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ON ASYMMETRIC DISTRIBUTIONS OF COPULA RELATED RANDOM VARIABLES WHICH INCLUDES THE SKEW-NORMAL ONES

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Assuming that $C_{X,Y}$ is the copula function of X and Y with marginal distribution functions $F_X(x)$ and $F_Y(y)$, in this work we study the selection distribution $Z \stackrel{\mathrm{d}}{=} (X|Y \in T)$. We present some special cases of our proposed distribution, among them, skew-normal distribution as well as normal distribution. Some properties such as moments and moment generating function are investigated. Also, some numerical analysis is presented for illustration.

Keywords: selection distribution, skew-normal, Gaussian copula

Classification: 62H05, 62Exx

1. INTRODUCTION

In last decades there is a growing interest in the literature on parametric distributions which represent a local departure from the symmetric distributions. In this studied there exist some skewness and kurtosis parameters in which they can yield symmetric distribution by regulating suitable values of these parameters. A general univariate form of these distributions, which is called skew-symmetric family, can be represented as a product of a cumulative distribution function and a density distribution function. Following Wang et al. (2004) [15], a random variable Z is said to have a skew-symmetric (SS) distribution if its probability density function (PDF) can be written as

$$f(z) = 2f_0(z)\,G(w(z)), \qquad z \in \mathbb{R}, \tag{1}$$

where $f_0(.)$ is a probability density function centrally symmetric about 0, $w(.) : \mathbb{R} \to \mathbb{R}$ is an odd real-valued function, i. e., $w(-x) = -w(x)$ for all $x \in \mathbb{R}$ and $G(w(.))$ is a cumulative distribution function on $\mathbb{R}$ such that $g = G'$ is an even density function, providing G is differentiable. The first and foremost special case of these families arises when f_0 follows a normal distribution and $g = f_0$ and is called the skew-normal distribution which has proposed independently by Roberts (1966) [11], Ainger et al. (1977) [1], Andel et al. (1984) [2] and Azzalini (1985) [4]. Including the location and scale parameters μ and σ respectively, it is said that the univariate random variable Z_{sn} has

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a skew-normal distribution with mean μ, variance σ^2 and the skewness parameter λ, if its density can be written as

$$f_\lambda(z) = 2\varphi(z;\mu,\sigma^2)\Phi\big(\lambda\frac{z-\mu}{\sigma}\big), \qquad z \in \mathbb{R}, \tag{2}$$

where $\varphi(.;\mu,\sigma^2)$ is the normal density function with mean μ and variance σ^2 and $\Phi(.)$ denotes the standard normal distribution function. We adopt the notation $Z_{sn} \sim SN(\mu,\sigma^2;\lambda)$ and it reduces to $Z_{sn} \sim N(\mu,\sigma^2)$ if $\lambda = 0$.

A most common scenario to constructing asymmetric distribution functions is based on conditional (or selection) random variables. Based on the definition (1) of Arellano - Valle et al. (2006), let $X, Y \in \mathbb{R}$ be two random variables, and denote by T a measurable subset of $\mathbb{R}$. The selection distribution is defined as the conditional distribution of X given $Y \in T$, i.e, it is said that a random variable $X \in \mathbb{R}$ has a selection distribution if $Z \stackrel{\mathrm{d}}{=} (X|Y \in T)$, denoted by $Z \sim SLCT(\theta)$ with parameter(s) θ depending on the characteristics of X, Y, and T. A well known special case of their definition is the skew-normal distribution. Suppose that (X, Y) has a bivariate normal distribution with standardized marginals and correlation ρ. The conditional density of X given $Y > 0$ is skew-normal $SN(\lambda)$ with $\lambda = \rho/\sqrt{(1-\rho^2)}$. See [3] for more details and the other scenarios.

