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L-FUZZY IDEAL DEGREES IN EFFECT ALGEBRAS

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In this paper, considering L being a completely distributive lattice, we first introduce the concept of L-fuzzy ideal degrees in an effect algebra E, in symbol $\mathfrak{D}_{ei}$. Further, we characterize L-fuzzy ideal degrees by cut sets. Then it is shown that an L-fuzzy subset A in E is an L-fuzzy ideal if and only if $\mathfrak{D}_{ei}(A) = \top$, which can be seen as a generalization of fuzzy ideals. Later, we discuss the relations between L-fuzzy ideals and cut sets (L_β-nested sets and L_α-nested sets). Finally, we obtain that the L-fuzzy ideal degree is an (L, L)-fuzzy convexity. The morphism between two effect algebras is an (L, L)-fuzzy convexity-preserving mapping.

Keywords: effect algebra, L-fuzzy ideal degree, cut set, (L, L)-fuzzy convexity

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1. INTRODUCTION

In 1994, Foulis and Bennett [5] introduced effect algebras to model unsharp quantum logics. We know that the ideals of effect algebras (pseudo-effect algebras) have attracted a lot of attention [13, 39, 42]. Since Zadeh introduced the concept of fuzzy sets, many branches of mathematics were discussed in fuzzy cases [17, 23, 34, 35]. In particular, Liu and Wang [11] proposed the concept of fuzzy ideals for effect algebras in the unit interval $[0, 1]$. Later, Liu [10] introduced and investigated fuzzy ideals and fuzzy filters in pseudo-effect algebras. In order to better study fuzzy sets, cut sets were introduced, which can be seen as a bridge between fuzzy sets and classic sets. The reader is referred to [9, 37] for more information of cut sets.

Many branches of mathematics have the concept of convexities [31], such as vector spaces, metric spaces, lattices, graphs, matroids and so on. At present, for the convex theory, the research has formed a system, as follows: Rosa [24] first proposed the concept of fuzzy convexities, which are called L-convex structures nowadays [2, 22, 25, 30, 36, 40, 45]. Afterwards, Shi and Xiu [28] gave a new approach to fuzzification of convexity and proposed the concept of M-fuzzifying convex structures [19, 20, 33]. Later, Shi and Xiu [29] further introduced the definition of (L, M)-fuzzy convex structures, which provided a more general framework of fuzzy convex structures [21, 43, 44].

Groups, rings and fields are important parts of algebra. Williams, Latha and Chandrasekeran discussed the fuzzification of bi-Γ-ideals in Γ-semigroups and studied some properties [38]. Öztürk, Jun and Yazarli introduced a kind of fuzzy Γ-ring and discussed

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some properties [18]. Malik and Mordeson [14] introduced the concepts of the fuzzy weak direct sum and the fuzzy complete direct sum of fuzzy subrings of commutative rings. Later, Mehmood, Shi and Hayat [16] introduced a new approach to the fuzzification of rings. Further, Mehmood and Shi [15] discussed the M-hazy vector spaces over M-hazy field. Recently, Shi and Xin [27] gave the concept of L-fuzzy subgroup degrees and L-fuzzy normal subgroup degrees, which generalized the notion of degrees to which a fuzzy subset is a fuzzy subgroup to L-fuzzy setting. Later, Li and Shi [9], Wen et al. [37] characterized L-fuzzy convex structures by L-convex fuzzy sublattice degrees and L-convex degrees on vector spaces, respectively. The L-fuzzy convexity is also called the (L, L)-fuzzy convex structure.

In this paper, considering L being a completely distributive lattice, we first introduce the definition of L-fuzzy ideal degrees and will characterize (L, L)-fuzzy convexities by L-fuzzy ideal degrees on an effect algebra. If the L-fuzzy ideal degree of an L-fuzzy subset equals to the maximum element in a lattice, then the L-fuzzy subset is an L-fuzzy ideal, which can be seen as a generalization of the fuzzy ideal on effect algebras. We further characterize L-fuzzy ideal degrees by four types of cut sets. We also discuss the relations between L-fuzzy ideals and their cut sets (L_β-nested sets and L_α-nested sets). Finally, we obtain that the L-fuzzy ideal degree is an (L, L)-fuzzy convex structure. The morphism between two effect algebras is an (L, L)-fuzzy convexity-preserving mapping.

2. PRELIMINARIES

2.1. Effect algebras

Definition 2.1. (Foulis and Bennett [5]) An effect algebra is a partial algebra $\big(E, +, 0, 1\big)$, where $0, 1$ are two different constants and $+$ is a partial binary operation satisfying the following:

(E1) If $x + y$ is defined, then $y + x$ is also defined, and $x + y = y + x$;

(E2) $x + y$ and $\big(x + y\big) + z$ are defined if and only if $y + z$ and $x + \big(y + z\big)$ are defined, and $\big(x + y\big) + z = x + \big(y + z\big)$;

(E3) For any $x \in E$, there exists a unique $y \in E$ such that $x+y$ is defined and $x+y = 1$;

(E4) If $x + 1$ is defined, then $x = 0$.

We often denote the effect algebra $\big(E, +, 0, 1\big)$ briefly by E. For any $x \in E$, we denote the unique y in condition (**E3**) by x'. The operation $+$ of an effect algebra $\big(E, +, 0, 1\big)$ can induce a partial order $\leq$ as follows: $x \leq y$ if and only if there exists $z \in E$ such that $x + z$ is defined and $x + z = y$. If $x + y$ is defined, then it is denoted by $x \perp y$.

In order to better understand effect algebras, we give the following examples, which are the most important effect algebras.

Example 2.2. (1) Let $\mathcal{H}$ be a complete Hilbert space and $B\big(\mathcal{H}\big)$ the set of all bounded linear operators on $\mathcal{H}$, $E\big(\mathcal{H}\big) = \big\{\, A \mid A \in B\big(\mathcal{H}\big), 0 \leq A \leq I \,\big\}$. For $A, B \in E\big(\mathcal{H}\big)$, if we define

$$A \perp B \Longleftrightarrow A + B \leq I,$$

then $\big(E(\mathcal{H}), +, 0, I\big)$ is an effect algebra.

(2) Let $E = [0,1]$. For any $x, y \in [0,1]$, $x \perp y$ if and only if $x+y \leq 1$, then $\big(E, +, 0, 1\big)$ is an effect algebra.

