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$\mathcal{T}$–SEMIRING PAIRS

JAIUNG JUN, KALINA MINCHEVA, AND LOUIS ROWEN

We develop a general axiomatic theory of algebraic pairs, which simultaneously generalizes several algebraic structures, in order to bypass negation as much as feasible. We investigate several classical theorems and notions in this setting including fractions, integral extensions, and Hilbert's Nullstellensatz. Finally, we study a notion of growth in this context.

Keywords: pair, semiring, system, triple, shallow, algebraic, integral, affine, Ore, negation map, congruence, module

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1. INTRODUCTION

Over recent years an effort has been made to understand tropical mathematics in terms of various algebraic structures. A major role has been taken by the max-plus algebra $\mathbb{R}_{\max}$, but with the drawback that its structure theory is quite limited by the absence of negation. Gaubert [14] addressed this issue in his dissertation, followed by [1], studying a construction which we call "symmetrization" in [2, 36]. Izhakian [20] introduced a modification, later called "supertropical algebra" and studied extensively by Izhakian and Rowen [21, 22], and later with Knebusch. Meanwhile, extensive literature developed around hyperstructures [5, 7, 16, 26, 40] and fuzzy rings [10, 11]. These constructions were unified by Lorscheid in "blueprints" [32, 33], and carried further into a "systemic" algebraic approach taken by Rowen [36], and developed in [2, 13, 25, 27] in order to unify classical algebra with the algebraic theories of supertropical algebra, symmetrized semirings, hyperfields, and fuzzy rings, with some success especially in obtaining theorems about matrices, polynomials, and linear algebra.

The idea of systems in brief is to consider a formal "negation map" $(-)$ on $\mathcal{A}$ satisfying all properties of negation except $a - a = 0$. The zero element no longer plays a role, but is replaced by the set of **quasi-zeros** $\mathcal{A}^\circ = \{b + ((-)b) : b \in \mathcal{A}\}$. In supertropical algebra the "negation map" actually is the identity, and in a semiring where a negation map is lacking, it can be provided in the symmetrization process. Another innovation was the "surpassing relation" $\preceq$, to extend equality in most results.

Although the negation map is very useful, this paper addresses the question as to how much semiring theory can be developed only with the "surpassing relation" given

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below in Definition 2.4, which is needed for our version of equations. In addition to the philosophical question of the minimal axiomatic framework needed to carry out the theory, we are motivated by new structures generalizing hyperfields which come up naturally in the study of semirings, given in §2.4.

Many of the relevant notions of systems are formulated without a negation map on $\mathcal{A}$, by elevating $\mathcal{A}^\circ$ to the principal structural role, via the embedding $\mathcal{A}^\circ \to \mathcal{A}$. Then we can formulate relative versions of algebraic concepts which could be applied to tropical (and other) situations. In other words, we continue the approach of systems, but with a different emphasis which may provide extra intuition, in an effort to obtain the greatest generality in which the algebraic structure theorems of the Artin–Krull–Noether theory are available. One such direction of the structure theory taken in [27, §3 and §4] was the spectrum of prime congruences.

Our main focus, extending [27, §3 and §4], is on the structure of **pairs** $(\mathcal{A}, \mathcal{A}_0)$ acted upon by a set $\mathcal{T}$. See Definition 2.2 for the formal set-up. In brief, $(\mathcal{A}, \mathcal{A}_0)$ is a **$\mathcal{T}$-semiring pair** when $\mathcal{A}_0$ and $\mathcal{A}$ are semirings with compatible $\mathcal{T}$-actions, and $\mathcal{T}$ is a multiplicative submonoid of $\mathcal{A}$, where $\mathcal{A}_0 \cap \mathcal{T} = \emptyset$ and $\mathcal{T}$ spans $\mathcal{A}$ additively. The $\mathcal{T}$-pair $(\mathcal{A}, \mathcal{A}_0)$ is **shallow** if $\mathcal{A} = \mathcal{T} \cup \mathcal{A}_0$.

Likewise, for modules (called "semimodules" in the semiring literature), we suppress the negation map, and simply consider pairs $(\mathcal{M}, \mathcal{N})$ of modules over $(\mathcal{A}, \mathcal{A}_0)$, together with a map $\phi_{\mathcal{M},\mathcal{N}} : \mathcal{N} \to \mathcal{M}$ where in applications $\phi_{\mathcal{M},\mathcal{N}}(\mathcal{N}) \subseteq \mathcal{M}$ could be identified with the quasi-zeroes. Ideals in algebra are replaced by congruences with the "twist product". We go through the basic structure theory, starting with "prime" and "semiprime" defined by means of the twist product. Then we compare notions of algebraicity, and compute the growth of some basic pairs. Throughout we search for the precise hypotheses necessary for the various theorems.

