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CHARACTERIZATION OF FUZZY ORDER RELATION BY FUZZY CONE

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In the present paper, fuzzy order relations on a real vector space are characterized by fuzzy cones. It is well-known that there is one-to-one correspondence between order relations, that a real vector space with the order relation is an ordered vector space, and pointed convex cones. We show that there is one-to-one correspondence between fuzzy order relations with some properties, which are fuzzification of the order relations, and fuzzy pointed convex cones, which are fuzzification of the pointed convex cones.

Keywords: fuzzy order relation, fuzzy cone

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1. INTRODUCTION

The concept of fuzzy sets has been primarily introduced for representing sets containing uncertainty or vagueness by Zadeh [8] as fuzzy set theory. Then, fuzzy set theory has been applied in various areas such as economics, management science, engineering, optimization theory, operations research, etc.

On the other hand, the concept of fuzzy relations has been primarily introduced by Zadeh [9], in which fuzzy order relation is defined by generalizing the reflexivity, antisymmetry, and transitivity of crisp order relation. Then, the fuzzy order relation has been extended in various ways; see, for example, Zhang et al. [10] and references therein.

In the present paper, fuzzy order relations on a real vector space are characterized by fuzzy cones. It is well-known that there is one-to-one correspondence between order relations, that a real vector space with the order relation is an ordered vector space, and pointed convex cones. We show that there is one-to-one correspondence between fuzzy order relations with some properties, which are fuzzification of the order relations, and fuzzy pointed convex cones, which are fuzzification of the pointed convex cones. Furthermore, three kinds of fuzzy preorder relations on the set of all subsets of a real vector space are constructed from a fuzzy convex cone.

For that purpose, this paper is organized as follows. In Section 2, some notations and the definition of fuzzy cone are presented. In Section 3, well-known results on ordered

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vector spaces are introduced. In Section 4, fuzzy order relations are characterized by fuzzy cones, and three kinds of fuzzy preorder relations are constructed from a fuzzy cone. Finally, conclusions are presented in Section 5.

2. PRELIMINARIES

In this section, some notations and the definition of fuzzy cone are presented.

Let $\mathbb{R}$ be the set of all real numbers. For $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, $[a, b[= \{x \in \mathbb{R} : a \leq x < b\}$, $]a, b] = \{x \in \mathbb{R} : a < x \leq b\}$, and $]a, b[= \{x \in \mathbb{R} : a < x < b\}$.

Let X be a set. Then, $\widetilde{A} : X \to [0, 1]$ is called a *fuzzy set* on X. Let $\mathcal{F}(X)$ be the set of all fuzzy sets on X. For $\widetilde{A} \in \mathcal{F}(X)$ and $\alpha \in]0, 1]$, the *α-level set* of $\widetilde{A}$ is defined as

$$[\widetilde{A}]_\alpha = \{x \in X : \widetilde{A}(x) \geq \alpha\}. \tag{1}$$

For a crisp set $S \subset X$, the indicator function of S is a function $c_S : X \to \{0, 1\}$ defined as

$$c_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S \end{cases} \tag{2}$$

for each $x \in X$.

Throughout the paper, E is a real vector space, and θ is a zero element of E. Let $\mathcal{P}(E)$ be the set of all subsets of E. For $A, B \subset E$ and $\lambda \in \mathbb{R}$, we define $A + B$, λA, and $A - B$ as follows:

$$A + B = \{x + y : x \in A,\ y \in B\}, \tag{3}$$
$$\lambda A = \{\lambda x : x \in A\}, \tag{4}$$
$$A - B = A + (-B) \tag{5}$$

where $-B = (-1)B$.

A nonempty set $K \subset E$ is called a *cone* if $\lambda x \in K$ for any $x \in K$ and any real number $\lambda \geq 0$. A cone $K \subset E$ is called a *convex cone* if K is a convex set. A cone $K \subset E$ is said to be *pointed* if $K \cap (-K) = \{\theta\}$.

A fuzzy set $\widetilde{K} \in \mathcal{F}(E)$ is called a *fuzzy cone* if $[\widetilde{K}]_\alpha \subset E$ is a cone for any $\alpha \in]0, 1]$; see Kon [3] and Kon and Kuwano [4]. A fuzzy cone $\widetilde{K} \in \mathcal{F}(E)$ is said to be *convex* if $[\widetilde{K}]_\alpha \subset E$ is a convex set for any $\alpha \in]0, 1]$. A fuzzy cone which is convex is called a fuzzy convex cone. A fuzzy cone $\widetilde{K} \in \mathcal{F}(E)$ is said to be *pointed* if $[\widetilde{K}]_\alpha \subset E$ is pointed for any $\alpha \in]0, 1]$. A fuzzy convex cone which is pointed is called a fuzzy pointed convex cone.

