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A PRINCIPAL TOPOLOGY OBTAINED FROM UNINORMS

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We obtain a principal topology and some related results. We also give some hints of possible applications. Some mathematical systems are both lattice and topological space. We show that a topology defined on the any bounded lattice is definable in terms of uninorms. Also, we see that these topologies satisfy the condition of the principal topology. These topologies can not be metrizable except for the discrete metric case. We show an equivalence relation on the class of uninorms on a bounded lattice based on equality of the topologies induced by uninorms.

Keywords: uninorm, closure operator, principal topology, bounded lattice

Classification: 03E72, 03B52, 06B30, 06F30, 08A72, 54A10

1. INTRODUCTION

Uninorms on the real unit interval $[0,1]$ have been studied by Yager and Rybalov [29]. These uninorms are the important aggregation operators which have interesting structures that generalize the notions of t-norms and t-conorms [7, 31], with more applications such as fuzzy logic, expert systems, neural networks, fuzzy system modelling [6, 10, 12, 20, 30]. The generalized problem of logical operators on the real unit interval $[0,1]$ for a complete lattice has been an attractive problem for many researchers [9, 13, 16, 18, 32, 33].

Principal spaces were first studied by Alexandroff [1]. It is a topological space in which an arbitrary intersection of open sets is open. Equivalently, each singletion has a minimal neighbourhood base. Principal spaces have important attentions in digital topology [21, 26]. By using the properties of finite spaces, they play an important role in image analysis and computer graphics [19, 22, 23]. Moreover, researchers focus on generating methods for new principal topologies by means of triangular norms and uninorms [13, 15].

This paper is organized as follows. In Section 2, we shortly recall some basic notions and results. We introduce a principal topology induced a uninorm in Section 3. We also show in this section that the uninorm is continuous and closed on a bounded lattice equipped with this topology. The preorder, denoted by $\preceq_U$, induced by this principal topology obtained from uninorms on a bounded lattice has been introduced. Furthermore, we obtain the order induced by t-norms (or t-conorms) on a bounded lattice. We

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introduce an equivalence on the class of uninorms on a bounded lattice based on the equality of the principal topologies induced by uninorms. We determine some relationships between the principal topologies induced by t-norms and their $N-$dual $t-$conorms. The last section is devoted to an application to the principal topological structure in dynamic Kripke frames. Also, principal topologies with uncountable cardinality of the same form uninorms on unit interval is obtained in this section.

2. PRELIMINARIES

Definition 1. (Birkhoff [5]) A lattice $(L, \leqslant)$ is bounded if L have top and bottom elements, which are denoted as 1 and 0, respectively, that is, there exists two elements $1, 0 \in L$ such that $0 \leqslant x \leqslant 1$, for all $x \in L$.

Definition 2. (Karaçal and Mesiar [16]) Let $(L, \leqslant, 0, 1)$ be a bounded lattice. An operation $U : L^2 \to L$ is called a uninorm on L, if it is commutative, associative, increasing with respect to the both variables and has a neutral element $e \in L$.

Definition 3. (Aşıcı and Karaçal [3] and Ma and Wu [25]) An operation T (S) on a bounded lattice L is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to the both variables and has a neutral element 1 (0).

Recall that, a uninorm U possessing a neutral element $e = 1$ is, in fact, a triangular norm. Similarly, a uninorm U with a neutral element $e = 0$ is, in fact, a triangular conorm.

Definition 4. (Baczyński and Jayaram [4]) Let $(L, \leqslant, 0, 1)$ be a bounded lattice. A decreasing operation $N : L \to L$ is called a negation if $N(0) = 1$ and $N(1) = 0$. A negation N on L is called strong if it is an involution, i. e., $N(N(x)) = x$, for all $x \in L$.

Definition 5. (Baczyński and Jayaram [4]) Let T be a t-norm on a bounded lattice L and N be a strong negation on L. The t-conorm S is defined by $S(x, y) = N(T(N(x), N(y))$ for all $x, y \in L$ is called the $N-$dual t-conorm to T on L.

Definition 6. (Baczyński and Jayaram [4]) If T is a t-norm on the unit interval $[0, 1]$ and $\phi : [0, 1] \to [0, 1]$ is an order-preserving bijection, then the operation $T_\phi : [0, 1]^2 \to [0, 1]$ given by $T_\phi(x, y) = \phi^{-1}(T(\phi(x), \phi(y))$ is also a t-norm. This t-norm is called $\phi-$conjugate of T.

The $\phi-$conjugate of any t-norm (or t-conorm) on a bounded lattice is defined as Definition 6.

Definition 7. (Kelley [17]) A topology on a set X is a collection τ of subsets of X such that;

1. $\emptyset$, $X \in \tau$,
2. The union of elements of any sub collection of τ is in τ,
3. The intersection of the elements of any finite sub collection of τ is in τ.

A set X together with a topology $\tau \subseteq \wp(X)$ is called a topological space (X, τ) and the set $G \in \tau$ is called open set of (X, τ). Here, $\wp(X)$ denote the power set of X.

Definition 8. (Kelley [17]) A subset F of X of a topological space (X, τ) is called closed if $X \setminus F$ is open.

Definition 9. (Arenas [2]) Let X be a topological space. Then X is a principal space (or Alexandroff space) if the intersection of arbitrary open sets is open.

Definition 10. (Kelley [17]) Let $f : (X, \tau_X) \to (Y, \tau_Y)$ be a function between the topological spaces X and Y. The map f is called continuous if the preimage of every open subset of Y is open in X.

Definition 11. (Kelley [17]) Let X and Y be topological spaces. A map $f : X \to Y$ is called an open map if the image $f(U) \subseteq Y$ is open for every open set $U \subseteq X$. Furthermore, f is said to be a closed map if the image $f(F) \subseteq Y$ is closed for every closed set $F \subseteq X$.

Proposition 1. (Kelley [17]) Let X and Y be topological spaces. A map $f : X \to Y$ is closed if and only if $\overline{f(A)} \subseteq f(\overline{A})$ for every set $A \subseteq X$.

Definition 12. (Kelley [17]) Let X be a topological space. X is called $T_0 -$ space if for each of points $x \neq y \in X$, there is a neighborhood of x, U_x such that $y \notin U_x$.

Proposition 2. (Kelley [17]) Let X be a topological space. Then, X is a $T_0 -$space if and only if either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$ for all $x, y \in X$ such that $x \neq y$.

Definition 13. (Kelley [17]) A closure operator on X is an operator which assings to each subset A of X a subset $C(A)$ of X such that the following four statements so-called the Kuratowski closure axioms, are true.

C1. $C(\emptyset) = \emptyset$,

C2. $A \subseteq C(A)$ for all $A \subseteq X$,

C3. $C(C(A)) = C(A)$ for all $A \subseteq X$,

C4. $C(A \cup B) = C(A) \cup C(B)$ for all $A, B \subseteq X$.