One also can manage without negation maps when studying polynomials (§3.4), as well as pairs of fractions. The Ore condition for semigroups is well known, cf. [31]. Following [34, p. 349] as a model, we also introduce the notion of Ore condition for pairs, under which one obtains the pair of fractions of a $\mathcal{T}$-semiring pair $(\mathcal{A}, \mathcal{A}_0)$ (Theorem 4.5).

One goal of any algebraic theory is to obtain some analog of Hilbert's Nullstellensatz, which says that every radical ideal of the affine polynomial algebra is a set of zeroes of polynomials. This is tricky in the semiring set-up because the structure is described in terms of congruences, not ideals, but we propose an approach in Appendix A.

1.1. Main Results

Theorem A. (Theorem 3.5) A congruence Φ on a $\mathcal{T}$-semiring pair $(\mathcal{A}, \mathcal{A}_0)$ is semiprime if and only if it is the intersection of a nonempty set of prime congruences.

Given a $\mathcal{T}$-pair $(\mathcal{A}, \mathcal{A}_0)$ with $\mathcal{T}$ satisfying the Ore condition with respect to S, one can define an equivalence relation on $\mathcal{A} \times S$, by $(b_1, s_1) \equiv (b_2, s_2)$ iff there are $c, c' \in \mathcal{A}$ for which $cb_1 = b_2c'$. Write $S^{-1}\mathcal{A}$ for $\{s^{-1}b := [(b, s)] : b \in \mathcal{A},\, s \in S\}$, $S^{-1}\mathcal{A}_0$ for $\{s^{-1}b := [(b, s)] : b \in \mathcal{A}_0,\, s \in S\}$, and $S^{-1}\mathcal{T}$ for $\{s^{-1}a := [(a, s)] : a \in \mathcal{T},\, s \in S\}$.

Theorem B. (Theorem 4.5) With the above notation, $(S^{-1}\mathcal{A}, S^{-1}\mathcal{A}_0)$ is an $S^{-1}\mathcal{T}$-pair.

See Definition 3.16 for "$(\preceq)$-base."

Theorem C. (Artin-Tate for pairs, Theorem 4.1) Suppose $(\mathcal{W}, \mathcal{W}_0)$ is an affine semialgebra pair over $\mathcal{A}$ and has a finite $(\preceq)$-base B over a central semialgebra $\mathcal{K} \subset \mathcal{W}$. Then $\mathcal{K}$ is affine over $\mathcal{A}$.

We need some more preparation for the next result.

A pair $(\mathcal{A}, \mathcal{A}_0)$ is $\preceq$-*nondegenerate* if $f(\mathcal{T}) \nsubseteq \mathcal{A}_0$ for any tangible polynomial f.

Theorem D. (Theorem 4.16) Suppose $(\mathcal{A}, \mathcal{A}_0)$ is a $\preceq_0$ -nondegenerate, shallow semiring pair, and $(\mathcal{W}, \mathcal{W}_0)$ is a centralizing extension of $(\mathcal{A}, \mathcal{A}_0)$, with $y \in \mathcal{T}_{\mathcal{W}}$ such that $(\mathcal{A}[y], \mathcal{A}_0[y])$ is tangibly separating.[1] Let $\mathcal{T}' = \{ay^i : a \in \mathcal{T}, i \in \mathbb{N}\}$. Let $(\mathcal{K}, \mathcal{K}_0)$ be the $\mathcal{T}'$-semifield of fractions of $(\mathcal{A}[y], \mathcal{A}_0[y])$. If $(\mathcal{K}, \mathcal{K}_0)$ is affine, then y is congruence algebraic[2] over the pair $(\mathcal{A}, \mathcal{A}_0)$.

The Nullstellensatz does not hold in this setting in general (Example 4.17), but we prove that in some circumstances (including the classical case and tropical case), a version of the Nullstellensatz holds.

Theorem E. (Theorem 4.18) Suppose $(\mathcal{W}, \mathcal{W}_0)$ is an affine $\mathcal{T}$-semifield pair over a shallow, $\preceq$-nondegenerate $\mathcal{T}$-semiring pair $(\mathcal{A}, \mathcal{A}_0)$, and also having the property that $\mathcal{T}$-algebraic implies integral. Then $(\mathcal{W}, \mathcal{W}_0)$ is integral over $(\mathcal{A}, \mathcal{A}_0)$.