3. ORDERED VECTOR SPACE

In this section, well-known results on ordered vector spaces are introduced.

A real vector space E is called an *ordered vector space* if E has an order relation (a binary relation on E that is reflexive, antisymmetric, and transitive) which satisfies the following conditions (O_1) and (O_2).

(O_1) $x, y, z \in E,\ x \leq y \ \Rightarrow\ x + z \leq y + z.$

(O_2) $x, y \in E,\ x \leq y,\ \lambda > 0\ \Rightarrow\ \lambda x \leq \lambda y.$

We often write $y \geq x$ when $x \leq y$.

When E is an ordered vector space, the set

$$\{x \in E : x \geq \theta\} \tag{6}$$

is called the *positive cone* of E. It can be seen that the positive cone of E is a pointed convex cone.

Conversely, in a real vector space E which is not necessarily an ordered vector space, let $K \subset E$, and

$$x \leq_K y \ \stackrel{\text{def}}{\Leftrightarrow}\ y - x \in K \tag{7}$$

for $x, y \in E$. Then, if K is a pointed convex cone, then $\leq_K$ is an order relation on E, E is an ordered vector space, and K is a positive cone of E. Moreover, if K is a convex cone, then $\leq_K$ is a preorder relation on E (a binary relation on E that is reflexive and transitive).

For a real vector space E, it is well-known that there is one-to-one correspondence between the set of all order relations on E which satisfies the conditions (O_1) and (O_2) and the set of all pointed convex cones by (6) and (7); see, for example, Khan et al. [2] and Peressini [6].

Let $K \subset E$ be a convex cone. For $A, B \in \mathcal{P}(E)$,

$$A \leq_K^l B \ \stackrel{\text{def}}{\Leftrightarrow}\ B \subset A + K, \tag{8}$$

$$A \leq_K^u B \ \stackrel{\text{def}}{\Leftrightarrow}\ A \subset B - K, \tag{9}$$

$$A \leq_K B \ \stackrel{\text{def}}{\Leftrightarrow}\ A \leq_K^l B,\ A \leq_K^u B. \tag{10}$$

Then, $\leq_K^l$, $\leq_K^u$, and $\leq_K$ are preorder relations on $\mathcal{P}(E)$ (binary relations on $\mathcal{P}(E)$ that are reflexive and transitive). The preorder relation $\leq_K$ has been primarily introduced by Young [7], and then $\leq_K^l$ and $\leq_K^u$, which are modifications of $\leq_K$, are introduced by Kuroiwa et al. [5]. For other order relations on $\mathcal{P}(E)$, see Jahn and Ha [1].

4. FUZZY ORDER RELATION

In this section, fuzzy order relations are characterized by fuzzy cones, and three kinds of fuzzy preorder relations are constructed from a fuzzy cone.

For sets X and Y, $\widetilde{R} \in \mathcal{F}(X \times Y)$ is called a *fuzzy relation* from X to Y. For a set X, a fuzzy relation from X to X is called a fuzzy relation on X.

Let X be a nonempty set, and let $\widetilde{R} \in \mathcal{F}(X \times X)$. The fuzzy relation $\widetilde{R}$ is called a *fuzzy order relation* on X if the following conditions (i), (ii), and (iii) are satisfied. The fuzzy relation $\widetilde{R}$ is called a *fuzzy preorder relation* on X if the following conditions (i) and (iii) are satisfied.

(i) *Reflexiveness*

$$\widetilde{R}(x, x) = 1, \quad \forall x \in X. \tag{11}$$

(ii) *Antisymmetricity*

$$x, y \in X,\ \widetilde{R}(x,y) > 0,\ \widetilde{R}(y,x) > 0 \ \Rightarrow\ x = y. \tag{12}$$

(iii) *Transitivity*

$$\widetilde{R}(x,y) \geq \bigvee_{z\in X} (\widetilde{R}(x,z) \wedge \widetilde{R}(z,y)), \quad \forall x, y \in X. \tag{13}$$

In (13),

$$\widetilde{R}(x,z) \wedge \widetilde{R}(z,y) = \min\{\widetilde{R}(x,z), \widetilde{R}(z,y)\}$$

and

$$\bigvee_{z\in X} (\widetilde{R}(x,z) \wedge \widetilde{R}(z,y)) = \sup_{z\in X}(\widetilde{R}(x,z) \wedge \widetilde{R}(z,y)).$$

For details on fuzzy order relations and other fuzzy relations, see, for example, Zadeh [9] and Zhang et al. [10].

