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# TOLERANCE PROBLEMS FOR GENERALIZED EIGENVECTORS OF INTERVAL FUZZY MATRICES

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Fuzzy algebra is a special type of algebraic structure in which classical addition and multiplication are replaced by maximum and minimum (denoted $\oplus$ and $\otimes$, respectively). The eigenproblem is the search for a vector $x$ (an eigenvector) and a constant $\lambda$ (an eigenvalue) such that $A \otimes x = \lambda \otimes x$, where $A$ is a given matrix. This paper investigates a generalization of the eigenproblem in fuzzy algebra. We solve the equation $A \otimes x = \lambda \otimes B \otimes x$ with given matrices $A, B$ and unknown constant $\lambda$ and vector $x$. Generalized eigenvectors have interesting and useful properties in the various computational tasks with inexact (interval) matrix and vector inputs. This paper studies the properties of generalized interval eigenvectors of interval matrices. Three types of generalized interval eigenvectors: strongly tolerable generalized eigenvectors, tolerable generalized eigenvectors and weakly tolerable generalized eigenvectors are proposed and polynomial procedures for testing the obtained equivalent conditions are presented.

*Keywords:* interval generalized eigenvector, fuzzy matrix

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## 1. INTRODUCTION

Fuzzy algebra is a triple $(\mathcal{I}, \oplus, \otimes)$, where $\mathcal{I} = [\mathrm{O}, \mathrm{I}]$ is a linearly ordered set with the least element O and the greatest element I and $\oplus, \otimes$ are binary operations defined as follows: $a \oplus b = \max(a, b)$ and $a \otimes b = \min(a, b)$.
Given natural numbers $m$ and $n$, we shall write $M = \{1, 2, \dots, m\}$ and $N = \{1, 2, \dots, n\}$. The set of $m \times n$ matrices ($m \times 1$ vectors) over $\mathcal{I}$ will be denoted by $\mathcal{I}(m, n)$, $(\mathcal{I}(m))$. Similarly to classical linear algebra the operations $\oplus, \otimes$ can be extended to the matrix-vector algebra over $\mathcal{I}$.

For $A, B \in \mathcal{I}(m, n)$, define $A \leq B$ if $a_{ij} \leq b_{ij}$ holds true for any $i \in M$, $j \in N$. Both operations in fuzzy algebra are idempotent, so no new numbers are created in the process of matrix-vector multiplication.
Let $L$ be a subset of $\mathcal{I}$. Then $L^{\oplus} \in L$ is called greatest element of $L$ if for every $x \in L$ the inequality $x \leq L^{\oplus}$ holds.

A vector $x \in \mathcal{I}(n)$ which satisfies $A \otimes x = \lambda \otimes B \otimes x$ for some $\lambda \in \mathcal{I}$ is called a generalized eigenvector of the matrices $A$, $B \in \mathcal{I}(m, n)$.

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The matrices in a fuzzy algebra and the investigation of the properties of generalized eigenvectors are useful for applications in knowledge engineering, scheduling, graph theory and modeling of fuzzy discrete dynamic systems [1, 8, 9, 10, 11]. They are helpful for describing diagnoses of technical devices [21], medical diagnoses [20] or fuzzy logic programming. In practice, the values of the vector and the matrix inputs can be considered as values in some intervals rather than exact numbers. The aim of this paper is to present equivalent conditions for an interval vector to be a generalized eigenvector. We also provide effective procedures for determining the strong tolerance, tolerance and weak tolerance of generalized interval eigenvectors.

The following example is a generalization of the example presented in [7] and is one motivation for studying the generalized eigenproblem.

**Example 1.1.** Consider a computer network consisting of three servers $S_1$, $S_2$ and $S_3$, three backup servers $S'_1$, $S'_2$ and $S'_3$, data storage units $D_1$, $D_2$, $D'_1$, $D'_2$, and a logical unit $L$. The lines $A_{ij}$, $i \in \{1,2,3\}, j \in \{1,2\}$ connect every $S_i$ with every $D_j$, and the lines $B_{ij}$, $i \in \{1,2,3\}, j \in \{1,2\}$ connect every $S'_i$ with every $D'_j$. Moreover, lines $L_j$ and $L'_j$ connect $L$ with every $D_j$ and $D'_j$, respectively. The security level of each line in the network is measured by values in the real interval $[O, I]$. The security of every $A_{ij}$ (data security) is denoted by $a_{ij}$, the security of $B_{ij}$ is denoted by $b_{ij}$, while the securities of $L_1$, $L_2$ and $L'_1$, $L'_2$ (logic securities) are denoted by $x_1$, $x_2$ and $y_1$, $y_2$, respectively. The maximal security levels $u_1$, $u_2$, $u_3$ from servers $S_1$, $S_2$, $S_3$ to the logical unit $L$ via $D_1$ and/or $D_2$ are given by

$$\begin{aligned} u_1 &= \max\big(\min(a_{11}, x_1), \min(a_{12}, x_2)\big) \\ u_2 &= \max\big(\min(a_{21}, x_1), \min(a_{22}, x_2)\big) \\ u_3 &= \max\big(\min(a_{31}, x_1), \min(a_{32}, x_2)\big) \end{aligned}$$

and the maximal security levels $v_1$, $v_2$, $v_3$ from servers $S'_1$, $S'_2$ and $S'_3$ to the logical unit $L$ via $D'_1$ and/or $D'_2$ are given by

$$\begin{aligned} v_1 &= \max(\min(b_{11}, y_1), \min(b_{12}, y_2)\big) \\ v_2 &= \max(\min(b_{21}, y_1), \min(b_{22}, y_2)\big) \\ v_3 &= \max(\min(b_{31}, y_1), \min(b_{32}, y_2)\big)\,. \end{aligned}$$

The aim of the model is to achieve the synchronization of the security levels of the lines connecting the data storage units $D_1, D_2$ with the logical unit $L$ preserved for the lines connecting the servers $S_1, S_2, S_3$ with $L$ and the security levels of the lines connecting the data storage units $D'_1, D'_2$ with the logical unit $L$ preserved for the lines connecting the servers $S'_1, S'_2$, $S'_3$ with $L$. Then the maximal security levels $u_1, u_2, u_3$ must be equal to another maximal security levels $v_1, v_2, v_3$. For practical reasons, the reduction to a fixed level $\lambda$ is usually used; that is, $y_1 = \min(\lambda, x_1)$, $y_2 = \min(\lambda, x_2)$. Then, in max-min notation we have

$$\begin{aligned} (a_{11} \otimes x_1) \oplus (a_{12} \otimes x_2) &= (b_{11} \otimes \lambda \otimes x_1) \oplus (b_{12} \otimes \lambda \otimes x_2) \\ (a_{21} \otimes x_1) \oplus (a_{22} \otimes x_2) &= (b_{21} \otimes \lambda \otimes x_1) \oplus (b_{22} \otimes \lambda \otimes x_2) \\ (a_{31} \otimes x_1) \oplus (a_{32} \otimes x_2) &= (b_{31} \otimes \lambda \otimes x_1) \oplus (b_{32} \otimes \lambda \otimes x_2)\,. \end{aligned}$$

Applying the matrix-vector multiplication, we get

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{23} & a_{32} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and hence

$$A \otimes x = \lambda \otimes B \otimes x \, . \tag{1}$$

The vector $x$ and number $\lambda$ satisfying (1) are called the generalized eigenvector and a generalized eigenvalue of $(A, B)$, respectively.

The paper is organised as follows. Section 2 is devoted to the definitions and basic properties of the generalized eigenproblem and to conditions for the existence of a generalized eigenvector. Section 3 gives the definitions of three types of interval generalized eigenvector. In Sections 4, 5 and 6, the equivalent conditions for a generalized eigenvector to be strongly tolerable, tolerable or weakly tolerable are presented. Based on the the conclusions we will obtain the polynomial complexity of procedures which check the equivalent conditions claimed in Theorem 3.2, Theorem 4.4 and Theorem 5.2.