Finally, we study growth of pairs and prove the following.

Theorem F. (Proposition 5.5) Any $\mathcal{T}$-semidomain pair $(\mathcal{A}, \mathcal{A}_0)$ with subexponential growth has the property that for any $a_1, a_1 \in \mathcal{T}$ there are $b_1, b_2 \in \mathcal{A} \setminus \mathcal{A}_0$ such that $b_1 a_1 + b_2 a_2 \in \mathcal{A}_0$.

2. BASIC NOTIONS

See [37] for a relatively brief introduction of systems; more details are given in [25], [27], and [36]. Throughout the paper, we let $\mathbb{N}$ be the additive monoid of nonnegative integers. Similarly, we view $\mathbb{Q}$ (resp. $\mathbb{R}$) as the additive monoid of rational numbers (resp. real numbers). $\mathcal{T}$ will always denote a multiplicative monoid with $\mathbb{1}$. We say that $\mathcal{T}$ **acts** on a set $\mathcal{A}$ if there is a binary operation $\mathcal{T} \times \mathcal{A} \to \mathcal{A}$ satisfying

$$(a_1 a_2)b = a_1(a_2 b), \qquad \forall a_1, a_2 \in \mathcal{T},\ b \in \mathcal{A}.$$

It is appropriate to turn to the context of universal algebra, where one is given sets, called "algebraic structures," or "Ω-algebras," with various operations, relations, and identities. The 0-ary operations can be thought of distinguished elements. Rather than stating the definitions formally, we refer the reader to [23], and give the main instances:

[1] (See Example 2.3(iv) below).
[2] (See Definition 2.3(ii) below).

Definition 2.1. 1. A **magma** is a set S with a binary operation denoted $(+)$ (addition) or $(\cdot)$ (multiplication). In this paper we also require a neutral element, written as $\mathbb{0}$ or $\mathbb{1}$ respectively.

A **semigroup** is a magma S whose given binary operation satisfies the law of associativity.

2. A **semiring** (cf. [9],[17]) $(\mathcal{A}, +, \cdot, \mathbb{0}, \mathbb{1})$ is an additive abelian semigroup $(\mathcal{A}, +, \mathbb{0})$ and multiplicative semigroup $(\mathcal{A}, \cdot, \mathbb{1})$ satisfying $\mathbb{0}b = b\mathbb{0} = \mathbb{0}$ for all $b \in \mathcal{A}$, as well as the usual distributive laws.

   The semiring predominantly used in tropical mathematics has been the max-plus algebra, where $\oplus$ designates max, and $\otimes$ designates $+$. However, we proceed with the familiar algebraic notation of addition and multiplication in the setting under consideration.

3. More generally, a **bimagma** is a semigroup $(S, +)$ which is also a magma $(S, \cdot)$.

4. A (left) $\mathcal{T}$**-module** over a set $\mathcal{T}$ is a semigroup $(\mathcal{A}, +, \mathbb{0})$, endowed with a $\mathcal{T}$-action satisfying the following axioms, for all $a \in \mathcal{T}$ and $b_j \in \mathcal{A}$ for $j = 1, \dots, u$:

   (a) $a\mathbb{0} = \mathbb{0}a = \mathbb{0}$.

   (b) $a(\sum_{j=1}^{u} b_j) = \sum_{j=1}^{u}(ab_j), \ a \in \mathcal{A}$.

5. Thus, a **semialgebra** over a commutative semiring $\mathcal{A}$ is an $\mathcal{A}$-module $\mathcal{M}$ which is also a bimagma which satisfies

$$(ay_1)y_2 = a(y_1y_2) = y_1(ay_2)$$

   for all $a \in \mathcal{A}$, $y_1, y_2 \in \mathcal{M}$. An example would be a Lie semialgebra, as defined in [19]; also cf. [36, Definition 10.4] and [8, Definition 3.4].

6. A $\mathcal{T}$-module $\mathcal{A}$ is $\mathcal{T}$**-spanned** if there is a given 1:1 map $\psi : \mathcal{T} \to \mathcal{A}$ whose image additively (with $\mathbb{0}$) generates $\mathcal{A}$. In this situation we identify $\mathcal{T}$ with $\psi(\mathcal{T})$, yielding a distinguished set $\mathcal{T} \subseteq \mathcal{A}$. We define $\mathcal{T}_0 = \mathcal{T} \cup \{\mathbb{0}\}$.