Theorem 4.1. Let $\widetilde{K} \in \mathcal{F}(E)$ be a fuzzy pointed convex cone. We define $\widetilde{R} \in \mathcal{F}(E\times E)$ as

$$\widetilde{R}(x,y) = \widetilde{K}(y-x) \tag{14}$$

for each $x, y \in E$. Then, the following statements hold.

(i) $\widetilde{R}$ is a fuzzy order relation on E.

(ii)

$$\widetilde{R}(x+z, y+z) = \widetilde{R}(x,y), \quad \forall x, y, z \in E. \tag{15}$$

(iii)

$$\widetilde{R}(\lambda x, \lambda y) = \widetilde{R}(x,y), \quad \forall x, y \in E,\ \forall \lambda > 0. \tag{16}$$

Proof. (i) First, we have

$$\widetilde{R}(x,x) = \widetilde{K}(x-x) = \widetilde{K}(\theta) = 1$$

for any $x \in E$.

Next, let $x, y \in E$, and suppose that $\widetilde{R}(x,y) > 0$ and $\widetilde{R}(y,x) > 0$. We put

$$\alpha = \widetilde{R}(x,y) \wedge \widetilde{R}(y,x) = \widetilde{K}(y-x) \wedge \widetilde{K}(x-y) > 0.$$

Since $y-x, x-y \in [\widetilde{K}]_\alpha$ and $[\widetilde{K}]_\alpha$ is pointed, it follows that $y-x=\theta$. Therefore, we have $x=y$.

Next, fix any $x, y, z \in E$. We put

$$\alpha = \widetilde{R}(x,z) \wedge \widetilde{R}(z,y) = \widetilde{K}(z-x) \wedge \widetilde{K}(y-z).$$

When $\alpha = 0$, we have

$$\widetilde{R}(x,y) \geq 0 = \alpha = \widetilde{R}(x,z) \wedge \widetilde{R}(z,y).$$

Suppose that $\alpha > 0$. Since $z - x, y - z \in [\widetilde{K}]_\alpha$ and $[\widetilde{K}]_\alpha$ is a convex cone, it follows that $y - x = (y - z) + (z - x) \in [\widetilde{K}]_\alpha$, and that

$$\widetilde{R}(x, y) = \widetilde{K}(y - x) \geq \alpha = \widetilde{R}(x, z) \wedge \widetilde{R}(z, y).$$

Therefore, we have

$$\widetilde{R}(x, y) \geq \bigvee_{z \in E} (\widetilde{R}(x, z) \wedge \widetilde{R}(z, y))$$

by the arbitrariness of $z \in E$.

(ii) Let $x, y, z \in E$. Then, we have

$$\widetilde{R}(x + z, y + z) = \widetilde{K}((y + z) - (x + z)) = \widetilde{K}(y - x) = \widetilde{R}(x, y).$$

(iii) Let $x, y \in E$, and let $\lambda > 0$. Then, it follows that

$$\widetilde{R}(\lambda x, \lambda y) = \widetilde{K}(\lambda y - \lambda x) = \widetilde{K}(\lambda(y - x))$$

and $\widetilde{R}(x, y) = \widetilde{K}(y - x)$. For any $\alpha \in]0, 1]$, since $[\widetilde{K}]_\alpha$ is a cone, it follows that $y - x \in [\widetilde{K}]_\alpha$ if and only if $\lambda(y - x) \in [\widetilde{K}]_\alpha$. Therefore, we have

$$\widetilde{R}(\lambda x, \lambda y) = \widetilde{K}(\lambda(y - x)) = \widetilde{K}(y - x) = \widetilde{R}(x, y).$$

□

In Theorem 4.1, (ii) and (iii) are conditions which correspond to (O_1) and (O_2) for crisp order relation, respectively.

