KYBERNETIKA — VOLUME *59* (2023), NUMBER 1, PAGES 110–129

ON FIXED FIGURE PROBLEMS IN FUZZY METRIC SPACES

Dhananjay Gopal, Juan Martínez-Moreno and Nihal Özgür

Fixed circle problems belong to a realm of problems in metric fixed point theory. Specifically, it is a problem of finding self mappings which remain invariant at each point of the circle in the space. Recently this problem is well studied in various metric spaces. Our present work is in the domain of the extension of this line of research in the context of fuzzy metric spaces. For our purpose, we first define the notions of a fixed circle and of a fixed Cassini curve then determine suitable conditions which ensure the existence and uniqueness of a fixed circle (resp. a Cassini curve) for the self operators. Moreover, we present a result which prescribed that the fixed point set of fuzzy quasi-nonexpansive mapping is always closed. Our results are supported by examples.

Keywords: fixed circle, Archimedean t-norm, M_h-triangular fuzzy metric

Classification: 54A40, 54E35

1. INTRODUCTION AND PRELIMINARIES

Geometric and topological properties of the fixed point set have been extensively studied for various aspects in the fixed point theory. For example, in [7], it was proved that the fixed point set of a quasi-nonexpansive self map of a metric space is always closed (see Lemma 1.1 on page 984). For more details, see [7, 26] and the references therein. Recently, geometric aspects of the fixed point set of a self-operator have been considered with various forms such as the fixed-circle, fixed-disc and fixed-ellipse problems. The most general form of these problems is the "fixed figure problem". Briefly, if the fixed point set $Fix(\mathcal{T}) = \{u \in \mathcal{X} : \mathcal{T}u = u\}$ of a self map $\mathcal{T}$ contains some geometric figure (a circle, an ellipse or a Cassini curve and so on) then this figure is called the fixed figure of $\mathcal{T}$ (a fixed circle, a fixed ellipse and so on) (see [23] and the references therein).

In [21, 22], Özgür and Taş examined the fixed circle issue in metric space. This topic generated much interest recently to fixed point community. One of the motivation for this interest is the profound applications of these results in neural network in terms of activation functions (see for instances [25, 31]).

Considering various advantages of generalized metric over regular metric, the fixed circle problems have been studied in various metric spaces including S-metric [20], rectangular metric [3], partial metric [30] and quasi-metric [4].

DOI: 10.14736/kyb-2023-1-0110

Fuzzy metric spaces [8, 17] is one of the notable generalization of regular metric concept. Indeed, fixed point theory of fuzzy metric spaces are more diverse than the regular metric fixed point theory. This is due to the pliability exhibited in the concept of fuzzy metric. But at the same time due to complexion involved in the nature of fuzzy metric one might need to use or develop new fuzzy mathematical tools to establish new results in this field (see for example [2, 13, 14, 15, 16, 29]). This pursuance of hunt make the new fuzzy results worthwhile.

In this article, we discuss various aspects of fixed figure problems in fuzzy metric setting. The organization of this article is as follows: In section 2, the concepts of M_h-triangular fuzzy metric and Archimedean t-norm are introduced and utilized to establish new fixed-circle theorems. It extends some existing results [1]. In section 3, we define fuzzy Cassini curve and prove corresponding fixed Cassini curve theorem. In section 4, we introduce a new class of self mappings defined on a fuzzy metric space and give a general theorem for the fixed point set of a self mapping that belongs in this new class. Finally, in Section 5, we conclude the article and jot down ideas for further research in this direction by putting forward some open questions.

Throughout this paper we consider the case $Fix(\mathcal{T}) \neq \emptyset$.

Here we quote some basic concepts and results which will be needed for the development of the present topic.

Definition 1.1. (Schweizer and Sklar [27]) A triangular norm (t-norm in short) is a continuous function $\diamondsuit\colon [0,1]\times[0,1] \to [0,1]$ which satisfies the following conditions: for all $u, v, w, x \in [0,1]$;

(*i*) $1\diamondsuit u = u$,

(*ii*) $u\diamondsuit v = v\diamondsuit u$;

(*iii*) $u\diamondsuit(v\diamondsuit w) = (u\diamondsuit v)\diamondsuit w$

(*iv*) $u\diamondsuit v \leq w\diamondsuit x$, whenever $u \leq w$ and $v \leq x$,

(*v*) $\diamondsuit$ is continuous.

Some basic and notable examples of t-norms are:

(*i*) the minimum t-norm $\diamondsuit_m$ defined as $u\diamondsuit_m v = \min\{u, v\}$,

(*ii*) the Hamacher class of t-norm defined as $u\diamondsuit_\lambda v = 0$ if $u = v = \lambda = 0$ and $u\diamondsuit_\lambda v = \frac{uv}{\lambda+(1-\lambda)(u+v-uv)}$ otherwise,

(*iii*) the product t-norm $\diamondsuit_p$ defined as $u\diamondsuit_p v = uv$,

(*iv*) the Łukasiewicz t-norm $\diamondsuit_L$ defined as $u\diamondsuit_L v = \max\{u + v - 1, 0\}$,

for all $u, v \in [0,1]$.

The t-norm $\diamondsuit$ is called Archimedean ([1]) if for each $u, v \in [0,1]$, $u \diamondsuit v \geq u$ implies $v = 1$.

All the above t-norms except the minimum t-norm are Archimedean.

Definition 1.2. (George and Veeramani [8]) A fuzzy metric space (GV-fuzzy metric space, for short) is an ordered triple $(\mathcal{X}, M, \diamondsuit)$ such that $\mathcal{X}$ is a (nonempty) set, $\diamondsuit$ is a continuous t-norm and M is a fuzzy set on $\mathcal{X} \times \mathcal{X} \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in \mathcal{X}$ and $\tau, \lambda > 0$;

(GV1) $M(x, y, \tau) > 0$;

(GV2) $M(x, y, \tau) = 1$ if and only if $x = y$;

(GV3) $M(x, y, \tau) = M(y, x, \tau)$;

(GV4) $M(x, z, \tau + \lambda) \geq M(x, y, \tau) \diamondsuit M(y, z, \lambda)$;

(GV5) $M(x, y, \cdot) \colon (0, \infty) \to [0,1]$ is continuous.

Remark 1.3. Note that in this context, condition (GV2) of Definition 1.2 is equivalent to the following:

$M(x, x, \tau) = 1$ for all $x \in \mathcal{X}$ and $\tau > 0$, and $M(x, y, \tau) < 1$ for all $x \neq y$ and $\tau > 0$.

Let $(\mathcal{X}, d)$ be a metric space and define $u \diamondsuit v = uv$ for all $u, v \in [0,1]$. Let $\mathcal{M}_d$ be the function on $\mathcal{X} \times \mathcal{X} \times (0, \infty)$ defined by

$$\mathcal{M}_d(u, v, \tau) = \frac{\tau}{\tau + d\,(u, v)}. \tag{1}$$

Then $(\mathcal{X}, \mathcal{M}_d, \diamondsuit)$ is a fuzzy metric space [8]. $\mathcal{M}_d$ is called the standard fuzzy metric induced by d.

Let $\mathcal{X} = (0, \infty)$ and $u \diamondsuit v = uv$ for all $u, v \in [0,1]$. Two of the well-known fuzzy metric examples on $(0, \infty)$ are defined by

$$\mathcal{M}(u, v, \tau) = \frac{\min\{u, v\}}{\max\{u, v\}} \tag{2}$$

and

$$\mathcal{M}(u, v, \tau) = \frac{\min\{u, v\} + \tau}{\max\{u, v\} + \tau}, \tag{3}$$

for all $u, v \in \mathcal{X}$ and for all $\tau > 0$. These fuzzy metrics have various advantages in front of classical metrics in the evaluation of images filtering process (for more details see [11] and [12]). Moreover, it has been pointed out in [8] that there exists no metric on $\mathcal{X}$ satisfying $M(u, v, \tau) = \frac{\tau}{\tau + d(u,v)}$, where $M(u, v, \tau)$ is defined by (2).