We often suppress the operations and distinguished elements in the notation. When $\mathcal{T}$ is ambiguous, we write $\mathcal{T}_{\mathcal{A}}$ to indicate that it is affiliated with $\mathcal{A}$.

A $\mathcal{T}$**-module homomorphism** $f : \mathcal{A}_1 \to \mathcal{A}_2$ is a function such that $f(ab_1) = af(b_1)$, $f(b_1 + b_2) = f(b_1) + f(b_2)$ for all $a \in \mathcal{T}$ and $b_1, b_2 \in \mathcal{A}_1$. Often we fix an action $f_{\mathcal{T}}$ on $\mathcal{T}$ and consider only those f whose restriction to $\mathcal{T}$ is $f_{\mathcal{T}}$.

2.1. Pairs

Definition 2.2.

(i) A **pair** $(\mathcal{A}, \mathcal{A}_0)$ is a pair of algebraic structures (which should be clear by the context) $\mathcal{A}$ and $\mathcal{A}_0$, together with a given homomorphism $\phi_{\mathcal{A},\mathcal{A}_0} : \mathcal{A}_0 \to \mathcal{A}$.

(ii) A $\mathcal{T}$**-pair** is a pair $(\mathcal{A}, \mathcal{A}_0)$ of $\mathcal{T}$-modules over the set $\mathcal{T}$. A $\mathcal{T}$-pair $(\mathcal{A}, \mathcal{A}_0)$ is **admissible** if $\mathcal{A}$ is a $\mathcal{T}$-spanned $\mathcal{T}$-module and $\phi_{\mathcal{A},\mathcal{A}_0}$ is an injection, and identifying $\mathcal{A}_0$ with the image of $\phi_{\mathcal{A},\mathcal{A}_0}$, we have $\mathcal{A}_0 \cap \mathcal{T} = \emptyset$. The elements of $\mathcal{T}$ are called **tangible**.

(iii) The admissible pair $\mathcal{T}$-pair $(\mathcal{A}, \mathcal{A}_0)$ is **shallow** if $\mathcal{A} = \mathcal{T} \cup \mathcal{A}_0$.

(iv) A $\mathcal{T}$**-semiring pair** is an admissible $\mathcal{T}$-pair $(\mathcal{A}, \mathcal{A}_0)$ where $\mathcal{A}$ is an (associative) semiring. A $\mathcal{T}$**-semifield pair** is a $\mathcal{T}$-semiring pair $(\mathcal{A}, \mathcal{A}_0)$ where $\mathcal{T}$ is a group.

(v) A $\mathcal{T}$-pair $(\mathcal{A}, \mathcal{A}_0)$ is **generated** by $S \subseteq \mathcal{A}$ if every sub-$\mathcal{T}$-pair of $(\mathcal{A}, \mathcal{A}_0)$ containing S and $\phi_{\mathcal{A},\mathcal{A}_0}(\mathcal{A}_0)$ is $\mathcal{A}$. $(\mathcal{A}, \mathcal{A}_0)$ is **finitely generated** if it is generated by a finite set.

We assume throughout this paper that $(\mathcal{A}, \mathcal{A}_0)$ is an admissible $\mathcal{T}$-semiring pair.

Example 2.3.

(i) (The classical case) $\mathcal{A}$ is an algebra and $\mathcal{A}_0 = \mathcal{I}$ is an ideal of $\mathcal{A}$. Then we can take $\phi_{\mathcal{A},\mathcal{A}_0}$ to be the identity $\mathcal{A}_0 \to \mathcal{A}$ and consider $(\mathcal{A}, \mathcal{A}_0)$ as $\mathcal{A}/\mathcal{I}$. If $\mathcal{I}$ is a prime ideal, we could have $\mathcal{T} = \mathcal{A} \setminus \mathcal{I}$, a monoid, and the pair $(\mathcal{A}, \mathcal{A}_0)$ is shallow

(ii) (Doubling; analogous to symmetrization in [36]) This is a way to create an admissible pair, for any additive semigroup $(A, +, 0)$. We define $\mathcal{A} = A \times A$, $\mathcal{A}_0 = \{(a, a) : a \in A\}$, $\mathcal{T} = (A \times 0) \cup (0 \times A)$ and $\phi_{\mathcal{A},\mathcal{A}_0}$ to be the identity map.

