KYBERNETIKA — VOLUME *58* (2022), NUMBER 6, PAGES 843–862

AN EFFICIENT *HP* SPECTRAL COLLOCATION METHOD FOR NONSMOOTH OPTIMAL CONTROL PROBLEMS

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One of the most challenging problems in the optimal control theory consists of solving the nonsmooth optimal control problems where several discontinuities may be present in the control variable and derivative of the state variable. Recently some extended spectral collocation methods have been introduced for solving such problems, and a matrix of differentiation is usually used to discretize and to approximate the derivative of the state variable in the particular collocation points. In such methods, there is typically no condition for the continuity of the state variable at the switching points. In this article, we propose an efficient *hp* spectral collocation method for the general form of nonsmooth optimal control problems based on the operational integration matrix. The time interval of the problem is first partitioned into several variable subintervals, and the problem is then discretized by considering the Legendre-Gauss-Lobatto collocation points. Here, the switching points are unknown parameters, and having solved the final discretized problem, we achieve some approximations for the optimal solutions and the switching points. We solve some comparative numerical test problems to support of the performance of the suggested approach.

Keywords: nonsmooth optimal control, *hp*-method, Lagrange interpolating polynomials, Legendre-Gauss-Lobatto points

Classification: MSC 49J15, MSC 93C10, 65M70

1. INTRODUCTION

Bang-Bang and singular optimal control problems [17, 25] are the most exciting problems in the optimal control (OC) theory. These problems are also called nonsmooth OC problems. The existence of a bound on the state variable and, or control variable and the presence of state path constraints and, or control path constraints in the OC problems usually cause non-smoothness in the optimal solution of such problems. Hence, the usual computational techniques for smooth OC problems may not apply to the optimal solutions for nonsmooth OC problems. Generally, nonsmooth OC problem occurs in a wide range of field of sciences which we can point out the following typical application areas: medicine (timing and dosage of medication [37, 38]), modern engineering (power systems [16]), hydraulics (hydraulic turbine [15]), financial mathematics (regime-switching models [33]), and many other dynamical systems which are nonsmooth.

DOI: 10.14736/kyb-2022-6-0843

Since the second half of the nineteenth century, nonsmooth OC problems have been raised and studied by many researchers.Because of the difficulty in nonsmooth OC problems, the numerical solution to these problems has received attention. Computational approaches and schemes to numerically solve the nonsmooth optimal control problems can divide into direct and indirect methods. Direct methods deal only with the dynamical system, constraints, and objective functional (or performance index). However, in indirect methods, the necessary optimal conditions and initial and boundary conditions are utilized to achieve a solution. Some indirect methods in [26, 11] are presented based on dynamic programming. Moreover, some methods directly transform the nonsmooth (bang-bang or singular) OC problem into a discrete-time (DT) problem or nonlinear programming (NLP) problem [14, 35]. In [2], Aly and Chen applied a modified quasilinearization to singular OC problems. In [29], a multiple shooting method is given numerically for solving the singular OC problem. Several other advanced methods have been proposed to solve some particular nonsmooth OC problems or general forms of these problems. Graichen and Petit [18] solved the Goddard problem [17, 34, 44] based on saturation functions. Sun [41] applied a method for solving the singular OC problems using the modified lin-up competition algorithm with a region-relaxing strategy. In [12], a new *hp*-adaptive scheme is extended for the OC problems, including discontinuous control. In [35], a modified Legendre pseudospectral approach for bang-bang OC problems is investigated where control is considered piecewise constant, and the state is assumed piecewise continuous polynomials. In [28], a hybrid non-uniform finite difference method is presented using Chebyshev polynomials and block-pulse functions for solving nonlinear OC problems. In [3], a shooting algorithm for a class of singular OC problems is proposed. In [14], a pseudospectral scheme is presented for the numerical solutions of continuous-time optimization problems, including the singular arc. In [45], a pseudospectral method to solve OC problems via the integration matrix is given. Hager et al. in [19] have introduced a Gauss orthogonal collocation method to OC problems with control constraints. In [42], a composite pseudospectral method using the composite interpolation operator and operational matrix of the derivative is presented. In [20], Pontryagins maximum principle (PMP) and discontinuous Galerkin (DG) method are applied to solve OC problems.

Generally, the methods for nonsmooth OC problems are classified into h-methods and hp-methods. In h-methods, the time-domain of the problem is parted into several fixed sub-domains. A polynomial with a fixed low-degree is utilized to estimate the solutions on each sub-domain. To achieve good approximations in these methods, we must increase the number of subdomains [4, 9]. But, in hp-methods, the problem is first divided into several segments. The solutions are then approximated on each part or subinterval by some basic functions such as Legendre, Chebyshev, and Lagrange polynomials [12]. The length of subintervals (marked by h) and degree of polynomials (marked by p) can be determined simultaneously, leading to solutions with high accuracy. A spectral or pseudospectral method, called the p-method, approximates the solution on each sub-interval. In p-methods, the three types of collocation points are used: Gauss points, Gauss-Radou (GR) points, and Gauss-Lobatto (GL) points [36]. By p-methods, we can achieve an accurate solution for smooth OC problems if we use a large-degree polynomial.

Notice that, in nonsmooth OC problems, it is possible to be available several discontinuities in the control variable (for bang-bang OC problems) and derivative of the state variable (for bang-bang and singular OC problems). Hence, applying p-methods for nonsmooth OC problems may be led to the Gibbs phenomenon [7], and increasing the number of collocation points causes an increase in the error of approximations. Thus, p-methods (for example, see [46]) cannot be singly utilized to solve nonsmooth OC problems, and these methods must be composited by an h-method to obtain an accurate solution.