The aim of this work is the study of conditional distribution of two copula related random variables X and Y. Based on the Sklar theorem [14] for any random vector (X, Y), there exists a grounded, uniformly marginal and $2-$increasing bivariate copula function $C : [0,1]^2 \to [0,1]$ such that

$$F_{X,Y}(x,y) = C(F_X(x), F_Y(y)), \tag{3}$$

where $F_{X,Y} : \mathbb{R}^2 \to [0,1]$ is the joint distribution function of the random vector (X, Y) and $F_X,\ F_Y : \mathbb{R} \to [0,1]$ are respectively distribution functions of random variables X and Y. We assume that both X and Y are continuous random variables which guarantees that copula C is unique and hence its density, if exists, is just function $c(.,.)$ such that

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) c(F_X(x), F_Y(y)), \tag{4}$$

where $f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}^+$ is the joint density function of X, Y and $f_X,\ f_Y : \mathbb{R} \to \mathbb{R}^+$ are respectively density functions of X and Y. We refer to [8, 12, 16] for more information about copulas and association measures. Denoting the copula coupling X and Y as $C_{X,Y}$, we study the distribution of $Z \stackrel{\mathrm{d}}{=} (X|Y \in T)$.

A special case of the copula function is the Gaussian copula. Denoting Φ the standard normal cumulative distribution and $\mathbf{\Phi}(.,\rho)$ the bivariate standard multivariate normal distribution function with correlation ρ, it is said that random variables X and Y are associated to a Gaussian copula with correlation matrix ρ if

$$C^{Ga}(u_1,u_2) = \mathbf{\Phi}(\Phi^{-1}(u_1), \Phi^{-1}(u_2), \rho), \tag{5}$$

where $\Phi^{-1}(.)$ is the inverse function of the standard normal distribution function [12]. We consider this copula as a connection function of our variables in this work. The rest of this paper is organized as follows. The main results are given in the next section. In

Section 3, we estimate the parameter of the proposed distribution. In section 4, we apply our results to do a simulation analysis. Also an application of our theoretical results in a real dataset is given. Finally, some concluding remarks are presented in section 5.

2. MAIN RESULTS

It is well known that conditioning a variable on a subset of another variable causes a skewness parameter. Consider two random variables X and Y are associated with the copula function $C_{X,Y}$. Also assume that the marginal distributions of these two random variables as well as their associated copula function are absolutely continuous. Then, from the classical probability, we have the distribution of $Z \stackrel{\mathrm{d}}{=} (X|Y \in T)$ as

$$\begin{aligned} F_Z(z) &= \frac{\int_{-\infty}^{z} \int_T f_{X,Y}(x,y)\,\mathrm{d}y\mathrm{d}x}{\int_T f_Y(y)\,\mathrm{d}y} \\ &= \frac{\int_{-\infty}^{x} \int_T f_X(x) f_Y(y) c(F_X(x), F_Y(y))\,\mathrm{d}y\mathrm{d}x}{\int_T f_Y(y)\mathrm{d}y} \\ &= m_T \int_{-\infty}^{x} \int_T f_X(x) f_Y(y) c(F_X(x), F_Y(y))\,\mathrm{d}y\mathrm{d}x, \end{aligned}$$

where $m_T = (\int_T f_Y(y)\mathrm{d}y)^{-1}$ and differentiating with respect to z we readily obtain the density of z as

$$f_Z(z) = m_T \int_T f_X(z) f_Y(y) c(F_X(z), F_Y(y))\,\mathrm{d}y, \qquad z \in \mathbb{R}, \tag{6}$$

where $c(\cdot,\cdot)$ is the density of copula.

The following theorem states a more specific version of (6) in which the selection is made on the positive values of Y.

Theorem 2.1. Assume that $C_{X,Y}$ is the copula function of random variables X and Y with marginal distribution functions $F_X(x)$ and $F_Y(y)$, then the density of $Z \stackrel{\mathrm{d}}{=} (X|Y > \mu_y)$ is given by

$$f_Z(z) = m_T f_X(z)(1 - D_1 C_{X,Y}), \qquad z \in \mathbb{R}, \tag{7}$$

where μ_y is the mean of Y and $D_1 C_{X,Y} = \frac{\partial C_{X,Y}}{\partial F_X(z)}$ if exist otherwise 0.