Definition 2.3. (Dvurečenskij and Pulmannová [3]) Let E and F be two effect algebras. A mapping $f : E \longrightarrow F$ is called a morphism provided that

(M1) $f(1_E) = 1_F$;

(M2) If $x, y \in E$ and $x \perp y$, then $f(x) \perp f(y)$ and $f(x) + f(y) = f(x+y)$.

Lemma 2.4. (Dvurečenskij and Pulmannová [3]) Let $f : E \longrightarrow F$ be a morphism between two effect algebras. Then

(1) f is order-preserving, i. e., $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in E$;

(2) $f(x') = f(x)'$ for all $x \in E$.

Definition 2.5. (Dvurečenskij and Pulmannová [3]) Let E and F be two effect algebras. A morphism $f : E \longrightarrow F$ is called a monomorphism provided that $f(x) \leq f(y)$ implies $x \leq y$ for all $x, y \in E$.

Definition 2.6. (Dvurečenskij and Pulmannová [3]) Let E be an effect algebra. A nonempty subset I of E is said to be an ideal provided that

(I1) If $x \in I$ and $y \in E$ with $y \leq x$, then $y \in I$;

(I2) If $x, y \in I$ and $x \perp y$, then $x + y \in I$.

2.2. Cut sets and (L, L)-fuzzy convexities

A partially ordered set $(L, \leq)$ [1] is said to be a lattice if any two elements λ and μ in L have a smallest upper bound, denoted by $\lambda \vee \mu$, as well as a greatest lower bound, denoted by $\lambda \wedge \mu$. Let $(L, \leq)$ be a partially ordered set and $\Lambda \subseteq L$ be a nonempty subset. If for any $\lambda, \mu \in \Lambda$, there always exists $\theta \in \Lambda$ such that $\lambda \leq \theta$ and $\mu \leq \theta$, then Λ is called upward directed.

Let L be a lattice. If for any $B \subseteq L$, $\bigvee B$ and $\bigwedge B$ exist, then L is called a complete lattice. An element λ in a complete lattice L is said to be a prime element if $\mu \wedge \theta \leq \lambda$ implies $\mu \leq \lambda$ or $\theta \leq \lambda$. An element λ is said to be co-prime if $\lambda \leq \mu \vee \theta$ implies $\lambda \leq \mu$ or $\lambda \leq \theta$ [6]. Every complete lattice is always a bounded lattice such that the unit is the top element and the zero is the bottom element. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $J(L)$.

The binary relation $\prec$ in a complete lattice L is defined as follows: for $\lambda, \mu \in L$, $\lambda \prec \mu$ if and only if for any subset $A \subseteq L$, such that $\mu \leq \bigvee A$ implies $\lambda \leq \theta$ for some $\theta \in A$ [4]. The set $\big\{ \lambda \,|\, \lambda \prec \mu \big\}$ is said to be the greatest minimal family of μ, denoted by $\beta(\mu)$ [32]. Moreover, for any $\mu \in L$, we define $\alpha(\mu) = \big\{ \lambda \in L \,|\, \lambda \prec^{op} \mu \big\}$. A complete lattice L is a completely distributive lattice if and only if $\mu = \vee\beta(\mu) = \wedge\alpha(\mu)$ for all $\mu \in L$ [32]. In a completely distributive lattice L, α is an $\wedge$-$\cup$ mapping and β is a union-preserving mapping. There also exists an implication operator $\rightarrow: L \times L \longrightarrow L$ as the right adjoint for the meet operator $\wedge$, which is defined by

$$\lambda \rightarrow \mu = \bigvee \Big\{ \theta \in L \,|\, \lambda \wedge \theta \leq \mu \Big\},$$

for all $\lambda, \mu \in L$.

In this paper, if not otherwise specified, we always assume that L is a completely distributive lattice, the smallest element and the largest element in L are denoted by $\perp$ and $\top$, respectively.

Lemma 2.7. (Höhle and Šostak [8]) Let L be a completely distributive lattice and the operation $\rightarrow$ be the implication operator corresponding to $\wedge$. For any $\lambda, \mu, \theta \in L$ and $\{\lambda_i\}_{i \in I} \subseteq L$, then the following statements hold:

(1) $\top \rightarrow \lambda = \lambda$;

(2) $\lambda \leq \theta \rightarrow \mu \Longleftrightarrow \lambda \wedge \theta \leq \mu$;

(3) $\lambda \rightarrow \mu = \top \Longleftrightarrow \lambda \leq \mu$;

(4) $\lambda \rightarrow \big(\bigwedge_{i \in I} \lambda_i\big) = \bigwedge_{i \in I} \big(\lambda \rightarrow \lambda_i\big)$, hence $\lambda \rightarrow \mu \leq \lambda \rightarrow \theta$ whenever $\mu \leq \theta$;

(5) $\big(\bigvee_{i \in I} \lambda_i\big) \rightarrow \mu = \bigwedge_{i \in I} \big(\lambda_i \rightarrow \mu\big)$, hence $\lambda \rightarrow \mu \leq \theta \rightarrow \mu$ whenever $\theta \leq \lambda$;

(6) $(\lambda \rightarrow \mu) \wedge (\mu \rightarrow \theta) \leq \lambda \rightarrow \theta$.

Lemma 2.8. (Li and Shi [9], Wen et al. [37]) Let L be a completely distributive lattice and $\lambda, \mu \in L$. Then the following statements are equivalent:

(1) $\lambda \leq \mu$;

(2) for any $\delta \in J(L)$, $\delta \leq \lambda$ implies $\delta \leq \mu$;

(3) for any $\delta \in P(L)$, $\lambda \not\leq \delta$ implies $\mu \not\leq \delta$;

(4) for any $\delta \in \beta(\top)$, $\delta \in \beta(\lambda)$ implies $\delta \in \beta(\mu)$;

(5) for any $\delta \in \alpha(\perp)$, $\delta \notin \alpha(\lambda)$ implies $\delta \notin \alpha(\mu)$.

In what follows, we will recall some famous examples of t-norms on interval $[0, 1]$.

Example 2.9. (1) The minimum t-norm $x * y = x \wedge y$. The corresponding implication is defined by

$$x \rightarrow y = \begin{cases} 1, & x \leq y; \\ y, & x > y. \end{cases}$$

(2) The product t-norm $x * y = x \cdot y$. The corresponding implication is defined by

$$x \rightarrow y = \begin{cases} 1\,, & x \leq y; \\ y/x\,, & x > y. \end{cases}$$

(3) The Łukasiewicz t-norm $x * y = \max\{x + y - 1, 0\}$. The corresponding implication is defined by $x \to y = \min\{1, 1 - x + y\}$.