In the hp-methods for nonsmooth OC problems, the matrix of differentiation is usually utilized on each interval to approximate the derivative of the state variable in the collocation points, and we do have not usually any condition for the continuity of the state variable at the boundary of subintervals which is called switching (or breaking) point. For example, in hp-method given by Darby et al. [12], there is not any condition on the continuity of the state variable of the OC problem. Also, in Agamawi et al. [1], there exist some constraints on the continuity of state variables at switching points. Still, conditions increase the complexity of the method and the time of implementation of the program. But, if we use the equivalent integral forms of the differential dynamics in the nonsmooth OC problem, we can enforce the continuity conditions of state variables in the switching points. This can be led to more accurate solutions.
Here, we use the equivalent integral form of the dynamical equation and apply the Legendre GL (LGL) points on each subinterval to achieve the integration matrix. The variables of obtained NLP problem are the switching points and coefficients of the approximations of control and state variables. In numerical and practical examples, we compare our results with those of other methods and show the efficiency and capability of the presented process. The proposed approach is easy to implement and has an excellent convergent rate. Also, this method needs fewer mesh points to improve the accuracy of the approximate solutions. Notice that our method differs from methods in [12, 35] because in [12] the matrix of differentiation is utilized, while in our method, we apply matrix of integration. Also, in [35], the method can only apply for bang-bang OC problems, while our method covers all smooth, bang-bang, and singular OC problems.

2. PROBLEM STATEMENT

The general form of nonsmooth OC problems is stated as

$$\text{Minimize} \quad J = \Phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\,\mathrm{dt}, \tag{1}$$

subject to

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), t), \qquad t_0 \le t \le t_f, \tag{2}$$

and mixed state-control path constraints

$$h(\mathbf{x}(t), \mathbf{u}(t), t) \le 0, \qquad t_0 \le t \le t_f, \tag{3}$$

and box constraints

$$\mathbf{x}^{\min} \le \mathbf{x}(t) \le \mathbf{x}^{\max}\ , \qquad t_0 \le t \le t_f, \tag{4}$$

$$\mathbf{u}^{\min} \le \mathbf{u}(t) \le \mathbf{u}^{\max}\ , \qquad t_0 \le t \le t_f, \tag{5}$$

with the initial and terminal conditions

$$\mathbf{x}(t_0) = \mathbf{x}_0, \tag{6}$$

$$\mathbf{x}(t_f) = \mathbf{x}_f, \tag{7}$$

where $\Phi : \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R} \to \mathbb{R}^r$ and $h : \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R} \to \mathbb{R}^p$ are given continuously differentiable functions, $\mathbf{x}_0$, $\mathbf{x}_f$, $\mathbf{x}^{\min}$, and $\mathbf{x}^{\max}$ are given vectors in $\mathbb{R}^r$ and, $\mathbf{u}^{\min}$, and $\mathbf{u}^{\max}$ are given vectors in $\mathbb{R}^s$. Notice that the optimal solution of the OC problem (1) – (7) has a nonsmooth nature because of the existence of constraints (3) – (5). Hence, some switching points may appear in the optimal solutions to such problems. We assume that there are finite number of switching points for the problem (1) – (7), and they are unknown. Here, we suggest a new and efficient *hp* method to solve the nonsmooth OC problem (1) – (7).

3. IMPLEMENTATION OF THE METHOD

We assume that the problem (1) – (7) has m indeterminate switching times $t_1, \ldots, t_m$ and $[t_0, t_f] = \bigcup_{k=1}^{m+1} [t_{k-1}, t_k]$ where $t_{m+1} = t_f$. We represent the state and control variables in the kth subinterval by $\mathbf{x}^k(t)$ and $\mathbf{u}^k(t)$, respectively. We express the equations 2 – 7 as follows

$$\dot{\mathbf{x}}^k(t) = f(\mathbf{x}^k(t), \mathbf{u}^k(t), t), \quad t_{k-1} \le t \le t_k, \quad k = 1, 2, \ldots, m+1, \tag{8}$$

$$h(\mathbf{x}^k(t), \mathbf{u}^k(t), t) \le 0, \qquad t_{k-1} \le t \le t_k, \quad k = 1, 2, \ldots, m+1, \tag{9}$$

$$\mathbf{x}^{\min} \le \mathbf{x}^k(t) \le \mathbf{x}^{\max}, \quad t_{k-1} \le t \le t_k, \tag{10}$$

$$\mathbf{u}^{\min} \le \mathbf{u}^k(t) \le \mathbf{u}^{\max}, \quad t_{k-1} \le t \le t_k, \tag{11}$$

$$\mathbf{x}^k(t_{k-1}) = \mathbf{x}^{k-1}(t_{k-1}), \quad k = 2, 3, \ldots, m+1, \tag{12}$$

$$\mathbf{x}^1(t_0) = \mathbf{x}_0, \tag{13}$$

$$\mathbf{x}^{m+1}(t_{m+1}) = \mathbf{x}_f. \tag{14}$$

Notice that relations (12) guarantee the continuity of state variables. By utilizing these points and integrating from both sides of equation (8), we can get

$$\mathbf{x}^k(t) = \mathbf{c}^{k-1} + \int_{t_{k-1}}^{t} f(\mathbf{x}^k(\tau), \mathbf{u}^k(\tau), \tau)\, \mathrm{d}\tau, \quad t_{k-1} \le t \le t_k, \quad k = 1, 2, \ldots, m+1, \tag{15}$$

where

$$\mathbf{c}^k = \begin{cases} \mathbf{x}_0\ , & k = 0, \\ \mathbf{x}^k(t_k), & k = 1, \ldots, m. \end{cases} \tag{16}$$

Now, we select $\{\tau_i\}_{i=0}^n$ are the LGL points on $[-1,1]$ where they are the roots of polynomial $Q_{n+1}(\tau)=(1-\tau^2)\dot{P}_n(\tau)$, and $P_n(\tau)$ is the Legendre polynomial of degree n defined by

$$(n+1)P_{n+1}(\tau)=(2n+1)\tau P_n(\tau)-nP_{n-1}(\tau), \quad n\geq 1,$$

where $P_0(\tau)=1$ and $P_1(\tau)=\tau$.