Proof. We have

$$\begin{aligned} F_Z(z) &= m_T P(X \leq z, y > \mu_y) = m_T[P(X \leq z) - P(X \leq z, y \leq \mu_y)] \\ &= m_T[F_X(z) - C_{X,Y}(F_X(z), F_Y(\mu_y))] \end{aligned}$$

and using some chain rules we readily obtain

$$\begin{aligned} f_Z(z) &= m_T\big[f_X(z) - \frac{\partial C_{X,Y}}{\partial F_X(z)} \frac{\partial F_X(z)}{\partial z}\big] = m_T\big[f_X(z) - f_X(z) D_1 C_{X,Y}\big] \\ &= m_T f_X(z)(1 - D_1 C_{X,Y}), \end{aligned}$$

which proves the assertion. □

The following corollary is a special case of the theorem and states a similar result of Arnold and Beaver (2002) [3].

Corollary 2.2. Let X and Y be two standard normal random variables with the Gaussian copula with correlation ρ (5) then the distribution of $Z \stackrel{\mathrm{d}}{=} (X|Y > 0)$ is $SN(\lambda)$ with

$$f_Z(z) = 2\varphi(z)\Phi(\lambda z), \quad z \in \mathbb{R}, \tag{8}$$

where $\lambda = \rho/\sqrt{(1-\rho^2)}$.

P r o o f. Since $m_T = 2$, similar to the proof of Theorem 2.1 we have

$$F_Z(z) \quad = \quad 2[\Phi(z) - C_{X,Y}(\Phi(z), \Phi(0))].$$

Regarding to differentiating with respect to z, we first note that

$$\frac{\partial C_{X,Y}(u, v, \rho)}{\partial u} = \Phi\big(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\big)$$

and again using some chain rules we have

$$\begin{aligned} f_Z(z) &= 2\big[\varphi(z) - \frac{\partial C_{X,Y}}{\partial \Phi(z)} \frac{\partial \Phi(z)}{\partial z}\big] = 2\big[\varphi(z) - \Phi\big(\frac{-\rho z}{\sqrt{1-\rho^2}}\big)\varphi(z)\big] \\ &= 2\varphi(z)\Phi\big(\frac{\rho z}{\sqrt{1-\rho^2}}\big), \end{aligned}$$

which is 8. □

We can generalize or simplify this distribution by changing related copula between the random variables and their marginals as well. For example, considering that the random variable X follows a standard normal and Y follows an arbitrary distribution then we may present the following results.

Corollary 2.3. Let X be a standard normal random variables and Y has a distribution function $F_Y(y)$ with mean 0 and they are associated with the Gaussian copula with correlation ρ then the distribution of $Z \stackrel{\mathrm{d}}{=} (X|Y > 0)$ is

$$f_Z(z) = m_T\varphi(z)\Phi\big(\frac{\rho z - h_0}{\sqrt{1-\rho^2}}\big), \quad z \in \mathbb{R}, \tag{9}$$

where $h_0 = \Phi^{-1}(F_Y(0))$.

P r o o f. The proof is similar to the proof of Corollary 2.2 and is omitted. □

Moreover, one may easily prove the following proposition which explains the properties of this new distribution, see e.g., [3].

Proposition 1. For the density (9) the following results hold.

i) If Z has the pdf (9) with parameter $\lambda \in \mathbb{R}^+$ then $-Z$ follows (9) with parameter $-\lambda$ and vise versa.

ii) If $\lambda = 0$ in (9) then $f_Z(z) = \varphi(z)$.

iii) $f_{Z^2}(z^2) = \frac{\varphi(z)}{z}$

iv) $f_{|Z|}(|z|) = 2\varphi(z)$

v) If $Z' = \xi + wZ$ so $Z' \sim N(\xi, w^2)$.