An L-fuzzy subset [7] of a set X is a mapping from X to L, and the family of all L-fuzzy subsets on X will be denoted by L^X, called the L-power set of X. $\top_X$ and $\bot_X$ denote the largest element and the smallest element in L^X, respectively.

Let $f : X \longrightarrow Y$ be a mapping between two nonempty sets. Define $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ and $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ by

$$f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x) \quad \text{and} \quad f_L^{\leftarrow}(B)(x) = B\big(f(x)\big),$$

for all $A \in L^X, B \in L^Y$, $x \in X$ and $y \in Y$. Then the L-fuzzy subset $f_L^{\rightarrow}(A)$ is called the image of A under f, and $f_L^{\leftarrow}(B)$ the preimage of B.

If L is a completely distributive lattice, then we can define

$$A_{[\lambda]} = \big\{\, x \in X \mid A(x) \geq \lambda \,\big\},\ A^{(\lambda)} = \big\{\, x \in X \mid A(x) \not\leq \lambda \,\big\},$$

$$A_{(\lambda)} = \big\{x \in X \mid \lambda \in \beta\big(A(x)\big)\,\big\},\ A^{[\lambda]} = \big\{x \in X \mid \lambda \notin \alpha\big(A(x)\big)\,\big\},$$

for all $A \in L^X$ and $\lambda \in L$.

In [29], Shi and Xiu introduced the notion of (L, M)-fuzzy convexities. When $L = M$, we called it (L, L)-fuzzy convex structure. In what follows, we will recall it.

Definition 2.10. A mapping $\mathfrak{C} : L^X \longrightarrow L$ is said to be an (L, L)-fuzzy convex structure on X if it satisfies the following three conditions:

(C1) $\mathfrak{C}(\top_X) = \mathfrak{C}(\bot_X) = \top$;

(C2) If $\{A_i\}_{i\in I} \subseteq L^X$, then $\mathfrak{C}\big(\bigwedge_{i\in I} A_i\big) \geq \bigwedge_{i\in I} \mathfrak{C}(A_i)$;

(C3) If $\{A_i\}_{i\in I} \subseteq L^X$ is nonempty and upward directed, then $\mathfrak{C}\big(\bigvee_{i\in I} A_i\big) \geq \bigwedge_{i\in I} \mathfrak{C}(A_i)$.

If $\mathfrak{C}$ is an (L, L)-fuzzy convex structure on X, then $(X, \mathfrak{C})$ is said to be an (L, L)-fuzzy convexity space. Every (L, L)-fuzzy convex structure on X is also called an (L, L)-fuzzy convexity on X.

Definition 2.11. Let $(X, \mathfrak{C}_x)$ and $(Y, \mathfrak{C}_y)$ be two (L, L)-fuzzy convexity spaces. Then a mapping $f : X \longrightarrow Y$ is called

(1) an (L, L)-fuzzy convexity-preserving mapping if $\mathfrak{C}_x(f_L^{\leftarrow}(B)) \geq \mathfrak{C}_y(B)$ for all $B \in L^Y$;

(2) an (L, L)-fuzzy convex-to-convex mapping if $\mathfrak{C}_y(f_L^{\rightarrow}(A)) \geq \mathfrak{C}_x(A)$ for all $A \in L^X$.

3. L-FUZZY IDEAL DEGREES

In this section, we will introduce the concept of L-fuzzy ideal degrees and investigate it by cut sets, further discuss some properties of L-fuzzy ideal degrees from the perspective of convexity. If not otherwise specified, E denotes an effect algebra and L is a completely distributive lattice with $\to$ (implication operator) corresponding to $\wedge$ (lattice infimum).

3.1. *L*-fuzzy ideal degrees

Definition 3.1. Let E be an effect algebra and A be an L-fuzzy subset in E. Then the L-fuzzy ideal degree $\mathfrak{D}_{ei}(A)$ of A is defined by

$$\mathfrak{D}_{ei}(A) = \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\leq y}} \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(z+w)\Big).$$

It is obvious that the $\mathfrak{D}_{ei}$ is a mapping from L^E to L. In [11], authors proposed the concept of L-fuzzy ideals with $L = [0,1]$, which are said to be fuzzy ideals.

A mapping $A : E \longrightarrow [0,1]$ is called a fuzzy ideal of an effect algebra E provided that

(II1) $A(y) \leq A(x)$ if $x \leq y$;

(II2) $A(x) \wedge A(y) \leq A(x+y)$ if $x \perp y$,

for all $x, y \in E$.

In what follows, we will generalize the concept of fuzzy ideals from [0,1] to a lattice.

Definition 3.2. Let E be an effect algebra and $\mathfrak{D}_{ei}(A)$ an L-fuzzy ideal degree of an L-fuzzy subset A in E. If $\mathfrak{D}_{ei}(A) = \top$, then the A is called an L-fuzzy ideal.

Remark 3.3. Let E be an effect algebra and $\mathfrak{D}_{ei}(A)$ an L-fuzzy ideal degree of an L-fuzzy subset A in E. If $\mathfrak{D}_{ei}(A) = \top$, then

$$\big(A(y) \to A(x)\big) \wedge \big(A(z) \wedge A(w) \to A(z+w)\big) = \top,$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. It follows that

$$A(y) \leq A(x) \text{ and } A(z) \wedge A(w) \leq A(z+w),$$

for $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. Hence, we could obtain that L-fuzzy ideals can be seen as generalizations of fuzzy ideals from [0,1] to a lattice L.

In the sequel, we will give some examples of L-fuzzy ideal degrees.

Example 3.4. Let $E = \{0, x, x', 1\}$ with $0 \leq x \leq x' \leq 1$, $x + x' = 1$ be an effect algebra.

(1) Let $A : E \longrightarrow [0,1]$ be an L-fuzzy subset with a minimum t-norm in $L = [0,1]$. If A is a constant value mapping on E, then

$$\mathfrak{D}_{ei}(A) = \top.$$

In this case, for any $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$, the L-fuzzy subset A satisfies

$$A(y) \leq A(x) \text{ and } A(z) \wedge A(w) \leq A(z+w).$$

Hence, the L-fuzzy subset A is an L-fuzzy ideal.