(iii) (The supertropical case) $3a = 2a$, in the sense that $a + a + a = a + a$ for all $a \in \mathcal{A}$, and $\mathcal{A}_0 = \{a + a : a \in \mathcal{T}\} = \{ma : m > 1,\ a \in \mathcal{T}\}$, written as A^ν in the literature. The pair $(\mathcal{A}, \mathcal{A}_0)$ is shallow.

2.2. Surpassing relations

We next provide the pair with a **surpassing relation** $\preceq$ ([36, Definition 1.31] and also described in [27, Definition 2.10]).

Definition 2.4. A **surpassing relation** on a $\mathcal{T}$-pair $(\mathcal{A}, \mathcal{A}_0)$, denoted $\preceq$, is a partial preorder satisfying the following, for elements $b_i \in \mathcal{A}$:

(i) $\mathbb{0} \preceq c$ for any $c \in \mathcal{A}_0$.

(ii) If $b_1 \preceq b_2$ and $b_1' \preceq b_2'$ for $i = 1, 2$ then $b_1 + b_1' \preceq b_2 + b_2'$.

(iii) If $a \in \mathcal{T}$ and $b_1 \preceq b_2$ then $ab_1 \preceq ab_2$.

(iv) $a \preceq b$ for $a, b \in \mathcal{T}$ implies $a = b$. (In other words, surpassing restricts to equality on tangible elements.)

A **strong surpassing relation** on $\mathcal{T}$-pair $(\mathcal{A}, \mathcal{A}_0)$ is a surpassing relation satisfying the following stronger version of (iv): If $b \preceq a$ for $a \in \mathcal{T}$ and $b \in \mathcal{A}$, then $b = a$.

The justification for these definitions is given in [36, Remark 1.34]. In brief, in proving theorems about pairs, we often need equations in tangible elements to be weakened, where $\mathcal{A}_0$ takes on the role of "zero."

Example 2.5. Our main example of a surpassing relation, denoted $\preceq_0$, is given by $b_1 \preceq b_2$ iff $b_2 = b_1 + y$ for some $y \in \mathcal{A}_0$.

This can be defined on any pair, and matches the definition of systems.

Lemma 2.6. If $b \preceq_0 a$ for $a \in \mathcal{T}$, then $b \notin \mathcal{A}_0$.

Consequently, if the admissible $\mathcal{T}$-pair $(\mathcal{A}, \mathcal{A}_0)$ is shallow, then $\preceq_0$ is a strong surpassing relation.

Proof. If $b \in \mathcal{A}_0$ then for some $y \in \mathcal{A}_0$, $a = b + y \in \mathcal{A}_0 \cap \mathcal{T}$, a contradiction.

Hence, for $(\mathcal{A}, \mathcal{A}_0)$ shallow, $b \in \mathcal{T}$, so $b = a$. □

2.3. Negation maps and Property N

In some cases we can define the negation map, the main tool of [36]. A **negation map** $(-)$ on $(\mathcal{A}, \mathcal{A}_0)$ is an additive automorphism of order ≤ 2 satisfying

$$(-)(bb') = ((-)b)b' = b((-)b'), \quad b + ((-)b) \in \mathcal{A}_0, \quad \forall b, b' \in \mathcal{A}, \tag{2.1}$$

and $(-)\mathcal{A}_0 = \mathcal{A}_0$. When $\mathcal{A}$ is a $\mathcal{T}$-module we also require $(-)$ to be defined on $\mathcal{T}$, such that

$$(-)(ab) = ((-)a)b = a((-)b), \qquad \forall a \in \mathcal{T}, \quad b \in \mathcal{A}.$$

We write $b(-)c$ for $b + ((-)c)$. Thus $\mathcal{A}_0$ contains the set of **quasi-zeros**, denoted as $\mathcal{A}^\circ := \{b(-)b : b \in \mathcal{A}\}$. One often has $\mathcal{A}_0 = \mathcal{A}^\circ$.

Lemma 2.7. ([25, Lemma 2.11]) If $b_1 \preceq_{\mathbb{0}} b_2$ in a pair with a negation map, then $b_2(-)b_1 = b_1(-)b_2 \succeq \mathbb{0}$.

Proof. Write $b_2 = b_1 + c^\circ$. Then $b_2(-)b_1 = (b_1 + c)^\circ = b_1(-)b_2$. □

Our main illustration having a surpassing relation of a different nature is as follows:

Example 2.8. Hypersystems $(\mathcal{A}, \mathcal{H}, (-), \subseteq)$ were defined in [2], where $\mathcal{A} \subseteq \mathcal{P}^*(\mathcal{H})$, $\mathcal{A}_0 = \{S : \mathbb{0} \in S\}$, and $\preceq = \subseteq$.