Proof. We only prove parts (iii) and (v). The proof of other parts are straightforward. In order to prove (iii), let $Y = Z^2$ then the density function of Y will be $F_Y(y) = F_Z(\sqrt{y}) - F_Z(\sqrt{-y})$, so we have $f_Y(y) = \dfrac{\varphi(\sqrt{y})}{\sqrt{y}}$ and hence $f_{Z^2}(z^2) = \dfrac{\varphi(z)}{z}$.
Regarding (v), we note that $F_{Z'}(z') = F_Z(\dfrac{z'-\xi}{w})$, so

$$f_{Z'}(z') = \frac{\partial F_{Z'}(z')}{\partial z'} = \frac{1}{w}\varphi(\frac{z'-\xi}{w}) = \frac{1}{\sqrt{2\pi w^2}} e^{\frac{-1}{2w^2}(z'-\xi)^2}.$$

□

A schematic representation of the properties of this distribution is displayed in Figure 1. As seen in this figure, density curve Z and $-Z$ matches, when Z and $-Z$ have the pdf (9) with the parameters $\lambda = 10$ and $\lambda = -10$, which are stated in the part (i) of Proposition 1. For example, if $Z \in \mathbb{R}^+$ has the pdf (9) with the parameter $\lambda \in \mathbb{R}^-$ then $-Z$ has the same distribution but with the parameter $-\lambda$ and in this case $f_Z(z) = f_{-Z}(-z) = 0$. In addition, density curve Z and $-Z$ matches, when Z and $-Z$ with the pdf (9) and the parameters $\lambda = -10$ and $\lambda = 10$, respectively, which are stated in the part (i) of Proposition 1.

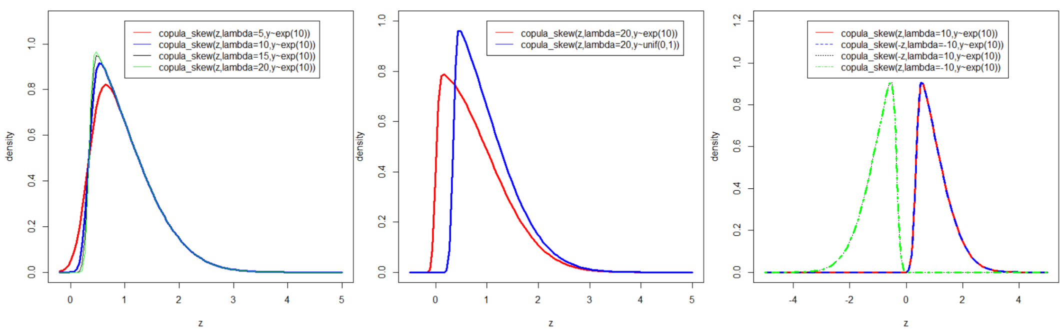

Fig. 1. Density plot of the proposed copula-skew distribution.

Regarding to investigate the moments of this distribution, we first find its Mgf as follows.

Corollary 2.4. Under the assumptions of corollary 2.3 the moment generating function of the random variable Z is

$$M_Z(t) = m_T e^{t^2/2}\Phi\big(\rho t - h_0\big), \tag{10}$$

where $h_0 = \Phi^{-1}(F_Y(0))$.

Proof.

$$\begin{aligned} M_Z(t) = E(e^{tz}) &= \int e^{tz} m_T \varphi(z)\Phi\Big(\frac{\rho z - h_0}{\sqrt{1-\rho^2}}\Big)\mathrm{d}z \\ &= m_T\ e^{t^2/2}\int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2}\Phi\Big(\frac{\rho z - h_0}{\sqrt{1-\rho^2}}\Big)\mathrm{d}z \\ &= m_T\ e^{t^2/2}\ \Phi\big(\rho t - h_0\big). \end{aligned}$$