Theorem 4.2. Let $\widetilde{R} \in \mathcal{F}(E \times E)$ be a fuzzy order relation on E which satisfies the conditions (15) and (16). Let $\widetilde{K} \in \mathcal{F}(E)$ satisfy

$$\widetilde{K}(x) = \widetilde{R}(\theta, x) \tag{17}$$

for each $x \in E$. Then, $\widetilde{K}$ is a fuzzy pointed convex cone.

Proof. Fix any $\alpha \in]0, 1]$. Then, we show that $[\widetilde{K}]_\alpha$ is a pointed convex cone.

First, since $\widetilde{K}(\theta) = \widetilde{R}(\theta, \theta) = 1$, it follows that $\theta \in [\widetilde{K}]_\alpha$, and that $[\widetilde{K}]_\alpha \neq \emptyset$.

Next, let $x \in [\widetilde{K}]_\alpha$, and let $\lambda \geq 0$. When $\lambda = 0$, we have $\lambda x = \theta \in [\widetilde{K}]_\alpha$. Suppose that $\lambda > 0$. Since $x \in [\widetilde{K}]_\alpha$, it follows that

$$\widetilde{K}(\lambda x) = \widetilde{R}(\theta, \lambda x) = \widetilde{R}(\theta, x) = \widetilde{K}(x) \geq \alpha.$$

Therefore, we have $\lambda x \in [\widetilde{K}]_\alpha$.

Next, let $x, y \in [\widetilde{K}]_\alpha$. Then, it follows that

$$\begin{aligned}\widetilde{K}(x + y) = \widetilde{R}(\theta, x + y) &\geq \bigvee_{z \in E} (\widetilde{R}(\theta, z) \wedge \widetilde{R}(z, x + y)) \\ &\geq \widetilde{R}(\theta, x) \wedge \widetilde{R}(x, x + y) = \widetilde{R}(\theta, x) \wedge \widetilde{R}(\theta, y) = \widetilde{K}(x) \wedge \widetilde{K}(y) \geq \alpha.\end{aligned}$$

Therefore, we have $x + y \in [\widetilde{K}]_\alpha$.

Finally, since $[\widetilde{K}]_\alpha \cap (-[\widetilde{K}]_\alpha) \supset \{\theta\}$, we need to show that $[\widetilde{K}]_\alpha \cap (-[\widetilde{K}]_\alpha) \subset \{\theta\}$. Let $x \in [\widetilde{K}]_\alpha \cap (-[\widetilde{K}]_\alpha)$. Since $x \in [\widetilde{K}]_\alpha$, it follows that

$$\widetilde{R}(\theta, x) = \widetilde{K}(x) \geq \alpha > 0.$$

Since $x \in -[\widetilde{K}]_\alpha$, there exists $y \in [\widetilde{K}]_\alpha$ such that $x = -y$. Since $-x = y \in [\widetilde{K}]_\alpha$, it follows that

$$\widetilde{R}(x, \theta) = \widetilde{R}(\theta, -x) = \widetilde{K}(-x) \geq \alpha > 0.$$

Therefore, we have $x = \theta$ by (12). □

The following theorem provides one-to-one correspondence between the set of all fuzzy order relations on E which satisfy the conditions (15) and (16) and the set of all fuzzy pointed convex cones.

Corollary 4.3. Let $\mathcal{FK}$ be the set of all fuzzy pointed convex cones on E, and let $\mathcal{FR}$ be the set of all fuzzy order relations on E which satisfy the conditions (15) and (16). We define $\Phi : \mathcal{FK} \to \mathcal{FR}$ as

$$\Phi(\widetilde{K}) = \widetilde{R}_{\widetilde{K}} \tag{18}$$

for each $\widetilde{K} \in \mathcal{FK}$, where

$$\widetilde{R}_{\widetilde{K}}(x, y) = \widetilde{K}(y - x) \tag{19}$$

for each $x, y \in E$. Then, Φ is a bijection.

P r o o f. First, we show that Φ is surjective. Let $\widetilde{R} \in \mathcal{FR}$. We define $\widetilde{K} \in \mathcal{F}(E)$ as $\widetilde{K}(x) = \widetilde{R}(\theta, x)$ for each $x \in E$. From Theorem 4.2, it follows that $\widetilde{K} \in \mathcal{FK}$. Since

$$\widetilde{R}_{\widetilde{K}}(x, y) = \widetilde{K}(y - x) = \widetilde{R}(\theta, y - x) = \widetilde{R}(x, y)$$

for any $x, y \in E$, we have $\widetilde{R}_{\widetilde{K}} = \widetilde{R}$. Therefore, Φ is surjective.