Note that this paper is related to [19] which deals with other types of interval generalized eigenvector and gives efficient equivalent conditions for checking them. [17] presents a polynomial procedure for computing the maximal element of the eigenspace.

A matrix $A$ is strongly robust if its greatest eigenvector can be achieved starting in any vector. Results on the strong robustness have been introduced in [12, 13, 14, 16, 18]. Polynomial procedures for checking the reachability of an eigenspace of $A$ starting only from the eigenspace of $A$ are described in [16]. The properties of $\mathbf{X}$-simple image eigenspace of $A$ are described in [18].

## 2. GENERALIZED EIGENVECTORS

For given $A$, $B \in \mathcal{I}(m, n)$, the generalized eigenproblem for the pair $(A, B)$ is defined as the task of finding $x \in \mathcal{I}(n)$ (generalized eigenvector) and $\lambda \in \mathcal{I}$ (generalized eigenvalue) such that (1) is satisfied. The generalized eigenspace of $(A, B)$ with the given generalized eigenvalue $\lambda$ is denoted by

$$V(A, B, \lambda) = \{x \in \mathcal{I}(n);\ A \otimes x = \lambda \otimes B \otimes x\}. \tag{2}$$

**Lemma 2.1.** (Plavka and Gazda [19]) Suppose $A$, $B \in \mathcal{I}(m, n)$ are given. Then the following assertions hold

(i) Any $\lambda \in \mathcal{I}$ is an eigenvalue of $(A, B)$.

(ii) $V(A, B, \lambda) \neq \emptyset$ for any $A, B \in \mathcal{I}(m, n)$ and for any $\lambda \in \mathcal{I}$.

Notice that the second assertion trivially holds since zero vector is still a solution of $A \otimes x = \lambda \otimes B \otimes x$.

Denote the greatest element of $V(A, B, \lambda)$ by $x^{\oplus}(A, B, \lambda)$:

$$x^{\oplus}(A, B, \lambda) = \bigoplus_{x \in V(A,B,\lambda)} x. \tag{3}$$

It has been shown in [2] that the greatest solution of the max-min linear system $A \otimes x = \lambda \otimes B \otimes x$ (which is equal to the greatest generalized eigenvector $x^{\oplus}(A, B, \lambda)$) exists for any pair of matrices $(A, B)$ and for any $\lambda \in \mathcal{I}$. The complexity of computing it is $O(mn \cdot \min(m, n))$.

### 2.1. Versions of generalized eigenvectors

Analogously to [5, 6, 7, 12, 13, 15] consider interval matrices $\boldsymbol{A}$ with bounds $\underline{A}, \overline{A} \in \mathcal{I}(m, n)$, $\boldsymbol{B}$ with bounds $\underline{B}, \overline{B} \in \mathcal{I}(m, n)$ and an interval vector with bounds $\underline{x}, \overline{x} \in \mathcal{I}(n)$ which are defined as follows

$$\boldsymbol{A} = [\underline{A}, \overline{A}] = \left\{ A \in \mathcal{I}(m, n);\ \underline{A} \le A \le \overline{A} \right\},$$

$$\boldsymbol{B} = [\underline{B}, \overline{B}] = \left\{ B \in \mathcal{I}(m, n);\ \underline{B} \le B \le \overline{B} \right\},$$

$$\boldsymbol{X} = [\underline{x}, \overline{x}] = \{ x \in \mathcal{I}(n);\ \underline{x} \le x \le \overline{x} \}.$$

We shall consider the following three versions of generalized eigenvectors.

**Definition 2.2.** Suppose given $\boldsymbol{A}$, $\boldsymbol{B} \subseteq \mathcal{I}(m, n)$ and $\boldsymbol{X} \subseteq \mathcal{I}(n)$. Then $\boldsymbol{X}$ is called

- *a strongly tolerable generalized eigenvector* of $(\boldsymbol{A}, \boldsymbol{B})$ if there exist $\lambda \in \mathcal{I}$, $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ such that $x \in V(A, B, \lambda)$ for each $x \in \boldsymbol{X}$;
- *a tolerable generalized eigenvector* of $(\boldsymbol{A}, \boldsymbol{B})$ if there exist $\lambda \in \mathcal{I}$ and $A \in \boldsymbol{A}$ such that for each $x \in \boldsymbol{X}$ there exists $B \in \boldsymbol{B}$ such that $x \in V(A, B, \lambda)$;
- *a weakly tolerable generalized eigenvector* of $(\boldsymbol{A}, \boldsymbol{B})$ if there exists $\lambda \in \mathcal{I}$ such that for each $x \in \boldsymbol{X}$ there exist $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ such that $x \in V(A, B, \lambda)$.

The investigated types can be interpreted as follows. In general, the interval vector $\boldsymbol{X}$ is a *tolerable* eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ if there exists eigenvalue $\lambda \in \mathcal{I}$ and $A \in \boldsymbol{A}$ such that every vector $x \in \boldsymbol{X}$ synchronizes the security levels of the model (see Example 1.1) with some matrix $B \in \boldsymbol{B}$ (in other words: $B$ tolerates $x$ with eigenvalue $\lambda$ and some matrix $A$).

If there are *common tolerating matrices* $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$ for all vectors $x \in \boldsymbol{X}$, then the interval vector $\boldsymbol{X}$ is called *strongly tolerable.* Otherwise, *the tolerating matrices* $A$, $B$ depend on $x$, then the interval eigenvector $\boldsymbol{X}$ is called *weakly tolerable.*

**Remark 2.3.** There are a number of different types of interval eigenvector in the literature, with different orderings of quantifiers. [19] is the first one to study generalized eigenvectors. This paper studies three other types of generalized eigenvector; the remaining types are not studied here, and they are left as a challenge for further research.

Suppose given $i \in M, j \in N$. Define the matrix $A^{(ij)} \in \mathcal{I}(m,n)$ and vector $x^{(i)} \in \mathcal{I}(n)$ (called generators) by setting

$$a_{kl}^{(ij)} = \begin{cases} \overline{a}_{ij}, & \text{for } \ k=i,\, l=j \\ \underline{a}_{kl}, & \text{otherwise} \end{cases} \quad , \qquad x_k^{(i)} = \begin{cases} \overline{x}_i, & \text{for } \ k=i \\ \underline{x}_k, & \text{otherwise} \end{cases}$$

for all $k \in M, l \in N$.

**Lemma 2.4.** (Plavka and Gazda [19]) Suppose $x \in \mathcal{I}(n)$ and $A \in \mathcal{I}(n,n)$ are given. Then

(i) $x \in \boldsymbol{X}$ if and only if $x = \bigoplus\limits_{i \in N} \beta_i \otimes x^{(i)}$ for some values $\beta_i \in \mathcal{I}$ with $\underline{x}_i \le \beta_i \le \overline{x}_i$ ,

(ii) $A \in \boldsymbol{A}$ if and only if $A = \bigoplus\limits_{i \in M,\, j \in N} \alpha_{ij} \otimes A^{(ij)}$ for some values $\alpha_{ij} \in \mathcal{I}$ with $\underline{a}_{ij} \le \alpha_{ij} \le \overline{a}_{ij}$ .