Also, this polynomial satisfies the Rodrigues' formula

$$P_n(\tau)=\frac{1}{2^n n!}[\frac{d^n}{d\tau^n}(\tau^2-1)^n].$$

The analytical form of Legendre polynomial $P_n(\cdot)$ is as follows

$$P_n(\tau)=\frac{1}{2^n}\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\frac{(2n-2l)!}{2^n l!(n-l)!(n-2l)!}\tau^{n-2l}.$$

We allot the LGL points $\{\tau_i\}_{i=0}^n$ correspondingly to points $\{\hat{\tau}_i^k\}_{i=0}^n$ (for $i=0,1,\ldots,n$ and $k=1,2,\ldots,m+1$) by transformation

$$\hat{\tau}_i^k=\psi_k(\tau_i)=\frac{(t_k-t_{k-1})(\tau_i+1)}{2}+t_{k-1}. \tag{17}$$

To approximate the control and state variables on $[t_{k-1},t_k]$, we use the following interpolating polynomials

$$\mathbf{x}^k(\tau)\simeq\sum_{i=0}^n\hat{\mathbf{x}}_i^k\hat{L}_i^k(\tau), \quad \mathbf{u}^k(\tau)\simeq\sum_{i=0}^n\hat{\mathbf{u}}_i^k\hat{L}_i^k(\tau), \quad k=1,2,\ldots,m+1, \tag{18}$$

where $\hat{\mathbf{x}}_i^k=\mathbf{x}^k(\hat{\tau}_i^k)$, $\hat{\mathbf{u}}_i^k=\mathbf{u}^k(\hat{\tau}_i^k)$ and $\hat{L}_i^k(\cdot)$ is defined according to the nodes $\{\hat{\tau}_i^k\}_{i=0}^n$, as

$$\hat{L}_i^k(\tau)=\prod_{j=0,i\neq j}^n\frac{\tau-\hat{\tau}_j^k}{\hat{\tau}_i^k-\hat{\tau}_j^k}.$$

Now, replacing approximations (18) into (15), we get

$$\sum_{i=0}^n\hat{\mathbf{x}}_i^k\hat{L}_i^k(t)=\mathbf{c}^{k-1}+\int_{t_{k-1}}^t\sum_{i=0}^n f(\hat{\mathbf{x}}_i^k,\hat{\mathbf{u}}_i^k,\hat{\tau}_i^k)\hat{L}_i^k(\tau)\,\mathrm{d}\tau, \tag{19}$$
$$t_{k-1}\leq t\leq t_k, \quad k=1,2,\ldots,m+1.$$

Also based on the approximation $\mathbf{x}(t)$, relation (16) is converted as follows

$$\mathbf{c}^k=\begin{cases}\hat{\mathbf{x}}_n^0\,, & k=0,\\ \hat{\mathbf{x}}_n^k, & k=1,2,\ldots,m+1.\end{cases}$$

By exchanging integral and summation, transformation $\hat{\tau} = \frac{2(\tau - t_{k-1})}{t_k - t_{k-1}} - 1$, and replacing t in (19) by $\hat{\tau}_i^k$, $i = 0, 1, \dots, n$, we get

$$\hat{\mathbf{x}}_i^k = \hat{\mathbf{x}}_n^{k-1} + \sum_{j=0}^{n} f(\hat{\mathbf{x}}_j^k, \hat{\mathbf{u}}_j^k, \hat{\tau}_j^k) \int_{t_{k-1}}^{\hat{\tau}_i^k} \hat{L}_j^k(\tau) \, \mathrm{d}\tau$$

$$= \hat{\mathbf{x}}_n^{k-1} + \sum_{j=0}^{n} f(\hat{\mathbf{x}}_j^k, \hat{\mathbf{u}}_j^k, \hat{\tau}_j^k) \frac{t_k - t_{k-1}}{2} \int_{-1}^{\tau_i} L_j(\hat{\tau}) \, \mathrm{d}\hat{\tau}, \quad k = 1, 2, \dots, m+1,$$

where $\{\tau_i\}_{i=0}^n$ are the LGL points on $[-1, 1]$ and

$$\mathcal{L}_{i,j} = \int_{-1}^{\tau_i} L_j(\hat{\tau}) \, \mathrm{d}\hat{\tau}, \quad i, j = 0, 1, \dots, n. \tag{20}$$

Here, $\mathcal{L}$ for $i, j = 0, 1, \dots, n$ are the components of the operational matrix of integration $\mathcal{L} = (\mathcal{L}_{ij})$ which is in [45] as

$$\mathcal{L}_{i,j} = \frac{\omega_j}{2} \{1 + \tau_i + \sum_{k=1}^{n} P_k(\tau_j)[P_{k+1}(\tau_i) - P_{k-1}(\tau_i)]\}, \quad i, j = 0, 1, \dots, n,$$

where weights ω_j's, $j = 0, 1, \dots, n$ satisfy

$$\omega_j = \frac{2}{n(n+1)[P_n(\tau_j)]^2}, \quad j = 0, 1, \dots, n. \tag{21}$$

This matrix for $n = 4, 6$ has the following form

$$\mathcal{L}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0.1354 & 0.2394 & -0.0434 & 0.0212 & -0.0074 \\ 0.0812 & 0.6063 & 0.3555 & -0.0619 & 0.0187 \\ 0.1074 & 0.5231 & 0.7545 & 0.3049 & -0.0354 \\ 0.1 & 0.5444 & 0.7111 & 0.5444 & 0.1 \end{bmatrix},$$

$$\mathcal{L}_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0656 & 0.1186 & -0.0215 & 0.0111 & -0.0069 & 0.0044 & -0.0016 \\ 0.0360 & 0.3154 & 0.2047 & -0.0369 & 0.0191 & -0.0113 & 0.0041 \\ 0.0550 & 0.2555 & 0.4749 & 0.2438 & -0.0432 & 0.0212 & -0.0074 \\ 0.0434 & 0.2881 & 0.4125 & 0.5245 & 0.2270 & -0.0385 & 0.0116 \\ 0.0492 & 0.2723 & 0.4387 & 0.4764 & 0.4532 & 0.1581 & -0.0180 \\ 0.0476 & 0.2768 & 0.4317 & 0.4876 & 0.4317 & 0.2768 & 0.0476 \end{bmatrix}.$$

Now, by the following lemma, we estimate the objective functional (1).