The fuzzy metric space $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ is said to be *non-Archimedean* or *strong* if it satisfies the following condition:

$$\mathcal{M}(u, v, \tau) \geq \mathcal{M}(u, w, \tau) \diamondsuit M(w, v, \tau),$$

for each $u, v, w \in \mathcal{X}$ and $\tau > 0$.

Example 1.4. Take $\mathcal{X} = \mathbb{N}$ and define the fuzzy set $\mathcal{M}$ on $\mathcal{X} \times \mathcal{X}$ by $\mathcal{M}(u, v, \tau) = 1$ if $u = v$ and $\mathcal{M}(u, v, \tau) = \frac{1}{2}$ if $u \neq v$, then $(\mathcal{X}, \mathcal{M}, \diamondsuit_m)$ is a strong GV-fuzzy metric space.

Remark 1.5. It is interesting to note that the example of this type of fuzzy metric is very useful in showing that the fuzzy distances are not necessarily equivalent to classical distances (see Example 4.5 given in page 15).

The readers are referred to [8, 1] for the definitions of a complete fuzzy metric space and an upper semi-continuous function.

Theorem 1.6. [1] Let $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ be a complete fuzzy metric space with $\diamondsuit$ is continuous and Archimedean, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a self mapping, $\varphi : \mathcal{X} \to [0, 1]$ be such that φ is non trivial (i. e. $u \in \mathcal{X}$ such that $\varphi(u) \neq 0$) and upper semi-continuous function. Assume that

$$\mathcal{M}(u, \mathcal{T}u, \tau) \diamondsuit \varphi(\mathcal{T}u) \geq \varphi(u) \tag{4}$$

for all $u \in \mathcal{X}$ and $\tau > 0$. Then $\mathcal{T}$ has a fixed point in $\mathcal{X}$.

2. THE FIXED-CIRCLE PROBLEM ON FUZZY METRIC SPACES

We begin with following:

Definition 2.1. Let $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ be a fuzzy metric space and let $u_0 \in \mathcal{X}$, $0 < r < 1$, $\tau > 0$. We define the circle of center u_0, radius r and parameter τ as

$$C_{r,\tau}(u_0) := \{u \in \mathcal{X} : \mathcal{M}(u_0, u, \tau) = 1 - r\}.$$

For a self mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$, if $\mathcal{T}u = u$ for all $u \in C_{r,\tau}(u_0)$ or $C_{r,\tau}(u_0) \subset Fix(\mathcal{T})$ then, we call the circle $C_{r,\tau}(u_0)$ as a fixed circle of $\mathcal{T}$.

Example 2.2. Let $\mathcal{X} = (0, \infty)$, $u \diamondsuit v = uv$ for all $u, v \in [0, 1]$ and let $\mathcal{M}(u, v, \tau)$ be as in (2). Consider the mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{T}u = \begin{cases} 1 - u & \text{if } u \in (0, 1) \\ \frac{u^2}{2} & \text{if } u \in [1, \frac{4}{3}) \cup (\frac{4}{3}, 2] \\ \frac{4}{3} & \text{if } u = \frac{4}{3} \\ 6 & \text{if } u \in (2, \infty). \end{cases}$$

Then the mapping $\mathcal{T}$ fixes the circle $C_{\frac{1}{2},\tau}(1) = \left\{\frac{1}{2}, 2\right\}$. However, $\mathcal{T}$ does not fix the circle $C_{\frac{1}{4},\tau}(1) = \left\{\frac{3}{4}, \frac{4}{3}\right\}$. Although the point $u = \frac{4}{3}$ is a fixed point of the mapping $\mathcal{T}$. Note that $Fix(\mathcal{T}) = \left\{\frac{1}{2}, \frac{4}{3}, 2, 6\right\}$.

Remark 2.3.
1. Let $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ be a fuzzy metric space and $C_{r,\tau}(u_0)$ be a circle on $\mathcal{X}$. Define a self mapping as

$$\mathcal{T}u = \begin{cases} u & \text{if } u \in C_{r,\tau}(u_0) \\ \\ u_0, & \text{otherwise.} \end{cases} \tag{5}$$

Clearly, $\mathcal{T}$ fixes the circle $C_{r,\tau}(u_0)$. Considering the identity map $I_{\mathcal{X}}$, defined by $I_{\mathcal{X}}u = u$ for each $u \in \mathcal{X}$, together with the map $\mathcal{T}$ defined in (5), we deduce that there exist at least two mappings that fix a given circle.

2. There are self mappings having fixed points but not fixed circles. For an example, let us consider the set of complex numbers $\mathbb{C}$ with the usual metric d and let $\mathcal{M}_d(u, v, \tau)$ be the fuzzy metric defined in (1). Define the self mapping $\mathcal{T}$ by

$$\mathcal{T}u = \begin{cases} u+1 & \text{if } |u| < 1 \\ \\ \frac{1}{u} & \text{if } |u| \geq 1 \end{cases} . \tag{6}$$

$\mathcal{T}$ has two fixed points -1 and 1. Although these fixed points lie on each of the circles $C_{r,\tau}(ai) = \left\{ u \in \mathbb{C} : |u - ai| = \frac{r\tau}{1-r} \right\}$, where $a = \pm\frac{\sqrt{r^2\tau^2-(1-r)^2}}{1-r}$ and $\tau \geq \frac{1-r}{r}$, the other points of such a circle are not fixed points of $\mathcal{T}$. That is, $\mathcal{T}$ has no any fixed circle but has two fixed points. For $\tau = 1$ and $r = \frac{1}{2}$, we have the circle $C_{\frac{1}{2},1}(0) = \{u \in \mathbb{C} : |u| = 1\}$ and observe that $\mathcal{T}$ maps the circle $C_{\frac{1}{2},1}(0)$ onto itself.

Now, we want to determine some necessary conditions to ensure the fixed point set of a self mapping contains a circle $C_{r,\tau}(u_0)$ by the use of the given parameters u_0 and τ.

Theorem 2.4. Let $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ be a fuzzy metric space with $\diamondsuit$ as Archimedean, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a self mapping, and $u_0 \in \mathcal{X}$, $\tau > 0$. Define the map $\varphi_{\tau,u_0} : \mathcal{X} \to (0, 1]$ by

$$\varphi_{\tau,u_0}(u) = \mathcal{M}(u, u_0, \tau),$$

for all $u \in \mathcal{X}$. Assume that there exists a number r with $0 < r < 1$ such that

P1) $\mathcal{M}(u, \mathcal{T}u, \tau) \diamondsuit \varphi_{\tau,u_0}(\mathcal{T}u) \geq \varphi_{\tau,u_0}(u)$,

P2) $\mathcal{M}(\mathcal{T}u, u_0, \tau) \leq 1 - r$,

for each $u \in C_{r,\tau}(u_0)$. Then, the set $Fix(\mathcal{T})$ contains the circle $C_{r,\tau}(u_0)$, that is, $C_{r,\tau}(u_0)$ is a fixed circle of $\mathcal{T}$.

Proof. Let us choose a number r satisfying the condition (P2). Consider the circle $C_{r,\tau}(u_0)$ and let $u \in C_{r,\tau}(u_0)$ be an arbitrary point. Then by (P1), (P2) and the monotonicity of $\diamondsuit$, we can write

$$\begin{aligned} (1-r) \diamondsuit \mathcal{M}(\mathcal{T}u, u, \tau) &\geq \mathcal{M}(Tu, u_0, \tau) \diamondsuit \mathcal{M}(\mathcal{T}u, u, \tau) \\ &\geq \mathcal{M}(u, u_0, \tau) \text{ (by P1)} \\ &= 1 - r, \end{aligned}$$

i. e.

$$(1-r)\diamondsuit\mathcal{M}(\mathcal{T}u,u,\tau)\geq 1-r.$$

Since $\diamondsuit$ is Archimedean, therefore, we must have

$$\mathcal{M}(\mathcal{T}u,u,\tau)=1,$$

and this implies $\mathcal{T}u=u$ by (GV2). Hence, we obtain $\mathcal{T}u=u$ for each $u\in C_{r,\tau}(u_0)$. This shows that $C_{r,\tau}(u_0)\subset Fix(\mathcal{T})$, that is, the circle $C_{r,\tau}(u_0)$ is a fixed circle of the self map $\mathcal{T}$. □

Remark 2.5.
1. The converse statement of Theorem 2.4 is also true because of the condition (GV2) of Definition 1.2 and the Archimedean property of $\diamondsuit$.

