, i = 1, 2, \ldots, n-1\}$.

Similarly, by Lemma 3.7,

$$\mathbb{P}\{\mathcal{D}\} \le n! \frac{1}{n!}(1-\eta)^n = (1-\eta)^n.$$

□

According to these theorems, it is illustrated that the convergence events would not happen a.s., it emerges possibly other kinds of events, for example, the opinion fluctuation. In the next section, we will demonstrate the probability bounds for the convergence events and find that the fluctuation events would happen with positive probabilities.

5. SIMULATIONS

In this section, we first simulate the probability bounds of Theorem 4.1, Theorem 4.2 and Theorem 4.3. Here we provide typical examples to show how the bounds changes as the agent number increases. To further understand the upper and lower bounds, some potential significance are talked among these simulations.

For Theorem 4.1, the convergence event is simplified as $\sum_{k=0}^{20} \sum_{i=1}^{n} (x_i(t-k) - x_i(t-k-1))^2 < 0.01$. Here we set $\eta = 0.05$, $\delta = 0.5$, the selection number $m = 17$ and the termination time $T = 1000$ in Figure 1. We count the average of 1000 simulations on the convergence events and find that the numerical frequencies of convergence events are beyond the lower probability bounds.

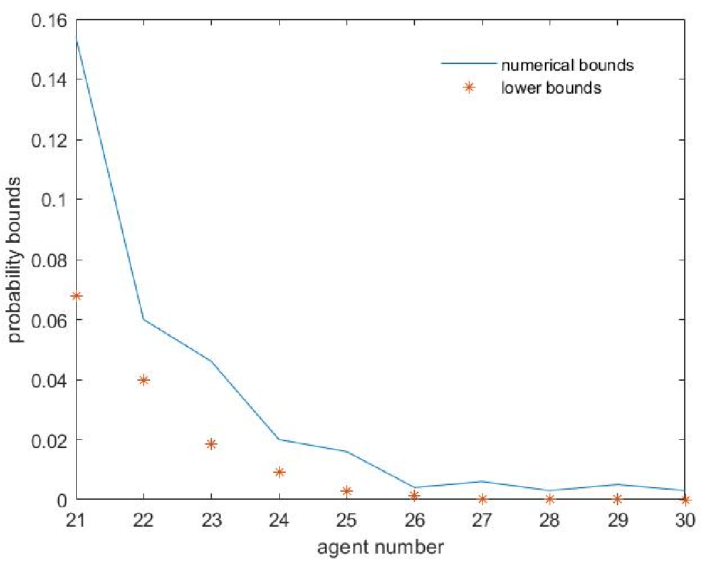


Figure 1: The lower bound of opinion convergence when $\eta = 0.05$, which illustrates individual opinions are more hard to reach stable if agent number increases and the confidence bound is small enough.

For Theorem 4.2 and Theorem 4.3, $\eta = 0.02$, $\delta = 0.5$, $m = 2$ and $T = 1000$ in Figure 2. The consensus event is simplified as $\{D_{[1,n]}(T) < 0.01\}$. Considering that 1000 simulations are far from enough to simulate the theoretical initial value equal probability coverage, we set agent number n changes from 91 to 100 in Figure 2, which further shows that the bound estimation of Theorem 4.2 and Theorem 4.3 is relatively loose, comparing to Figure 1.

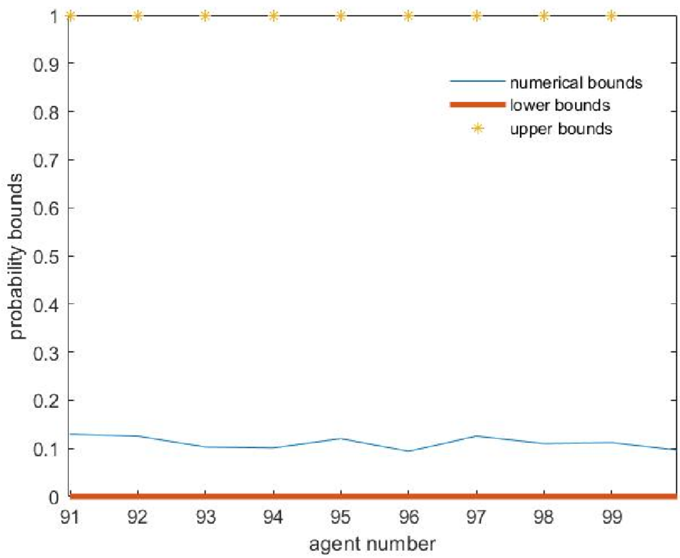
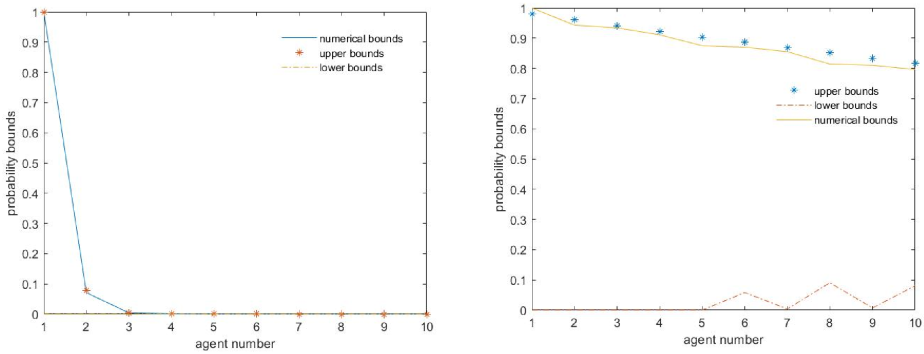