The following theorem of Kuratowski shows that these four statements are actually characteristic of closure. The topology defined below is the topology associated with a closure operator.

Theorem 3. (Kelley [17]) Let C be a topological closure operator on X, Let $\mathfrak{F}$ be the family of all subsets A of X for which $C(A) = A$, and let $\mathfrak{T}$ be the family of complements of members of $\mathfrak{F}$. That is, $\mathfrak{T} = \{A^c : A \in \mathfrak{F}\}$. Then $\mathfrak{T}$ is a topology for X, and $C(A)$ is the $\mathfrak{T}_C$-closure of A for each subset A of X.

3. PRINCIPAL TOPOLOGY GENERATED BY UNINORMS

Proposition 4. Let L be a bounded lattice and U be a uninorm on L with the neutral element e . Let A be a subset of L such that $e \in A$ and $U(A, A) \subseteq A$. Then the operator $C_{U,A} : \wp(L) \to \wp(L)$ defined for all $X \in \wp(L)$ by $C_{U,A}(X) := U(X, A)$ is a topological closure operator on L.

Throughout this paper, the operator $C_{U,A}$ is just denoted by the letter C unless otherwise stated. Proof. Let X, Y be subsets of L.

C1: $C(\emptyset) = U(\emptyset, A) = \emptyset$.

C2: $x = U(x, 1) \in X = \{U(x, 1) : x \in X\} \subseteq \{U(x, a) : x \in X,\ a \in A\} = C(X)$ for all $x \in X$.

C3:

$$\begin{aligned} C(C(X)) &= U(C(X), A) = U(U(X, A), A) && \text{(by def. } C) \\ &= U(U(X, A), A) && \text{(by uninorm properties)} \\ &= U(X, U(A, A)) && \text{(by uninorm properties)} \\ &= U(X, A) = C(X). \end{aligned}$$

C4: We verify the statement $X \subseteq Y \Rightarrow C(X) \subseteq C(Y)$. This is clear, because of $C(X) = U(X, A) = \{U(x, a) : x \in X, a \in A\} \subseteq \{U(y, a) : y \in Y, a \in A\} = U(Y, A) = C(Y)$. Now we show that C4 is valid. $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y$ imply $C(X) \subseteq C(X \cup Y)$ and $C(Y) \subseteq C(X \cup Y)$. Then, $C(X) \cup C(Y) \subseteq C(X \cup Y)$ is true. On the other hand, $C(X \cup Y) \subseteq C(X) \cup C(Y)$ is clear from the elementary mathematics. Hence, $C(X \cup Y) = C(X) \cup C(Y)$ is obtained.

□

Definition 14. A subset X of L is called a closed subset if $C(X) = X$. The family of all closed subsets of L is denoted by $\mathfrak{F}_C = \{C(X) : X \subseteq L\}$.

Proposition 5. Let $C(X)$ be the closure of $X \subseteq L$. Then, $\mathfrak{T}_C = \{L \backslash C(X) : X \in \wp(L)\}$ is a topology on L.

Proof. The proof follows from Proposition 4 and Theorem 3. □

Proposition 6. The family $\mathfrak{F}_C$ satisfies arbitrary intersection property, i. e.

$$C\left(\bigcap_{\lambda \in I} X_\lambda\right) = \bigcap_{\lambda \in I} (X_\lambda)$$

for arbitrary subfamily $\{X_\lambda : \lambda \in I\}$ of $\mathfrak{F}_C$.

P r o o f. Since $X_\lambda \in \mathfrak{F}_C$ for all $\lambda \in I$, we have $C(X_\lambda) = X_\lambda$. Then, by C4 we get,

$$\bigcap_{\lambda\in I} C(X_\lambda) = \bigcap_{\lambda\in I} X_\lambda \subseteq C\left(\bigcap_{\lambda\in I} X_\lambda\right).$$

On the other hand, since $\bigcap_{\lambda\in I} X_\lambda \subseteq X_\lambda$ and $X_\lambda \in \mathfrak{F}_C$ for all $\lambda \in I$, by C4 we get ,

$$C\left(\bigcap_{\lambda\in I} X_\lambda\right) \subseteq \bigcap_{\lambda\in I} C(X_\lambda).$$

Therefore we have,

$$C\left(\bigcap_{\lambda\in I} X_\lambda\right) = \bigcap_{\lambda\in I} C(X_\lambda) = \bigcap_{\lambda\in I} X_\lambda$$

and this completes the proof. □

Proposition 7. Let L be a bounded lattice and U be a uninorm on L with the neutral element e . Let A be a subset of L such that $e \in A$ and $U(A, A) \subseteq A$. Then, $(L, \mathfrak{T}_C)$ is a principal topological space.

P r o o f. From Proposition 5, $\mathfrak{T}_C$ is a topology, so it is sufficient to show the arbitrary intersection property. Let $\{G_\lambda : \lambda \in I\}$ be an arbitrary subfamily of the topology $\mathfrak{T}_C$. Then we have,

$$\begin{aligned}
C\left((\bigcap_{\lambda\in I} G_\lambda)^c\right) &= C\left(\bigcup_{\lambda\in I}(G_\lambda)^c\right) && \text{(by De-Morgan rule)}\\
&= U\left(\bigcup_{\lambda\in I}(G_\lambda)^c, A\right) && \text{(by def. } C)\\
&= \bigcup_{\lambda\in I}\{U((G_\lambda)^c, A)\} && \text{(by set theory)}\\
&= \bigcup_{\lambda\in I} C\left((G_\lambda)^c\right) && \text{(by def. } C)\\
&= \bigcup_{\lambda\in I}(G_\lambda)^c = \left(\bigcap_{\lambda\in I}(G_\lambda)\right)^c \in \mathfrak{F}_C && (G_\lambda \in \mathfrak{T}_C \text{ and De-Morgan rule})
\end{aligned}$$

It follows that in a principal space L, for each point x, there is a smallest neighbourhood which is contained in each other neighbourhood of x. For each $x \in L$,

$$\bigcap\{V : V \text{ is an open set containing } x\}$$

is the smallest open set containing x since L is a principal space. □

Proposition 8. Let U be a uninorm with the neutral element e on a bounded lattice L and a subset A of L such that $U(A, A) \subseteq A$, $e \in A$. Then, the set A is closed (i. e., $A \in \mathfrak{F}_C$).

Proof. $C(\{e\}) = U(\{e\}, A) = \{U(e, a) : a \in A\} = A$ □

Theorem 9. Let U be a uninorm with the neutral element e on a bounded lattice L and a subset A of L such that $U(A, A) \subseteq A$, $e \in A$. The following statements are equivalent for the $\mathfrak{T}_C$ topology.