Next, we show that Φ is injective. Let $\widetilde{K}, \widetilde{K}' \in \mathcal{FK}$, and suppose that

$$\Phi(\widetilde{K}) = \widetilde{R}_{\widetilde{K}} = \widetilde{R}_{\widetilde{K}'} = \Phi(\widetilde{K}').$$

Since

$$\widetilde{K}(x) = \widetilde{R}_{\widetilde{K}}(\theta, x) = \widetilde{R}_{\widetilde{K}'}(\theta, x) = \widetilde{K}'(x)$$

for any $x \in E$, we have $\widetilde{K} = \widetilde{K}'$. Therefore, Φ is injective. □

Let $K \subset E$ be a pointed convex cone. Suppose that the fuzzy order relation $\widetilde{R}$ on E is defined by the fuzzy pointed convex cone $c_K \in \mathcal{F}(E)$. That is, $\widetilde{R}$ is defined by (14) when $\widetilde{K} = c_K$. Then, it follows that

$$x \leq_K y \;\Leftrightarrow\; y - x \in K \;\Leftrightarrow\; \widetilde{R}(x, y) = c_K(y - x) = 1,$$

$$x \not\leq_K y \;\Leftrightarrow\; y - x \notin K \;\Leftrightarrow\; \widetilde{R}(x, y) = c_K(y - x) = 0$$

for $x, y \in E$, where $\leq_K$ is defined by (7). Thus, the fuzzy order relation $\widetilde{R}$ can be identified with the crisp order relation $\leq_K$. Therefore, fuzzy order relation on E is an extension of crisp order relation on E.

We say that an order relation $R \subset E \times E$ satisfies the conditions (O_1) and (O_2) if $\leq$ satisfies the conditions (O_1) and (O_2) when $\leq$ is defined for any $x, y \in E$ as follows:

$$x \leq y \overset{\text{def}}{\Leftrightarrow} (x, y) \in R.$$

We say that an order relation $R \subset E \times E$ corresponds to a pointed convex cone $K \subset E$ if the condition

$$(x, y) \in R \;\Leftrightarrow\; y - x \in K$$

holds for any $x, y \in E$.

Theorem 4.4. Let $\widetilde{K} \in \mathcal{F}(E)$ be a fuzzy pointed convex cone. It is assumed that $\widetilde{R} \in \mathcal{F}(E \times E)$ is a fuzzy order relation on E defined by (14) for $\widetilde{K}$. Then, for any $\alpha \in]0, 1]$, $[\widetilde{R}]_\alpha$ is an order relation on E which satisfies the conditions (O_1) and (O_2), and it corresponds to a pointed convex cone $[\widetilde{K}]_\alpha$.

Proof. Fix any $\alpha \in]0, 1]$.

First, we show that $[\widetilde{R}]_\alpha$ is an order relation on E.

Let $x \in E$. Since $\widetilde{R}(x, x) = 1 \geq \alpha$, we have $(x, x) \in [\widetilde{R}]_\alpha$.

Let $x, y \in E$, and suppose that $(x, y), (y, x) \in [\widetilde{R}]_\alpha$. Since $\widetilde{R}(x, y) \geq \alpha > 0$ and $\widetilde{R}(y, x) \geq \alpha > 0$, we have $x = y$.

Let $x, y, z \in E$, and suppose that $(x, y), (y, z) \in [\widetilde{R}]_\alpha$. Since $\widetilde{R}(x, y) \geq \alpha$ and $\widetilde{R}(y, z) \geq \alpha$, it follows that

$$\widetilde{R}(x, z) \geq \widetilde{R}(x, y) \wedge \widetilde{R}(y, z) \geq \alpha.$$

Therefore, we have $(x, z) \in [\widetilde{R}]_\alpha$.

Next, we show that $[\widetilde{R}]_\alpha$ satisfies the condition (O_1). Let $x, y, z \in E$, and suppose that $(x, y) \in [\widetilde{R}]_\alpha$. Since

$$\widetilde{R}(x + z, y + z) = \widetilde{R}(x, y) \geq \alpha,$$

we have $(x + z, y + z) \in [\widetilde{R}]_\alpha$.