## 3. STRONGLY TOLERABLE GENERALIZED EIGENVECTORS

**Theorem 3.1.** Suppose $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{X}$ are given. Then $\boldsymbol{X}$ is a strongly tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ if and only if there are $\lambda \in \mathcal{I}$, $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ such that

$$A \otimes x^{(k)} = \lambda \otimes B \otimes x^{(k)} \quad \text{for every } k \in N. \tag{4}$$

Proof. Suppose that $\lambda \in \mathcal{I}$, $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ and (4) is fulfilled. Let $x \in \boldsymbol{X}$. Then by Lemma 2.4 (i), for each $k \in N$ there exists $\gamma_k \in \mathcal{I}$ such that $\underline{x}_k \le \gamma_k \le \overline{x}_k$, $x = \bigoplus_{k \in N} \gamma_k \otimes x^{(k)}$ and

$$\begin{aligned} A \otimes x = A \otimes \left( \bigoplus_{k \in N} \gamma_k \otimes x^{(k)} \right) &= \bigoplus_{k \in N} \left( A \otimes \gamma_k \otimes x^{(k)} \right) \\ = \bigoplus_{k \in N} \gamma_k \otimes \left( A \otimes x^{(k)} \right) &= \bigoplus_{k \in N} \gamma_k \otimes (\lambda \otimes B \otimes x^{(k)}) = \lambda \otimes B \otimes x. \end{aligned}$$

Hence, $\boldsymbol{X}$ is a strongly tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$. The converse implication trivially holds. □

To decide on the existence of $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$ satisfying (4), for each $i \in M, j \in N$ define the vectors $\tilde{C}^{(ij)} \in \mathcal{I}(mn)$ and $\tilde{D}^{(ij)} \in \mathcal{I}(mn)$ as follows:

$$\tilde{C}^{(ij)} = \begin{pmatrix} A^{(ij)} \otimes x^{(1)} \\ A^{(ij)} \otimes x^{(2)} \\ \vdots \\ A^{(ij)} \otimes x^{(n)} \end{pmatrix}, \quad \tilde{D}^{(ij)} = \begin{pmatrix} B^{(ij)} \otimes x^{(1)} \\ B^{(ij)} \otimes x^{(2)} \\ \vdots \\ B^{(ij)} \otimes x^{(n)} \end{pmatrix}. \tag{5}$$

Write

$$\tilde{C} = \big(\tilde{C}^{(11)}, \dots, \tilde{C}^{(1n)}, \tilde{C}^{(21)}, \dots, \tilde{C}^{(2n)}, \dots, \tilde{C}^{(mn)}\big),$$
$$\tilde{D} = (\tilde{D}^{(11)}, \dots, \tilde{D}^{(1n)}, \tilde{D}^{(21)}, \dots, \tilde{D}^{(2n)}, \dots, \tilde{D}^{(mn)})$$

and consider the fuzzy linear system

$$\tilde{C} \otimes y = \lambda \otimes \tilde{D} \otimes z \tag{6}$$

where $\lambda \in \mathcal{I}$ is fixed, the columns of $\tilde{C} \in \mathcal{I}(mn, mn)$ are $\tilde{C}^{(ij)}$, the columns of $\tilde{D} \in \mathcal{I}(mn, mn)$ are $\tilde{D}^{(ij)}$ and the variable vectors $y$ $(z) \in \mathcal{I}(mn, 1)$ are built from the variables $y_{(ij)}$ $(z_{(ij)})$ $i \in M, j \in N$.

**Theorem 3.2.** Suppose given $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{X}$. An interval vector $\boldsymbol{X}$ is a strongly tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ if and only there exists $\lambda \in \mathcal{I}$ such that the system $\tilde{C} \otimes y = \lambda \otimes \tilde{D} \otimes z$ has a solution $(y, z)$ satisfying inequalities

$$\underline{a}_{ij} \le y_{(ij)} \le \overline{a}_{ij}, \quad \underline{b}_{ij} \le z_{(ij)} \le \overline{b}_{ij} \tag{7}$$

for any $i \in M, j \in N$.

Proof. Suppose that there are $y$ and $z$ satisfying (6) and (7). Then for the matrices $A \in \mathcal{I}(m, n)$, $B \in \mathcal{I}(m, n)$ defined by

$$A = \bigoplus_{i \in M, j \in N} y_{(ij)} \otimes A^{(ij)}, \quad B = \bigoplus_{i \in M, j \in N} z_{(ij)} \otimes B^{(ij)}, \tag{8}$$

we obtain $A \in \boldsymbol{A}$ and $B \in \boldsymbol{B}$, according to Lemma 2.4 (ii).

Therefore, from (6), for every fixed $k \in M$ we obtain

$$\bigoplus_{i \in M, j \in N} \Big(A^{(ij)} \otimes x^{(k)}\Big) \otimes y_{(ij)} = \lambda \otimes \bigoplus_{i \in M, j \in N} \Big(B^{(ij)} \otimes x^{(k)}\Big) \otimes z_{(ij)},$$
$$\left(\bigoplus_{i \in M, j \in N} y_{(ij)} \otimes A^{(ij)}\right) \otimes x^{(k)} = \lambda \otimes \left(\bigoplus_{i \in M, j \in N} z_{(ij)} \otimes B^{(ij)}\right) \otimes x^{(k)},$$
$$A \otimes x^{(k)} = \lambda \otimes B \otimes x^{(k)}.$$

Hence, according to Theorem 3.1, $\boldsymbol{X}$ is a strongly tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$.

To prove the converse implication, a strongly tolerable generalized eigenvector $\boldsymbol{X}$ of $(\boldsymbol{A}, \boldsymbol{B})$ implies the existence of $\lambda \in \mathcal{I}$, $A \in [\underline{A}, \overline{A}])$ and $B \in [\underline{B}, \overline{B}])$ such that $A \otimes x^{(k)} = \lambda \otimes B \otimes x^{(k)}$ for any $k \in N$. By Lemma 2.4(ii), there are $\alpha_{ij}, \beta_{ij} \in \mathcal{I}$, $i \in M, j \in N$ such that $A = \bigoplus_{i \in M, j \in N} \alpha_{ij} \otimes A^{(ij)}$, $B = \bigoplus_{i \in M, j \in N} \beta_{ij} \otimes B^{(ij)}$ and $\underline{a}_{ij} \le \alpha_{ij} \le \overline{a}_{ij}$, $\underline{b}_{ij} \le \beta_{ij} \le \overline{b}_{ij}$. Then $y, z \in \mathcal{I}(mn)$ satisfy (6) and (7), where $y_{(ij)} = \alpha_{ij}$, $z_{(ij)} = \beta_{ij}$ for any $i \in M, j \in N$. □

In view of [4], the system $\tilde{C} \otimes y = \lambda \otimes \tilde{D} \otimes z$ can be transformed to a system with the same variables on both sides as follows:

$$(\tilde{C}, \tilde{O}) \otimes \begin{pmatrix} y \\ z \end{pmatrix} = \lambda \otimes (\tilde{O}, \tilde{D}) \otimes \begin{pmatrix} y \\ z \end{pmatrix}, \tag{9}$$

where $(\tilde{C}, \tilde{O}) \in \mathcal{I}(mn, 2mn)$ and $(\tilde{O}, \tilde{D}) \in \mathcal{I}(mn, 2mn)$ and $\tilde{O}$ is $mn \times mn$ matrix with every entry equal to $O$.

The efficient algorithm for solvability of the system $R \otimes x = \lambda \otimes S \otimes y$ for $R,\ S \in \mathcal{I}(k, l)$ with the computational complexity $O(kl \cdot \min(k, l))$ has been presented in [4]. Taking into consideration that the dimension of the coefficient matrices $(\tilde{C}, \tilde{O})\,(\tilde{O}, \tilde{D})$ in (9) is $mn \times 2mn$, we get the following result.

**Theorem 3.3.** The complexity of a procedure for checking whether $\boldsymbol{X}$ is a strongly tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ for $|\mathcal{I}| = \ell$ is equal to $O(\ell \cdot m^3 n^3)$.