Lemma 3.1. (Shen et al. [36]) Assume that $\{\tau_i\}_{i=0}^n$ (on interval $[-1, 1]$) and $\{\omega_i\}_{i=0}^n$ (defined by (21)) are the LGL points and weights, respectively. Then for any polynomials $\eta(\cdot)$ of degree at most $2n - 1$ we achieve

$$\int_{-1}^{1} \eta(\tau) \, \mathrm{d}\tau = \sum_{i=0}^{n} \omega_i \eta(\tau_i).$$

By Lemma (3.1), we have

$$\begin{aligned}\int_{t_0}^{t_f} g(\mathbf{x}(t),\mathbf{u}(t),t)\,\mathrm{dt} &= \sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k} g(\mathbf{x}(t),\mathbf{u}(t),t)\,\mathrm{dt}\\ &= \sum_{k=1}^{m+1}\frac{t_k-t_{k-1}}{2}\int_{-1}^{1} g(\mathbf{x}(\psi(\tau)),\mathbf{u}(\psi(\tau)),\psi(\tau))\,\mathrm{dt}\\ &\simeq \sum_{k=1}^{m+1}\frac{t_k-t_{k-1}}{2}\sum_{i=0}^{n}\omega_i g(\mathbf{x}(\psi_k(\tau_i)),\mathbf{u}(\psi_k(\tau_i)),\psi_k(\tau_i))\\ &= \sum_{k=1}^{m+1}\frac{t_k-t_{k-1}}{2}\sum_{i=0}^{n}\omega_i g(\hat{\mathbf{x}}_i^k,\hat{\mathbf{u}}_i^k,\hat{\tau}_i^k),\end{aligned}$$

where $\psi_k(\tau_i)$ is defined in (17).

Hence, the problem is converted into the following NLP problem:

$$\text{Minimize}\quad J=\Phi(\hat{\mathbf{x}}_n^{m+1},t_{m+1})+\sum_{k=1}^{m+1}\frac{t_k-t_{k-1}}{2}\sum_{i=0}^{n}\omega_i g(\hat{\mathbf{x}}_i^k,\hat{\mathbf{u}}_i^k,\hat{\tau}_i^k), \tag{22}$$

subject to

$$\hat{\mathbf{x}}_i^k=\hat{\mathbf{x}}_n^{k-1}+\frac{t_k-t_{k-1}}{2}\sum_{j=0}^{n}\mathcal{L}_{ij}f(\hat{\mathbf{x}}_j^k,\hat{\mathbf{u}}_j^k,\hat{\tau}_j^k), \tag{23}$$

$$h(\hat{\mathbf{x}}_i^k,\hat{\mathbf{u}}_i^k,\hat{\tau}_i^k)\le 0, \tag{24}$$

$$\mathbf{x}^{\min}\le\hat{\mathbf{x}}_i^k\le\mathbf{x}^{\max}, \tag{25}$$

$$\mathbf{u}^{\min}\le\hat{\mathbf{u}}_i^k\le\mathbf{u}^{\max}, \tag{26}$$

$$\hat{\mathbf{x}}_n^{m+1}-\mathbf{x}_f=0, \tag{27}$$

$$t_i-t_{i-1}\ge\varepsilon, \tag{28}$$

$$i=0,1,\dots,n;\ k=1,2,\dots,m+1,$$

where $\hat{\mathbf{x}}_n^{m+1}=\mathbf{x}^{m+1}(t_f)$ and ε has a small value. Relation (28) is a guarantee to prevent the overlapping of sub-intervals. The variables of problem are $\left(\hat{\mathbf{x}}_i^k,\hat{\mathbf{u}}_i^k,t_k\right)$ for $i=0,1,\dots,n$ and $k=1,2,\dots,m+1$.

Remark 3.2. Related to the selection of value of m, we first consider $m=1$ and solve the problem in the main interval. According to the obtained results and the drawn graph if needed, we increase m. In the following, consider $m=k$ and solve the problem. In the next step, consider $m=k+1$ and solve the problem again. If the results and trajectories obtained for $m=k$ and $m=k+1$ are very similar, then we can conclude that the number of switching points is equal to $m=k$, and we do not need to increase m.

4. NUMERICAL SIMULATION

Here, we implement our approach for five test problems to show its superiority compared with other methods. Here, we utilize the FMINCON command in MATLAB software to solve the NLP problem (22) – (28). We use a Core i3 PC Laptop with a 2.5 GHz CPU and 4 GB RAM for computations. We also use the sequential quadratic programming (SQP) algorithm in the option of fmincon command.

Example 4.1. Influence of the gravitational field and atmospheric drag on a vertically ascending rocket and maximizing its final altitude, is known as Goddard problem [17, 18]. This problem is formulated as the following nonsmooth OC problem

$$\text{Maximize} \quad J = h(t_f), \tag{29}$$

subject to

$$\dot{h} = v, \tag{30}$$

$$\dot{v} = \frac{u - D(h,v)}{m} - \frac{1}{h^2}, \tag{31}$$

$$\dot{m} = -\frac{u}{c}, \tag{32}$$

$$m(t_f) = 0.6, \tag{33}$$

$$q(h,v) \leq 10, \tag{34}$$

$$h(0) = 1, \quad v(0) = 0, \quad m(0) = 1, \tag{35}$$

where states m, v and h show the mass, speed and altitude of the rocket, respectively. Here, the final time t_f is free and control u is thrust. Function $D(h,v)$ is formulated as

$$D(h,v) = q(h,v)\frac{C_D A}{m_0 g}, \tag{36}$$

where g is the earth's gravitational acceleration and $q(h,v)$ shows the dynamic pressure defined as

$$q(h,v) = \frac{1}{2}\rho_0 v^2 exp(\beta(1-h)). \tag{37}$$

In the equations (32) – (37), C_D is drag coefficient, A is reference area, m_0 is initial mass, ρ_0 is air density at sea level, β is the density decay rate and c is exhaust velocity. According to [18], we take the constants as follows

$$\beta = 500, \quad c = 0.5, \quad \frac{\rho_0 C_D A}{m_0 g} = 620.$$

Notice that, inequality (34) is suggested in [34] on the dynamic pressure.