(2) Let $A : E \longrightarrow [0,1]$ be an L-fuzzy subset with a minimum t-norm in $L = [0,1]$.

$$A = \frac{0.1}{0} + \frac{0.3}{x} + \frac{0.7}{x'} + \frac{1}{1}.$$

We obtain $\mathfrak{D}_{ei}(A) = 0.1$ and $A(z) \wedge A(w) \leq A(z+w)$ if $z \perp w$. Since $x \leq x'$, but we have

$$A(x') = 0.7 \nleq 0.3 = A(x).$$

Hence, the L-fuzzy subset A is not an L-fuzzy ideal.

(3) Let $A : E \longrightarrow [0,1]$ be an L-fuzzy subset with a minimum t-norm in $L = [0,1]$.

$$A = \frac{1}{0} + \frac{0.7}{x} + \frac{0.3}{x'} + \frac{0.1}{1}.$$

We obtain $\mathfrak{D}_{ei}(A) = 0.1$ with $A(y) \leq A(z)$ if $z \leq y$. Since $x + x' = 1$, but we have

$$A(x) \wedge A(x') = 0.7 \wedge 0.3 \nleq 0.1 = A(1).$$

Hence, the L-fuzzy subset A is not an L-fuzzy ideal.

(4) Let $A : E \longrightarrow [0,1]$ be an L-fuzzy subset with a minimum t-norm in $L = [0,1]$.

$$A = \frac{0}{0} + \frac{0.7}{x} + \frac{1}{x'} + \frac{0.3}{1}.$$

We obtain $\mathfrak{D}_{ei}(A) = \perp$. In this case, we have $0 \leq x$ and $x + x' = 1$. But

$$A(x) = 0.7 \nleq 0 = A(0) \text{ and } A(x) \wedge A(x') = 0.7 \wedge 1 \nleq 0.3 = A(1).$$

Hence, the L-fuzzy subset A is not an L-fuzzy ideal.

Lemma 3.5. Let E be an effect algebra and A be an L-fuzzy subset in E.

(1) If $A(x) = \top$ for all $x \in E$, then $\mathfrak{D}_{ei}(A) = \top$;

(2) If $A(x) = \perp$ for all $x \in E$, then $\mathfrak{D}_{ei}(A) = \top$.

Proof. It is easy and omitted. □

Lemma 3.6. Let E be an effect algebra and A an L-fuzzy subset in E. For any $\lambda \in L$, $\lambda \leq \mathfrak{D}_{ei}(A)$ if and only if for any $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$, then

$$\lambda \wedge A(y) \leq A(x) \text{ and } \lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Proof. Necessity: Take any $\lambda \in L$. If $\lambda \leq \mathfrak{D}_{ei}(A)$, then

$$\lambda \leq \bigwedge_{\substack{x,y,z,w \in E \\ z \perp w, x \leq y}} \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(z+w)\Big).$$

Hence, it follows that

$$\lambda \leq \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(z+w)\Big),$$

which means

$$\lambda \leq A(y) \to A(x) \text{ and } \lambda \leq A(z) \wedge A(w) \to A(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. Hence, we obtain

$$\lambda \wedge A(y) \leq A(x) \text{ and } \lambda \wedge A(z) \wedge A(w) \leq A(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$.

Sufficiency: Take any $\lambda \in L$. For any $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$, by the assumption, we have

$$\lambda \wedge A(y) \leq A(x) \text{ and } \lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Then it follows that

$$\lambda \leq A(y) \to A(x) \text{ and } \lambda \leq A(z) \wedge A(w) \to A(z+w),$$

which means

$$\lambda \leq \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(z+w)\Big),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. That is to say, we obtain

$$\lambda \leq \bigwedge_{\substack{x,y,z,w \in E \\ z \perp w, x \leq y}} \Big(A(y) \to A(x)\Big) \wedge \Big(A(z) \wedge A(w) \to A(z+w)\Big),$$

as desired. □

Theorem 3.7. Let E be an effect algebra and A be an L-fuzzy subset in E. Then

$$\mathfrak{D}_{ei}(A) = \bigvee \Big\{\lambda \in L | \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{for any } x \leq y,\ z \perp w\Big\}.$$

Proof. Take any $t \in L$. Then it follows that

$$\begin{aligned} t \leq \mathfrak{D}_{ei}(A) &\Longleftrightarrow t \wedge A(y) \leq A(x) \text{ and } t \wedge A(z) \wedge A(w) \leq A(z+w), \text{ for any } x, y, z, w \in E \\ &\qquad\qquad \text{with } x \leq y \text{ and } z \perp w \text{ (by Lemma 3.6)} \\ &\Longrightarrow t \leq \bigvee \Big\{\lambda \in L \,|\, \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \forall\, x \leq y, z \perp w\Big\}. \end{aligned}$$

On the other hand, assume that

$$t \leq \bigvee \Big\{\lambda \in L \,|\, \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \forall\ x \leq y, z \perp w\Big\}.$$

For any $\alpha \prec t$, it follows from the definition of binary relation $\prec$ that $\alpha \leq \lambda$ for some $\lambda \in L$ satisfying

$$\lambda \wedge A(y) \leq A(x) \text{ and } \lambda \wedge A(z) \wedge A(w) \leq A(z+w),$$

for all $x \leq y$ and $z \perp w$. Then we know

$$\alpha \leq \lambda \leq A(y) \to A(x) \text{ and } \alpha \leq \lambda \leq \big(A(z) \wedge A(w)\big) \to A(z+w),$$

for all $\alpha \in L$ with $\alpha \prec t$. By $t = \bigvee \{ \alpha \in L \mid \alpha \prec t \}$, we know

$$t \leq A(y) \to A(x) \text{ and } t \leq \big(A(z) \wedge A(w)\big) \to A(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. That is to say,

$$t \wedge A(y) \leq A(x) \text{ and } t \wedge A(z) \wedge A(w) \leq A(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$. Then it follows from Lemma 3.6 that $t \leq \mathfrak{D}_{ei}(A)$. Hence, we obtain

$$\mathfrak{D}_{ei}(A) = \bigvee \Big\{ \lambda \in L | \lambda \wedge A(y) \leq A(x), \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{for any } x \leq y,\ z \perp w \Big\},$$

as desired. □

In the following, cut sets of L-fuzzy subset A may be empty. If cut sets of A are empty, then we still consider the empty set as a special ideal of E, when we discuss that cut sets of L-fuzzy subset A are ideals. That is to say, the empty set is a special ideal of E.