Next, we show that $[\widetilde{R}]_\alpha$ satisfies the condition (O_2). Let $x, y \in E$, and let $\lambda > 0$. Suppose that $(x, y) \in [\widetilde{R}]_\alpha$. Since

$$\widetilde{R}(\lambda x, \lambda y) = \widetilde{R}(x, y) \geq \alpha,$$

we have $(\lambda x, \lambda y) \in [\widetilde{R}]_\alpha$.

Finally, we show that $[\widetilde{R}]_\alpha$ corresponds to $[\widetilde{K}]_\alpha$. For $x, y \in E$, we have

$$(x, y) \in [\widetilde{R}]_\alpha \;\Leftrightarrow\; \widetilde{R}(x, y) = \widetilde{K}(y - x) \geq \alpha \;\Leftrightarrow\; y - x \in [\widetilde{K}]_\alpha.$$

□

Theorem 4.5. Let $\{R_\alpha\}_{\alpha\in]0,1]}$ be a family of order relations on E which satisfy the conditions (O_1) and (O_2). It is assumed that

$$\alpha,\beta\in\,]0,1]\,,\ \alpha<\beta\ \Rightarrow\ R_\alpha\supset R_\beta.$$

Let $\{K_\alpha\}_{\alpha\in]0,1]}$ be a family of pointed convex cones in E. It is assumed that

$$\alpha,\beta\in\,]0,1]\,,\ \alpha<\beta\ \Rightarrow\ K_\alpha\supset K_\beta.$$

For any $\alpha\in\,]0,1]$, it is assumed that R_α corresponds to K_α. If $\widetilde{R}\in\mathcal{F}(E\times E)$ and $\widetilde{K}\in\mathcal{F}(E)$ are given by

$$\widetilde{R}=\sup_{\alpha\in]0,1]}\alpha c_{R_\alpha},\quad \widetilde{K}=\sup_{\alpha\in]0,1]}\alpha c_{K_\alpha},$$

then $\widetilde{R}$ is a fuzzy order relation on E, $\widetilde{K}$ is a fuzzy pointed convex cone on E, and $\widetilde{R}$ and $\widetilde{K}$ satisfy (14).

P r o o f. First, we show that $\widetilde{R}$ is a fuzzy order relation on E.

Let $x\in E$. Since $(x,x)\in R_\alpha$ for any $\alpha\in\,]0,1]$, we have

$$\widetilde{R}(x,x)=\sup_{\alpha\in]0,1]}\alpha c_{R_\alpha}(x,x)=\sup_{\alpha\in]0,1]}\alpha=1.$$

Let $x,y\in E$, and suppose that $\widetilde{R}(x,y)>0$ and $\widetilde{R}(y,x)>0$. Since

$$\widetilde{R}(x,y)=\sup_{\alpha\in]0,1]}\alpha c_{R_\alpha}(x,y)=\sup\{\alpha\in\,]0,1]:(x,y)\in R_\alpha\}>0$$

and

$$\widetilde{R}(y,x)=\sup_{\alpha\in]0,1]}\alpha c_{R_\alpha}(y,x)=\sup\{\alpha\in\,]0,1]:(y,x)\in R_\alpha\}>0,$$

it follows that $(x,y),(y,x)\in R_\beta$ for any $\beta\in\left]0,\widetilde{R}(x,y)\wedge\widetilde{R}(y,x)\right[$. Therefore, we have $x=y$.

Fix any $x,y,z\in E$, and we put $\beta=\widetilde{R}(x,z)\wedge\widetilde{R}(z,y)$. When $\beta=0$, it follows that

$$\widetilde{R}(x,y)\geq 0=\beta=\widetilde{R}(x,z)\wedge\widetilde{R}(z,y).$$

Suppose that $\beta>0$. Since

$$\widetilde{R}(x,z)=\sup_{\alpha\in]0,1]}\alpha c_{R_\alpha}(x,z)=\sup\{\alpha\in\,]0,1]:(x,z)\in R_\alpha\}\geq\beta>0$$

and

$$\widetilde{R}(z,y)=\sup_{\alpha\in]0,1]}\alpha c_{R_\alpha}(z,y)=\sup\{\alpha\in\,]0,1]:(z,y)\in R_\alpha\}\geq\beta>0,$$

it follows that for any $\gamma\in\,]0,\beta[$, $(x,z),(z,y)\in R_\gamma$, and we have $(x,y)\in R_\gamma$. Since

$$\widetilde{R}(x,y)=\sup_{\alpha\in]0,1]}\alpha c_{R_\alpha}(x,y)=\sup\{\alpha\in\,]0,1]:(x,y)\in R_\alpha\}\geq\gamma$$

for any $\gamma \in]0, \beta[$, it follows that

$$\widetilde{R}(x, y) \geq \beta = \widetilde{R}(x, z) \wedge \widetilde{R}(z, y).$$

Therefore, we have

$$\widetilde{R}(x, y) \geq \bigvee_{z \in E} (\widetilde{R}(x, z) \wedge \widetilde{R}(z, y))$$

by the arbitrariness of $z \in E$.