Proof. Using Theorem 3.2, the computation of $\tilde{C}$, $\tilde{D}$ in (6) requires computing $A^{(ij)} \otimes x^{(k)}$, $B^{(ij)} \otimes x^{(k)}$ for all $i \in M, j, k \in N$ in $O(mn)$ time each. Thus, the computation of $\tilde{C}$, $\tilde{D}$ requires $O(m^2 n^3)$ time similarly as computing the remaining data in (6), (7). Hence, if $|\mathcal{I}| = \ell$, then the solvability of (9) can be computed in $\ell \cdot O(m^3 n^2) + \ell \cdot O(m^3 n^3) = O(\ell \cdot m^3 n^3)$ time since each $\lambda \in \mathcal{I}$ is checked independently. □

## 4. TOLERABLE GENERALIZED EIGENVECTORS

For given matrices $A = (a_{ij}) \in \boldsymbol{A}$, $B = (b_{ij}) \in \boldsymbol{B}$, a given vector $x = (x_1, \dots, x_n)^T \in \mathcal{I}(n)$, $i \in M$ and a given value $\lambda \in \mathcal{I}$ we shall use the notation

$$a_i(x) = \max_{j \in N}(a_{ij} \otimes x_j) = \big[A \otimes x\big]_i,$$

$$b_i(x) = \max_{j \in N}(b_{ij} \otimes x_j) = \big[B \otimes x\big]_i.$$

In particular, for $A = \overline{A}$ and $B = \overline{B}$

$$\overline{a}_i(x) = \max_{j \in N}(\overline{a}_{ij} \otimes x_j) = \big[\,\overline{A} \otimes x\big]_i,$$

$$\overline{b}_i(x) = \max_{j \in N}(\overline{b}_{ij} \otimes x_j) = \big[\,\overline{B} \otimes x\big]_i.$$

Furthermore, define $A^x = \big(a^x_{ij}\big) \in \mathcal{I}(m, n)$ and $B^x = \big(b^x_{ij}\big) \in \mathcal{I}(m, n)$ by putting

$$a^x_{ij} = \begin{cases} \overline{a}_{ij} & \text{if } \ \overline{a}_{ij} \otimes x_j \le \lambda \otimes \max\limits_{j \in N}(\overline{b}_{ij} \otimes x_j), \\ \lambda \otimes \max\limits_{j \in N}(\overline{b}_{ij} \otimes x_j) & \text{otherwise,} \end{cases} \tag{10}$$

$$b^x_{ij} = \begin{cases} \overline{b}_{ij} & \text{if } \ \lambda \otimes \overline{b}_{ij} \otimes x_j \le \max\limits_{j \in N}(\overline{a}_{ij} \otimes x_j), \\ \max\limits_{j \in N}(\overline{a}_{ij} \otimes x_j) & \text{otherwise.} \end{cases} \tag{11}$$

**Lemma 4.1.** Assume $x \in \mathcal{I}(n)$ and $\lambda \in \mathcal{I}$. Then $A^x \otimes x = \lambda \otimes B^x \otimes x$.

Proof. Let $x \in \mathcal{I}(n)$ and $\lambda \in \mathcal{I}$ be fixed. We have to show $\left[A^x \otimes x\right]_i = \lambda \otimes \left[B^x \otimes x\right]_i$, for all $i \in M$. Take fixed $i \in M$ and consider two cases.

First, suppose $\overline{a}_i(x) \le \lambda \otimes \overline{b}_i(x)$. Then

$$a_i^x(x) = \bigoplus_{j\in N} a_{ij}^x \otimes x_j = \Big( \bigoplus_{\overline{a}_{ij}\otimes x_j \le \lambda\otimes\overline{b}_i(x)} a_{ij}^x \otimes x_j \Big) \oplus \Big( \bigoplus_{\overline{a}_{ij}\otimes x_j > \lambda\otimes\overline{b}_i(x)} a_{ij}^x \otimes x_j \Big) \tag{12}$$

$$= \Big( \bigoplus_{\overline{a}_{ij}\otimes x_j \le \lambda\otimes\overline{b}_i(x)} \overline{a}_{ij} \otimes x_j \Big) \oplus \Big( \bigoplus_{\overline{a}_{ij}\otimes x_j > \lambda\otimes\overline{b}_i(x)} \lambda \otimes \overline{b}_i(x) \otimes x_j \Big) \tag{13}$$

$$= \bigoplus_{\overline{a}_{ij}\otimes x_j \le \lambda\otimes\overline{b}_i(x)} \overline{a}_{ij} \otimes x_j = \bigoplus_{j\in N} \overline{a}_{ij} \otimes x_j = \overline{a}_i(x) = \lambda \otimes \overline{a}_i(x) \tag{14}$$

since $\{j \in N;\ \overline{a}_{ij}\otimes x_j > \lambda\otimes\overline{b}_i(x)\} = \emptyset$ ($\max \emptyset = O$) and the assumption $\overline{a}_i(x) \le \lambda\otimes\overline{b}_i(x)$ implies $\overline{a}_i(x) \le \lambda$.

We also get

$$\lambda \otimes b_i^x(x) = \lambda \otimes \bigoplus_{j\in N} b_{ij}^x \otimes x_j \tag{15}$$

$$= \lambda \otimes \Big( \bigoplus_{\lambda\otimes\overline{b}_{ij}\otimes x_j \le \overline{a}_i(x)} b_{ij}^x \otimes x_j \Big) \oplus \lambda \otimes \Big( \bigoplus_{\lambda\otimes\overline{b}_{ij}\otimes x_j > \overline{a}_i(x)} b_{ij}^x \otimes x_j \Big) \tag{16}$$

$$= \Big( \bigoplus_{\lambda\otimes\overline{b}_{ij}\otimes x_j \le \overline{a}_i(x)} \lambda \otimes \overline{b}_{ij} \otimes x_j \Big) \oplus \Big( \lambda \otimes \bigoplus_{\lambda\otimes\overline{b}_{ij}\otimes x_j > \overline{a}_i(x)} \overline{a}_i(x) \otimes x_j \Big) \tag{17}$$

$$= \lambda \otimes \overline{a}_i(x), \tag{18}$$

because if $\{j \in N;\ \lambda \otimes \overline{b}_{ij} \otimes x_j > \overline{a}_i(x)\} = \emptyset$, then $\{j \in N;\ \lambda \otimes \overline{b}_{ij} \otimes x_j \le \overline{a}_i(x)\} = N$,

$$\bigoplus_{\lambda\otimes\overline{b}_{ij}\otimes x_j \le \overline{a}_i(x)} \lambda \otimes \overline{b}_{ij} \otimes x_j = \lambda \otimes \overline{b}_i(x) \le \overline{a}_i(x)$$

and together with the assumption $\overline{a}_i(x) \le \lambda \otimes \overline{b}_i(x)$ we get $\lambda \otimes \overline{b}_i(x) = \overline{a}_i(x) = \lambda \otimes \overline{a}_i(x)$

else $\{j \in N;\ \lambda \otimes \overline{b}_{ij} \otimes x_j > \overline{a}_i(x)\} \ne \emptyset$, so

$$\bigoplus_{\lambda\otimes\overline{b}_{ij}\otimes x_j \le \overline{a}_i(x)} \lambda \otimes \overline{b}_{ij} \otimes x_j \le \overline{a}_i(x) \quad \text{and} \quad (\lambda \otimes \bigoplus_{\lambda\otimes\overline{b}_{ij}\otimes x_j > \overline{a}_i(x)} \overline{a}_i(x) \otimes x_j) = \lambda \otimes \overline{a}_i(x),$$

(summing over the set $\{j \in N;\ \lambda \otimes \overline{b}_{ij} \otimes x_j > \overline{a}_i(x)\}$ implies $x_j > \overline{a}_i(x)$).
As a consequence we obtain $a_i^x(x) = \lambda \otimes b_i^x(x)$.