We solve the NLP problem for $m = 2$ and arbitrary values of n. The corresponding NLP problem (22)–(28) can be written as follow

$$\text{Maximize} \quad J = \hat{x}_{1_n}^{m+1}(t_{m+1}),$$

subject to

$$\hat{x}_{1_i}^k = \hat{x}_{1_n}^{k-1} - \frac{t_k - t_{k-1}}{2}\sum_{j=0}^{n}\mathcal{L}_{ij}\hat{x}_{2_j}^k, \quad k = 1, 2, \; i = 1, \dots, n,$$

$$\hat{x}_{2_i}^k = \hat{x}_{2_n}^{k-1} - \frac{t_k - t_{k-1}}{2}\sum_{j=0}^{n}\mathcal{L}_{ij}\big[\hat{u}_j^k - D(\hat{x}_{1_j}^k, \hat{x}_{2_j}^k) - \frac{1}{(\hat{x}_{1_j}^k)^2}\big],$$

$$\hat{x}_{3_i}^k = \hat{x}_{3_n}^{k-1} - \frac{t_k - t_{k-1}}{2}\sum_{j=0}^{n}\mathcal{L}_{ij}\big[-\frac{\hat{u}_j^k}{c}\big],$$

$$q(\hat{x}_{1_i}^k, \hat{x}_{2_i}^k) - 10 \le 0, \quad k = 1, 2; \; i = 1, \dots, n,$$

$$\hat{x}_{3_n}^{m+1} - 0.6 = 0,$$

$$t_i - t_{i-1} \ge 10^{-3},$$

where $x_1(t) = h(t)$, $x_2(t) = v(t)$ and $x_3(t) = m(t)$, for $0 \le t \le t_f$. Having solved above NLP problem, Table 1 show the obtained optimal values for J and t_f. The achieved results, comparing with those of work [18], show that the presented method acts more accurate and more effective to solve these nonsmooth OC problem. The optimal solutions are illustrated in Figure 1.

		t_f	J
Presented method			
$m = 2$	$n = 4$	0.1998	1.01289034
	$n = 5$	0.1990	1.01284797
	$n = 6$	0.1994	1.01284672
	$n = 7$	0.1990	1.01284423
	$n = 10$	0.1989	1.01283742
	$n = 14$	0.1988	1.01283695
Method [18]			
		0.2040	1.01271727

Tab. 1. The gained values of t_f and J for Example 1.

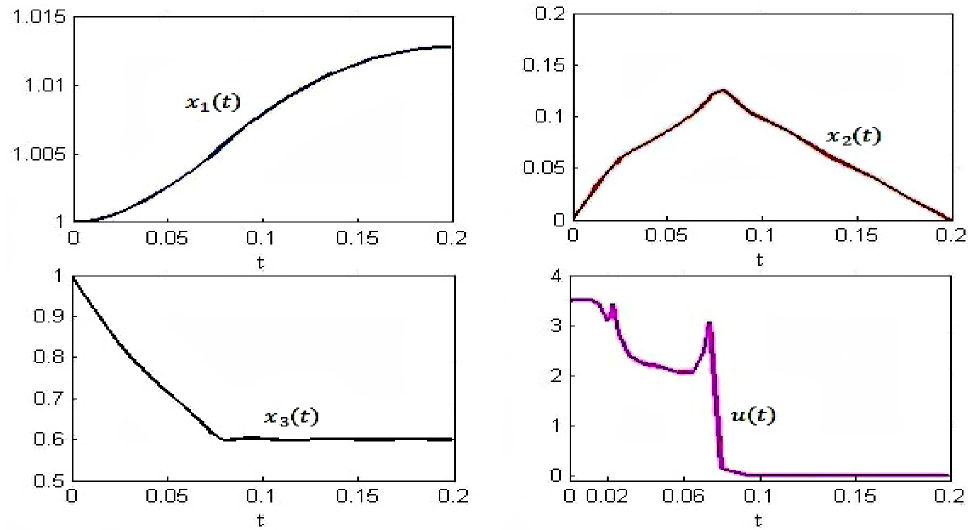

Fig. 1. The gained approximate solutions with $m = 2$ and $n = 8$, for Example 1.

Example 4.2. In this example, the aim is to minimize the following performance index (see [19])

$$J = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))\,\mathrm{dt}, \tag{38}$$

subject to

$$\dot{x}(t) = u(t), \quad 0 \le t \le 1, \tag{39}$$

$$x(0) = \frac{1+3e}{2(1-e)}, \tag{40}$$

$$u(t) \le 1, \quad 0 \le t \le 1. \tag{41}$$

The exact solutions are

$$J^* = 2.7939778112$$

$$x^*(t) = \begin{cases} t + \frac{1+3e}{2(1-e)}, & 0 \le t \le \frac{1}{2}, \\ \frac{e^t + e^{2-t}}{\sqrt{e}(1-e)}, & \frac{1}{2} \le t \le 1. \end{cases}, \quad u^*(t) = \begin{cases} 1, & 0 \le t \le \frac{1}{2}, \\ \frac{e^t - e^{2-t}}{\sqrt{e}(1-e)}, & \frac{1}{2} \le t \le 1. \end{cases}$$

We apply the corresponding NLP (22)–(28) to solve the problem (38)–(41) for $\varepsilon = 10^{-1}$. In Table 2, the results are given for switching point t_1, index J, and *CPU Time*. The achieved approximate optimal solutions and the absolute errors are illustrated in Figures 2 and 3, respectively. Numerical solutions approve that our method is convergent and stable, and errors go to zero as n increases.

		t_1	J	$\mid J^* - J \mid$	$CPU\ Time(Sec)$
$m=1$	$n=3$	0.542	**2.793**86315	1.14×10^{-4}	0.63
	$n=5$	0.536	**2.7939**6047	1.73×10^{-5}	3.44
	$n=7$	0.511	**2.79397**624	1.56×10^{-6}	5.36
	$n=8$	0.509	**2.79397**696	1.07×10^{-6}	5.98
	$n=9$	0.507	**2.797977**23	5.79×10^{-7}	6.01
	$n=11$	0.500	**2.79397781**	1.21×10^{-9}	6.28

Tab. 2. The results of t_1, J and $CPU\ Time$ for Example 2.