Next, we show that $\widetilde{K}$ is a fuzzy pointed convex cone. For any $\alpha \in]0, 1]$, we have

$$[\widetilde{K}]_\alpha = \bigcap_{\beta \in]0, \alpha[} K_\beta$$

since

$$\begin{aligned} x \in [\widetilde{K}]_\alpha \quad &\Leftrightarrow \quad \widetilde{K}(x) = \sup\{\beta \in]0, 1] : x \in K_\beta\} \geq \alpha \\ &\Leftrightarrow \quad x \in K_\beta, \ \forall \beta \in]0, \alpha[\\ &\Leftrightarrow \quad x \in \bigcap_{\beta \in]0, \alpha[} K_\beta. \end{aligned}$$

Therefore, since $[\widetilde{K}]_\alpha$ is a pointed convex cone for any $\alpha \in]0, 1]$, $\widetilde{K}$ is a fuzzy pointed convex cone.

Finally, we show that $\widetilde{R}$ and $\widetilde{K}$ satisfy (14). For $x, y \in E$, we have

$$\begin{aligned} \widetilde{R}(x, y) &= \sup_{\alpha \in]0,1]} \alpha c_{R_\alpha}(x, y) \\ &= \sup\{\alpha \in]0, 1] : (x, y) \in R_\alpha\} \\ &= \sup\{\alpha \in]0, 1] : y - x \in K_\alpha\} \\ &= \sup_{\alpha \in]0,1]} \alpha c_{K_\alpha}(y - x) \\ &= \widetilde{K}(y - x). \end{aligned}$$

□

The following theorem provides three kinds of fuzzy preorder relations on $\mathcal{P}(E)$ which are constructed from a fuzzy convex cone.

Theorem 4.6. Let $\widetilde{K} \in \mathcal{F}(E)$ be a fuzzy convex cone. We define $\widetilde{R}_l, \widetilde{R}_u, \widetilde{R} \in \mathcal{F}(\mathcal{P}(E) \times \mathcal{P}(E))$ as

$$\widetilde{R}_l(A, B) = \sup\{\alpha \in]0, 1] : B \subset A + [\widetilde{K}]_\alpha\}, \tag{20}$$

$$\widetilde{R}_u(A, B) = \sup\{\alpha \in]0, 1] : A \subset B - [\widetilde{K}]_\alpha\}, \tag{21}$$

$$\widetilde{R}(A, B) = \sup\{\alpha \in]0, 1] : B \subset A + [\widetilde{K}]_\alpha, \ A \subset B - [\widetilde{K}]_\alpha\} \tag{22}$$

for each $A, B \in \mathcal{P}(E)$, where $\sup \emptyset = 0$. Then, $\widetilde{R}_l$, $\widetilde{R}_u$, and $\widetilde{R}$ are fuzzy preorder relations on E.

P r o o f . We show only that $\widetilde{R}$ is a fuzzy preorder relation on E. It can be shown that $\widetilde{R}_l$ and $\widetilde{R}_u$ are fuzzy preorder relations on E in the similar way.

First, let $A \in \mathcal{P}(E)$. Since $A \subset A + [\widetilde{K}]_\alpha$ and $A \subset A - [\widetilde{K}]_\alpha$ for any $\alpha \in]0,1]$, we have

$$\widetilde{R}(A,A) = \sup\{\alpha \in]0,1] : A \subset A + [\widetilde{K}]_\alpha,\ A \subset A - [\widetilde{K}]_\alpha\} = 1.$$

Next, fix any $A, B, C \in \mathcal{P}(E)$. We put

$$\begin{aligned} \alpha &= \widetilde{R}(A,C) \wedge \widetilde{R}(C,B) \\ &= (\sup\{\alpha \in]0,1] : C \subset A + [\widetilde{K}]_\alpha,\ A \subset C - [\widetilde{K}]_\alpha\}) \\ &\quad \wedge(\sup\{\alpha \in]0,1] : B \subset C + [\widetilde{K}]_\alpha,\ C \subset B - [\widetilde{K}]_\alpha\}). \end{aligned}$$