2. In view of Example 2.2, Theorem 2.4 is stronger than Theorem 1.6 (in the sense that Theorem 2.4 can be used to produce self mappings which fixes a given circle). However, Theorem 2.4 is a special case of Theorem 1.6 for the cases where the circle $C_{r,\tau}(u_0)$ has only one element.

3. The condition $(P1)$ of Theorem 2.4 guarantee that $\mathcal{T}u$ is not in the interior of the circle $C_{r,\tau}(u_0)$ for each $u\in C_{r,\tau}(u_0)$, while the condition $(P2)$ guarantee that $\mathcal{T}u$ is not in the exterior of the circle $C_{r,\tau}(u_0)$ for each $u\in C_{r,\tau}(u_0)$. Combining these, we obtain $\mathcal{T}u\in C_{r,\tau}(u_0)$ for each $u\in C_{r,\tau}(u_0)$. We note that the circle $C_{r,\tau}(u_0)$ need not to be fixed even if $\mathcal{T}(C_{r,\tau}(u_0))=C_{r,\tau}(u_0)$ (see Remark 2.3 (2) or Example 2.8).

4. If the conditions $(P1)$ and $(P2)$ are satisfied by $\mathcal{T}$ for all $x\in\mathcal{X}$, then it is clear from the proof of Theorem 2.4 that $\mathcal{T}u=u$ for each $u\in\mathcal{X}$, that is, we have $\mathcal{T}=I_{\mathcal{X}}$, the identity map on $\mathcal{X}$.

Now, we give some illustrative examples.

Example 2.6. Let $\mathcal{X}=(0,\infty)$, $u\diamondsuit v=uv$ for all $u,v\in[0,1]$ and $\mathcal{M}(u,v,\tau)$ be as in (3). Consider the mapping $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ defined by

$$Tu=\begin{cases} u & \text{if } u\geq\frac{1}{4}\\ \\ \frac{1}{4} & \text{if } 0<u<\frac{1}{4}.\end{cases}$$

Let $u_0=1,\tau=1$. Then, $\mathcal{T}$ satisfies the conditions of Theorem 2.4 for $r=\frac{1}{8}$ and we have the fixed circle $C_{\frac{1}{8},1}(1)=\left\{\frac{3}{4},\frac{9}{7}\right\}$. Clearly, $Fix(\mathcal{T})=\left[\frac{1}{4},\infty\right)$, and $C_{\frac{1}{8},1}(1)\subset Fix(\mathcal{T})$.

Now, let $u_0=2,\tau=1$. Then, $\mathcal{T}$ satisfies the conditions of Theorem 2.4 for $r=\frac{1}{6}$ and we get another fixed circle $C_{\frac{1}{6},1}(2)=\left\{\frac{3}{2},\frac{13}{5}\right\}$ of $\mathcal{T}$.

On the other hand, it is easy to check that $\mathcal{T}$ also satisfies all hypotheses of Theorem 1.6 with the function $\varphi:(0,\infty)\to[0,1]$ defined by

$$\varphi(x)=\begin{cases} x & \text{if } 0<x<1\\ \\ \frac{1}{x} & \text{if } x\geq 1.\end{cases}$$

Evidently, Theorem 1.6 guarantees the existence of a fixed point whereas Theorem 2.4 characterizes the existence of fixed circles via the conditions $(P1)$ and $(P2)$.

Example 2.7. Let

$$\mathcal{X} = \left\{ \frac{3}{4}, 1, 2, 4 \right\}$$

and $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ is same as in Example 2.2. Consider the self mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{T}u = \begin{cases} \frac{3}{4} & \text{if } u \in \{\frac{3}{4}, 4\} \\ \\ 2u & \text{otherwise.} \end{cases}$$

Then, $\mathcal{T}$ fulfils all the requirements of Theorem 2.4 only for the circle $C_{\frac{1}{4},\tau}(1) = \left\{\frac{3}{4}\right\}$. Hence, $\mathcal{T}$ has a unique fixed circle with one element.

Example 2.8. Let

$$\mathcal{X} = \left\{ \frac{1}{2}, 1, 2, 3, 4 \right\},$$

and $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ is same as in Example 2.2. Consider the self mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{T}u = \begin{cases} 3 & \text{if } u \in \{1, 3, 4\} \\ \\ \frac{1}{u} & \text{if } u \in \{\frac{1}{2}, 2\}. \end{cases}$$

Consider the circle $C_{\frac{1}{2},\tau}(1) = \left\{\frac{1}{2}, 2\right\}$. We have $\mathcal{T}\left(C_{\frac{1}{2},\tau}(1)\right) = C_{\frac{1}{2},\tau}(1)$. But the points of the circle $C_{\frac{1}{2},\tau}(1)$ is not fixed by $\mathcal{T}$.

The following two examples illustrates the necessity of the conditions $(P1)$ and $(P2)$ in Theorem 2.4.

Example 2.9. Let

$$\mathcal{X} = \{1, 2, 3\},$$

and $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ is same as in Example 2.2. Consider the self mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{T}u = \begin{cases} u + 1 & \text{if } u \in \{1, 2\} \\ \\ 1 & \text{if } u = 3. \end{cases}$$

Let $u_0 = 1$ and $\tau > 0$. Then, $\mathcal{T}$ satisfies the condition (P1) of Theorem 2.4 for the circle $C_{\frac{2}{3},\tau}(1) = \{3\}$. Indeed, for $u = 3$, we have

$$\begin{aligned} \mathcal{M}(u, \mathcal{T}u, \tau) \diamondsuit \mathcal{M}(\mathcal{T}u, u_0, \tau) &= \mathcal{M}(3, 1, \tau) \diamondsuit \mathcal{M}(1, 1, \tau) = \frac{1}{3} \\ &\geq \mathcal{M}(u, u_0, \tau) = \mathcal{M}(3, 1, \tau) = \frac{1}{3}. \end{aligned}$$

But, $\mathcal{T}$ does not satisfy the condition (P2) since we have

$$\mathcal{M}(\mathcal{T}u, u_0, \tau) = \mathcal{M}(1, 1, \tau) = 1 > 1 - \frac{2}{3} = \frac{1}{3}.$$

Notice that $\mathcal{T}$ has no any fixed circle (resp. fixed point). This example shows the necessity of the condition (P2).

Example 2.10. Let $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ and the self mapping $\mathcal{T}$ be same as in Example 2.9. Choose $u_0 = 2$, $r = \frac{1}{3}$ and $\tau > 0$. Then $(P2)$ holds for the circle $C_{\frac{1}{3},\tau}(2) = \{3\}$. Indeed, for $u = 3$, we have

$$\mathcal{M}(\mathcal{T}u, u_0, \tau) = \mathcal{M}(1, 2, \tau) = \frac{1}{2} \leq 1 - \frac{1}{3} = \frac{2}{3}.$$

However, for $u = 3$, we have

$$\begin{aligned}\mathcal{M}(\mathcal{T}u, u_0, \tau)\diamondsuit\mathcal{M}(\mathcal{T}u, u, \tau) &= \mathcal{M}(1, 2, \tau)\diamondsuit\mathcal{M}(1, 3, \tau) = \frac{1}{2}.\frac{1}{3} = \frac{1}{6} \\ &< \mathcal{M}(u, u_0, \tau) = \mathcal{M}(3, 2, \tau) = \frac{2}{3},\end{aligned}$$

i. e. $(P1)$ is not satisfied by $\mathcal{T}$.