Figure 2: The bound of opinion consensus when $\eta = 0.02$, and the numerical bounds show that opinion consensus probability would be not too closely related to the individual number.

Secondly, we simulate how the probability bounds of Theorem 4.4 change. Here $\eta = 0.02$, $\delta = 0.5$ and the termination time $T = 10000$. According to Theorem 4.4, the consensus event is simplified as $\{D_{[1,n]}(T) < 0.00001\}$ in this demonstration. We count the average of 1000 simulations on the events $\{D_{[1,n]}(T) < 0.00001\}$ and $\{|B_i - B_j| > \eta, i \neq j\}$, respectively. In Figure 3, we find that as the agent number n increases, the consensus probability bounds decrease and the probability upper bounds for the limit distances larger than the confidence bound decrease.

Next, we simulate how opinions evolve for general cases. Here $n = 30$, the selection number $m = 12$, $\delta = 0.5$, the confidence bound $\eta = 0.1$ and the termination time $T = 1000$. Figure 4 implies that opinions always fluctuate between the relative extremal opinions on both sides. This phenomenon needs further mathematical proving, whose mathematical analysis will be our further work.

Finally, based on the simulation of Figure 4, probability bounds of opinion fluctuations are demonstrated where $n = 100$, $\delta = 0.8$ and the termination time $T = 1000$. For this demonstration, the opinion fluctuation is assumed to occur if $\sum_{k=0}^{20} \sum_{i=1}^{n} (x_i(t -$



(a) the opinion consensus bounds for the global selection rule.

(b) the pair wise convergence bounds for the global selection rule.

Figure 3: Bounds of multiply limit states change as n increases when $\eta = 0.02$.

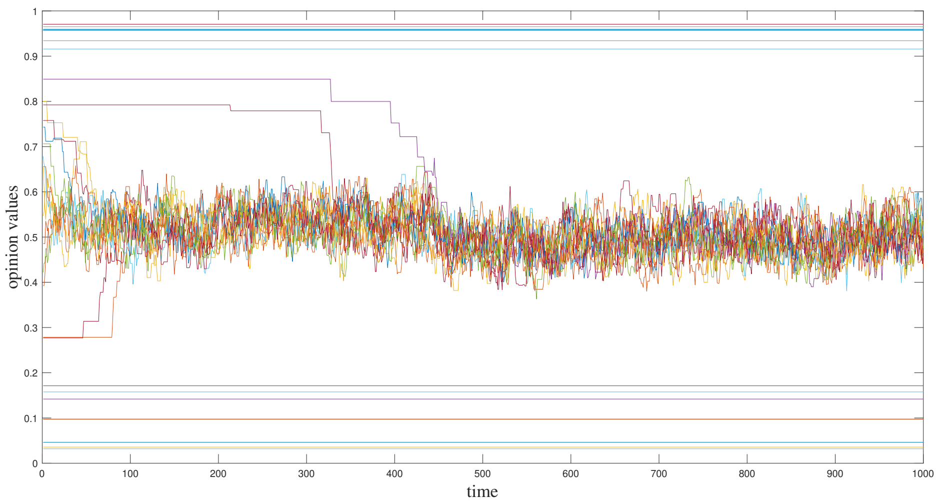


Figure 4: opinion dynamics when $m = 12$ and $n = 30$.

$k) - x_i(t-k-1))^2 > 0.01$ and opinion convergence happens otherwise. In Figure 5, we simulate the experiment for 100 times. The frequency of opinion fluctuations is defined as the proportion of the opinion fluctuation events for these 100 experiments. Results show that opinion fluctuation would happen with positive probabilities if the confidence bound η is small enough.