(i) $\{x\} \in \mathfrak{F}_C$ for all $x \in L$.

(ii) $\{e\} \in \mathfrak{F}_C$.

(iii) $A = \{e\}$

(iv) $\mathfrak{T}_C$ is metrizable with discrete metric.

Proof. $(i) \Rightarrow (ii)$: Trivial.
$(ii) \Rightarrow (iii)$: Since, $\{e\} \in \mathfrak{F}_C$ then, $\{e\} = C(\{e\}) = U(\{e\}, A) = \{U(e, a) : a \in A\} = A$.
$(iii) \Rightarrow (iv)$: Let $A = \{e\}$. For all $x \in L$, we obtain that $C(\{x\}) = \{U(x, e)\} = \{x\}$. Hence $\{x\} \in \mathfrak{F}_C$. Therefore $\mathfrak{T}_C$ is metrizable with discrete metric.
$(iv) \Rightarrow (i)$: We suppose that $\mathfrak{T}_C$ is metrizable with discrete metric. Then, the set $\{x\}$ is closed for all $x \in L$. That is $\{x\} \in \mathfrak{F}_C$ for all $x \in L$. □

Remark 1. The Euclid topology on the real unit interval can not be generated by any uninorm and any special subset with this method.

Definition 15. Let U be a uninorm with the neutral element e on a bounded lattice L and a subset A of L such that $U(A, A) \subseteq A$, $e \in A$. The preorder $x \preceq_U y \Leftrightarrow x \in C(\{y\})$ is called a U-preorder for U.

Remark 2. Let $L = [0, 1]$ be the real unit interval and $U = U_{\min,\max}$ be a uninorm on L with the neutral element e, where $U_{\min,\max}$ is defined by,

$$U_{\min,\max} := \begin{cases} \min(x, y) & \max(x, y) \le e \\ \max(x, y) & \text{otherwise.} \end{cases}$$

Let be $x < y < e$ and $A = [e, 1]$. Then,

$$\begin{aligned} C(\{x\}) &= \{U(x, a) : a \in [e, 1]\} && \text{(by def. of C)} \\ &= \{\max(x, a) : a \in [e, 1]\} && \text{(by def. of U)} \\ &= \{a : a \in [e, 1]\} = A, \end{aligned}$$

and similarly,

$$\begin{aligned} C(\{y\}) &= \{U(y, a) : a \in [e, 1]\} && \text{(by def. of C)} \\ &= \{\max(y, a) : a \in [e, 1]\} && \text{(by def. of U)} \\ &= \{a : a \in [e, 1]\} = A. \end{aligned}$$

Hence, $C(\{x\}) = C(\{y\})$ is obtained although $x \neq y$. Then the topology $\mathfrak{T}_C$ may not be $T_0 -$ space. If we considering the relation $x \preceq_U y \Leftrightarrow x \in C(\{y\})$ on L, we understand that $(L, \preceq_U)$ is a pre order. Because it satisfies reflexive and transitive properties.

We conclude from Remark 2 that $(L, \preceq_U)$ may not be a partial order on L. The following example illustrates this case.

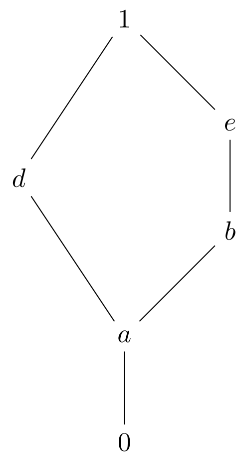

Fig. 1: A lattice correspond to $L = \{0, a, b, e, d, 1\}$.

U	0	a	b	e	d	1
0	0	0	0	0	d	1
a	0	0	0	a	d	1
b	0	0	0	b	d	1
e	0	a	b	e	d	1
d	d	d	d	d	1	1
1	1	1	1	1	1	1

Tab. 1: The Uninorm U on L.

Example 1. We consider the bounded lattice L in Figure 1 and the uninorm U in Table 1. Let $A = \{e, 1\} \subseteq L$.

$$C_{U,A}(\{0\}) = \{U(0, e), U(0, 1)\} = \{0, 1\}$$

$$C_{U,A}(\{a\}) = \{U(a, e), U(a, 1)\} = \{a, 1\}$$

$$C_{U,A}(\{b\}) = \{U(b, e), U(b, 1)\} = \{b, 1\}$$

$$C_{U,A}(\{1\}) = \{U(1, e), U(1, 1)\} = \{1\}$$

$$C_{U,A}(\{e\}) = \{U(e, e), U(e, 1)\} = \{e, 1\}$$

$$C_{U,A}(\{d\}) = \{U(d,e), U(d,1)\} = \{d,1\}$$
$$C_{U,A}(\{a,b\}) = \{U(a,e), U(a,1), U(b,e), U(b,1)\} = \{a,b,1\}$$

It is seen that $C_{U,A}(X) = X \cup \{1\}$, for the subset X of L and the topology generated by U and A is written as,

$$\tau = \{X^c \cap \{0,a,b,d,e\} : X \subseteq L\}.$$

Then, the uniqe open set including the element 1 is the set L. Also we can consider the preorder

$$\preceq_U = \{(0,0),(1,0),(b,b),(1,b),(e,e),(1,e),(d,d),(1,d),(a,a),(1,a)\}$$

defined as in Definition 15. The hasse diagram of this preorder can be shown as Figure 2.

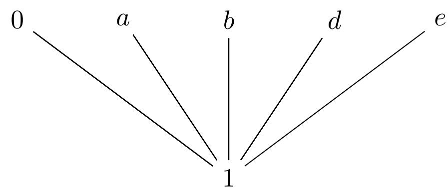

Fig. 2: The preorder of $(L, \preceq_U)$.

Proposition 10. If U is a t-norm (or t-conorm) on a bounded lattice L and a subset A of L such that $U(A,A) \subseteq A$, $e \in A$, then $\preceq_U$ is an order.

Proof. Let U be a t-norm on a bounded lattice L. We suppose $x \preceq_U y$ and $y \preceq_U x$ for $x, y \in L$. Then, $x \in C_{U,A}(\{y\})$ and $y \in C_{U,A}(\{x\})$ are valid. Then, there exist elements $a_1, a_2 \in A$ such that $x = U(y, a_1)$ and $y = U(x, a_2)$. Further, since U is a t-norm we have $x = U(y,a_1) \leq y \wedge a_1 \leq y$ and $y = U(x,a_2) \leq x \wedge a_2 \leq x$. From these we get $x = y$. $\square$

Corollary 11. If U is a t-norm (or t-conorm) on a bounded lattice L and a subset A of L such that $U(A,A) \subseteq A$, $1 \in A$ (or $0 \in A$), then the topological space $(L, \mathfrak{T}_{U,A})$ is a T_0-space.