Example 2.9 and Example 2.10 indicate that Theorem 2.4 characterizes the existence of fixed circles by means of the conditions $(P1)$ and $(P2)$. On the other hand, Theorem 1.6 guarantees the existence of a fixed point, but not a fixed circle. The following example illustrates this fact.

Example 2.11. Let $\mathcal{X} = \left\{\frac{1}{2}, 1, 2, 3\right\}$, $u\diamondsuit v = uv$ for all $u, v \in [0, 1]$ and let $\mathcal{M}(u, v, \tau)$ be as in (2). Consider the mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{T}\left(\frac{1}{2}\right) = 2, \mathcal{T}(1) = \mathcal{T}(2) = \mathcal{T}(3) = 3,$$

and the function $\varphi : \mathcal{X} \to [0, 1]$ defined by

$$\varphi(u) = \frac{u}{4}.$$

It is easy to verify that $\mathcal{T}$ satisfies the condition (4) of Theorem 1.6 for all $u \in \mathcal{X}$ and $\tau > 0$. Clearly, $\mathcal{T}$ has unique fixed point i. e. $T3 = 3$. Theorem 1.6 is very weak in comparison of Theorem 2.4. Actually, Theorem 1.6 assure the existence of a fixed point but not fixed circles (for all circles) in the space satisfying condition of Theorem 1.6 whereas Theorem 2.4 do so for all circles satisfying conditions of Theorem 2.4. We can check that the above map $\mathcal{T}$ satisfy the condition (4) of Theorem 1.6 for all circles but the circle $C_{\frac{1}{2},\tau}(1) = \{2\}$ is not fixed by $\mathcal{T}$. However, $\mathcal{T}$ satisfies the conditions $(P1)$ and $(P2)$ for all possible circles $C_{\frac{1}{3},\tau}(2)$, $C_{\frac{2}{3},\tau}(1)$, $C_{\frac{5}{6},\tau}(\frac{1}{2})$ in the fuzzy metric space and these are fixed circles of $\mathcal{T}$. In fact, we have $C_{\frac{1}{3},\tau}(2) = C_{\frac{2}{3},\tau}(1) = C_{\frac{5}{6},\tau}(\frac{1}{2}) = \{3\}$.

The next example elucidate that Theorem 2.4 may not true when $\diamondsuit$ is the minimum t-norm.

Example 2.12. Take $\mathcal{X} = \{1, 2, 3, \ldots\}$ and define the fuzzy set $\mathcal{M}$ on $\mathcal{X}\times\mathcal{X}$ by $\mathcal{M}(u, v, \tau) = 1$ if $u = v$ and $\mathcal{M}(u, v, \tau) = \frac{1}{3}$ if $u \neq v$, then $(\mathcal{X}, \mathcal{M}, \diamondsuit_m)$ is a strong GV-fuzzy metric space. Let $\mathcal{T}$ be defined by $\mathcal{T}u = u + 1, \forall u \in \mathcal{X}$. Then the conditions $(P1)$ and $(P2)$ are satisfied for the circle $C_{\frac{2}{3},\tau}(1) = \{2, 3, \ldots\}$ but $\mathcal{T}$ does not fixes the circle. For any circle with center $u_0 \neq 1$, the condition $(P2)$ is not satisfied. Because, for the point $u = u_0 - 1$ we have $\mathcal{T}u = u_0$ and so $\mathcal{M}(Tu, u_0, \tau) = \mathcal{M}(u_0, u_0, \tau) = 1$. In fact, $\mathcal{T}$ is a fixed point free mapping.

The following example justifies superiority of Theorem 2.4 over Abbasi Theorem 3.1 [1].

Example 2.13. Let $\mathcal{X} = \{u_n = (1 - \frac{1}{n+1}) \colon n \in \mathbb{N}\} \cup \{1\}$ and $u \diamondsuit_{\frac{1}{2}} v = \frac{2uv}{1+u+v-uv}$ for all $u, v \in [0,1]$, consider a fuzzy set $\mathcal{M}$ on $\mathcal{X} \times \mathcal{X} \times (0, \infty)$ by:

$$\mathcal{M}(1,1,\tau) = 1 = \mathcal{M}(u_n, u_n, \tau),$$

for each $n \in \mathbb{N}$, and

$$\mathcal{M}(1, \frac{1}{2}, \tau) = \mathcal{M}(\frac{1}{2}, 1, \tau) = \frac{1}{2},$$

$$\mathcal{M}(1, u_n, \tau) = \mathcal{M}(u_n, 1, \tau) = \frac{1}{4},$$

for $n = 2, 3, \ldots$,

$$\mathcal{M}(u_1, u_3, \tau) = \mathcal{M}(u_3, u_1, \tau) = \ldots = \mathcal{M}(u_2, u_3, \tau) = \mathcal{M}(u_3, u_2, \tau) = \mathcal{M}(u_2, x_4, \tau) = \ldots = \frac{1}{4}.$$

Then $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ is a fuzzy metric space. Consider the circle $C_{\frac{1}{2},\tau}(\frac{1}{2}) = \{1\}$ in $\mathcal{X}$ and designate $\mathcal{T} \colon \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{T} u_n = u_{n+1} \text{ for all } n \in \mathbb{N} \text{ and } \mathcal{T} 1 = 1.$$

Then $\mathcal{T}$ fulfils all the requirements of Theorem 2.4, and consequently $\mathcal{T}$ fixes the circle $C_{\frac{1}{2},\tau}(\frac{1}{2})$.

However, $\mathcal{T}$ does not satisfy the Abbasi condition (3.1) (see [1] page 933). Suppose to the contrary that $\mathcal{T}$ satisfy the Abbasi condition (3.1), then

$$\limsup_{n\to\infty} \mathcal{M}(u_n, \mathcal{T} u_n, \tau) \diamondsuit \limsup_{n\to\infty} \varphi(\mathcal{T} u_n) \geq \limsup_{n\to\infty} \varphi(u_n)$$

i. e.

$$\frac{1}{4} \limsup_{n\to\infty} \varphi(u_{n+1}) \geq \varphi(u_n)$$

or

$$\frac{1}{2} \diamondsuit \limsup_{n\to\infty} \varphi(u_{n+1}) \geq \varphi(u_n).$$

Since φ is u.s.c. and hence $k = \limsup_{n\to\infty} \varphi(u_{n+1}) \leq \varphi(u)$, where $\lim_{n\to\infty} u_n = u$. So the above inequality reduces to

$$\frac{1}{4} \diamondsuit k \geq k$$

(or $\frac{1}{2} \diamondsuit k \geq k$) which is a contradiction to the Archimedean condition of $\diamondsuit$.

Obviously, we have $Fix(\mathcal{T}) = \mathcal{X}$ for the identity map $I_{\mathcal{X}}$. Now, we give a characterization of $I_{\mathcal{X}}$ in the setting of a fuzzy metric space. We begin with the following definition.

Definition 2.14. Let $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ be a fuzzy metric space. The fuzzy metric $\mathcal{M}$ is called $\mathcal{M}_h$*-triangular* if there exists some $h > 1$ such that

$$h\left(\frac{1}{\mathcal{M}(u,w,\tau)}-1\right) \leq \left(\frac{1}{\mathcal{M}(u,v,\tau)}-1\right)+h\left(\frac{1}{\mathcal{M}(v,w,\tau)}-1\right) \tag{7}$$

for all $u, v, w \in \mathcal{X}$ such that $v \neq w$ and $\tau > 0$.

Example 2.15. Let $\mathcal{X} = \{1, 2, 3\}$, $u \diamondsuit v = uv$ for all $u, v \in [0, 1]$ and $\mathcal{M}(u, v, \tau) = \frac{\min\{u,v\}}{\max\{u,v\}}$ for all $u, v \in \mathcal{X}$ and for all $\tau > 0$. Clearly, $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ is $\mathcal{M}_h$-triangular for $h = 4$.

Example 2.16. The fuzzy metric defined in (1) is the $\mathcal{M}_h$-triangular if d is a discrete metric.