Second, suppose $\overline{a}_i(x) \ge \lambda \otimes \overline{b}_i(x)$. Then

$$a_i^x(x) = \bigoplus_{j\in N} a_{ij}^x \otimes x_j = \Big( \bigoplus_{\overline{a}_{ij}\otimes x_j \le \lambda\otimes\overline{b}_i(x)} a_{ij}^x \otimes x_j \Big) \oplus \Big( \bigoplus_{\overline{a}_{ij}\otimes x_j > \lambda\otimes\overline{b}_i(x)} a_{ij}^x \otimes x_j \Big) \tag{19}$$

$$= \Big( \bigoplus_{\overline{a}_{ij}\otimes x_j \leq \lambda\otimes \overline{b}_i(x)} \overline{a}_{ij} \otimes x_j \Big) \oplus \Big( \bigoplus_{\overline{a}_{ij}\otimes x_j > \lambda\otimes \overline{b}_i(x)} \lambda \otimes \overline{b}_i(x) \otimes x_j \Big) = \lambda \otimes \overline{b}_i(x), \quad (20)$$

because if $\{j \in N;\ \overline{a}_{ij} \otimes x_j > \lambda \otimes \overline{b}_i(x)\} = \emptyset$, then $\{j \in N;\ \overline{a}_{ij} \otimes x_j \leq \lambda \otimes \overline{b}_i(x)\} = N$,

$$\bigoplus_{\overline{a}_{ij}\otimes x_j \leq \lambda\otimes \overline{b}_i(x)} \overline{a}_{ij} \otimes x_j = \bigoplus_{j\in N} \overline{a}_{ij} \otimes x_j = \overline{a}_i(x) \leq \lambda \otimes \overline{b}_i(x)$$

and together with the assumption $\overline{a}_i(x) \geq \lambda \otimes \overline{b}_i(x)$ we get $\overline{a}_i(x) = \lambda \otimes \overline{b}_i(x)$

else $\{j \in N;\ \overline{a}_{ij} \otimes x_j > \lambda \otimes \overline{b}_i(x)\} \neq \emptyset$, so

$$\bigoplus_{\overline{a}_{ij}\otimes x_j \leq \lambda\otimes \overline{b}_i(x)} \overline{a}_{ij} \otimes x_j \leq \lambda \otimes \overline{b}_i(x) \quad \text{and} \quad \bigoplus_{\overline{a}_{ij}\otimes x_j > \lambda\otimes \overline{b}_i(x)} \lambda \otimes \overline{b}_i(x) \otimes x_j = \lambda \otimes \overline{b}_i(x)$$

(summing over the set $\{j \in N;\ \overline{a}_{ij} \otimes x_j > \lambda \otimes \overline{b}_i(x)\}$ implies $x_j > \lambda \otimes \overline{b}_i(x)$).
As a consequence we obtain $a^x_i(x) = \lambda \otimes b^x_i(x)$.

On the other hand, we get

$$\lambda \otimes b^x_i(x) = \lambda \otimes \bigoplus_{j\in N} b^x_{ij} \otimes x_j \quad (21)$$

$$= \lambda \otimes \Big( \bigoplus_{\lambda\otimes\overline{b}_{ij}\otimes x_j \leq \overline{a}_i(x)} b^x_{ij} \otimes x_j \Big) \oplus \lambda \otimes \Big( \bigoplus_{\lambda\otimes\overline{b}_{ij}\otimes x_j > \overline{a}_i(x)} b^x_{ij} \otimes x_j \Big) \quad (22)$$

$$= \lambda \otimes \Big( \bigoplus_{\lambda\otimes\overline{b}_{ij}\otimes x_j \leq \overline{a}_i(x)} \overline{b}_{ij} \otimes x_j \Big) \oplus \lambda \otimes \Big( \bigoplus_{\lambda\otimes\overline{b}_{ij}\otimes x_j > \overline{a}_i(x)} \overline{a}_i(x) \otimes x_j \Big) \quad (23)$$

$$= \lambda \otimes \Big( \bigoplus_{\lambda\otimes\overline{b}_{ij}\otimes x_j \leq \overline{a}_i(x)} \overline{b}_{ij} \otimes x_j \Big) = \lambda \otimes \bigoplus_{j\in N} \overline{b}_{ij} \otimes x_j = \lambda \otimes \overline{b}_i(x), \quad (24)$$

since $\{j \in N;\ \lambda \otimes \overline{b}_{ij} \otimes x_j > \overline{a}_i(x)\} = \emptyset$. Thus, we have $a^x_i(x) = \lambda \otimes b^x_i(x)$. □

**Lemma 4.2.** For $\boldsymbol{A} = [\underline{A}, \overline{A}]$, $\boldsymbol{B} = [\underline{B}, \overline{B}]$, $x \in \mathcal{I}(n)$ and $\lambda \in \mathcal{I}$, the following implications hold

(i) $A^x \leq \overline{A}$, $B^x \leq \overline{B}$,

(ii) if $A \in \boldsymbol{A}$, $B \in \boldsymbol{B}$ and $A \otimes x = \lambda \otimes B \otimes x$, then $A \leq A^x$ and $B \leq B^x$,

(iii) $\underline{A} \leq A^x$ if and only if $\underline{A} \otimes x \leq \lambda \otimes \overline{B} \otimes x$,

(iv) $\underline{B} \leq B^x$ if and only if $\lambda \otimes \underline{B} \otimes x \leq \overline{A} \otimes x$.

P r o o f. (i) Trivially follows from the definitions of $A^x$ and $B^x$.

(ii) Let $i, j \in N$ and $A \otimes x = \lambda \otimes B \otimes x$. If $\overline{a}_{ij} \otimes x_j \leq \lambda \otimes \overline{b}_i(x)$, then we have $a^x_{ij} = \overline{a}_{ij} \geq a_{ij}$. If the opposite inequality holds, i. e. $\overline{a}_{ij} \otimes x_j > \lambda \otimes \overline{b}_i(x)$, then

$$a^x_{ij} = \lambda \otimes \overline{b}_i(x) \geq \lambda \otimes b_i(x) = \bigoplus_{k \in N} a_{ik} \otimes x_k \geq a_{ij} \otimes x_j = a_{ij}$$

because $\overline{a}_{ij} \otimes x_j > \lambda \otimes \overline{b}_i(x) \Rightarrow x_j > \lambda \otimes \overline{b}_i(x) = a^x_{ij}$.

If $\lambda \otimes \overline{b}_{ij} \otimes x_j \leq \overline{a}_i(x)$, then $b^x_{ij} = \overline{b}_{ij} \geq b_{ij}$. On the other hand, if $\lambda \otimes \overline{b}_{ij} \otimes x_j > \overline{a}_i(x)$, then

$$b^x_{ij} = \overline{a}_i(x) \geq a_i(x) = \lambda \otimes b_i(x) = \lambda \otimes \bigoplus_{k \in N} b_{ik} \otimes x_k \geq \lambda \otimes b_{ij} \otimes x_j = b_{ij}$$

because $\lambda \otimes \overline{b}_{ij} \otimes x_j > \overline{a}_i(x) \Rightarrow \lambda \otimes x_j > \overline{a}_i(x) = b^x_{ij}$.

(iii) If $\underline{A} \leq A^x$, then $\underline{A} \otimes x \leq A^x \otimes x = \lambda \otimes B^x \otimes x \leq \lambda \otimes \overline{B} \otimes x$, according to Lemma 4.1 and (i).

To prove the reverse implication, suppose that $\underline{A} \otimes x \leq \lambda \otimes \overline{B} \otimes x$. Let $i \in M, j \in N$ be fixed. If $\overline{a}_{ij} \otimes x_j \leq \lambda \otimes \overline{b}_i(x)$, then $\underline{a}_{ij} \leq \overline{a}_{ij} = a^x_{ij}$. If $\overline{a}_{ij} \otimes x_j > \lambda \otimes \overline{b}_i(x)$, then $a^x_{ij} = \lambda \otimes \overline{b}_i(x)$ and $x_j \geq \overline{a}_{ij} \otimes x_j > \lambda \otimes \overline{b}_i(x)$. By assumption, we also have $\underline{a}_{ij} \otimes x_j \leq \lambda \otimes \overline{b}_i(x)$, which is only possible if $\underline{a}_{ij} \leq \lambda \otimes \overline{b}_i(x) = a^x_{ij}$. Hence, $\underline{A} \leq A^x$.