Proposition 12. Let T be a t-norm on bounded lattice L, N be a strong negation on L and S be a $N-$dual t-conorm to T on L. Then following stetements are true for $A \subseteq L$.

(i) $T(A,A) \subseteq A$ and $1 \in A$ iff $S(N(A), N(A)) \subseteq N(A)$ and $0 \in N(A)$.

(ii) The spaces $(L, \mathfrak{T}_{T,A})$ and $(L, \mathfrak{T}_{S,N(A)})$ are homeomorphic.

Proof. (i) Let, $T(A,A) \subseteq A$ and $1 \in A$. It is clear $0 \in N(A)$. Then, we have $S(N(A), N(A)) = N(T(A,A)) \subseteq N(A)$. The converse is similar.

(ii) $N : L \to L$ is the desired homeomorphism. Because $x \preceq_{T,A} y \iff N(x) \preceq_{S,N(A)} N(y)$ is satisfied. Indeed, if $x \preceq_{T,A} y$ then, there exists an element $a \in A$ such that $x = T(y,a)$, $N(x) = N(T(y,a)) = N(T(N(N(y)), N(N(a))) = S(N(y), N(a))$. Hence, we have $N(x) \preceq_{S,N(A)} N(y)$. The converse is similar. $\square$

Proposition 13. Let T be a t-norm on bounded lattice L, N be a strong negation on L and S be a $N-$dual t-conorm to T on L. Then following statements are equivalent.

(i) N is increasing with respect to $\preceq_{T,A}$.

(ii) $\mathfrak{T}_{T,A}$ is a discrete topology.

(iii) $\mathfrak{T}_{S,N(A)}$ is a discrete topology.

(iv) $\preceq_{T,A}=\preceq_{S,N(A)}$.

Proof. $(i) \Rightarrow (ii)$: Suppose that $A \neq \{1\}$. Then, there exists an element $a' \in A$ such that $a' \neq 1$. By $C_{T,A}(\{1\}) = A$, we have $a' \preceq_{T,A} 1$. Since N is increasing with respect to $\preceq_{T,A}$, we have $N(a') \preceq_{T,A} N(1) = 0$. Hence, we obtain $N(a') = T(0, a') = 0$ and $a' = 1$. Thus $A = \{1\}$. By Theorem 9, we obtain $\mathfrak{T}_{C,T}$ is a discrete topology.

$(ii) \Rightarrow (iii)$: Since $C_{T,A}(\{x\}) = \{x\}$ for all $x \in L$, we have the following equalities,

$$\begin{aligned} C_{S,N(A)}(\{x\}) &= \{S(x, N(a)) : a \in A\} \\ &= \{N(T(N(x), a)) : a \in A\} \\ &= \{N(N(x))\} \\ &= \{x\}\,. \end{aligned}$$

Thus, by considering $C_{T,A}(X) = X$ for all $X \subseteq L$, we get:

$$\begin{aligned} C_{S,N(A)}(X) &= S(X, N(A)) \\ &= S\left(\bigcup_{x\in X}\{x\}\,, N(A)\right) \\ &= \bigcup_{x\in X} S(\{x\}\,, N(A)) \\ &= \bigcup_{x\in X} C_{S,N(A)}(\{x\}) \\ &= \bigcup_{x\in X}\{x\} = X. \end{aligned}$$

$(iii) \Rightarrow (iv)$: Let $\mathfrak{T}_{S,N(A)}$ be a discrete topology. Then, both of $\preceq_{T,A}$ and $\preceq_{S,N(A)}$ are the equality relations. Hence, $\preceq_{T,A}=\preceq_{S,N(A)}$.

$(iv) \Rightarrow (i)$: Let $\preceq_{T,A}=\preceq_{S,N(A)}$. Then, by the proof of Proposition 12 (ii), $x \preceq_{T,A} y \iff N(x) \preceq_{S,N(A)} N(y)$. Finally, we have $x \preceq_{T,A} y \iff N(x) \preceq_{S,N(A)} N(y) \iff N(x) \preceq_{T,A} N(y)$. □

The proof of following proposition can be easily obtained.

Proposition 14. Let T be a t-norm on a lattice L and ϕ be an order preserving bijection and $T(A, A) \subseteq A$ and $1 \in A$ then,

(i) $T_\phi(\phi^{-1}(A), \phi^{-1}(A)) \subseteq \phi^{-1}(A)$ and $1 \in \phi^{-1}(A)$,

(ii) ϕ is order preserving with respect to $\preceq_{T,A} \Longleftrightarrow$ ϕ is an order preserving with respect to $\preceq_{T_\phi,\phi^{-1}(A)}$,

(iii) ϕ^{-1} is an order preserving with respect to $\preceq_{T,A} \Longleftrightarrow$ ϕ^{-1} is an order preserving with respect to $\preceq_{T_{T_\phi,\phi^{-1}(A)}}$.

Proof. (i) The proof is clear.

(ii)($\Rightarrow$:)Let ϕ be order preserving with respect to $\preceq_{T,A}$ and $x \preceq_{T_\phi,\phi^{-1}(A)} y$ for all $x, y \in L$. We must show that $\phi(x) \preceq_{T_\phi,\phi^{-1}(A)} \phi(y)$. Then, there exist $x_1, y_1 \in L$ such that $x = \phi^{-1}(x_1)$ and $y = \phi^{-1}(y_1)$. Therefore, we get the following equalities:

$$
\begin{aligned}
\phi^{-1}(x_1) \preceq_{T_\phi,\phi^{-1}(A)} \phi^{-1}(y_1) &\Rightarrow \phi^{-1}(x_1) = T_\phi(\phi^{-1}(y_1), \phi^{-1}(a)) \text{ for some } a \in A \\
&\Rightarrow \phi^{-1}(x_1) = \phi^{-1}(T(\phi(\phi^{-1}(y_1)), \phi(\phi^{-1}(a))) \\
&\Rightarrow \phi^{-1}(x_1) = \phi^{-1}(T(y_1, a)) \\
&\Rightarrow x_1 = T(y_1, a) \\
&\Rightarrow x_1 \preceq_{T,A} y_1 \\
&\Rightarrow \phi(x_1) \preceq_{T,A} \phi(y_1) \\
&\Rightarrow \phi(x_1) = T(\phi(y_1), a') \text{ for some } a' \in A \\
&\Rightarrow x_1 = \phi^{-1}(T(\phi(y_1), a')) = T_\phi(y_1, \phi^{-1}(a')) \\
&\Rightarrow x_1 \preceq_{T_\phi,\phi^{-1}(A)} y_1 \\
&\Rightarrow \phi(x) \preceq_{T_\phi,\phi^{-1}(A)} \phi(y).
\end{aligned}
$$

($\Leftarrow$:)