□

As a quick result of the previous corollary, derivation of mgf and getting the torques, we have the mean and variance of the random variable Z, respectively as

$$\mu = m_T \rho\ \phi(h_0),$$

$$\sigma^2 = m_T \Phi\big(-h_0\big) + m_T \rho^2 h_0 \phi(h_0) - (m_T \rho\ \phi(h_0))^2.$$

Also, its skewness and kurtosis are obtained respectively as

$$s = \frac{\mu_3}{\sigma^3} \quad \text{and} \quad \kappa = \frac{\mu_4}{\sigma^4} - 3,$$

where
$\mu_3 = (3 - \rho^2 + \rho^2 h_0^2)k - 3m_T\phi(h_0) - 3\rho h_0 k^2 + 2k^3$ with $k = m_T\rho\phi(h_0)$ and
$\mu_4 = (\frac{3}{\rho} + 6\rho h_0 - 3\rho^3 h_0 + \rho^3 h_0^3)k + (4\rho^2 h_0^2 + 4\rho^2 + 6m_T\phi(h_0) - 12)k^2 + 6\rho h_0 k^3 - 3k^4$.
Special cases of these four moments yield when Y follows standard normal distribution and with the assumptions of corollary 2.3, the Mgf of Z will be [4]:

$$M_Z(t) = 2e^{t^2/2}\Phi\Big(\frac{\lambda t}{\sqrt{1+\lambda^2}}\Big)$$

and hence, its mean, variance, skewness and kurtosis, are respectively
$\mu = \sqrt{\frac{2}{\pi}}\rho$
$\sigma^2 = 1 - (\sqrt{\frac{2}{\pi}}\rho)^2$
$s = \dfrac{2(\sqrt{\frac{2}{\pi}}\rho)^3 - \sqrt{\frac{2}{\pi}}\rho^3}{(\sigma^2)^{\frac{3}{2}}}$
$\kappa = \dfrac{3 - 4(\frac{6}{\sqrt{2\pi}}\rho - \frac{2}{\sqrt{2\pi}}\rho^3)\sqrt{\frac{2}{\pi}}\rho + 6(\sqrt{\frac{2}{\pi}}\rho)^2 - 3(\sqrt{\frac{2}{\pi}}\rho)^4}{(\sigma^2)^2} - 3.$

Evidently, by switching the role of X and Y in corollary 2.3, we may conclude that if X is a random variables with distribution function $F_X(x)$ and Y has a standard normal distribution and they are associated to the Gaussian copula with correlation ρ (5) then the distribution of $Z \stackrel{\mathrm{d}}{=} (X|Y>0)$ is

$$f_Z(z) = 2f_X(z)\Phi\big(\frac{\rho\Phi^{-1}(F_X(z))}{\sqrt{1-\rho^2}}\big), \quad z \in \mathbb{R}. \tag{11}$$

3. PARAMETER ESTIMATION

Regarding to estimate the parameter of the proposed distribution, without loss of generality and in order to obtain a closed form, we first consider a simple case of copula skew-normal distribution under the assumption of Corollary 2.

Let $z_1,\ z_2,\dots,z_n$ be a sample of size n from distribution (9) in which the distribution of Y is free of parameter. The likelihood function is given by

$L_n(\lambda;\mathbf{z}) = f_n(\mathbf{z};\lambda) = m_T^n \Pi_{i=1}^n \varphi(z_i)\Pi_{i=1}^n \Phi(\lambda z_i - \sqrt{1+\lambda^2}\Phi^{-1}(F_Y(0)))$.

So,

$\ell(\lambda;\mathbf{z}) = lnL_n(\lambda;\mathbf{z}) = nlnm_T + \Sigma_{i=1}^n ln\varphi(z_i) + \Sigma_{i=1}^n ln\Phi(\lambda z_i - \sqrt{1+\lambda^2}\Phi^{-1}(F_Y(0)))$.