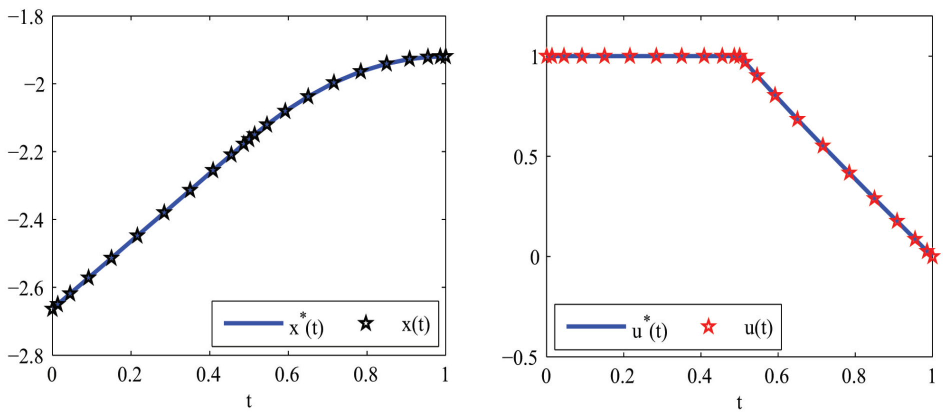

Fig. 2. The obtained solutions with $m = 1$ and $n = 11$ for Example 2.

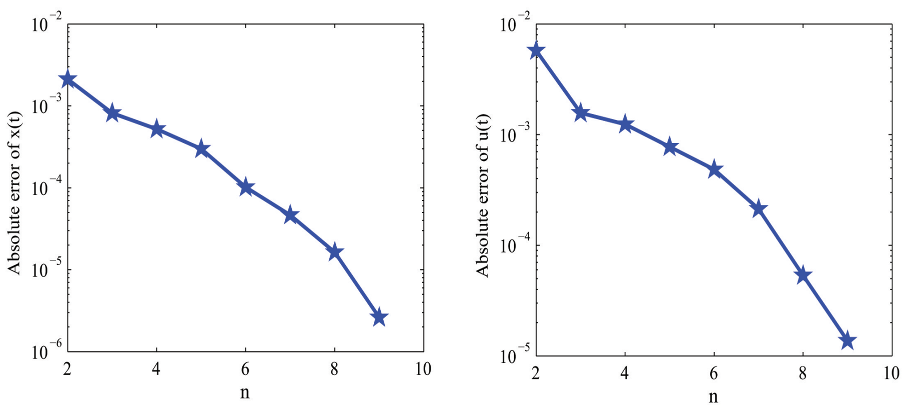

Fig. 3. The absolute error of obtained solution for Example 2.

Example 4.3. Consider the following OC problem [42]

$$\text{Minimize} \quad J = \int_{-\frac{5}{4}}^{\frac{5}{4}} \sqrt{1 + u(t)^2}\,\mathrm{dt}, \tag{42}$$

subject to

$$\dot{x}(t) = u(t), \quad -\frac{5}{4} \leq t \leq \frac{5}{4}, \tag{43}$$

$$x(-\frac{5}{4}) = x(\frac{5}{4}) = 0, \tag{44}$$

$$x(t) + t^2 \geq 1, \quad -\frac{5}{4} \leq t \leq \frac{5}{4}. \tag{45}$$

The obtained value of performance is

$$J^* = 2\sqrt{2} - \frac{1}{2} ln(\sqrt{2} - 1) = 3.2691139282\ldots,$$

and optimal solutions are as

$$x^*(t) = \begin{cases} t + \frac{5}{4}, & -\frac{5}{4} \leq t \leq -\frac{1}{2}, \\ 1 - t^2, & -\frac{1}{2} \leq t \leq \frac{1}{2}, \\ -t + \frac{5}{4}, & \frac{1}{2} \leq t \leq \frac{5}{4}. \end{cases} , \quad u^*(t) = \begin{cases} 1, & -\frac{5}{4} \leq t \leq -\frac{1}{2}, \\ -2t, & -\frac{1}{2} \leq t \leq \frac{1}{2}, \\ -1, & \frac{1}{2} \leq t \leq \frac{5}{4}. \end{cases}$$

We utilize the corresponding NLP (22)–(28) for solving (42)–(45) for $\varepsilon = 10^{-1}$. Table 3 gives the switching points t_1, t_2, and performance index J. Also, the achieved results are compared with the method [42], which shows the superiority of the suggested method. Moreover, the approximate and exact solutions are illustrated in Figure 4. We observe, by increasing n, obtained solutions approach to the exact solutions, and our method has a stable treatment.

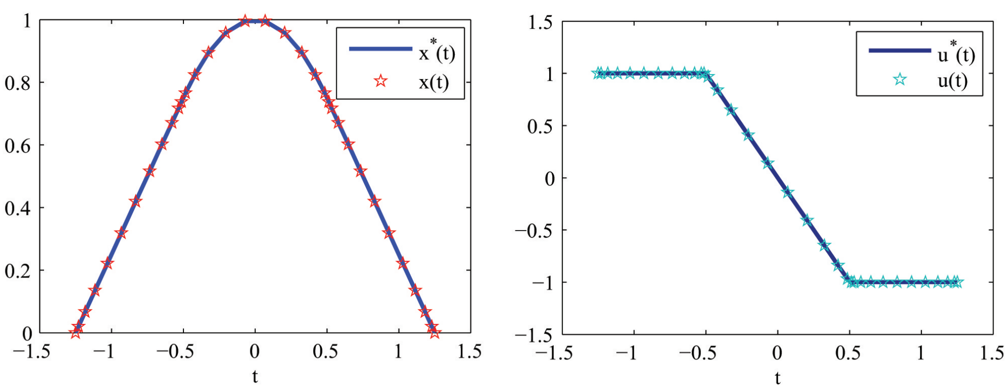

Fig. 4. The obtained approximate solutions with $m = 2$ and $n = 11$ for Example 3.