2.3.1. Weaker versions of negation maps

Other properties may suffice when we do not have a negation map at our disposal.

Definition 2.9.

(i) We say that an admissible $\mathcal{T}$-pair $(\mathcal{A}, \mathcal{A}_0)$ satisfies **Property N** if for each $a \in \mathcal{T}$, there is $a' \in \mathcal{T}$ such that $a + a' \in \mathcal{A}_0$.

(ii) A pair satisfying Property N is **tangibly separating** if it satisfies the condition:

(a) For each $c \neq a \in \mathcal{T}$, there is $a' \in \mathcal{T}$ such that $c + a' \in \mathcal{T}$ and $a + a' \in \mathcal{A}_0$.

The set-up here is more general, and our main objective is to recast the theory of systems to see when negation maps (and "systemic") can be omitted, and only to utilize the surpassing relation.

Here are some versions of classical structures which need not even satisfy Property N, but at times one can obtain a negation map on a $\mathcal{T}$-submodule S of $\mathcal{A}$, where (2.1) holds on S, and $(-)(S \cap \mathcal{A}_0) = (S \cap \mathcal{A}_0)$. In this case we call $(-)$ a **partial negation map** (on S). We give some examples motivated by [8].

Example 2.10.

(i) An **exterior pair** is an admissible $\mathcal{T}$-pair $(\mathcal{A}, \mathcal{A}_0)$ with $x^2 \in \mathcal{A}_0$, $xy + yx \in \mathcal{A}_0$, for each $x, y \in \mathcal{A}$.

In [8, §4], a partial negation map is defined on the tensors of degree ≥ 2 in the tensor semialgebra for the exterior pair, by $(-)v \otimes w = w \otimes v$.

(ii) A **Lie pair** is a pair $(\mathcal{A}, \mathcal{A}_0)$ with **Lie multiplication** $[xy]$ satisfying for all $x, y, z \in \mathcal{A}$, $y_0 \in \mathcal{A}_0$:

- $[xx] \in \mathcal{A}_0$,
- $[xy_0] \in \mathcal{A}_0$,
- $[xy] + [yx] \in \mathcal{A}_0$,
- $[[xy]z] = [z[yx]]$,
- $[[xy]z] \preceq [x[yz]] + [[xz]y]$.

2.4. Semi-hyperrings

Furthermore we can weaken the requirement of having the hyperring negation.

Definition 2.11. A **semi-hypergroup** is a set $(\mathcal{H}, \boxplus)$ where

1. $\boxplus$ is a commutative binary operation $\mathcal{H} \times \mathcal{H} \to \mathcal{P}^*(\mathcal{H})$ (the set of non-empty subsets of $\mathcal{H}$), extended elementwise to $\mathcal{P}^*(\mathcal{H})$, i. e.,

$$S_1 \boxplus S_2 = \{a_1 + a_2 : \ a_1 \in S_1, \quad a_2 \in S_2\}, \tag{2.2}$$

which also is associative in the sense that if we define

$$a \boxplus S = S \boxplus a = \bigcup_{s \in S} a \boxplus s,$$

then $(a_1 \boxplus a_2) \boxplus a_3 = a_1 \boxplus (a_2 \boxplus a_3)$ for all a_i in $\mathcal{H}$.

2. We adjoin an element $\mathbb{0}_{\mathcal{H}}$ called **the hyperzero**, which is the neutral element: $\mathbb{0}_{\mathcal{H}} \boxplus a = a$.

3. A **semi-hyperring** is a semi-hypergroup $\mathcal{H}$ including an absorbing hyperzero $\mathbb{0}_{\mathcal{H}}$, with multiplication by $\mathcal{H}$ distributes over addition, that is, for all $a_1, a_2, a_3 \in \mathcal{H}$,

$$a_1(a_2 \boxplus a_3) := \{a_1 c : c \in a_2 \boxplus a_2\} = (a_1 a_2) \boxplus (a_1 a_3).$$

Example 2.12. Suppose $\mathcal{T}$ is a semi-hypergroup $\mathcal{H} = (\mathcal{H}, \boxplus, \mathbb{0})$, and $\mathcal{A}$ the subset of the power set of $\mathcal{H}$ additively generated by $\{\{a\} : a \in \mathcal{H}\}$, addition defined elementwise. We define $\preceq$ on $\mathcal{A}$ by putting $S_1 \preceq S_2$ iff $S_1 \subseteq S_2$. There are two choices for $\mathcal{A}_0$:

(i) $\mathcal{A}_0 = \{S : \mathbb{0} \in S\}$.