When $\alpha = 0$, we have

$$\widetilde{R}(A,B) \geq 0 = \alpha = \widetilde{R}(A,C) \wedge \widetilde{R}(C,B).$$

Suppose that $\alpha > 0$, and fix any $\beta \in]0,\alpha[$. Since $C \subset A + [\widetilde{K}]_\beta$, $A \subset C - [\widetilde{K}]_\beta$, $B \subset C + [\widetilde{K}]_\beta$, $C \subset B - [\widetilde{K}]_\beta$, and $[\widetilde{K}]_\beta$ is a convex cone, it follows that

$$B \subset C + [\widetilde{K}]_\beta \subset A + [\widetilde{K}]_\beta + [\widetilde{K}]_\beta = A + [\widetilde{K}]_\beta$$

and

$$A \subset C - [\widetilde{K}]_\beta \subset B - [\widetilde{K}]_\beta - [\widetilde{K}]_\beta = B - [\widetilde{K}]_\beta.$$

Thus, it follows that

$$\widetilde{R}(A,B) = \sup\{\alpha \in]0,1] : B \subset A + [\widetilde{K}]_\alpha,\ A \subset B - [\widetilde{K}]_\alpha\} \geq \beta.$$

Therefore, we have

$$\widetilde{R}(A,B) \geq \alpha = \widetilde{R}(A,C) \wedge \widetilde{R}(C,B)$$

by the arbitrariness of $\beta \in]0,\alpha[$. Moreover, we have

$$\widetilde{R}(A,B) \geq \bigvee_{C \in \mathcal{P}(E)} (\widetilde{R}(A,C) \wedge \widetilde{R}(C,B))$$

by the arbitrariness of $C \in \mathcal{P}(E)$. □

Let $K \subset E$ be a convex cone. Suppose that the fuzzy preorder relations $\widetilde{R}_l$, $\widetilde{R}_u$, and $\widetilde{R}$ on $\mathcal{P}(E)$ are defined by the fuzzy convex cone $c_K \in \mathcal{F}(E)$. That is, $\widetilde{R}_l$, $\widetilde{R}_u$, and $\widetilde{R}$ are defined by (20), (21), and (22) when $\widetilde{K} = c_K$, respectively. Then, it follows that

$$\begin{aligned} A \leq_K^l B &\Leftrightarrow \widetilde{R}_l(A,B) = 1, \\ A \not\leq_K^l B &\Leftrightarrow \widetilde{R}_l(A,B) = 0, \end{aligned}$$

$$\begin{aligned} A \leq_K^u B &\Leftrightarrow \widetilde{R}_u(A,B) = 1, \\ A \not\leq_K^u B &\Leftrightarrow \widetilde{R}_u(A,B) = 0, \end{aligned}$$

$$\begin{aligned} A \leq_K B &\Leftrightarrow \widetilde{R}(A,B) = 1, \\ A \not\leq_K B &\Leftrightarrow \widetilde{R}(A,B) = 0 \end{aligned}$$

for $A, B \in \mathcal{P}(E)$, where $\leq_K^l$, $\leq_K^u$, and $\leq_K$ are defined by (8), (9), and (10), respectively. Thus, the fuzzy preorder relations $\widetilde{R}_l$, $\widetilde{R}_u$, and $\widetilde{R}$ can be identified with the crisp preorder relations $\leq_K^l$, $\leq_K^u$, and $\leq_K$, respectively. Therefore, the fuzzy preorder relations on $\mathcal{P}(E)$ defined by (20), (21), and (22) are extensions of the crisp preorder relations on $\mathcal{P}(E)$ defined by (8), (9), and (10), respectively.

5. CONCLUSION

In the present paper, fuzzy order relations on a real vector space were characterized by fuzzy cones. It was shown that there is one-to-one correspondence between fuzzy order relations on a real vector space with some properties and fuzzy pointed convex cones. Moreover, three kinds of fuzzy preorder relations on the set of all subsets of a real vector space were constructed from a fuzzy convex cone.

The similar results as Theorems 4.1, 4.2, 4.4, 4.5 and Corollary 4.3 can be derived for fuzzy preorder relation on a real vector space, where the fuzzy preorder relations correspond to fuzzy convex cones rather than fuzzy pointed convex cones.

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