		t_1	t_2	J	$\mid J^* - J \mid$
Presented method					
$m = 0$	$n = 4$	–	–	**3.2**4236754	2.67×10^{-2}
	$n = 8$	–	–	**3.26**540696	3.70×10^{-3}
	$n = 9$	–	–	**3.26910**596	7.96×10^{-6}
Method[42]	$n = 16$	–	–	3.26918101	6.70×10^{-5}
Presented method					
$m = 1$	$n = 4$	−0.502	–	**3.26**339021	5.72×10^{-3}
	$n = 8$	−0.750	–	**3.269**25501	1.41×10^{-4}
	$n = 10$	−0.750	–	**3.2691**7897	6.50×10^{-5}
	$n = 11$	−0.665	–	**3.2691**2968	1.57×10^{-5}
Method[42]	$n = 16$	−0.001	–	3.26915219	3.80×10^{-5}
Presented method					
$m = 2$	$n = 4$	−0.540	0.540	**3.269**02079	9.13×10^{-5}
	$n = 8$	−0.534	0.534	**3.2691**0945	4.46×10^{-6}
	$n = 10$	−0.533	0.533	**3.26911**159	2.33×10^{-6}
	$n = 11$	−0.512	0.512	**3.2691139**5	3.00×10^{-8}
Method[42]	$n = 16$	−0.523	0.523	3.26911301	9.10×10^{-7}

Tab. 3. The results of switching points t_1, t_2 and performance index J for Example 3.

Example 4.4. Consider the following OC problem [39]

$$\text{Minimize} \quad J = \int_0^1 \frac{1}{2}(e^{-t} - 2t)x(t)\,\mathrm{dt}, \tag{46}$$

subject to

$$\dot{x}(t) = -tx(t) + ln(u(t) + t + 3), \quad 0 \le t \le 1, \tag{47}$$

$$x(0) = 0, \quad x(1) = 0.8, \tag{48}$$

$$-1 \le u(t) \le 1, \quad 0 \le t \le 1. \tag{49}$$

We solve the problem for $\varepsilon = 10^{-3}$. In Table 4, the results are given for switching points t_1, t_2, and performance index J. Moreover, comparing results with those of the method [39] is observed in this Table. The approximate and exact solutions are illustrated in Figure 5.

		t_1	t_2	J
Presented method				
$m = 2$	$n = 3$	0.1942	0.5057	-0.183061
	$n = 5$	0.1951	0.5125	-0.183056
	$n = 7$	0.1957	0.5151	-0.183052
	$n = 9$	0.1959	0.5152	-0.183050
	$n = 10$	0.1972	0.5161	-0.183044
Method [39]				
$m = 0$	$n = 100$	–	–	-0.1830

Tab. 4. The results of switching points t_1, t_2 and cost functional J for Example 4.

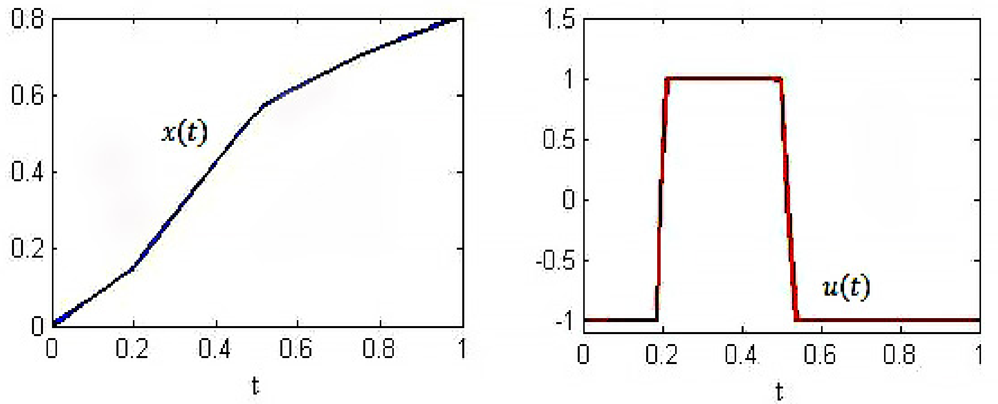

Fig. 5. The gained approximate solutions with $m = 2$ and $n = 8$, for Example 4.

Example 4.5. In this example, we solve the Robot Arm problem [1, 13]. In this example, the aim is to minimize.

$$J = t_f, \tag{50}$$

subject to

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = \frac{u_1(t)}{L}, \quad \dot{x}_3(t) = x_4(t), \tag{51}$$

$$\dot{x}_4(t) = \frac{u_2(t)}{I_\theta}, \quad \dot{x}_5(t) = x_6(t), \quad \dot{x}_6(t) = \frac{u_3(t)}{I_\phi}, \tag{52}$$

$$x_1(0) = \frac{9}{2}; \quad x_2(0) = 0; \quad x_3(0) = 0, \tag{53}$$

$$x_4(0) = 0; \quad x_5(0) = \frac{\pi}{4}; \quad x_6(0) = 0, \tag{54}$$

$$x_1(t_f) = \frac{9}{2}; \quad x_2(t_f) = 0; \quad x_3(t_f) = \frac{2\pi}{3}, \tag{55}$$

$$x_4(t_f) = 0; \quad x_5(t_f) = \frac{\pi}{4}; \quad x_6(t_f) = 0, \tag{56}$$

$$-1 \leq u_i(t) \leq 1, \quad i = 1, 2, 3, \tag{57}$$

where $L = 5$ and

$$I_\theta = \frac{((L - x_1(t))^3 + x_1(t)^3)}{3} \sin^2(x_5(t)),$$

$$I_\phi = \frac{(L - x_1(t))^3 + x_1(t)^3}{3}.$$

We solve the problem for $\varepsilon = 0.3$. In Table 5, the gained values for $J = t_f$ with different is shown. The achieved solutions are represented in Figures 6, 8 and 7. Also, the gained switching points are

$$t_1 = 2.1716,\ t_2 = 2.8298,\ t_3 = 4.5652,\ t_4 = 6.4366,\ t_5 = 6.9437,$$

for $n = 11$. Comparing our results with those of work [13], shows that we achieve to $t_f = 9.1413$ with $n = 11$ while this value in work [13] is given for $n = 200$. So, presented work is more accurate and more effective.

		t_f
Presented method		
$m = 5$	$n = 4$	**9.1**3556016
	$n = 6$	**9.1**3982078
	$n = 8$	**9.14**093774
	$n = 10$	**9.141**26867
	$n = 11$	**9.1413**6236
	$n = 12$	**9.14138**462
Method [13]		
$m = 0$	$n = 200$	9.14138

Tab. 5. The obtained values of t_f for Example 5.