(ii) $\mathcal{A}_0 = \{S : \ |S| \geq 2\}$.

(i) is the customary definition, following [27, Definition 2.16], [36, Definition 1.70], and yielding a system, called a hypersystem, in which $(-)a = -a$. However (ii) has the advantage that the pair $(\mathcal{A}, \mathcal{A}_0)$ is shallow.

We have the following extension of an idea of Krasner:

Proposition 2.13. (cf. [29]) Suppose R is a commutative semiring having a multiplicative subgroup G. Then the set of cosets $\mathcal{H} := R/G = \{[r] = rG : r \in R\}$, equipped with the multivalued addition

$$[r] \boxplus [r'] = \{[x + x'] : x \in rG, \ x' \in r'G\},$$

and the multiplication inherited from R, is a semi-hyperring. In particular, $\mathcal{H}$ is a semi-hyperfield if R is a semifield.

Proof. The proofs of [29] do not use negation:

$$(a_1 + a_2)([r])) = \{(a_1 + a_2)[x] : x \in rG\} = \{[(a_1 + a_2)x] : x \in rG\} = [(a_1 + a_2)[r]];$$

$$\begin{aligned} a([r] \boxplus [r'])) &= \{a[x + x'] : x \in rG, \ x' \in r'G\} \\ &= \{[ax + ax'] : x \in rG, \ x' \in r'G\} = [a[r] \boxplus a[r']); \end{aligned} \tag{2.3}$$

associativity also is clear. □

Let us take this one step further.

Proposition 2.14. Suppose $\mathcal{H}$ is a commutative semi-hyperring having a multiplicative subgroup G. For $S \subseteq \mathcal{P}(\mathcal{H})$, define the coset $GS = \{as : a \in \mathcal{H}, \ s \in S\}$. Then the set of cosets $\mathcal{H} := \mathcal{H}/G = \{[a] = aG : a \in \mathcal{H}\}$, equipped with the multivalued addition

$$[a] \boxplus [a'] = \{[x + x'] : x \in aG, \ x' \in a'G\},$$

and the multiplication inherited from R, is a semi-hyperring $\mathcal{H}$, which we denote as $\mathcal{H}/_{\mathrm{hyp}}G$. In particular, $\mathcal{H}$ is a semi-hyperfield if $\mathcal{H}$ is a semi-hyperfield.

Furthermore, if $G \subseteq \hat{G}$ are subgroups of $\mathcal{H}$, then

$$\mathcal{H}/_{\mathrm{hyp}}\hat{G} \cong (\mathcal{H}/_{\mathrm{hyp}}G)/_{\mathrm{hyp}}\hat{G}.$$

P r o o f. The first assertion is as in Proposition 2.13. Clearly $\mathcal{H}$ is a monoid, and associativity and distributivity are easy to check.

For the second assertion, take the map sending $S \mapsto \{ag : a \in S, g \in \hat{G}\}$. This yields the composition

$$S \mapsto S_G := \{ag : a \in S, g \in \hat{G}\} \mapsto \cup_{\bar{g} \in \hat{G}/G} S_G \bar{g}.$$

□

2.5. Congruences

Classically, one defines homomorphic images by defining a **congruence** on an algebraic structure $\mathcal{A}$ to be an equivalence relation Φ, which viewed as a set of ordered pairs, is a subalgebra of $\mathcal{A} \times \mathcal{A}$ which we require to be disjoint from $\mathcal{T} \times \mathcal{A}_0$.

Definition 2.15. A **congruence** on a pair $(\mathcal{A}, \mathcal{A}_0)$ is a pair (Φ, Φ_0) of congruences on $\mathcal{A}$ and $\mathcal{A}_0$ such that we get an induced map $\phi_{\mathcal{A},\mathcal{A}_0} : \Phi_0 \to \Phi$. (In the case of admissible pairs, we consider $\Phi_0 \subseteq \Phi$.)

Remark 2.16. Any congruence (Φ, Φ_0) on $(\mathcal{A}, \mathcal{A}_0)$ can be applied to produce a pair $(\overline{\mathcal{A}}, \overline{\mathcal{A}_0})$ where $\overline{\mathcal{A}} = \mathcal{A}/\Phi$ and $\overline{\mathcal{A}_0} = \mathcal{A}_0/(\mathcal{A}_0 \cap \Phi_0)$, and $\overline{\phi}_{\mathcal{A},\mathcal{A}_0} : \overline{\mathcal{A}_0} \to \overline{\mathcal{A}}$ is the induced homomorphism.