Theorem 3.8. Let E be an effect algebra and A be an L-fuzzy subset in E. Then

$$\mathfrak{D}_{ei}(A) = \bigvee \Big\{ \lambda \in L \mid \forall \mu \leq \lambda,\ A_{[\mu]} \text{ is an ideal of E} \Big\}.$$

Proof. Assume that $A_{[\mu]}$ is an ideal of E for $\lambda \in L$ with $\mu \leq \lambda$. Take any $x, y \in E$ with $x \leq y$. Let $\theta = \lambda \wedge A(y)$. Then we have $\theta \leq \lambda$ and $\theta \leq A(y)$, which imply $y \in A_{[\theta]}$. By the assumption, we know that $A_{[\theta]}$ is an ideal of E. Then it shows that

$$x \in A_{[\theta]},$$

which means $\theta \leq A(x)$. It follows that

$$\lambda \wedge A(y) \leq A(x).$$

Similarly, for any $z, w \in E$ with $z \perp w$, we obtain

$$\lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Hence, it follows that

$$\mathfrak{D}_{ei}(A) = \bigvee\Big\{\lambda\in L|\ \lambda\wedge A(y)\leq A(x),\ \lambda\wedge A(z)\wedge A(w)\leq A(z+w),\ \text{for any}\, x\leq y, z\perp w\Big\}$$
$$\geq \bigvee\Big\{\ \lambda\in L\mid \forall\,\mu\leq\lambda,\ A_{[\mu]}\ \text{is an ideal of E}\Big\}.$$

Conversely, assume that $\lambda\wedge A(y)\leq A(x)$ and $\lambda\wedge A(z)\wedge A(w)\leq A(z+w)$ for all $x,y,z,w\in E$ with $z\perp w$ and $x\leq y$. For any $\mu\leq\lambda$, we need to prove $A_{[\mu]}$ is an ideal of E.

(**I1**) If $y\in A_{[\mu]}$ with $x\leq y$, then $\mu\leq A(y)$. It implies that

$$\mu\leq\lambda\wedge A(y)\leq A(x).$$

Then it follows that $x\in A_{[\mu]}$.

(**I2**) If $z,w\in A_{[\mu]}$ and $z\perp w$, then

$$\mu\leq\lambda\wedge A(z)\wedge A(w)\leq A(z+w).$$

Hence, it follows that

$$z+w\in A_{[\mu]}.$$

That is to say, $A_{[\mu]}$ is an ideal of E. Then it implies that

$$\mathfrak{D}_{ei}(A) = \bigvee\Big\{\lambda\in L\mid\lambda\wedge A(y)\leq A(x),\ \lambda\wedge A(z)\wedge A(w)\leq A(z+w),\text{for any}\, x\leq y,\ z\perp w\Big\}$$
$$\leq \bigvee\Big\{\ \lambda\in L\mid \forall\,\mu\leq\lambda,\ A_{[\mu]}\ \text{is an ideal of E}\Big\},$$

as desired. □

Theorem 3.9. Let E be an effect algebra and A be an L-fuzzy subset in E. Then

$$\mathfrak{D}_{ei}(A) = \bigvee\Big\{\ \lambda\in L\mid\mu\notin\alpha(\lambda),\ A^{[\mu]}\ \text{is an ideal of E}\Big\}.$$

Proof. Assume that $\lambda\in L$ with $\lambda\wedge A(y)\leq A(x)$ and $\lambda\wedge A(z)\wedge A(w)\leq A(z+w)$ for all $x,y,z,w\in E$ with $x\leq y$ and $z\perp w$. For $\mu\notin\alpha(\lambda)$, we need to prove that $A^{[\mu]}$ is an ideal of E.

(**I1**) If $x\leq y$ and $y\in A^{[\mu]}$, then $\mu\notin\alpha(A(y))$. It follows from

$$\lambda\wedge A(y)\leq A(x)$$

that

$$\alpha(A(x))\subseteq\alpha(\lambda\wedge A(y))=\alpha(\lambda)\cup\alpha(A(y)),$$

which means $\mu\notin\alpha(A(x))$. Hence, we obtain $x\in A^{[\mu]}$.

(**I2**) If $z,w\in A^{[\mu]}$ and $z\perp w$, then

$$\mu\notin\alpha(A(z))\cup\alpha(A(w))\cup\alpha(\lambda)=\alpha(\lambda\wedge A(z)\wedge A(w)).$$

It follows from

$$\lambda \wedge A(z) \wedge A(w) \leq A(z+w)$$

that

$$\alpha(A(z+w)) \subseteq \alpha(\lambda \wedge A(z) \wedge A(w)).$$

Hence, we obtain

$$\mu \notin \alpha(A(z+w)).$$

Then it follows that $z+w \in A^{[\mu]}$, which means

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee\Big\{\lambda \in L \mid \lambda \wedge A(y \leq A(x),\ \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{for any } x \leq y, z \perp w\Big\}\\ &\leq \bigvee\Big\{\ \lambda \in L \mid \mu \notin \alpha(\lambda),\ A^{[\mu]} \text{ is an ideal of E}\Big\}.\end{aligned}$$

Conversely, assume that $A^{[\mu]}$ is an ideal of E for $\lambda \in L$ with $\mu \notin \alpha(\lambda)$. In the sequel, for any $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$, we need to prove

$$\lambda \wedge A(y) \leq A(x) \text{ and } \lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Suppose that $\mu \notin \alpha(\lambda \wedge A(y))$. It follows from

$$\alpha(\lambda \wedge A(y)) = \alpha(\lambda) \cup \alpha(A(y))$$

that

$$\mu \notin \alpha(\lambda) \text{ and } \mu \notin \alpha(A(y)).$$

It implies that $y \in A^{[\mu]}$. By the assumption, we know that $A^{[\mu]}$ is an ideal of E, which means $x \in A^{[\mu]}$. Then it follows that

$$\mu \notin \alpha(A(x)).$$

Hence, we obtain

$$\lambda \wedge A(y) \leq A(x).$$

Similarly, for any $z, w \in E$ with $z \perp w$, we obtain

$$\lambda \wedge A(z) \wedge A(w) \leq A(z+w).$$

Then it shows that

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee\Big\{\ \lambda \in L \mid \lambda \wedge A(y) \leq A(x),\ \lambda \wedge A(z) \wedge A(w) \leq A(z+w), \text{for any } x \leq y, z \perp w\Big\}\\ &\geq \bigvee\Big\{\ \lambda \in L \mid \mu \notin \alpha(\lambda),\ A^{[\mu]} \text{ is an ideal of E}\Big\},\end{aligned}$$

as desired. □

Theorem 3.10. Let E be an effect algebra and A be an L-fuzzy subset in E. Then

$$\mathfrak{D}_{ei}(A) = \bigvee \Big\{ \lambda \in L \mid \forall \mu \in P(L), \lambda \nleq \mu,\ A^{(\mu)} \text{ is an ideal of E} \Big\}.$$

Proof. Assume that $\lambda \in L$ with $\lambda \wedge A(y) \le A(x)$ and $\lambda \wedge A(z) \wedge A(w) \le A(z+w)$ for all $x, y, z, w \in E$ with $x \le y$ and $z \perp w$. If $\mu \in P(L)$ and $\lambda \nleq \mu$, then we need to prove that $A^{(\mu)}$ is an ideal of E.