Remark 2.17. The fuzzy metrics $\mathcal{M}$ given in Examples 2.15 and 2.16 are actually strong fuzzy metrics. So, the following question arises naturally:

Question 1: Does there exists a non-strong $\mathcal{M}_h$-triangular fuzzy metric?

Now, we are ready to state and prove our next result of this section.

Theorem 2.18. Let $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ be a $\mathcal{M}_h$-triangular fuzzy metric space and u_0 is a point of $\mathcal{X}$. Then, the self mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfies the condition

$$\left(\frac{1}{\mathcal{M}(\mathcal{T}u,u_0,\tau)}-1\right) \geq \left(\frac{1}{\mathcal{M}(u,u_0,\tau)}-1\right)+h\left(\frac{1}{\mathcal{M}(u,\mathcal{T}u,\tau)}-1\right) \tag{8}$$

for all $u \in \mathcal{X}, \tau > 0$ and some $h > 1$ if and only if $\mathcal{T} = I_{\mathcal{X}}$.

P r o o f. Let u be any arbitrary point of $\mathcal{X}$ and assume that $Tu \neq u$. By (GV2), we have $\mathcal{M}(u, \mathcal{T}u, \tau) \neq 1$. Then, using (8) and $\mathcal{M}_h$-triangularity of fuzzy metric $\mathcal{M}$, we get

$$\begin{aligned}\left(\frac{1}{\mathcal{M}(\mathcal{T}u,u_0,\tau)}-1\right) &\geq \left(\frac{1}{\mathcal{M}(u,u_0,\tau)}-1\right)+h\left(\frac{1}{\mathcal{M}(u,\mathcal{T}u,\tau)}-1\right)\\ &\geq h\left(\frac{1}{\mathcal{M}(u_0,Tu,\tau)}-1\right),\end{aligned}$$

a contradiction as $h > 1$. This means that $\mathcal{T}u = u$ for all $u \in \mathcal{X}$, and so $\mathcal{T} = I_{\mathcal{X}}$.

Clearly, the identity map $I_{\mathcal{X}}$ satisfies the condition (8). □

Remark 2.19. In Theorem 2.4, the fixed circle $C_{r,\tau}(u_0)$ is not necessarily unique. Example 2.6 illustrates this situation. However, there are cases where the fixed circle is unique (see Example 2.7). Hence, the investigation of some uniqueness conditions for fixed circles of self mappings appears a natural problem.

In the following theorem, we give a uniqueness condition for fixed circles.

Theorem 2.20. Let $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ be a fuzzy metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a self mapping such that $C_{r,\tau}(u_0) \subset Fix(\mathcal{T})$. If the contractive condition

$$\mathcal{M}(u, \mathcal{T}v, \tau) > \mathcal{M}(u, v, \tau) \tag{9}$$

is satisfied for all $u \in C_{r,\tau}(u_0)$ and $v \in \mathcal{X} \setminus C_{r,\tau}(u_0)$ then $C_{r,\tau}(u_0)$ is the unique fixed circle of $\mathcal{T}$, that is, we have $Fix(\mathcal{T}) = C_{r,\tau}(u_0)$.

Proof. Assume that there exists a point $v \in Fix(\mathcal{T}) \setminus C_{r,\tau}(u_0)$. For any $u \in C_{r,\tau}(u_0)$, there exists $\lambda > 0$ such that $0 < \mathcal{M}(u, v, \lambda) < 1$. Using the condition (9), we have

$$\mathcal{M}(u, v, \lambda) = \mathcal{M}(u, \mathcal{T}v, \lambda) > \mathcal{M}(u, v, \lambda),$$

a contradiction. This shows that $Fix(\mathcal{T}) = C_{r,\tau}(u_0)$, that is, $C_{r,\tau}(u_0)$ is the unique fixed circle. □

Example 2.21. Let

$$\mathcal{X} = \left\{ \frac{3}{4}, \frac{4}{5}, 1, 2 \right\}$$

and $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ is same as in Example 2.2. Consider the self mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{T}u = 2.$$

For $u = 2 \in C_{\frac{1}{2},\tau}(1)$ and $v \in \left\{\frac{3}{4}, \frac{4}{5}, 1\right\}$, we have

$$\mathcal{M}(u, \mathcal{T}v, \tau) = \mathcal{M}(2, 2, \tau) = 1 > \mathcal{M}(u, v, \tau) = \mathcal{M}(2, v, \tau) = \frac{v}{2}.$$

Then, the condition (9) is satisfied, and hence the circle $C_{\frac{1}{2},\tau}(1)$ is the unique fixed circle of $\mathcal{T}$.

3. THE FIXED-CASSINI CURVE PROBLEM ON FUZZY METRIC SPACES

Cassini curves have profound applications in various scientific disciplines such as: nuclear physics, biosciences and computational sciences. Besides modelling human red blood cells; population growth etc.

On the other hand fuzzy metric is a useful tool to describe imprecise information and process in terms of "degree of closedness" given by fuzzy metric $\mathcal{M}$. These facts compelled us to look in to the possibility of having fuzzy type Cassini Curves. In this section, we examine the same.

Let $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ be a fuzzy metric space and let $u_1, u_2 \in \mathcal{X}$, $0 < \gamma < 1, \tau > 0$. We call the set $C_{\gamma,\tau}(u_1, u_2)$ defined by

$$C_{\gamma,\tau}(u_1, u_2) := \{u \in \mathcal{X} : \mathcal{M}\left(u_1, u, \tau\right) \diamondsuit \mathcal{M}\left(u_2, u, \tau\right) = 1 - \gamma\}$$

as a Cassini curve on $\mathcal{X}$. Then, the set $C_\gamma(u_1, u_2)$ is called a fixed Cassini curve of the self mapping $\mathcal{T} : \mathcal{X} \longrightarrow \mathcal{X}$ if $\mathcal{T}u = u$ for all $u \in C_{\gamma,\tau}(u_1, u_2)$ or $C_{\gamma,\tau}(u_1, u_2) \subset Fix(\mathcal{T})$.

Theorem 3.1. Let $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ be a fuzzy metric space with $\diamondsuit$ as Archimedean, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a self mapping, and $u_1, u_2 \in \mathcal{X}$, $\tau > 0$. Define the map $\varphi_{\tau,u_1,u_2} : \mathcal{X} \to (0,1]$ by

$$\varphi_{\tau,u_1,u_2}(u) = \mathcal{M}(u, u_1, \tau) \diamondsuit \mathcal{M}(u, u_2, \tau), \tag{10}$$

for all $u \in \mathcal{X}$. Assume that there exists a number γ with $0 < \gamma < 1$ such that

C1) $\mathcal{M}(u, \mathcal{T}u, \tau) \diamondsuit \varphi_{\tau,u_1,u_2}(\mathcal{T}u) \geq \varphi_{\tau,u_1,u_2}(u)$ and

C2) $\varphi_{\tau,u_1,u_2}(\mathcal{T}u) \leq 1 - \gamma$,

for each $u \in C_{\gamma,\tau}(u_1, u_2)$. Then, the set $C_{\gamma,\tau}(u_1, u_2)$ is a fixed Cassini curve of $\mathcal{T}$.