(iv) If $\underline{B} \leq B^x$ then, according to Lemma 4.1 and (i), we obtain $\lambda \otimes \underline{B} \otimes x \leq \lambda \otimes B^x \otimes x = A^x \otimes x \leq \overline{A} \otimes x$. To prove the converse implication, suppose that $\lambda \otimes \underline{B} \otimes x \leq \overline{A} \otimes x$ and that $i \in M, j \in N$ are fixed. If $\lambda \otimes \overline{b}_{ij} \otimes x_j \leq \overline{a}_i(x)$, then $\underline{b}_{ij} \leq \overline{b}_{ij} = b^x_{ij}$. If $\lambda \otimes \overline{b}_{ij} \otimes x_j > \overline{a}_i(x)$, then $b^x_{ij} = \overline{a}_i(x)$ and $\lambda \otimes x_j > \overline{a}_i(x)$. By assumption, we also have $\lambda \otimes \underline{b}_{ij} \otimes x_j \leq \overline{a}_i(x)$, which is only possible if $\underline{b}_{ij} \leq \overline{a}_i(x) = b^x_{ij}$. Hence, $\underline{B} \leq B^x$. □

**Theorem 4.3.** Suppose given $\boldsymbol{A} = [\underline{A}, \overline{A}]$, $\boldsymbol{B} = [\underline{B}, \overline{B}]$ and $\boldsymbol{X} = [\underline{x}, \overline{x}]$. An interval vector $\boldsymbol{X}$ is a tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ if and only if there exist $\lambda \in \mathcal{I}$ and $A \in \boldsymbol{A}$ such that

$$\lambda \otimes \underline{B} \otimes x \leq A \otimes x \leq \lambda \otimes \overline{B} \otimes x \tag{25}$$

is fulfilled for any $x \in \boldsymbol{X}$.

Proof. Suppose that there exist $\lambda \in \mathcal{I}$ and $A \in \boldsymbol{A}$ such that (25) holds for any $x \in \boldsymbol{X}$. Put $\hat{\boldsymbol{A}} = [\hat{\underline{A}}, \hat{\overline{A}}] = [A, A]$. Observe that the following equivalence easily holds:

$$(\exists \lambda \in B)(\exists A \in \boldsymbol{A})(\forall x \in \boldsymbol{X})(\exists B \in \boldsymbol{B})\ \lambda \otimes \underline{B} \otimes x \leq A \otimes x \leq \lambda \otimes \overline{B} \otimes x \Leftrightarrow$$

$$\Leftrightarrow (\exists \lambda \in B)(\exists \hat{A} \in \hat{\boldsymbol{A}})(\forall x \in \boldsymbol{X})(\exists B \in \boldsymbol{B})\ \lambda \otimes \underline{B} \otimes x \leq \hat{A} \otimes x \leq \lambda \otimes \overline{B} \otimes x.$$

From $\lambda \otimes \underline{B} \otimes x \leq \hat{A} \otimes x \leq \lambda \otimes \overline{B} \otimes x$ and by Lemma 4.2(iii),(iv) we get:

$$\lambda \otimes \underline{B} \otimes x \leq \hat{A} \otimes x = \overline{\hat{A}} \otimes x \Rightarrow \underline{B} \leq B^x,$$

$$\underline{\hat{A}} \otimes x \leq \hat{A} \otimes x \leq \lambda \otimes \overline{B} \otimes x \Rightarrow \underline{\hat{A}} \leq \hat{A}^x (= A).$$

By Lemma 4.1 for each $x \in \boldsymbol{X}$ the equality $A \otimes x = \lambda \otimes B^x \otimes x$ is satisfied, thus we have proved that

$$(\exists \lambda \in B)(\exists A \in \boldsymbol{A})(\forall x \in \boldsymbol{X})(\exists B = B^x \in \boldsymbol{B})\ A \otimes x = \lambda \otimes B^x \otimes x.$$

For the converse implication, suppose that $\boldsymbol{X}$ is a tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$, i.e.,

$$(\exists \lambda \in B)(\exists A \in \boldsymbol{A})(\forall x \in \boldsymbol{X})(\exists B \in \boldsymbol{B})\ A \otimes x = \lambda \otimes B \otimes x$$

which straightforwardly implies

$$(\exists \lambda \in B)(\exists A \in \boldsymbol{A})(\forall x \in \boldsymbol{X})(\exists B \in \boldsymbol{B})\ \lambda \otimes \underline{B} \otimes x \leq \lambda \otimes B \otimes x = A \otimes x \leq \lambda \otimes \overline{B} \otimes x.$$

□

**Theorem 4.4.** Suppose given $\boldsymbol{A} = [\underline{A}, \overline{A}]$, $\boldsymbol{B} = [\underline{B}, \overline{B}]$ and $\boldsymbol{X} = [\underline{x}, \overline{x}]$. Then $\boldsymbol{X}$ is a tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ if and only if there are $\lambda \in \mathcal{I}$ and $A \in \boldsymbol{A}$ such that for each $i \in N$ the system of inequalities

$$\lambda \otimes \underline{B} \otimes x^{(i)} \leq A \otimes x^{(i)} \leq \lambda \otimes \overline{B} \otimes x^{(i)} \tag{26}$$

is satisfied.

Proof. Suppose that (26) holds and $x$ is an arbitrary vector of $\boldsymbol{X}$. By Lemma 2.4, $x$ can be written as a max-min linear combination of $x^{(i)}$, i.e. $x = \bigoplus_{i \in N} \beta_i \otimes x^{(i)}$, where each $\beta_i \in \mathcal{I}$ and $\underline{x}_i \leq \beta_i \leq \overline{x}_i$. Then we have

$$\lambda \otimes \underline{B} \otimes x = \lambda \otimes \underline{B} \otimes \bigoplus_{i \in N} \beta_i \otimes x^{(i)} = \bigoplus_{i \in N} \beta_i \otimes \lambda \otimes \underline{B} \otimes x^{(i)} \tag{27}$$

$$\leq \bigoplus_{i \in N} \beta_i \otimes A \otimes x^{(i)} = A \otimes \bigoplus_{i \in N} \beta_i \otimes x^{(i)} = A \otimes x. \tag{28}$$

Similarly, we can prove $A \otimes x \leq \lambda \otimes \overline{B} \otimes x$.
The converse assertion follows trivially. □

Note that Theorem 4.4 transforms the task of recognizing whether a given interval vector is tolerable generalized eigenvector to the solvability problem of (26). Further reduction can be done using the following basic result from [21].

Suppose given $A \in \mathcal{I}(m,n)$ and $b \in \mathcal{I}(m)$. Consider the system of inequalities of the form

$$A \otimes x \leq b. \tag{29}$$

Define a principal solution $\hat{x}(A, b)$ of (29) as follows:

$$\hat{x}_j(A, b) = \min_{i \in M} \{b_i : a_{ij} > b_i\}, \tag{30}$$

where $\min \emptyset = I$.

**Lemma 4.5.** (Zimmermann [21]) Suppose given $A \in \mathcal{I}(m,n)$ and $b \in \mathcal{I}(m)$. Then the following assertions hold:

(i) if $A \otimes x \leq b$ for some $x \in \mathcal{I}(n)$, then $x \leq \hat{x}(A,b)$;

(ii) $A \otimes \hat{x}(A,b) \leq b$.

**Theorem 4.6.** Suppose given $C, D \in \mathcal{I}(m,n)$ and $e, f \in \mathcal{I}(m)$. Then the system of inequalities

$$C \otimes x \leq e, \tag{31}$$

$$D \otimes x \geq f \tag{32}$$

is solvable if and only if

$$D \otimes \hat{x}(C,e) \geq f. \tag{33}$$

Proof. ($\Leftarrow$) According to Lemma 4.5 (ii), $\hat{x}(C,e)$ satisfies (31). If (33) is fulfilled, then $\hat{x}(C,e)$ satisfies (32). Hence, the system (31), (32) is solvable.

($\Rightarrow$) Suppose that the system (31), (32) has a solution $y$. Since $y \leq \hat{x}(C,e)$, according to Lemma 4.5 (i) and according to the monotonicity of $\otimes$ we obtain

$$D \otimes \hat{x}(C,e) \geq D \otimes y \geq f,$$

therefore (33) is fulfilled. □

Observe that checking the solvability of the system (31), (32) needs $O(mn)$ arithmetic operations.