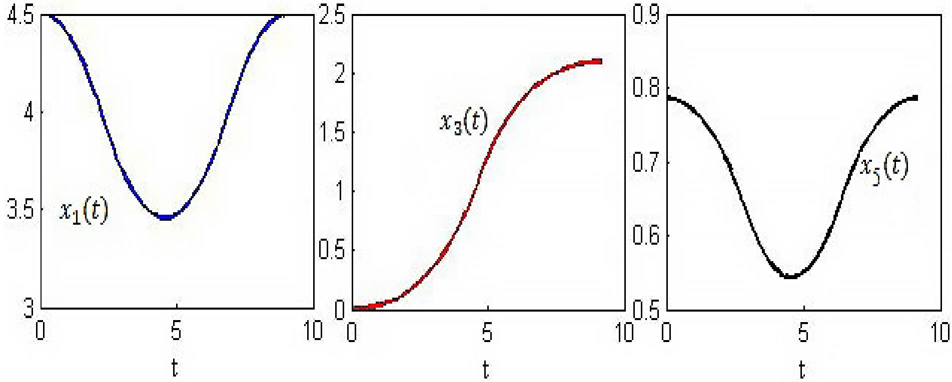

Fig. 6. The approximate solutions $x_1(t)$, $x_3(t)$, $x_5(t)$ with $m = 5$ and $n = 10$, for Example 5.

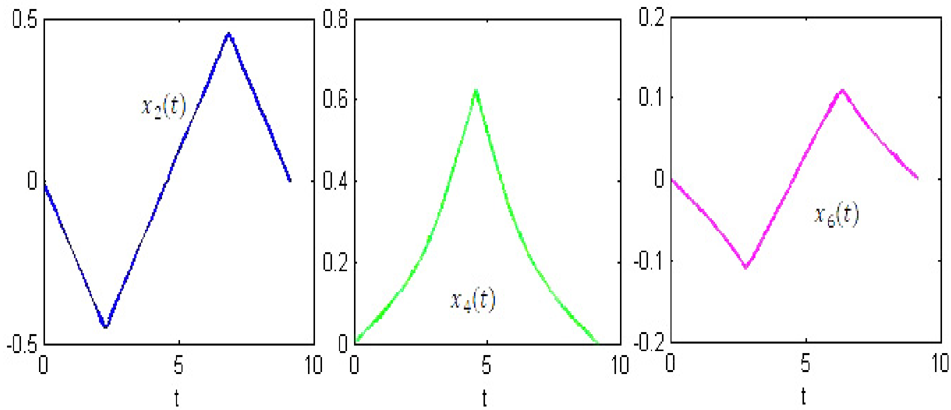

Fig. 7. The approximate solutions $x_2(t)$, $x_4(t)$, $x_6(t)$ with $m = 5$ and $n = 10$, for Example 5.

5. CONCLUSIONS AND SUGGESTIONS

In this article, we extended a highly accurate and efficient *hp* spectral collocation method via the operational integration matrix to numerically solve the general form of the non-smooth control problems. Utilizing the integration matrix instead of the differentiation matrix reduces the method's complexity. Unlike methods based on the operational matrix of differentiation, the condition of continuity of the state variable is considered in the proposed method, and the interpolating polynomials of the control and state variables on the considered sub-intervals, especially at the switching points, provide high accuracy of the method compared with other methods. We showed the proposed approach could be utilized for both bang-bang and singular optimal control problems. Moreover,

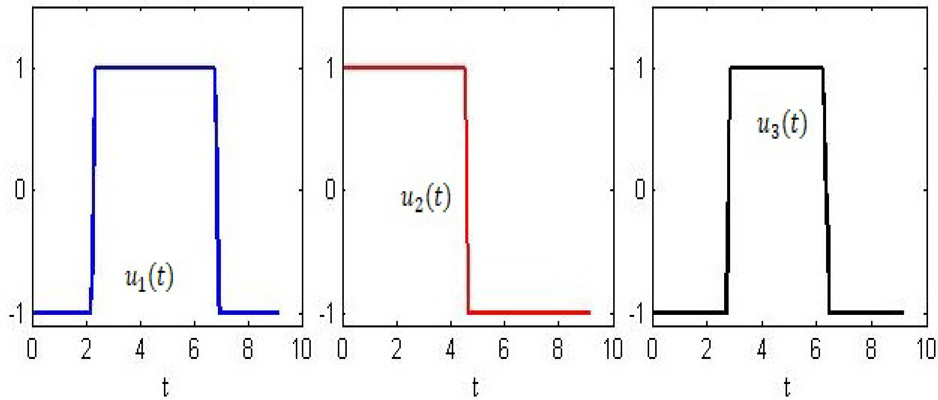

Fig. 8. The approximate control variables with $m = 5$ and $n = 10$, for Example 5.

the switching points are unknown parameters, and they can be gained having solved a nonlinear programming problem. Further, the numerical simulation showed that our method for nonsmooth OC problems has a higher convergence rate than other methods, such as methods given in [13, 39, 42].

For future works, we suggest the given method for nonsmooth fractional and non-fractional delay optimal control problems. Also, we can extend the current approach for optimal control problems under delay or non-delay partial differential equations with nonsmooth treatment.

ACKNOWLEDGEMENT

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

(Received December 24, 2021)

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