By differentiating with respect to λ, we obtain the following equality.

$$\frac{(z_1 - \frac{\lambda\Phi^{-1}(F_Y(0))}{\sqrt{1+\lambda^2}})\phi(\lambda z_1 - \sqrt{1+\lambda^2}\Phi^{-1}(F_Y(0)))}{\Phi(\lambda z_1 - \sqrt{1+\lambda^2}\Phi^{-1}(F_Y(0)))} + \ldots + \frac{(z_n - \frac{\lambda\Phi^{-1}(F_Y(0))}{\sqrt{1+\lambda^2)}})\phi(\lambda z_n - \sqrt{1+\lambda^2}\Phi^{-1}(F_Y(0)))}{\Phi(\lambda z_n - \sqrt{1+\lambda^2}\Phi^{-1}(F_Y(0)))} = 0$$

So, $\hat{\lambda}$ is the solution of following simultaneous equations.

$$z_i = \frac{\lambda\Phi^{-1}(F_Y(0))}{\sqrt{1+\lambda^2}} \quad or \quad \phi(\lambda z_i - \sqrt{1+\lambda^2}\Phi^{-1}(F_Y(0))) = 0, \quad i = 1,2,\dots,n.$$

Hence,

$$\hat{\lambda} = \frac{z_i}{\sqrt{\Phi^{-1}(F_Y(0))^2 - z_i^2}} \quad i = 1,2,\dots,n \quad or \quad \hat{\lambda} = \pm\infty.$$

Now, according to $\lambda = \frac{\rho}{\sqrt{1-\rho^2}}$, i. e., $\rho = \frac{\lambda}{\sqrt{(1+\lambda^2)}}$, we readily estimate $\hat{\rho} = \frac{z_i}{\Phi^{-1}(F_Y(0))}$, $i = 1,2,\dots,n$ or $\hat{\rho} = \pm 1$. In the other hand, $\hat{\rho} = \pm 1$ is unacceptable because $\rho \in (-1,1)$. Hence,

$$\hat{\rho} = \frac{z_i}{\Phi^{-1}(F_Y(0))} \quad i = 1,2,\dots,n.$$

As another more general case of corollary 2.3, in which $\mu_y \neq 0$, let $Y \sim beta(\alpha,1)$, then from 7 we again have the density 9 except $h_{\mu_y} = \Phi^{-1}(F_Y(\mu_y))$ instead of h_0, i. e.,

$$f_Z(z) = m_T\varphi(z)\Phi\big(\frac{\rho z - \Phi^{-1}(F_Y(\mu_y))}{\sqrt{1-\rho^2}}\big), \quad z \in \mathbb{R}. \tag{12}$$

Therefore, from $F_Y(\mu_y) = (\frac{\alpha}{\alpha+1})^\alpha$ we have the log-likelihood function as

$\ell(\alpha,\lambda;\mathbf{z}) = lnL_n(\alpha,\lambda;\mathbf{z}) = nlnm_T + \Sigma_{i=1}^n ln\varphi(z_i) + \Sigma_{i=1}^n ln\Phi(\lambda z_i - \sqrt{1+\lambda^2}\Phi^{-1}((\frac{\alpha}{\alpha+1})^\alpha))$

and by derivation with respect to λ and α, the estimated values of these parameters will be the solution of the following two simultaneous equations:

$$\begin{cases} \sum_{i=1}^{n} \frac{(z_i - \frac{\lambda\Phi^{-1}(\frac{\alpha}{\alpha+1})^{\alpha}}{\sqrt{1+\lambda^2}})\phi(\lambda z_i - \sqrt{1+\lambda^2}\Phi^{-1}((\frac{\alpha}{\alpha+1})^{\alpha})))}{\Phi(\lambda z_i - \sqrt{1+\lambda^2}\Phi^{-1}((\frac{\alpha}{\alpha+1})^{\alpha})))} = 0 \\ \frac{\alpha^{\alpha}(ln\frac{\alpha}{\alpha+1}+\frac{1}{\alpha+1})\sqrt{1+\lambda^2}}{(\alpha+1)^{\alpha}\phi(\Phi^{-1}((\frac{\alpha}{\alpha+1})^{\alpha}))} \sum_{i=1}^{n} \frac{\phi(\lambda z_i - \sqrt{1+\lambda^2}\Phi^{-1}((\frac{\alpha}{\alpha+1})^{\alpha}))}{\Phi(\lambda z_i - \sqrt{1+\lambda^2}\Phi^{-1}((\frac{\alpha}{\alpha+1})^{\alpha}))} = \frac{n\alpha^{\alpha}(ln\frac{\alpha}{\alpha+1}+\frac{1}{\alpha+1})}{(\alpha+1)^{\alpha}-(\alpha)^{\alpha}} \end{cases}$$

4. NUMERICAL ANALYSIS

4.1. Simulation study

In order to assess and visualize our proposed distribution, in the section we used a Monte Carlo simulation study. We generated 1000 random pairs $(X_i, Y_i), i = 1, 2, \ldots, 1000$, in such a way that X followed a standard normal distribution and Y came from an exponential distribution with λ and they are connected via a *Gaussian*-copula with correlation ρ_{XY}[1]. Without loss of generality, we considered four values for λ as $\lambda = 2, 5, 10, 20$ and ρ_{XY} varies within values of $\rho_{XY} = 0.8$, $\rho_{XY} = 0.9$ and $\rho_{XY} = 0.99$ and from 12 we obtain the density of $Z = X|Y > \mu_y$. We repeated this procedure 5000 times. Tables 1-3 summarized the AIC and BIC of our proposed copula-skew-normal distribution against the skew-normal ones. As seen from these tables, increasing value of correlation between two copula-connected random variables, yields a significant difference between goodness of fit of these two distribution in favor of copula-skew-normal distribution, see also Figure 2. Also, high values of the exponential rate of the distribution of Y cause more distinct the goodness of fit between the copula-skew-normal distribution and skew-normal ones.

$\rho = 0.8$		
Estimation	copula-skew	skew-normal
AIC	818	819
BIC	833	833
$\rho = 0.9$		
Estimation	copula-skew	skew-normal
AIC	750	751
BIC	765	766
$\rho = 0.99$		
Estimation	copula-skew	skew-normal
AIC	532	534
BIC	546	549

Tab. 1. AIC and BIC of copula-skew and skew-normal when $Y \sim exp(\lambda = 2)$.

[1]We use the R software and all codes are available upon request.

$\rho = 0.8$		
Estimation	copula-skew	skew-normal
AIC	814	814
BIC	828	829
$\rho = 0.9$		
Estimation	copula-skew	skew-normal
AIC	716	717
BIC	731	732
$\rho = 0.99$		
Estimation	copula-skew	skew-normal
AIC	528	532
BIC	543	547

Tab. 2. AIC and BIC of copula-skew and skew-normal when $Y \sim exp(\lambda = 10)$.

$\rho = 0.8$		
Estimation	copula-skew	skew-normal
AIC	813	814
BIC	828	828
$\rho = 0.9$		
Estimation	copula-skew	skew-normal
AIC	727	728
BIC	742	743
$\rho = 0.99$		
Estimation	copula-skew	skew-normal
AIC	526	531
BIC	541	546

Tab. 3. AIC and BIC of copula-skew and skew-normal when $Y \sim exp(\lambda = 20)$.