Proof. Choose a number γ satisfying the condition (C2) and consider the Cassini curve $C_{\gamma,\tau}(u_1, u_2)$. Let $u \in C_{\gamma,\tau}(u_1, u_2)$ be an arbitrary point. Using (C1) and (10), we have

$$\mathcal{M}(u, \mathcal{T}u, \tau) \diamondsuit \left(\mathcal{M}(\mathcal{T}u, u_1, \tau) \diamondsuit \mathcal{M}(\mathcal{T}u, u_2, \tau) \right) \geq \mathcal{M}(u, u_1, \tau) \diamondsuit \mathcal{M}(u, u_2, \tau) = 1 - \gamma. \tag{11}$$

Then, by (C2), monotonicity of $\diamondsuit$ and (11), we obtain

$$(1-\gamma) \diamondsuit \mathcal{M}(u, \mathcal{T}u, \tau) \geq \left(\mathcal{M}(\mathcal{T}u, u_1, \tau) \diamondsuit \mathcal{M}(\mathcal{T}u, u_2, \tau) \right) \diamondsuit \mathcal{M}(u, \mathcal{T}u, \tau) \geq 1 - \gamma,$$

and so

$$(1-\gamma) \diamondsuit \mathcal{M}(u, \mathcal{T}u, \tau) \geq 1 - \gamma.$$

Since $\diamondsuit$ is Archimedean, we must have

$$\mathcal{M}(\mathcal{T}u, u, \tau) = 1 \implies \mathcal{T}u = u,$$

by (GV2). Therefore, we obtain $\mathcal{T}u = u$ for each $u \in C_{\gamma,\tau}(u_1, u_2)$. Consequently, the self map $\mathcal{T}$ fixes the Cassini curve $C_{\gamma,\tau}(u_1, u_2)$. □

Now, we give two examples in which (C1) is satisfied but not (C2) and vice-versa. Consequently, the Cassini curve is not fixed.

Example 3.2. Let $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ and the self mapping $\mathcal{T}$ are same as in Example 2.9. Then, $\mathcal{T}$ satisfies the condition (C2) of Theorem 3.1 but $\mathcal{T}$ does not satisfy the condition (C1) for the Cassini curve $C_{\frac{1}{2},\tau}(1,2) = \{1,2\}$. Notice that $\mathcal{T}$ has no any fixed point. This example shows the necessity of the condition (C1).

Example 3.3. Let $\mathcal{X} = \left\{ \frac{3}{4}, 1, \frac{9}{8} \right\}$, $u \diamondsuit v = uv$ for all $u, v \in [0,1]$ and consider the fuzzy metric defined in (2). Consider the self map $\mathcal{T}$ defined by $\mathcal{T}u = 1$ for each $u \in \mathcal{X}$ and the Cassini curve $C_{\frac{1}{2},\tau}(1, \frac{9}{8}) = \left\{ \frac{3}{4} \right\}$. Then, $\mathcal{T}$ satisfies the condition (C1) of Theorem 3.1 but does not satisfy the condition (C2) for the Cassini curve $C_{\frac{1}{2},\tau}(1, \frac{9}{8})$. Clearly, the Cassini curve $C_{\frac{1}{2},\tau}(1, \frac{9}{8})$ is not fixed by $\mathcal{T}$. This example shows the necessity of the condition (C2).

Remark 3.4. Example 3.2 and Example 3.3 indicate that Theorem 3.1 characterizes the existence of fixed Cassini curves by means of the conditions (C1) and (C2).

Example 3.5. Let $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ be the fuzzy metric space defined in Example 2.2. Define the mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{T}u = |u-1| + |u-2| + u - 1. \tag{12}$$

Then, for $\gamma = \frac{1}{2}$ and $u_1 = 1$, $u_2 = 2$, $\mathcal{T}$ satisfies the conditions $(C1)$ and $(C2)$ of Theorem 3.1 for the Cassini curve $C_{\frac{1}{2},\tau}(1,2) = [1,2]$. Clearly, we have $Fix(\mathcal{T}) = [1,2]$ and the set $C_{\frac{1}{2},\tau}(1,2)$ is a fixed Cassini curve of $\mathcal{T}$.

Remark 3.6. The uniqueness condition (9) given in Theorem 2.20 can also be used for a fixed Cassini curve, in general, for any geometric figure contained in the set $Fix(\mathcal{T})$.

4. FIXED POINT SETS OF FUZZY QUASI-NONEXPANSIVE MAPS

In [7] Chaoha et. al. established some interesting results concerning the topological properties of the fixed point sets of a quasi-nonexpansive mapping.

Precisely, we quote the following amazing results:

1. (Lemma 1.1 on page 984 [7]) The fixed point set of a quasi-nonexpansive self map of a metric space is always closed.
2. (Theorem 2.1 on page 985 [7]) Let A be a nonempty subset of a $CAT(0)$ space $(\mathcal{X}, d)$. Then there exists a continuous map $f : \mathcal{X} \to \mathcal{X}$ such that $F(f) = \overline{A}$.

Motivated from these results, we launch quasi-nonexpansive self mapping in fuzzy setting and establish a general result for the fixed point set of a fuzzy quasi-nonexpansive self map.

Recall that an open ball $B_{r,\tau}(u_0)$ with centre $u_0 \in \mathcal{X}$, radius r $(0 < r < 1)$ and parameter $\tau > 0$ is defined as

$$B_{r,\tau}(u_0) = \{u \in \mathcal{X} : \mathcal{M}(u_0, u, \tau) > 1 - r\},$$

and a closed ball $B_{r,\tau}[u_0]$ is defined as

$$B_{r,\tau}[u_0] = \{u \in \mathcal{X} : \mathcal{M}(u_0, u, \tau) \geq 1 - r\}.$$

The sequence $\{x_n\}$ is called convergent and converges to x if, for each $\varepsilon \in (0,1)$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$, for all $n \geq n_0$.

For more details one can see [8].

Definition 4.1. Let $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ be a fuzzy metric space. The mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is called fuzzy quasi-nonexpansive if

$$\mathcal{M}(\mathcal{T}u, p, \tau) \geq \mathcal{M}(u, p, \tau), \tag{13}$$

for each $u \in \mathcal{X}$ and $p \in Fix(\mathcal{T})$.

Theorem 4.2. Let $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ be a fuzzy metric space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a fuzzy quasi-nonexpansive self map. Then the fixed point set $Fix(\mathcal{T})$ of $\mathcal{T}$ is closed.

Proof. Assume that $\mathcal{T}$ is a fuzzy quasi-nonexpansive self map. Let (u_n) be a sequence in the set $Fix(\mathcal{T})$ converging to u. Then for $0 < r < 1$ and each $\tau > 0$ there exists $n_0 \in \mathbb{N}$ such that $u_n \in B_{r,\tau}(u_0)$ for all $n \geq n_0$. It follows that $\mathcal{M}(u_n, u, \tau) > 1 - r$ and hence $1 - \mathcal{M}(u_n, u, \tau) < r$. By the definition of a fuzzy quasi-nonexpansive self map, we have $\mathcal{M}(u, u_n, \tau) \leq \mathcal{M}(\mathcal{T}u, u_n, \tau)$ and using this, we find

$$1 - \mathcal{M}(\mathcal{T}u, u_n, \tau) \leq 1 - \mathcal{M}(u, u_n, \tau) < r$$

and so

$$1 - \mathcal{M}(\mathcal{T}u, u_n, \tau) < r.$$

This implies $\mathcal{M}(\mathcal{T}u, u_n, \tau) \to 1$ as $n \to \infty$. Then from (GV4), we can write

$$\mathcal{M}(u, \mathcal{T}u, \tau) \geq \mathcal{M}\left(u, u_n, \frac{\tau}{2}\right) \diamondsuit \mathcal{M}\left(u_n, \mathcal{T}u, \frac{\tau}{2}\right).$$

Letting $n \to \infty$, we get

$$\mathcal{M}(u, \mathcal{T}u, \tau) \geq 1$$

and so,

$$\mathcal{M}(u, \mathcal{T}u, \tau) = 1,$$

by (GV2). This last equality implies $\mathcal{T}u = u$. Consequently, $u \in Fix(\mathcal{T})$. □

Example 4.3. Take $\mathcal{X} = (0, \infty)$, $u \diamondsuit v = uv$ for all $u, v \in [0, 1]$ and consider the fuzzy metric defined in (3). Define the self map $\mathcal{T}$ as

$$\mathcal{T}u = \begin{cases} u & ; \quad u \geq u_0 \\ u_0 & ; \quad 0 < u < u_0. \end{cases} \tag{14}$$

Then, it is easy to check that $\mathcal{T}$ is a fuzzy quasi-nonexpansive self map. Clearly, we have $Fix(\mathcal{T}) = [u_0, \infty)$ and this is a closed set. This closed set can contain several geometric figures. For example, it is easy to see that the set $Fix(\mathcal{T}) = [u_0, \infty)$ contains all of the circles $C_{\frac{1}{2},\tau}(u_0 + t) = \left\{2u_0 + 2t + \tau, \frac{u_0+t-\tau}{2}\right\}$ (where t is chosen such that $t - \tau \geq u_0$) and Cassini curves $C_{\frac{1}{2},\tau}(u_0, u_0 + t) = [u_0, u_0 + t]$ (where t is chosen such that $t = u_0 + \tau$).