Assume that $y \in A^{(\mu)}$. If $x \notin A^{(\mu)}$, then $A(x) \le \mu$. It follows from

$$\lambda \wedge A(y) \le A(x)$$

that

$$\lambda \wedge A(y) \le \mu.$$

By $\mu \in P(L)$ and $y \in A^{(\mu)}$, i.e., $A(y) \nleq \mu$, we have $\lambda \le \mu$. This is a contradiction. Hence, it follows that

$$x \in A^{(\mu)}.$$

Similarly, for any $z, w \in E$ with $z \perp w$, we obtain

$$z, w \in A^{(\mu)} \text{ implies } z + w \in A^{(\mu)}.$$

Then it follows that

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee \Big\{\lambda \in L \mid \lambda \wedge A(y) \le A(x),\ \lambda \wedge A(z) \wedge A(w) \le A(z+w), \text{for any } x \le y, z \perp w\Big\}\\ &\le \bigvee \Big\{ \lambda \in L \mid \forall \mu \in P(L), \lambda \nleq \mu,\ A^{(\mu)} \text{ is an ideal of E } \Big\}.\end{aligned}$$

Conversely, assume that $A^{(\mu)}$ is an ideal of E for $\lambda \in L$ and $\mu \in P(L)$ with $\lambda \nleq \mu$. In what follows, for any $x, y, z, w \in E$ with $x \le y$ and $z \perp w$, we need to prove

$$\lambda \wedge A(y) \le A(x) \text{ and } \lambda \wedge A(z) \wedge A(w) \le A(z+w).$$

For any $x, y \in E$ with $x \le y$, let $\mu \in P(L)$ and $\lambda \wedge A(y) \nleq \mu$. Then we have

$$\lambda \nleq \mu \text{ and } A(y) \nleq \mu.$$

It follows that $y \in A^{(\mu)}$. By the assumption, we know $A^{(\mu)}$ is an ideal of E, then $x \in A^{(\mu)}$. Further, it implies that

$$A(x) \nleq \mu.$$

Hence, we have

$$\lambda \wedge A(y) \le A(x).$$

Similarly, for any $z, w \in E$ with $z \perp w$, we obtain

$$\lambda \wedge A(z) \wedge A(w) \le A(z+w).$$

Then it follows that

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee \Big\{\lambda \in L \mid \lambda \wedge A(y) \le A(x), \lambda \wedge A(z) \wedge A(w) \le A(z+w), \text{ for any } x \le y, z \perp w\Big\}\\ &\ge \bigvee \Big\{ \lambda \in L \mid \forall \mu \in P(L), \lambda \nleq \mu,\ A^{(\mu)} \text{ is an ideal of E } \Big\},\end{aligned}$$

as desired. □

Theorem 3.11. Let E be an effect algebra and A an L-fuzzy subset in E. If $\beta(\lambda\wedge\mu) = \beta(\lambda)\cap\beta(\mu)$ for all $\lambda,\mu\in L$, then

$$\mathfrak{D}_{ei}(A) = \bigvee\Big\{\lambda\in L \mid \forall\,\mu\in\beta(\lambda),\ A_{(\mu)} \text{ is an ideal of E}\Big\}.$$

P r o o f. Assume that $\lambda\in L$ such that $\lambda\wedge A(y)\le A(x)$ and $\lambda\wedge A(z)\wedge A(w)\le A(z+w)$ for all $x,y,z,w\in E$ with $x\le y$ and $z\perp w$. For any $\mu\in\beta(\lambda)$, we need to prove that $A_{(\mu)}$ is an ideal of E.

(**I1**) If $y\in A_{(\mu)}$ and $x\le y$, then

$$\mu\in\beta(A(y))\cap\beta(\lambda)=\beta(A(y)\wedge\lambda)\subseteq\beta(A(x)).$$

It follows that $x\in A_{(\mu)}$.

(**I2**) If $z,w\in A_{(\mu)}$ and $z\perp w$, then

$$\mu\in\beta(A(z))\cap\beta(A(w))\cap\beta(\lambda)=\beta(A(z)\wedge A(w)\wedge\lambda)\subseteq\beta(A(z+w)).$$

Hence, we obtain

$$z+w\in A_{(\mu)}.$$

Then it follows that

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee\Big\{\lambda\in L \mid \lambda\wedge A(y)\le A(x),\ \lambda\wedge A(z)\wedge A(w)\le A(z+w), \text{ for any } x\le y, z\perp w\Big\}\\ &\le \bigvee\Big\{\lambda\in L\mid\forall\,\mu\in\beta(\lambda),\ A_{(\mu)} \text{ is an ideal of E}\Big\}.\end{aligned}$$

Conversely, assume that $A_{(\mu)}$ is an ideal of E for $\lambda\in L$ with $\mu\in\beta(\lambda)$. For any $x,y,z,w\in E$ with $x\le y$ and $z\perp w$, we need to prove

$$\lambda\wedge A(y)\le A(x) \text{ and } \lambda\wedge A(z)\wedge A(w)\le A(z+w).$$

(i) Assume that $x,y\in E$ with $x\le y$. Let $\mu\in\beta(\lambda\wedge A(y))$. Then it follows from

$$\beta(\lambda\wedge A(y))=\beta(A(y))\cap\beta(\lambda)$$

that

$$\mu\in\beta(\lambda) \text{ and } \mu\in\beta(A(y)).$$

It implies $y\in A_{(\mu)}$. By the assumption, we know that $A_{(\mu)}$ is an ideal of E. Then it shows $x\in A_{(\mu)}$. Hence, we have

$$\mu\in\beta(A(x)).$$

It follows that $\lambda\wedge A(y)\le A(x)$.