4.2. Real data

Regarding to apply our previous material so far in a real data set analysis, we consider the well known data set collected by Australian Institute of sports (AIC) [7]. This data set is compressing of biological characteristics of 202 Australian athletes in both sexes (102 males and 100 females) and was considered as an example of skew-normal data in the literature [6, 5]. Considering two variables height (Ht) and sum of skin folds (SSF) and by selecting a random sample of size 150 from this data set, we observed that "Ht" followed a normal distribution with mean 180 and variance 100 and "SSF" followed an exponential distributions with an average of 68. Moreover we found that these two

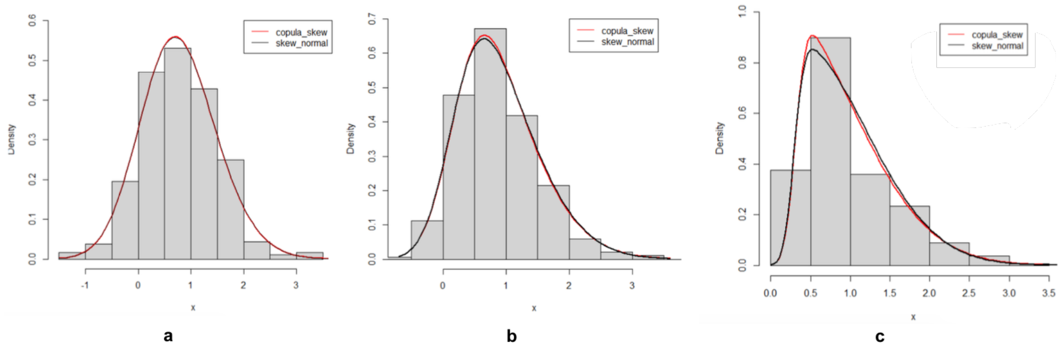

Fig. 2. Density plot of skew-normal and copula-skew distributions in simulated data when $Y \sim exp(\lambda = 20)$ and a) $\rho = 0.80$, b) $\rho = 0.90$, c) $\rho = 0.99$.

variables were connected via a *Gaussian*-copula with correlation $\rho = -0.06$. By conditioning the distribution of HT given SSF greater than 68, from 12, the distribution of $Z = Ht|SSF > 68$, was fitted as

$$f_Z(z) = m_T \varphi(z) \Phi\big(1.002(-0.06z - \Phi^{-1}(F_Y(68))\big), \quad z \in \mathbb{R}.$$

Comparing the AIC and BIC values of the Table 4, we conclude that implementing the correlation between two variables Ht and SSF yields an improvement to the fitting the distribution of $Z = Ht|SSF > 68$, in contrast to the traditional school normal distribution, see also Figure 3.

Estimation	copula-skew	skew-normal
AIC	155	158
BIC	170	173

Tab. 4. AIC and BIC of copula-skew and skew-normal.

5. CONCLUSION

Based on the idea of the selection distributions, in this work we have proposed a conditional distribution of two copula related random variables which has asymmetric behaviors. Using a simulation analysis we have shown that our proposed distribution has a better performance in contrary of other skew distributions.

The idea of this work may be extended in several manners. Although, we have considered the Gaussian copula as a connection between random variables, other copula functions one may be assumed elsewhere. Also, we have only considered two related

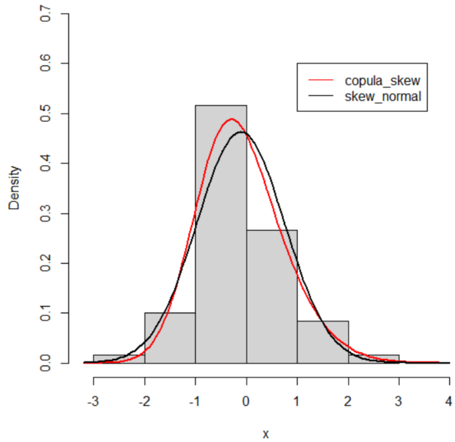

Fig. 3. Performance of skew-normal and copula-skew in real data.

random variables, so more than two variables variables would be of interest. In our ongoing work we are extending this distribution to its multivariate version.

Finally, it is well known that the skew distributions are frequently used to model the behavior of order statistics, see e. g., Loperfido (2008) [10] and Sheikhi and Tata (2013) [13]. Our copula skew distribution will be used in this subject as well as in concomitants of order statistics.

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