Let ϕ be order preserving with respect to $\preceq_{T_\phi,\phi^{-1}(A)}$ and $x \preceq_{T,A} y$ for $x, y \in L$. Then, there exist $x_1, y_1 \in L$ such that $x = \phi(x_1)$ and $y = \phi(y_1)$. Thus, we get the following equalities:

$$
\begin{aligned}
\phi(x_1) \preceq_{T,A} \phi(y_1) &\Rightarrow \phi(x_1) = T(a_1, \phi(y_1)) \text{ for some } a_1 \in A \\
&\Rightarrow x_1 = \phi^{-1}(T(a_1, \phi(y_1))) = T_\phi(\phi^{-1}(a_1), y_1) \\
&\Rightarrow x_1 \preceq_{T_\phi,\phi^{-1}(A)} y_1 \\
&\Rightarrow \phi^{-1}(x) \preceq_{T_\phi,\phi^{-1}(A)} \phi^{-1}(y) \\
&\Rightarrow \phi(\phi^{-1}(x)) \preceq_{T_\phi,\phi^{-1}(A)} \phi(\phi^{-1}(y)) \\
&\Rightarrow x \preceq_{T_\phi,\phi^{-1}(A)} y \\
&\Rightarrow x = T_\phi(y, \phi^{-1}(a)) \text{ for some } a \in A \\
&\Rrightarrow x = \phi^{-1}(T(\phi(y), a)) \\
&\Rightarrow \phi(x) = T(\phi(y), a) \\
&\Rightarrow \phi(x) \preceq_{T,A} \phi(y).
\end{aligned}
$$

(iii) The proof is similar to the proof of (ii). $\square$

Proposition 15. Let T be a t-norm on lattice L and ϕ be an order preserving bijection. If $T(A,A) \subseteq A$ and $1 \in A$ then, the following statements are equivalent:

(i) $\mathfrak{T}_{T,A} = \mathfrak{T}_{T_\phi,\phi^{-1}(A)}$.

(ii) For $x, y \in L$, $x \preceq_{T,A} y \iff \phi(x) \preceq_{T,A} \phi(y)$.

Proof. (i)⇒(ii): Let $x \preceq_T y$ for some $x, y \in L$. Then, it is obtained that $x \in C_{T,A}(\{y\}) = C_{T_{\phi,\phi^{-1}(A)}}(\{y\})$. Hence, for some $\phi^{-1}(a) \in \phi^{-1}(A)$ we get,

$$x = T_\phi(y, \phi^{-1}(a)) = \phi^{-1}(T(\phi(y), \phi(\phi^{-1}(a))) = \phi^{-1}(T(\phi(y), a)).$$

Then, we have for some $a \in A$, $\phi(x) = T(\phi(y), a)$. Thus, $\phi(x) \preceq_{T,A} \phi(y)$.

(ii)⇒(i): For any $x \in C_{T,A}(\{y\})$ we have,

$$\begin{aligned} x = T(y,a) &\iff x \preceq_{T,A} y \\ &\iff \phi(x) \preceq_{T,A} \phi(y) \\ &\iff \phi(x) = T(\phi(y), a') \text{ for some } a' \in A \\ &\iff x = \phi^{-1}(T(\phi(y), \phi(\phi^{-1}(a))) = T_\phi(y, \phi^{-1}(a)), \text{ for some } \phi^{-1}(a) \in \phi^{-1}(A) \\ &\iff x \in C_{T_{\phi,\phi^{-1}(A)}}(\{y\}). \end{aligned}$$

□

Proposition 16. Let T be a t-norm on a bounded lattice L, S be its $N-$dual t-conorm and let $\phi : L \to L$ be an order-preserving bijection. Then,

$$\mathfrak{T}_{T,A} = \mathfrak{T}_{T_\phi,\phi^{-1}(A)} \iff \mathfrak{T}_{S,N(A)} = \mathfrak{T}_{S_\psi,\psi^{-1}(N(A))},$$

where $\psi = N \circ \phi \circ N$.

Proof. Let $\mathfrak{T}_{T,A} = \mathfrak{T}_{T_\phi,\phi^{-1}(A)}$. By Proposition 14 and Proposition 15 for any $a, b \in L$, $a \preceq_{T,A} b$ iff $\phi(a) \preceq_{T,A} \phi(b)$. Now, let us prove that $\mathfrak{T}_{S_\psi,\psi^{-1}(N(A))}$. For any $x, y \in L$, the following equivalences are hold:

$$\begin{aligned} x \in C_{S,N(A)}(\{y\}) &\iff x = S(y, N(a)) \text{ for some } N(a) \in N(A) \\ &\iff x = N(T(N(y), N(N(a))) \\ &\iff N(x) = T(N(y), a) \\ &\iff N(x) \preceq_{T,A} N(y) \\ &\iff \phi(N(x)) \preceq_{T,A} \phi(N(y)) \\ &\iff N(\phi(N(x))) \preceq_{S,N(A)} N(\phi(N(y))) \\ &\iff (N \circ \phi \circ\ N)(x) \preceq_{S,N(A)} (N \circ \phi \circ\ N)(y) \\ &\iff \psi(x) \preceq_{S,N(A)} \psi(y) \\ &\iff \psi(x) = S(\psi(y), N(a')), \text{ for some } N(a') \in N(A) \\ &\iff x = \psi^{-1}(S(\psi(y), N(a')) \end{aligned}$$

$$\begin{aligned}
&\iff x = \psi^{-1}(S(\psi(y), \psi(\psi^{-1}(N(a')))\\
&\iff x = S_\psi(y, \psi^{-1}(N(a')))\\
&\iff x \in C_{S_\psi, \psi^{-1}(N(A))}(\{y\}).
\end{aligned}$$

Conversely, let $\mathfrak{T}_{S,N(A)} = \mathfrak{T}_{S_\psi,\psi^{-1}(N(A))}$. Then, ψ is order-preserving with respect to $\preceq_{S,N(A)}$. For any $x, y \in L$ we get

$$\begin{aligned}
x \in C_T(\{y\}) &\iff x \preceq_{T,A} y\\
&\iff N(x) \preceq_{S,N(A)} N(y)\\
&\iff \psi(N(x)) \preceq_{S,N(A)} \psi(N(y))\\
&\iff (N \circ \phi \circ N)(N(x)) \preceq_{S,N(A)} (N \circ \phi \circ N)(N(y))\\
&\iff (N \circ \phi)(x) \preceq_{S,N(A)} (N \circ \phi)(y)\\
&\iff N(N \circ \phi)(x) \preceq_{T,A} N(N \circ \phi)(y)\\
&\iff \phi(x) \preceq_{T,A} \phi(y)\\
&\iff \phi(x) \in C_T(\{\phi(y)\})\\
&\iff \phi(x) = T(\phi(y), a), \text{ for some } a \in A\\
&\iff x = \phi^{-1}(T(\phi(y), \phi(\phi^{-1}(a))) = T_\phi(y, \phi^{-1}(a))\\
&\iff x \in C_{T_\phi}(\{y\}).
\end{aligned}$$

Thus, we get $\mathfrak{T}_{C,T} = \mathfrak{T}_{C,T_\phi}$. □

The following proposition is a matter of the direct verification.