We can check the conditions of Theorem 4.4 in practice using the last theorem. The inequalities can be joined into two systems according to Theorem 4.6 as follows:

$$\begin{array}{c} (\forall k \in N) \bigoplus_{i \in M, j \in N} \alpha_{ij} \otimes A^{(ij)} \otimes x^{(k)} \leq \lambda \otimes \overline{B} \otimes x^{(k)}, \\ \alpha_{ij} \leq \overline{a}_{ij} \end{array},$$

$$\begin{array}{c} (\forall k \in N) \bigoplus_{i \in M, j \in N} \alpha_{ij} \otimes A^{(ij)} \otimes x^{(k)} \geq \lambda \otimes \underline{B} \otimes x^{(k)}, \\ \alpha_{ij} \geq \underline{a}_{ij}. \end{array}$$

and $\alpha_{ij} \in \mathcal{I}$.

Suppose given $\boldsymbol{A}, \boldsymbol{B} \subseteq \mathcal{I}(m,n)$ and $\boldsymbol{X} \subseteq \mathcal{I}(n)$. Define the block matrix $C \in \mathcal{I}(2mn, mn)$, vectors $e, f \in \mathcal{I}(2mn)$ and $\alpha \in \mathcal{I}(mn)$ as follows:

$$C = \begin{pmatrix} A^{(11)} \otimes x^{(1)} & \dots & A^{(1n)} \otimes x^{(1)} & A^{(21)} \otimes x^{(1)} & \dots & A^{(mn)} \otimes x^{(1)} \\ A^{(11)} \otimes x^{(2)} & \dots & A^{(1n)} \otimes x^{(2)} & A^{(21)} \otimes x^{(2)} & \dots & A^{(mn)} \otimes x^{(2)} \\ \vdots & & & & & \\ A^{(11)} \otimes x^{(n)} & \dots & A^{(1n)} \otimes x^{(n)} & A^{(21)} \otimes x^{(n)} & \dots & A^{(mn)} \otimes x^{(n)} \\ I & O & O & O & \dots & O \\ O & I & O & O & \dots & O \\ \vdots & & & & & \\ O & O & O & O & \dots & I \end{pmatrix},$$

$$e = \begin{pmatrix} \lambda \otimes \overline{B} \otimes x^{(1)} \\ \lambda \otimes \overline{B} \otimes x^{(2)} \\ \vdots \\ \lambda \otimes \overline{B} \otimes x^{(n)} \\ \overline{a}_{11} \\ \vdots \\ \overline{a}_{1n} \\ \vdots \\ \overline{a}_{21} \\ \vdots \\ \overline{a}_{mn} \end{pmatrix}, \quad f = \begin{pmatrix} \lambda \otimes \underline{B} \otimes x^{(1)} \\ \lambda \otimes \underline{B} \otimes x^{(2)} \\ \vdots \\ \lambda \otimes \underline{B} \otimes x^{(n)} \\ \underline{a}_{11} \\ \vdots \\ \underline{a}_{1n} \\ \vdots \\ \underline{a}_{21} \\ \vdots \\ \underline{a}_{mn} \end{pmatrix}$$

and $\alpha = (\alpha_{11}, \ldots, \alpha_{1n}, \alpha_{21}, \ldots, \alpha_{2n}, \ldots, \alpha_{m1}, \ldots, \alpha_{mn})^T$.

**Theorem 4.7.** Suppose given $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{X}$ and $\lambda$. Then $\boldsymbol{X}$ is a tolerable generalized eigenvector of ($\boldsymbol{A}$, $\boldsymbol{B}$) if and only if the system of inequalities $C \otimes \alpha \leq e$, $C \otimes \alpha \geq f$ is solvable.

P r o o f. Using Theorem 4.4 and Theorem 4.6 the assertion follows. $\square$

**Theorem 4.8.** Suppose given $\boldsymbol{A}$, $\boldsymbol{B} \subseteq \mathcal{I}(m,n)$, $\boldsymbol{X} \subseteq \mathcal{I}(n)$ and $\lambda \in \mathcal{I}$. There is an $O(m^2n^3)$ procedure for checking whether $\boldsymbol{X}$ is a tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$.

P r o o f. Checking whether $\boldsymbol{X}$ is a tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ is equivalent to determining the solvability of (26) for some $\lambda \in \mathcal{I}$. Computing the products $A^{(ij)} \otimes x^{(k)}$, $\underline{B} \otimes x^{(k)}$, $\overline{B} \otimes x^{(k)}$ needs $O(mn)$ elementary operations for each $i \in M$ and for each $j, k \in N$. The solvability of the system $C \otimes \alpha \leq e$, $C \otimes \alpha \geq f$ can be determined in $O(2mn \cdot mn)$ time according to (30) and (33). Thus, determining the solvability of (26) needs $mn^2 O(mn) + O(2mn \cdot mn) = O(m^2n^3)$ time for a given $\lambda \in \mathcal{I}$. $\square$

Note that to check (26) for all $\lambda \in \mathcal{I}$ for $|\mathcal{I}| = \ell$ needs $\ell \cdot O(m^2n^3)$ time.

## 5. WEAKLY TOLERABLE GENERALIZED EIGENVECTORS

We will now present the main properties of weakly tolerable generalized eigenvectors and a polynomially recognizable characterization for them.

**Theorem 5.1.** Suppose given $\boldsymbol{A} = [\underline{A}, \overline{A}]$, $\boldsymbol{B} = [\underline{B}, \overline{B}]$ and $\boldsymbol{X} = [\underline{x}, \overline{x}]$. Then $\boldsymbol{X}$ is a weakly tolerable generalized eigenvector of ($\boldsymbol{A}$, $\boldsymbol{B}$) if and only if

$$(\exists \lambda \in \mathcal{I})(\forall x \in \boldsymbol{X})\big[\underline{A} \otimes x \leq \lambda \otimes \overline{B} \otimes x \,\wedge\, \lambda \otimes \underline{B} \otimes x \leq \overline{A} \otimes x\big]. \tag{34}$$

Proof. Suppose that (34) holds. From inequalities $\underline{A}\otimes x \le \lambda\otimes\overline{B}\otimes x$, $\lambda\otimes\underline{B}\otimes x \le \overline{A}\otimes x$ by Lemma 4.2 (iii), (iv) we get $\underline{A} \le A^x$, $\underline{B} \le B^x$ and the assertion follows from Lemma 4.1.

The converse assertion follows trivially. □

**Theorem 5.2.** Suppose given $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{X}$. A vector $\boldsymbol{X}$ is a weakly tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$ if and only if there is $\lambda \in \mathcal{I}$ such that for each $i \in M$ the system of inequalities

$$\begin{aligned} \underline{A} \otimes x^{(i)} &\le \lambda \otimes \overline{B} \otimes x^{(i)} \\ \lambda \otimes \underline{B} \otimes x^{(i)} &\le \overline{A} \otimes x^{(i)} \end{aligned} \tag{35}$$

is satisfied.

Proof. We will prove that (34) is equivalent to (35). Let $x \in \boldsymbol{X}$ be arbitrary.

(35)⇒(34). By Lemma 2.4 we have $x = \bigoplus_{i\in N} \beta_i \otimes x^{(i)}$, where each $\beta_i \in \mathcal{I}$ and $\underline{x}_i \le \beta_i \le \overline{x}_i$. Then

$$\underline{A} \otimes x = \underline{A} \otimes \bigoplus_{i\in N} \beta_i \otimes x^{(i)} = \bigoplus_{i\in N} \beta_i \otimes \underline{A} \otimes x^{(i)} \le$$

$$\bigoplus_{i\in N} \beta_i \otimes \lambda \otimes \overline{B} \otimes x^{(i)} = \lambda \otimes \overline{B} \otimes \bigoplus_{i\in N} \beta_i \otimes x^{(i)} = \lambda \otimes \overline{B} \otimes x.$$

Similarly we can prove that $\lambda \otimes \underline{B} \otimes x \le \overline{A} \otimes x$.