Example 4.4. Take $\mathcal{X} = \mathbb{R}$, $u \diamondsuit v = uv$ for all $u, v \in [0, 1]$ and consider the fuzzy metric defined in (1) where $d(u, v) = |u - v|$. Consider the mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{T}u = \begin{cases} u & ; \quad u \geq 0 \\ \alpha u & ; \quad u < 0, \end{cases} \tag{15}$$

where $0 < \alpha < 1$. Then, $\mathcal{T}$ is a fuzzy quasi-nonexpansive self map and we have $Fix(\mathcal{T}) = [0, \infty)$.

On the other hand, consider the mapping $\mathcal{S} : \mathcal{X} \to \mathcal{X}$ defined by

$$\mathcal{S}u = \begin{cases} 5u & ; \quad u > 0 \\ u & ; \quad u \leq 0. \end{cases}$$

For any $p \in Fix(\mathcal{S}) = (-\infty, 0]$ and $u \in (0, \infty)$, we have

$$\mathcal{M}(\mathcal{T}u, p, \tau) = \mathcal{M}(5u, p, \tau) = \frac{\tau}{\tau + |5u - p|} < \mathcal{M}(u, p, \tau) = \frac{\tau}{\tau + |u - p|}.$$

That is, $\mathcal{S}$ is not a fuzzy quasi-nonexpansive map. However, the fixed point set $Fix(\mathcal{S})$ is closed. This example shows that the converse statement of Theorem 4.2 does not hold in general.

Example 4.5. Let $\mathcal{X} = \{1, 2, 3\}$ and define the fuzzy set $\mathcal{M}$ on $\mathcal{X} \times \mathcal{X}$ by $\mathcal{M}(u, v, \tau) = 1$ if $u = v$ and $\mathcal{M}(u, v, \tau) = \frac{1}{2}$ if $u \neq v$, then $(\mathcal{X}, \mathcal{M}, \diamondsuit_m)$ is a strong GV-fuzzy metric space. Now consider the self mapping $\mathcal{T}$ as $\mathcal{T}1 = 1, \mathcal{T}2 = 3, \mathcal{T}3 = 3$. Then $\mathcal{T}$ is a fuzzy quasi-nonexpansive and we have $Fix(\mathcal{T}) = \{1, 3\}$ which is closed.

However, the same mapping $\mathcal{T}$ fails to be fuzzy quasi-nonexpansive (at $p = 1$ and $u = 2$) if we consider the standard fuzzy metric i. e. $M(u, v, \tau) = \frac{\tau}{\tau + d(u,v)}$.

Now, we investigate the case in which the fixed point set of a fuzzy quasi-nonexpansive map contains the closed ball $B_{\rho,\tau}[p]$ for a given fixed point p and a chosen parameter $\tau > 0$.

Theorem 4.6. Let $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ be a fuzzy metric space, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a fuzzy quasi-nonexpansive self map and $p \in Fix(\mathcal{T})$. Suppose that there exists a parameter $\tau > 0$ such that

D1) $\mathcal{M}(u, \mathcal{T}u, \tau) \diamondsuit \mathcal{M}(\mathcal{T}u, p, \tau) < (1 - \rho) \diamondsuit \rho$,

for each $u \in \mathcal{X} \setminus Fix(\mathcal{T})$ where the number ρ is defined as

$$\rho := \inf \{\mathcal{M}(\mathcal{T}u, u, \tau) : u \in \mathcal{X} \setminus Fix(\mathcal{T})\}. \tag{16}$$

Then we have

$$B_{\rho,\tau}[p] \subset Fix(\mathcal{T}).$$

Proof. Let $v \in B_{\rho,\tau}[p]$ be an arbitrary point. Conversely, assume that $v \notin Fix(\mathcal{T})$. Then by the definition of the number ρ, we can write

$$\rho \leq \mathcal{M}(\mathcal{T}v, v, \tau). \tag{17}$$

Since $\mathcal{T}$ is a fuzzy quasi-nonexpansive self map and $p \in Fix(\mathcal{T})$, we obtain

$$\mathcal{M}(\mathcal{T}v, p, \tau) \geq \mathcal{M}(v, p, \tau) \geq 1 - \rho. \tag{18}$$

This means that $\mathcal{T}v \in B_{\rho,\tau}[p]$. Then, by (17), (18) and the monotonicity of $\diamondsuit$, we get

$$\mathcal{M}(Tv, p, \tau) \diamondsuit \mathcal{M}(v, \mathcal{T}v, \tau) \geq \mathcal{M}(v, p, \tau) \diamondsuit \mathcal{M}(v, \mathcal{T}v, \tau) \geq (1 - \rho) \diamondsuit \rho,$$

a contradiction with the condition (D1). Hence, we have $\mathcal{T}v = v$ for each $v \in B_{\rho,\tau}[p]$. That is, the closed ball $B_{\rho,\tau}[p]$ is contained in the set $Fix(\mathcal{T})$. □

Example 4.7. Consider the fuzzy metric space used in Example 4.3 and the self map $\mathcal{T}$ defined in (14) for $u_0 = 2$. For a number $\tau > 0$ we have

$$\begin{aligned} \rho &= \inf\{\mathcal{M}(\mathcal{T}u, u, \tau) : 0 < u < 2\} \\ &= \inf\{\mathcal{M}(2, u, \tau) : 0 < u < 2\} \\ &= \inf\left\{\frac{u+\tau}{2+\tau} : 0 < u < 2\right\} \\ &= \frac{\tau}{2+\tau}. \end{aligned}$$

Let $p = 13$ and $\tau = 1$. Then, $\rho = \frac{1}{3}$ and for each $u \in (0,2)$, we have

$$\begin{aligned} \mathcal{M}(u, \mathcal{T}u, \tau)\diamondsuit\mathcal{M}(\mathcal{T}u, p, \tau) &= \mathcal{M}(u,2,1)\diamondsuit\mathcal{M}(2,13,1) \\ &= \frac{u+1}{2+1}\frac{2+1}{13+1} = \frac{u+1}{14} \\ &< \frac{2+1}{14} = \frac{3}{14} \\ &< \left(1-\frac{1}{3}\right)\frac{1}{3} = \frac{2}{9}. \end{aligned}$$

Hence, the condition (D1) is satisfied by $\mathcal{T}$. Clearly, the closed ball $B_{\frac{1}{3},1}[13] = \left[\frac{25}{3}, 20\right]$ is contained in the fixed point set $Fix(\mathcal{T}) = [2, \infty)$.

Remark 4.8. We note that the converse statement of Theorem 4.6 does not hold in general. For an instance, consider Example 4.7. If we choose $p = 3$ and $\tau = \frac{1}{2}$, then we have $\rho = \frac{1}{5}$ and it is easy to check that the condition (D1) is satisfied only for each $u \in \left(0, \frac{3}{50}\right)$. However, the closed ball $B_{\frac{1}{5},\frac{1}{2}}[3] = \left[\frac{23}{10}, \frac{31}{8}\right]$ is contained in the fixed point set $Fix(\mathcal{T}) = [2, \infty)$.

Now, we investigate the case in which the fixed point set of a fuzzy quasi-nonexpansive map $\mathcal{T}$ contains the set $C_{\gamma,\tau}[p_1, p_2] := \{u \in \mathcal{X} : \mathcal{M}(p_1, u, \tau)\diamondsuit\mathcal{M}(p_2, u, \tau) \geq 1 - \gamma\}$ for the given fixed points p_1, p_2 and a chosen parameter $\tau > 0$.