(ii) Assume that $z,w\in E$ and $z\perp w$. Let $\mu\in\beta(\lambda\wedge A(z)\wedge A(w))$. It follows from

$$\beta(\lambda\wedge A(z)\wedge A(w))=\beta(\lambda)\cap\beta(A(z))\cap\beta(A(w))$$

that

$$\mu\in\beta(\lambda),\mu\in\beta(A(z)) \text{ and } \mu\in\beta(A(w)).$$

It implies that

$$z, w \in A_{(\mu)}.$$

By the assumption, we know that $A_{(\mu)}$ is an ideal of E and $z \perp w$. Then it shows that

$$z + w \in A_{(\mu)},$$

which means $\mu \in \beta(A(z + w))$. It follows that

$$\lambda \wedge A(z) \wedge A(w) \leq A(z + w).$$

Hence, we have

$$\begin{aligned}\mathfrak{D}_{ei}(A) &= \bigvee \Big\{\lambda \in L \,|\, \lambda \wedge A(y) \leq A(x),\ \lambda \wedge A(z) \wedge A(w) \leq A(z + w), \text{for any } x \leq y, z \perp w\Big\} \\ &\geq \bigvee \Big\{\lambda \in L | \forall\, \mu \in \beta(\lambda),\ A_{(\mu)} \text{ is an ideal of E}\Big\},\end{aligned}$$

as desired. □

Zhang [41] discussed the relations between fuzzy ideals and fuzzy filters in dual effect algebras. Liu [10], Liu and Wang [11] studied the connections between a fuzzy filter $\mathcal{F}$ and its cut sets $\mathcal{F}_{[\lambda]}$ for all $\lambda \in [0, 1]$ in effect algebras and pseudo-effect algebras, respectively. In the sequel, on one hand, we investigate L-fuzzy ideals by cut sets $A_{[\lambda]}$ for all $\lambda \in L$, which generalizes the unit interval $[0, 1]$ to a lattice L. On the other hand, we characterize L-fuzzy ideals by another three kinds of cut sets. In particular, we think that the empty set is a special ideal of an effect algebra E. By [9, 37], we obtain the following corollaries immediately.

Corollary 3.12. Let E be an effect algebra and A an L-fuzzy subset in E. Then the following statements are equivalent:

(1) A is an L-fuzzy ideal of E;

(2) for every $\lambda \in L$, $A_{[\lambda]}$ is an ideal;

(3) for every $\lambda \in J(L)$, $A_{[\lambda]}$ is an ideal;

(4) for every $\lambda \in L$, $A^{[\lambda]}$ is an ideal;

(5) for every $\lambda \in P(L)$, $A^{[\lambda]}$ is an ideal;

(6) for every $\lambda \in P(L)$, $A^{(\lambda)}$ is an ideal.

Corollary 3.13. Let E be an effect algebra and A an L-fuzzy subset in E. Then the following statements are equivalent when $\beta(\lambda \wedge \mu) = \beta(\lambda) \cap \beta(\mu)$ for all $\lambda, \mu \in L$.

(1) A is an L-fuzzy ideal of E;

(2) for every $\lambda \in J(L)$, $A_{(\lambda)}$ is an ideal;

(3) for every $\lambda \in L$, $A_{(\lambda)}$ is an ideal.

In what follows, we will characterize L-fuzzy ideals by L_β-nested sets and L_α-nested sets. We also can refer to [12, 26] for more information on nested sets. By [26], we can immediately obtain the following Theorems 3.14 and 3.15.

Theorem 3.14. Let E be an effect algebra and $\{A(\lambda) \mid \lambda \in L\}$ be an L_β-nest of ideals of E. Then there exists an L-fuzzy ideal A such that

(1) $A_{(\lambda)} \subseteq A(\lambda) \subseteq A_{[\lambda]}$ for all $\lambda \in L$;

(2) $A_{(\lambda)} = \bigcup\limits_{\lambda \in \beta(\nu)} A(\nu)$ for all $\lambda \in L$;

(3) $A_{[\nu]} = \bigcap\limits_{\lambda \in \beta(\nu)} A(\lambda)$ for all $\nu \in L$.

Theorem 3.15. Let E be an effect algebra and $\{A(\lambda) \mid \lambda \in L\}$ be an L_α-nest of ideals of E. Then there exists an L-fuzzy ideal A such that

(1) $A^{(\lambda)} \subseteq A(\lambda) \subseteq A^{[\lambda]}$ for all $\lambda \in L$;

(2) $A^{(\lambda)} = \bigcup\limits_{\nu \in \alpha(\lambda)} A(\nu)$ for all $\lambda \in P(L)$;

(3) $A^{[\lambda]} = \bigcap\limits_{\lambda \in \alpha(\nu)} A(\nu)$ for all $\lambda \in P(L)$.

3.2. (L, L)-fuzzy convexities are induced by L-fuzzy ideal degrees

In this subsection, we will characterize convex properties of L-fuzzy ideal degrees. By morphisms between effect algebras, we obtain one kind of mappings between convexity spaces. Firstly, we will investigate the structural properties of convexity spaces by L-fuzzy ideal degrees.

Theorem 3.16. Let E be an effect algebra and $\mathfrak{D}_{ei}$ an L-fuzzy ideal degree. Then $\mathfrak{D}_{ei}$ is an (L, L)-fuzzy convexity on E.

Proof. By Lemma 3.5, we only need to prove (**C2**) and (**C3**).

(**C2**) Let $\{A_i\}_{i \in I}$ be a family of L-fuzzy subsets in E. Then it follows that

$$\begin{aligned}
\mathfrak{D}_{ei}\Big(\bigwedge_{i\in I} A_i\Big) &= \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\le y}} \Big(\bigwedge_{i\in I} A_i(y) \to \bigwedge_{i\in I} A_i(x)\Big) \wedge \Big(\bigwedge_{i\in I} A_i(z) \wedge \bigwedge_{i\in I} A_i(w) \to \bigwedge_{i\in I} A_i(z+w)\Big)\\
&= \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\le y}} \bigwedge_{i\in I} \Big(\bigwedge_{j\in I} A_j(y) \to A_i(x)\Big) \wedge \bigwedge_{i\in I} \Big(\bigwedge_{j\in I} A_j(z) \wedge \bigwedge_{j\in I} A_j(w) \to A_i(z+w)\Big)\\
&= \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\le y}} \bigwedge_{i\in I} \Big(\bigwedge_{j\in I} A_j(y) \to A_i(x)\Big) \wedge \Big(\bigwedge_{j\in I} A_j(z) \wedge \bigwedge_{j\in I} A_j(w) \to A_i(z+w)\Big)\\
&\ge \bigwedge_{i\in I} \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\le y}} \Big(A_i(y) \to A_i(x)\Big) \wedge \Big(A_i(z) \wedge A_i(w) \to A_i(z+w)\Big)\\
&= \bigwedge_{i\in I} \mathfrak{D}_{ei}(A_i).
\end{aligned}$$