Proposition 17. Consider the uninorms U_1 and U_2 on bounded lattices L_1 and L_2 with the neutral elements e_1, e_2, respectively. Then the direct product $U_1 \times U_2$ of U_1 and U_2, defined by

$$U_1 \times U_2((x_1, y_1), (x_2, y_2)) = (U_1(x_1, x_2), U_2(y_1, y_2))$$

is a uninorm with the neutral element (e_1, e_2) on the product lattice $L_1 \times L_2$.

Theorem 18. Let U_1 U_2 be uninorms with the neutral elements e_1 and e_2 on the bounded lattices L_1 and L_2 respectively and subsets A_1 and A_2 of L_1 and L_2 such that $U_1(A_1, A_1) \subseteq A_1$ and $U_2(A_2, A_2) \subseteq A_2$, $e_1 \in A_1$, $e_2 \in A_2$. Then $\mathfrak{T}_{U_1,A_1} \times \mathfrak{T}_{U_2,A_2} = \mathfrak{T}_{U_1 \times U_2, A_1 \times A_2}$.

Proof. Let $G_1 \times G_2 \in \mathfrak{T}_{U_1,A_1} \times \mathfrak{T}_{U_2,A_2}$.

$$\begin{aligned}
&C_{U_1 \times U_2, A_1 \times A_2}((L_1 \times L_2) \setminus (G_1 \times G_2))\\
&= C_{U_1 \times U_2, A_1 \times A_2}((L_1 \setminus G_1) \times L_2) \cup (L_1 \times (L_2 \setminus G_2))\\
&= (U_1 \times U_2)((L_1 \setminus G_1) \times L_2, A_1 \times A_2)\\
&\quad \cup (U_1 \times U_2)(L_1 \times (L_2 \setminus G_2), A_1 \times A_2)\\
&= (U_1(L_1 \setminus G_1, A_1) \times U_2(L_2, A_2)) \cup (U_1(L_1, A_1) \times U_2(L_2 \setminus G_2, A_2))
\end{aligned}$$

$$= ((L_1 \setminus G_1) \times L_2) \cup ((L_1 \times (L_2 \setminus G_2))$$
$$= (L_1 \times L_2) \setminus (G_1 \times G_2) \in \mathfrak{F}_{U_1\times U_2, A_1\times A_2}.$$

Then, $G_1 \times G_2 \in \mathfrak{T}_{U_1\times U_2, A_1\times A_2}$ i. e. $\mathfrak{T}_{U_1,A_1} \times \mathfrak{T}_{U_1\times U_2, A_1\times A_2} \subseteq \mathfrak{T}_{U_1\times U_2, A_1\times A_2}$.

Conversely, let $O \in \mathfrak{T}_{U_1\times U_2, A_1\times A_2}$, Then, we have $K = (L_1\times L_2)\setminus O \in \mathfrak{F}_{U_1\times U_2, A_1\times A_2}$.

$$K = C_{U_1\times U_2, A_1\times A_2}(K) = C_{U_1\times U_2, A_1\times A_2}\left(\bigcup_{(x,y)\in K}\{(x,y)\}\right)$$
$$= \bigcup_{(x,y)\in K} C_{U_1\times U_2, A_1\times A_2}(\{(x,y)\})$$
$$= \bigcup_{(x,y)\in K} (U_1 \times U_2)((x,y), A_1 \times A_2) \qquad \text{(by def. of } C_{U_1\times U_2, A_1\times A_2})$$
$$= \bigcup_{(x,y)\in K} (U_1((x, A_1) \times U_2(y, A_2)) \qquad \text{(by def. of } U_1 \times U_2)$$
$$= \bigcup_{(x,y)\in K} (C_{U_1,A_1}(\{x\}) \times C_{U_2,A_2}(\{y\})) \qquad \text{(by def. of closer op.)}$$

Thus, we have $K = \bigcup_{(x,y)\in K}(C_{U_1,A_1}(\{x\}) \times C_{U_2,A_2}(\{y\}))$. On the other hand,

$$O = (L_1 \times L_2) \setminus K$$
$$= (L_1 \times L_2) \setminus \bigcup_{(x,y)\in K} (C_{U_1,A_1}(\{x\}) \times C_{U_2,A_2}(\{y\}))$$
$$= \left(\bigcup_{(x,y)\in K} (C_{U_1,A_1}(\{x\}) \times C_{U_2,A_2}(\{y\}))\right)^c$$
$$= \bigcap_{(x,y)\in K} (C_{U_1,A_1}(\{x\}) \times C_{U_2,A_2}(\{y\}))^c \qquad \text{(by De-Morgan)}$$
$$= \bigcap_{(x,y)\in K} \{[(L_1 \setminus C_{U_1,A_1}(\{x\})) \times L_2] \cup [L_1 \times (L_2 \setminus C_{U_2,A_2}(\{y\}))]\}$$

Since $(L_1 \setminus C_{U_1,A_1}(\{x\})) \times L_2$ and $L_1 \times (L_2 \setminus C_{U_2,A_2}(\{y\}))$ is open with respect to the topology $\mathfrak{T}_{U_1,A_1} \times \mathfrak{T}_{U_2,A_2}$ and by considering Proposition 7 we have $O \in \mathfrak{T}_{U_1,A_1} \times \mathfrak{T}_{U_2,A_2}$. □

Corollary 19. Let U be a uninorm with the neutral element e on a bounded lattice L and let A be a subset of L such that $U(A, A) \subseteq A$, $e \in A$. Then, $\mathfrak{T}_{U,A} \times \mathfrak{T}_{U,A}$ is a principal topology.

Theorem 20. Let L be a bounded lattice and U be a uninorm on L with the neutral element e .Let A be a subset of L such that $e \in A$ and $U(A, A) \subseteq A$. Then, the uninorm U is continuous with respect to the $\mathfrak{T}_C$ topology.