Through these results, we find that the opinion dynamics of the model (1) are more

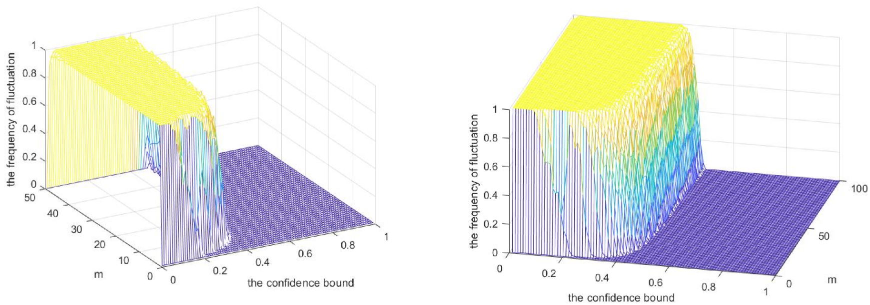


Figure 5: The frequency of opinion fluctuation changes as m and the confidence bound increase.

complicated than the one in [14], especially for the parameter conditions. Furthermore, different from the case in [14], simulations show that opinion fluctuation would happen even without the influence of stubborn agents.

6. CONCLUSIONS

In this paper, we analyzed the multiple selection bounded confidence model (1) and estimated probability bounds of opinion convergence and consensus, respectively. The results showed that both the multi limit events and the consensus events are non-trivial with certain parameter conditions. Simulations indicated that opinion fluctuation would happen with positive probability or occur a.s. asymptotically if the confidence bound is less enough. The complete mathematical analysis for opinion fluctuation of the model (1) will be our future work.

ACKNOWLEDGEMENT

This work was partially supported by the Program of Science and Technology of Sichuan Province of China under grant 2021ZYD0012, 2022NSFSC0532, the MOE Project of Humanities and Social Sciences under grant 21YJA630122.

(Received August 3, 2022)

References

[1] C. Altafini: Dynamics of opinion forming in structurally balanced social networks. Plos One *7* (2012), 6, e38135. DOI:10.1371/journal.pone.0038135

[2] V. D. Blondel, J. M. Hendrickx, and J. N. Tsitsiklis: On Krause's multi-agent consensus model with state-dependent connectivity. In: IEEE Conference on Decision and Control 2009.

[3] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah: Randomized gossip algorithms. IEEE Trans. Inform. Theory *52* (2006), 6, 2508–2530. DOI:10.1109/TIT.2006.874516

[4] E. David and J. Kleinberg: Networks, Crowds, and Markets: Reasoning About a Highly Connected World. Cambridge University Press, New York 2010.

[5] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch: Mixing beliefs among interacting agents. Adv. Compl. Syst. *3* (2000), 87–98. DOI:10.1023/A:1009927321376

[6] M. H. Degroot: Reaching a consensus. J. Amer. Statist. Assoc. *69* (1974), 345, 118–121. DOI:10.1080/01621459.1974.10480137

[7] R. Durrett: Probability Theory: Theory and Examples (Fourth edition). Cambridge University Press, New York 2010.

[8] Y. Hatano and M. Mesbahi: Agreement over random networks. IEEE Trans. Automat. Control *50* (2005), 11, 1867–1872. DOI:10.1109/TAC.2005.858670

[9] R. Hegselmann and U. Krause: Opinion dynamics and bounded confidence models, analysis and simulation. J. Artific. Soc. Social Simul. (JASSS) *5* (2002), 3.

[10] J. Liu, X. Chen, T. Basar, and M. A. Belabbas: Exponential convergence of the discrete-time altafini model. Preprint, arXiv [math.OC]:1512.0715, 2015.

[11] S. A. Marvel, J. Kleinberg, R. D. Kleinberg, and S. H. Strogatz: Continuous-time model of structural balance. Proc. National Academy of Sciences *108* (2011), 5, 1751–1752. DOI:10.1073/pnas.1018901108

[12] Y. Peng, Y. Zhao, and J. Hu: On the role of community structure in evolution of opinion formation: A new bounded confidence opinion dynamics. Inform. Sci. *621* (2023), 672–690. DOI:10.1016/j.ins.2022.11.101

[13] E. Qualman: Socialnomics: How Social Media Transforms the Way We Live and Do Business. Wiley, New York 2009.

[14] J. Zhang, Y. Hong, and X. Hu: Multi-agent opinion dynamics of bounded confidence with non-local aggregative interaction. SIAM J. Control Optim. *55* (2017), 4, 2543–2573. DOI:10.1137/16M1083268

[15] Y. Zhao, M. Xu, Y. Dong, and Y. Peng: Fuzzy inference based Hegselmann–Krause opinion dynamics for group decision-making under ambiguity. Inform. Process. Management *58* (2021), 5, 102671. DOI:10.1016/j.ipm.2021.102671

Jiangbo Zhang, School of Science, The Southwest Petroleum University, Chengdu, 610500, Sichuan P. R. China.
e-mail: jbzhang@amss.ac.cn

Yiyi Zhao, School of Business Administration, Faculty of Business Administration, Southwestern University of Finance and Economics, Chengdu,611130, Sichuan. P. R. China.
e-mail: zhaoyy@swufe.edu.cn