(**C3**) Let $\{A_i\}_{i\in I}$ be a upward directed family of L-fuzzy subsets in E. Then we need to prove

$$\mathfrak{D}_{ei}\big(\bigvee_{i\in I} A_i\big) \geq \bigwedge_{i\in I} \mathfrak{D}_{ei}\big(A_i\big).$$

Take any $\lambda \in L$ with $\lambda \leq \bigwedge_{i\in I} \mathfrak{D}_{ei}\big(A_i\big)$. Then it follows that $\lambda \leq \mathfrak{D}_{ei}\big(A_i\big)$ for all $i \in I$. By Lemma 3.6, we know

$$\lambda \wedge A_i(y) \leq A_i(x) \text{ and } \lambda \wedge A_i(z) \wedge A_i(w) \leq A_i(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$, $z \perp w$ and $i \in I$. In what follows, we need to prove

$$\lambda \wedge \big(\bigvee_{i\in I} A_i(y)\big) \leq \bigvee_{i\in I} A_i(x) \text{ and } \lambda \wedge \big(\bigvee_{i\in I} A_i(z)\big) \wedge \big(\bigvee_{i\in I} A_i(w)\big) \leq \bigvee_{i\in I} A_i(z+w),$$

for all $x, y, z, w \in E$ with $x \leq y$ and $z \perp w$.

For any $\eta \prec \lambda \wedge \big(\bigvee_{i\in I} A_i(z)\big) \wedge \big(\bigvee_{i\in I} A_i(w)\big)$, there exist $i \in I$ and $j \in I$ such that

$$\eta \leq A_i(z),\ \eta \leq A_j(w) \text{ and } \eta \leq \lambda.$$

Since $\{A_i\}_{i\in I}$ is upward directed, there exists $k \in I$ such that $A_i \leq A_k$ and $A_j \leq A_k$. Then it follows that

$$A_i(z) \leq A_k(z) \text{ and } A_j(w) \leq A_k(w),$$

which means that

$$\eta \leq \lambda \wedge A_k(z) \wedge A_k(w) \leq A_k(z+w) \leq \bigvee_{i\in I} A_i(z+w),$$

for all $z, w \in E$ with $z \perp w$. Hence, we obtain

$$\lambda \wedge \big(\bigvee_{i\in I} A_i(z)\big) \wedge \big(\bigvee_{i\in I} A_i(w)\big) \leq \bigvee_{i\in I} A_i(z+w),$$

for all $z, w \in E$ with $z \perp w$. Similarly, we obtain

$$\lambda \wedge \big(\bigvee_{i\in I} A_i(y)\big) \leq \bigvee_{i\in I} A_i(x),$$

for all $x, y \in E$ with $x \leq y$. Then it follows from Lemma 3.6 that

$$\lambda \leq \mathfrak{D}_{ei}\big(\bigvee_{i\in I} A_i\big),$$

which implies $\mathfrak{D}_{ei}\big(\bigvee_{i\in I} A_i\big) \geq \bigwedge_{i\in I} \mathfrak{D}_{ei}\big(A_i\big)$. Hence, we obtain that $\mathfrak{D}_{ei}$ is an (L, L)-fuzzy convexity, as desired. □

Theorem 3.17. Let E and F be two effect algebras and $f : E \longrightarrow F$ be an effect algebra morphism. Then $f : (E, \mathfrak{D}_{ei}) \longrightarrow (F, \mathfrak{D}_{fi})$ is an (L, L)-fuzzy convexity-preserving mapping.

Proof. Take any L-fuzzy subset A in F. Then

$$\begin{aligned}
&\mathfrak{D}_{ei}\big(f_L^{\leftarrow}(A)\big)\\
&= \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\le y}} \big(f_L^{\leftarrow}(A)(y) \to f_L^{\leftarrow}(A)(x)\big) \wedge \big(f_L^{\leftarrow}(A)(z) \wedge f_L^{\leftarrow}(A)(w) \to f_L^{\leftarrow}(A)(z+w)\big)\\
&= \bigwedge_{\substack{x,y,z,w\in E\\ z\perp w, x\le y}} \Big(A(f(y)) \to A(f(x))\Big) \wedge \Big(A(f(z)) \wedge A(f(w)) \to A(f(z)+f(w))\Big)\\
&\ge \bigwedge_{\substack{x_1,y_1,z_1,w_1\in F\\ z_1\perp w_1, x_1\le y_1}} \big(A(y_1) \to A(x_1)\big) \wedge \big(A(z_1) \wedge A(w_1) \to A(z_1+w_1)\big)\\
&= \mathfrak{D}_{fi}(A).
\end{aligned}$$

Hence, we obtain that f is an (L, L)-fuzzy convexity-preserving mapping, as desired. □

In the sequel, we will discuss the relations between L-fuzzy ideals and their inverse images by L-fuzzy ideal degrees.

Theorem 3.18. Let E and F be two effect algebras and $f : E \longrightarrow F$ a monomorphism. If B is an L-fuzzy ideal of F, then $f_L^{\leftarrow}(B)$ is an L-fuzzy ideal of E.

Proof. It can be obtained from Theorem 3.17. □

Remark 3.19. In this paper, we first introduce the concept of L-fuzzy ideal degrees and further investigate it. In order to highlight the idea of fuzzy mathematics, we discuss L-fuzzy ideal degrees, which emphasize the ideal of many-valued logics. The concept can reveal essential characterizations of different mathematical structures. There are some papers for different mathematical structures on degrees of mathematical structures, such as [9, 27, 37, 44] and (Y.-Y. Dong, F.-G. Shi, L-fuzzy Sub-Effect Algebras).

4. CONCLUSIONS

In this paper, considering L being a completely distributive lattice, we first introduce the concept of L-fuzzy ideal degrees. Then, we characterize L-fuzzy ideal degrees by four types of cut sets. By L-fuzzy ideal degrees, we could give the concept of L-fuzzy ideals, which can be seen as generalizations of fuzzy ideals. We also discuss the relations between L-fuzzy ideals and cut sets (L_β-nested sets and L_α-nested sets). Finally, we obtain that the L-fuzzy ideal degree is an (L, L)-fuzzy convexity. These morphisms between effect algebras are (L, L)-fuzzy convexity-preserving mappings.

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