P r o o f. Since the continuity of the function defined between two principal topological spaces is equivalent to the monotonicity in the sense of preorder between those topological spaces, it is sufficient to show the monotonicity. Let $(x,y) \preceq_{U^2,A^2} (z,t)$ for all $(x,y),(z,t) \in L^2$. We have to show $U(x,y) \preceq_{U,A} U(z,t)$. In this case, there exists $(a_1,a_2) \in A^2$ such that $(x,y) = U^2((z,t),(a_1,a_2)) = (U(z,a_1),U(t,a_2))$. On the other hand, $U(x,y) = U(U(z,a_1),U(t,a_2)) = U(U(z,t),U(a_1,a_2))$, $U(a_1,a_2) \in A$. □

Definition 16. Let L be a bounded lattice, U_1 and U_2 are uninorms with the same neutral element e on L and $U_1(A,A) \subseteq A$, $U_2(A,A) \subseteq A$, $e \in A$. Also, let C_1 and C_2 be the closure operators obtained from the set A and the uninorms U_1 and U_2, respectively. Define a relation "$\sim$" on the class of all uninorms on L by

$$U_1 \sim U_2 \;\Leftrightarrow\; C_1(\{x\}) = C_2(\{x\})$$

for all $x \in L$.

The next result is obvious.

Proposition 21. The relation "$\sim$" given in Definition 16 is an equivalence relation on the set of all uninorms U on L with the neutral element e such that $U(A,A) \subseteq A$ and $e \in A$.

Now, we define the underlying t-norm $T_U : [0,1]^2 \to [0,1]$ and t-conorm $S_U : [0,1]^2 \to [0,1]$ induced by a uninorm $U : [0,1]^2 \to [0,1]$ on the real unit interval as taken in [11]:

$$T_U(x,y) = \frac{1}{e} U(ex, ey),$$

$$S_U(x,y) = \frac{1}{1-e} U(e + (1-e)x, e + (1-e)y).$$

Proposition 22. If $U_1 \sim U_2$ then $T_{U_1} \sim T_{U_2}$ and $S_{U_1} \sim S_{U_2}$, where T_{U_i}, S_{U_i} are underlying t-norms and t-conorms respectively for U_i uninorms $(i = 1,2)$.

P r o o f. For $x, y \in A$ we have,

$$T_U\left(\frac{1}{e}x, \frac{1}{e}y\right) = \frac{1}{e} U(x,y) \in \frac{1}{e} A.$$

Hence $T_U\left(\frac{1}{e}A, \frac{1}{e}A\right) \subseteq \frac{1}{e}A$. On the other hand; for all $a \in A$ we obtain, $\dfrac{1}{e}a = T(1, \dfrac{1}{e}a) \in T_U\left(\frac{1}{e}A, \frac{1}{e}A\right)$ then, $\frac{1}{e}A \subseteq T_U\left(\frac{1}{e}A, \frac{1}{e}A\right)$. Then, $T_U\left(\frac{1}{e}A, \frac{1}{e}A\right) = \frac{1}{e}A$. It is trivial that $1 \in \frac{1}{e}A$. Therefore we have $C_{T_U, \frac{1}{e}A}$ for $x \in [0,1]$ as follows:

$$\begin{aligned} C_{T_U, \frac{1}{e}A}(\{x\}) &= T_U(\{x\}, \frac{1}{e}A) = \{T_U(x, \frac{1}{e}a) : a\frac{1}{e} \in a\frac{1}{e}A\} \\ &= \left\{\frac{1}{e} U(ex, a) : a \in A\right\} = \frac{1}{e} C_{U,A}(\{ex\}). \end{aligned}$$

It can be seen that the following equalities are hold:

$$\begin{aligned}
C_{T_{U_1},\frac{1}{e}A}(\{x\}) &= \{T_{U_1}(x, \frac{1}{e}a) : a \in A\} \\
&= \left\{\frac{1}{e}U_1(ex, e\frac{1}{e}a) : a \in A\right\} \\
&= \left\{\frac{1}{e}U_1(ex, a) : a \in A\right\} \\
&= \frac{1}{e}C_{U_1,A}(\{ex\}) = \frac{1}{e}C_{U_2,A}(\{ex\}) \\
&= \left\{\frac{1}{e}U_2(ex, a) : a \in A\right\} \\
&= \left\{\frac{1}{e}U_2(ex, e\frac{1}{e}a) : \frac{1}{e}a \in \frac{1}{e}A\right\} \\
&= \left\{\frac{1}{e}U_2(ex, \frac{1}{e}a) : \frac{1}{e}a \in \frac{1}{e}A\right\} \\
&= \left\{T_{U_2}(x, \frac{1}{e}a) : a \in A\right\} \\
&= C_{T_{U_2},\frac{1}{e}A}(\{x\}).
\end{aligned}$$

Thus, we obtain $T_{U_1} \sim T_{U_2}$. Similarly, $S_{U_1} \sim S_{U_2}$ can be proven.

□

Theorem 23. The set of principal topologies generated by uninorms defined on the real unit interval $[0,1]$ has uncountable cardinality.

P r o o f. Let $U_a : [0,1]^2 \to [0,1]$ be a uninorm with the same underlying t-norm (and t-conorm) defined as follows [11]:

$$U_a(x,y) = \begin{cases} eT_D\left(\frac{x}{e}, \frac{y}{e}\right) & (x,y) \in [0,e]^2, \\ e + (1-e)S_D\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & (x,y) \in [e,1]^2, \\ 1 & x = 1 \text{ or } y = 1, \\ a & (x,y) \in [0,e) \times (a,1) \cup (a,1) \times [0,e), \\ \max(x,y) & \text{otherwise,} \end{cases}$$

where $e \in (0,1)$, $e < a < 1$ and

$$T_D(x,y) = \begin{cases} 0 & (x,y) \in [0,1)^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

$$S_D(x,y) = \begin{cases} 1 & (x,y) \in (0,1]^2, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

If we take $e < a < b < 1$ and $A = [e,1]$, it can be seen that $U_a \nsim U_b$ although they have the same underlying t-norm and t-conorm. Indeed, for $x < e$ we have,

$C_a(\{x\}) = U_a(\{x\}, [e,a]) \cup U_a(\{x\}, (a,1)) \cup U_a(\{x\}, \{1\})$

$$
\begin{aligned}
&= U_a(\{x\},\{e\}) \cup U_a(\{x\},(e,a]) \cup U_a(\{x\},(a,1)) \cup U_a(\{x\},\{1\})\\
&= U_a(\{x\},\{e\}) \cup U_a(\{x\},\{a\}) \cup U_a(\{x\},(e,a)) \cup U_a(\{x\},(a,1)) \cup U_a(\{x\},\{1\})\\
&= \{x\} \cup \{a\} \cup (e,a) \cup \{a\} \cup \{1\}\\
&= \{x\} \cup (e,a] \cup \{1\} \in \mathfrak{F}_{C_a}.
\end{aligned}
$$