Theorem 4.9. Let $(\mathcal{X}, \mathcal{M}, \diamondsuit)$ be a fuzzy metric space, $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a fuzzy quasi-nonexpansive self map and $p_1, p_2 \in Fix(\mathcal{T})$. Suppose that there exists a parameter $\tau > 0$ such that

$$\mathcal{M}(u, \mathcal{T}u, \tau)\diamondsuit\left(\mathcal{M}(\mathcal{T}u, p_1, \tau)\diamondsuit\mathcal{M}(\mathcal{T}u, p_2, \tau)\right) < \rho\diamondsuit(1-\rho), \tag{19}$$

for each $u \in \mathcal{X} \setminus Fix(\mathcal{T})$ where the number ρ is defined as in (16). Then we have

$$C_{\rho,\tau}[p_1, p_2] \subset Fix(\mathcal{T}).$$

Proof. Let $\tau > 0$ be chosen such that the condition (19) is satisfied and $v \in C_{\rho,\tau}[p_1, p_2]$ be an arbitrary point. Conversely, assume that $v \notin Fix(\mathcal{T})$. Then by the definition of the number ρ, we can write

$$\rho \leq \mathcal{M}(\mathcal{T}v, v, \tau). \tag{20}$$

Since $\mathcal{T}$ is a fuzzy quasi-nonexpansive self map and $p_1, p_2 \in Fix(\mathcal{T})$, we obtain

$$\mathcal{M}(\mathcal{T}v, p_1, \tau) \geq \mathcal{M}(v, p_1, \tau),$$

and

$$\mathcal{M}(\mathcal{T}v, p_2, \tau) \geq \mathcal{M}(v, p_2, \tau).$$

By the monotonicity of $\diamondsuit$, we obtain

$$\mathcal{M}(\mathcal{T}v, p_1, \tau)\diamondsuit\mathcal{M}(\mathcal{T}v, p_2, \tau) \geq \mathcal{M}(v, p_1, \tau)\diamondsuit\mathcal{M}(v, p_2, \tau) \geq (1-\rho). \tag{21}$$

Then, by (20), (21) and the monotonicity of $\diamondsuit$, we get

$$\mathcal{M}(u, \mathcal{T}u, \tau)\diamondsuit\left(\mathcal{M}(\mathcal{T}v, p_1, \tau)\diamondsuit\mathcal{M}(\mathcal{T}v, p_2, \tau)\right) \geq \rho\diamondsuit(1-\rho),$$

a contradiction with the condition (19). Hence, we have $\mathcal{T}v = v$ for each $v \in C_{\rho,\tau}[p_1, p_2]$. That is, the set $C_{\rho,\tau}[p_1, p_2]$ is contained in the set $Fix(\mathcal{T})$. □

Example 4.10. Let the fuzzy metric space and the self map $\mathcal{T}$ are the same as in Example 4.7. Let us choose the fixed points $p_1 = 5$ and $p_2 = 6$. Then, for $\tau = 1$ and $u \in (0, 2)$, we have $\rho = \frac{1}{3}$ and

$$\begin{aligned}
\mathcal{M}(u, \mathcal{T}u, \tau)\diamondsuit\left(\mathcal{M}(\mathcal{T}u, p_1, \tau)\diamondsuit\mathcal{M}(\mathcal{T}u, p_2, \tau)\right) &= \mathcal{M}(u, 2, 1)\diamondsuit\left(\mathcal{M}(2, 5, 1)\diamondsuit\mathcal{M}(2, 6, 1)\right) \\
&= \frac{u+1}{2+1}\frac{2+1}{5+1}\frac{2+1}{6+1} = \frac{u+1}{14} \\
&< \frac{2+1}{14} = \frac{3}{14} \\
&< \left(1-\frac{1}{3}\right)\frac{1}{3} = \frac{2}{9}.
\end{aligned}$$

Hence, the condition (19) is satisfied by $\mathcal{T}$. Clearly, the set $C_{\frac{1}{3},1}[5, 6] = \left[\sqrt{28}-1, \sqrt{63}-1\right]$ is contained in the fixed point set $Fix(\mathcal{T}) = [2, \infty)$.

Remark 4.11. The converse statement of Theorem 4.9 does not hold in general. For an instance, consider Example 4.10. If we choose $\tau = 2$, then we have $\rho = \frac{1}{2}$ and it is easy to check that the condition (19) is satisfied only for each $u \in \left(0, \frac{3}{2}\right)$. However, the set $C_{\frac{1}{2},2}[5, 6] = \left[\sqrt{28}-2, \sqrt{112}-2\right]$ is contained in the fixed point set $Fix(\mathcal{T}) = [2, \infty)$.

5. CONCLUSION AND FUTURE SCOPE

The fuzzy metric fixed point theory have various advantages over the regular metric fixed point theory. This is due to the pliancy which the fuzzy concepts inherently possess. Even than it is not easy to translate the classical metric contractions and corresponding fixed point theorems in fuzzy setting. Such issues are discussed in [9, 10, 13, 29]. Also, it is well known that Caristi's [6] fixed point theorem is considered as one of the most beautiful extension of Banach contraction theorem which also characterizes metric completeness. The Caristi's fixed point theorem is extended by Abbasi et al. in the setting of fuzzy metric spaces. Most recently, an interesting idea of fixed circle was introduced by Özgür and Taş [21, 22] which also have applications in the area of neural network in terms of activation functions. Pursuing this approach of research, we introduce the notion of fixed circle in fuzzy setting and then utilize the ideas of Archimedean t-norm and M_h-triangular fuzzy metric to obtain fixed circle theorems. Moreover, we present a result which prescribed that the fixed point set of fuzzy quasi-nonexpansive mapping is always closed. Our results could be considered as an extension of fixed circle theory and quasi-nonexpansive mapping in the setting of fuzzy metric spaces.

It is known that theoretical fixed point results are important for the study of artificial neural networks. Some fixed point theorems (for instance, Brouwer's fixed point theorem and Banach fixed point theorem) have been intensively used (for example, see [19, 32]). On the other hand, activation functions play a significant role to design a neural network. In [19], globally Lipschitzian activation functions was used in the study of global exponential stability of delayed cellular neural networks. In [32], the following activation function was used in the numerical example:

$$\mathcal{T}u = \frac{1}{2}\left(|u+1| - |u-1|\right).$$

Let $X = \mathbb{R}$, $a * b = ab$ for all $a, b \in [0,1]$ and consider the fuzzy metric defined in (1) with the usual metric $d(x,y) = |x-y|$. Then, it is easy to see that $\mathcal{T}$ is a fuzzy quasi-nonexpansive self map and we have $Fix(\mathcal{T}) = [-1,1]$. In addition, we note that the fuzzy quasi-nonexpansive self map $\mathcal{T}$, defined in (15), is one of the most popular activation functions used in the neural networks. This function known as the Parametrized ReLU function. For the case $\alpha = 0.01$, the corresponding function $\mathcal{T}$ is known as the Leaky ReLU function (see [5] and [28] for more details about the frequently used activation functions). These examples show the effectiveness of the obtained results.

Before closing the article, we pose the following question:

Question 2: Is it possible to generalize Theorem 2.1 of [7] in frame work of fuzzy metric?

ACKNOWLEDGEMENT

The authors would like to thank Editor-in-Chief/Area Editors and Referees for their insightful comments and suggestions, which were very much useful to improve the paper significantly.

(Received December 4, 2021)

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Dhananjay Gopal, Department of Mathematics Guru Ghasidas Vishwavidyalaya, Koni, Bilaspur 495-009, Chhattisgarh, India.
e-mail: gopaldhananjay@yahoo.in

Juan Martínez-Moreno, Department of Mathematics, University of Jaén, Campus las Lagunillas s/n, 23071, Jaén, Spain.
e-mail: jmmoreno@ujaen.es

Nihal Özgür, Izmir Democracy University, Department of Mathematics, 35140, Karabağlar, Izmir, Türkiye.
e-mail: nihal.ozgur@idu.edu.tr