(34)⇒(35). The implication trivially follows. □

**Theorem 5.3.** There is an $O(m^2n^3)$ procedure which checks whether $\boldsymbol{X}$ is a weakly tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$.

Proof. The computation of (35) needs to compute $\underline{A}\otimes x^{(i)}$, $\overline{B}\otimes x^{(i)}$, $\underline{B}\otimes x^{(i)}$, $\overline{A}\otimes x^{(i)}$ for all $i \in N$ in $O(mn)$ time each. Thus, the verification of (35) requires $O(mn^2)$ time for a given $\lambda \in \mathcal{I}$. In a process of the calculation of $\underline{A}\otimes x^{(i)}$, $\overline{B}\otimes x^{(i)}$, $\underline{B}\otimes x^{(i)}$, $\overline{A}\otimes x^{(i)}$ for any $i \in M$, the elements are in $\Lambda = \{\underline{a}_{ij}, \overline{a}_{ij}, \underline{b}_{ij}, \overline{b}_{ij}, \underline{x}_i, \overline{x}_i;\ i \in M,\ j \in N\}$ and for this reason it suffices to consider $\lambda \in \Lambda$ for determining the solvability of (35). Hence it follows that the solvability of (35) can be recognized in $mn \cdot O(mn^2) = O(m^2n^3)$ time. □

**Example 5.4.** Put $\mathcal{I} = [0, 10]$, $\lambda = 10$ and

$$\underline{A} = \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \quad \overline{A} = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \\ 3 & 6 & 2 \end{pmatrix},$$

$$\underline{B} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} 1 & 2 & 2 \\ 5 & 4 & 5 \\ 4 & 3 & 4 \\ 3 & 4 & 2 \end{pmatrix},$$

$$\underline{x} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad \overline{x} = \begin{pmatrix} 4 \\ 5 \\ 4 \end{pmatrix}.$$

We have

$$x^{(1)} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix},\ x^{(2)} = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix},\ x^{(3)} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$$

$$\underline{A} \otimes x^{(1)} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 2 \end{pmatrix},\ \underline{A} \otimes x^{(2)} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 2 \end{pmatrix},\ \underline{A} \otimes x^{(3)} = \begin{pmatrix} 2 \\ 2 \\ 3 \\ 2 \end{pmatrix},$$

$$\overline{B} \otimes x^{(1)} = \begin{pmatrix} 2 \\ 4 \\ 4 \\ 3 \end{pmatrix},\ \overline{B} \otimes x^{(2)} = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 4 \end{pmatrix},\ \overline{B} \otimes x^{(3)} = \begin{pmatrix} 2 \\ 4 \\ 4 \\ 2 \end{pmatrix},$$

$$\underline{B} \otimes x^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix},\ \underline{B} \otimes x^{(2)} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix},\ \underline{B} \otimes x^{(3)} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix},$$

$$\overline{A} \otimes x^{(1)} = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 3 \end{pmatrix},\ \overline{A} \otimes x^{(2)} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 5 \end{pmatrix},\ \overline{A} \otimes x^{(3)} = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 2 \end{pmatrix}.$$

Since $\underline{A} \otimes x^{(i)} \leq \lambda \otimes \overline{B} \otimes x^{(i)}$, $\lambda \otimes \underline{B} \otimes x^{(i)} \leq \overline{A} \otimes x^{(i)}$ for each $i \in N$, by Theorem 5.2, the given interval vector $\boldsymbol{X}$ is a weakly tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$.

In what follows we will show that $\boldsymbol{X}$ is a tolerable generalized eigenvector of $(\boldsymbol{A}, \boldsymbol{B})$. We have to solve the system of inequalities

$$\begin{pmatrix}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 3 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 2 \\
2 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 2 & 4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 5 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 5 & 2 \\
2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10
\end{pmatrix} \otimes \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \\ \alpha_{21} \\ \alpha_{22} \\ \alpha_{23} \\ \alpha_{31} \\ \alpha_{32} \\ \alpha_{33} \\ \alpha_{41} \\ \alpha_{42} \\ \alpha_{43} \end{pmatrix} \leq \begin{pmatrix} 2 \\ 4 \\ 4 \\ 3 \\ 2 \\ 4 \\ 3 \\ 4 \\ 2 \\ 4 \\ 4 \\ 2 \\ 1 \\ 3 \\ 4 \\ 3 \\ 4 \\ 5 \\ 4 \\ 5 \\ 6 \\ 3 \\ 6 \\ 2 \end{pmatrix} \tag{36}$$

and

$$\begin{pmatrix}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 3 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 2 \\
2 & 3 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 2 & 4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 5 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 5 & 2 \\
2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 3 & 3 & 3 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10
\end{pmatrix} \otimes \begin{pmatrix} \alpha_{11} \\ \alpha_{12} \\ \alpha_{13} \\ \alpha_{21} \\ \alpha_{22} \\ \alpha_{23} \\ \alpha_{31} \\ \alpha_{32} \\ \alpha_{33} \\ \alpha_{41} \\ \alpha_{42} \\ \alpha_{43} \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 1 \\ 1 \\ 2 \\ 1 \\ 1 \\ 2 \\ 2 \\ 1 \\ 0 \\ 2 \\ 2 \\ 1 \\ 0 \\ 2 \\ 2 \\ 1 \\ 3 \\ 1 \\ 2 \\ 1 \end{pmatrix} \tag{37}$$

To obtain a solution of the system (36), (37) we will use the Theorem 4.6: $\hat{x}(C,e) = (1,2,2,3,4,5,4,3,6,3,4,2)^T$ and

$$C \otimes \hat{x}(C,e) = (2,3,4,3,2,4,3,4,2,4,4,2,1,2,2,3,4,5,4,3,6,3,4,2)^T$$
$$\geq f = (1,1,2,1,1,1,2,1,1,2,2,1,0,2,2,1,0,2,2,1,3,1,2,1)^T.$$

The vector $\hat{x}(C,e)$ is the greatest solution of the above system of inequalities and corresponds to the following matrix:

$$A = 1 \otimes A^{(11)} \oplus 2 \otimes A^{(12)} \oplus 2 \otimes A^{(13)} \oplus 3 \otimes A^{(21)} \oplus 4 \otimes A^{(22)} \oplus 5 \otimes A^{(23)} \oplus$$
$$4 \otimes A^{(31)} \oplus 3 \otimes A^{(32)} \oplus 6 \otimes A^{(33)} \oplus 3 \otimes A^{(41)} \oplus 4 \otimes A^{(42)} \oplus 2 \otimes A^{(43)} =$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 & 2 \\ 3 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 & 2 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix} \oplus$$
$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 4 \\ 2 & 1 & 3 \\ 2 & 1 & 2 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 2 \\ 4 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix} \oplus$$
$$\begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & 6 \\ 1 & 2 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & 4 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix} =$$
$$\begin{pmatrix} 1 & 2 & 2 \\ 3 & 3 & 4 \\ 4 & 3 & 6 \\ 3 & 4 & 2 \end{pmatrix}.$$

Hence $\boldsymbol{X}$ is tolerable generalized eigenvector of ($\boldsymbol{A}$, $\boldsymbol{B}$).

## 6. CONCLUSIONS

In this paper, three different concepts of an interval generalized eigenvector have been studied. The results obtained are useful in many practical applications. Replacing the exact values of the elements of a matrix and a vector with intervals opens the possibility of defining several types of generalized eigenvectors according to the choice of quantifiers and their order. Three notions of an interval generalized eigenvector of a given interval matrix have been treated, namely: strongly tolerable generalized eigenvectors, tolerable generalized eigenvectors and weakly tolerable generalized eigenvectors. Results based on the properties of these types of generalized eigenvectors have allowed us to formulate efficient necessary and sufficient conditions. In addition, algorithms have been found that recognize these types of generalized eigenvectors. The results have been illustrated by numerical examples (in the cases of tolerable and weakly tolerable generalized eigenvectors).

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