Similarly, we obtain $C_b(\{x\}) = \{x\} \cup (e,b] \cup \{1\} \in \mathfrak{F}_{C_b}$. On the other hand,

$$
\begin{aligned}
&C_b(\{x\} \cup (e,a] \cup \{1\})\\
&= U_b(\{x\} \cup (e,a] \cup \{1\}, [e,1])\\
&= U_b(\{x\},[e,1]) \cup U_b((e,a],[e,1])) \cup U_b(\{1\},\{1\})\\
&= U_b(\{x\},\{e\}) \cup U_b(\{x\},(e,1]) \cup U_b((e,a],[e,1)) \cup U_b((e,a],\{1\}) \cup U_b(\{1\},\{1\})\\
&= \{x\} \cup U_b(\{x\},(e,b]) \cup U_b(\{x\},(b,1)) \cup U_b(\{x\},\{1\})\\
&\cup U_b((e,a],[e,1)) \cup U_b(\{1\},\{1\})\\
&= \{x\} \cup (e,b] \cup \{b\} \cup \{1\} \cup \{e + (1-e)S_D\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) : e < x \leq a,\ e \leq y < 1\}\\
&= \{x\} \cup (e,b] \cup \{b\} \cup \{e + (1-e)S_D\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) : e < x \leq a,\ e < y < 1\}\\
&\cup \left\{e + (1-e)S_D\left(\frac{x-e}{1-e}, 0\right)\right\} \cup \{1\}\\
&= \{x\} \cup (e,b] \cup \{e + (1-e)\} \cup \{1\}\\
&= \{x\} \cup (e,b] \cup \{1\}.
\end{aligned}
$$

Hence, $C_b(\{x\} \cup (e,a] \cup \{1\}) \neq \{x\} \cup (e,a] \cup \{1\}$ for arbitrary $x < e$. That is, $\{x\} \cup (e,a] \cup \{1\} \notin \mathfrak{F}_{C_b}$. Thus, the topologies generated from U_a and U_b are different. Furthermore, from this example we show that there are uncountable many principal topologies on the real unit interval $[0,1]$ generated from uninorms which are not equivalent. $\square$

Proposition 24. Let U be a uninorm with the neutral element e on L and a subset A of L such that $U(A,A) \subseteq A$, $e \in A$. Then, U is closed map from $L \times L$ to L with respect to $\mathfrak{T}_C$ topology.

P r o o f. Let $K \subseteq L \times L$ be arbitrary set. By using the definition of the closure operators $C_{U,A}$, C_{U^2,A^2} and properties of the uninorm U we have,

$$
\begin{aligned}
C_{U,A}(U(K)) = U(U(K),A) &= \{U(U(k_1,k_2),a) : (k_1,k_2) \in K, a \in A\}\\
&= \{U(U(k_1,k_2),U(a,e)) : (k_1,k_2) \in K, a \in A\}\\
&= \{U(k_1,U(k_2,U(a,e))) : (k_1,k_2) \in K, a \in A\}\\
&= \{U(k_1,U(U(k_2,a),e)) : (k_1,k_2) \in K, a \in A\}\\
&= \{U(k_1,U(U(a,k_2),e)) : (k_1,k_2) \in K, a \in A\}\\
&= \{U(k_1,U(e,U(k_2,a))) : (k_1,k_2) \in K, a \in A\}\\
&= \{U(U(k_1,e),U(k_2,a)) : (k_1,k_2) \in K, a \in A\}\\
&\subseteq \{U(U(k_1,a_1),U(k_2,a_2)) : (k_1,k_2) \in K, a_1,a_2 \in A\}
\end{aligned}
$$

$$= \{U(U^2((k_1,k_2),(a_1,a_2))) : (k_1,k_2) \in K, (a_1,a_2) \in A \times A\}$$
$$= U(U^2(K, A \times A)) = U(C_{U^2,A^2}(K)).$$

Thus, we obtain that U is closed function. $\square$

Remark 3. a) In Theorem 23, U_a and U_b have the same underlying t-norm and t-conorm. But $U_a \nsim U_b$. Therefore, we consider that the converse of Proposition 22 may not be satisfied.

b) The set of all equivalence classes with respect to the relation $\sim$ on the real unit interval $[0,1]$ is uncountable.

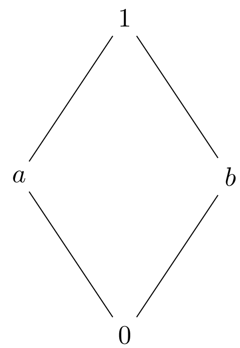

Fig. 3: Dimond lattice correspond to $L = \{0, a, b, 1\}$.

The following example shows that the uninorm U may not be open in general.

Example 2. Consider the lattice $(L = \{0, a, b, 1\}, \leq, 0, 1)$ whose diagram is displayed in Figure 3. We take the weak t-norm T_W as a uninorm on L with $e = 1$. As a subset of L, we take $A = L$. Then, we can define a principal topology on L as Proposition 4. We denote this topology by $\mathfrak{T}_{W,L}$. Here,

$$T_W(x,y) = \begin{cases} x & y = 1 \\ y & x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

In this case, since $C(\{a\}) = \{0, a\}$ then $\{b, 1\} \in \mathfrak{T}_{W,L}$. But,

$$T_W(\{b,1\} \times \{b,1\}) = T_W(\{(b,b),(b,1),(1,b),(1,1)\})$$
$$= \{0, b, 1\} \notin \mathfrak{T}_{W,L}$$

4. AN APPLICATION

In this section, we give an application to the principal topological structure in dynamic Kripke frames.

Definition 17. A set $A \subseteq X$ is called an upset of (X, R) if for each $x, y \in X$, xRy and $x \in A$ imply $y \in A$, where R is a relation on A.

Definition 18. A dynamic Kripke frame is a triple (W, R, G) where W is a set R is a reflexive, transitive relation on W and $G : W \to W$ is function, that is $R-$monotone in the following sense for any $u, v \in W$, if uRv, then $G(u)RG(v)$.

There is one-to-one correspondence between reflexive and transitive Kripke frames and principal spaces. More precisely, given a reflexive and transitive Kripke frame $F = \rangle X, R\langle$ we can construct a topological spaces, indeed a principal space $X = (X, \tau_R)$ by defining τ_R to be the set of all upset of F. Moreover, the evaluation of modal formulas in a reflexive and transitive Kripke model coincides with their evaluation in the corresponding principal topological space (see e. g., [[27], p. 306])

Proposition 25. Let L be a bounded lattice, U be a uninorm on L with the neutral element e and $x_0 \in L$ be arbitrary. Then $(L, \preceq_U, f_{x_0})$ is a dynamic Kripke frame, where $f_{x_0} : L \to L$, $f_{x_0}(x) = U(x, x_0)$

Proof. The proof is obtained from that $f_{x_0} : L \to L$ is a $\preceq_U$ $-$monotone function. By Theorem 20 we obtain, f_{x_0} is continuous with respect to the topology $\mathfrak{T}$. □

From this point of view, this topology obtained from uninorms will be applicable to the dynamic topology, dynamic programming and modal logic.

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