The **congruence kernel** $\ker f$ of a homomorphism $f : (\mathcal{A}, \mathcal{A}_0) \to (\mathcal{A}', \mathcal{A}'_0)$ is

$$\{(y_1, y_2) \in \mathcal{A} \times \mathcal{A} : f(y_1) = f(y_2)\}$$

and its restriction to $\mathcal{A}_0$; $\ker f$ is easily seen to be a congruence. Conversely, every congruence (Φ, Φ_0) is the congruence kernel of the natural homomorphism $y \mapsto [y]$, where $[y]$ is the equivalence class of $y \in \mathcal{A}$ under Φ. (When $y \in \mathcal{A}_0$ then we can view $[y]$ as the equivalence class of y under Φ_0).

Often we delete Φ_0 from the notation, when it is the restriction to $\mathcal{A}_0$. Certain congruences play a fundamental role.

Definition 2.17.

(i) Diag$\mathcal{A}$ denotes the diagonal congruence $\{(b, b) : b \in \mathcal{A}\}$, also called the **trivial** congruence.

(ii) When $\mathcal{A}$ is commutative and associative, for any $a \in \mathcal{A}$ the congruence $\Phi_a(\mathcal{A})$ generated by a is $\{(a_1 a, a_2 a) : a_i \in \mathcal{A}\}$. In the image of $\mathcal{A}$ under this congruence, a is identified with $\mathbb{0} a = \mathbb{0}$, so we have the natural homomorphism sending $a \mapsto \mathbb{0}$.

Definition 2.18. $\mathcal{A} \times \mathcal{A}$ has the **switch map** given by $(a_1, a_2) \mapsto (a_2, a_1)$, and the **twist product** given by

$$(a_1, a'_1) \cdot_{\mathrm{tw}} (a_2, a'_2) = (a_1 a_2 + a'_1 a'_2, a_1 a'_2 + a'_1 a_2).$$

Lemma 2.19. Any congruence is closed under the switch map and the twist product.

Proof. Any congruence is closed under the switch map since any congruence is symmetric.

For the twist product,

$$(a_1a_2 + a_1'a_2', a_1a_2' + a_1'a_2) = (a_1, a_1)(a_2, a_2') + (a_1', a_1')(a_2', a_2).$$

□

We would want to identify $\phi_{\mathcal{A},\mathcal{A}_0}(\mathcal{A}_0)$ as "zero" in some factor set of $\mathcal{A}$ with respect to $\phi_{\mathcal{A},\mathcal{A}_0}(\mathcal{A}_0)$, but the congruence generated by $\{(b, b + \phi_{\mathcal{A},\mathcal{A}_0}(b_0)) : \ b \in \mathcal{A},\ b_0 \in \mathcal{A}_0\}$ could be larger than one wants.

3. STRUCTURE OF ASSOCIATIVE PAIRS

Note. For simplicity, for the remainder of this paper we assume throughout that $(\mathcal{A}, \mathcal{A}_0)$ is an admissible $\mathcal{T}$-semiring pair. The interaction between $\mathcal{A}$, $\mathcal{A}_0$, and $\mathcal{T}$ is crucial.

Much of the classic structure theory works for pairs since we can avoid negation by using the twist product.

Lemma 3.1. The twist product is associative.

Proof.

$$\begin{aligned}((a_1, a_1') \cdot_{\mathrm{tw}} (a_2, a_2')) \cdot_{\mathrm{tw}} (a_3, a_3') &= (a_1a_2 + a_1'a_2', a_1a_2' + a_1'a_2) \cdot_{\mathrm{tw}} (a_3, a_3') \\ &= (a_1a_2 + a_1'a_2')a_3 + (a_1a_2' + a_1'a_2)a_3', (a_1a_2 + a_1'a_2')a_3' + (a_1a_2' + a_1'a_2)a_3)) \\ &= a_1a_2a_3 + a_1'a_2'a_3' + a_1a_2'a_3' + a_1'a_2a_3', a_1a_2a_3' + a_1'a_2'a_3' + a_1a_2'a_3 + a_1'a_2a_3 \\ &= (a_1, a_1') \cdot_{\mathrm{tw}} ((a_2, a_2') \cdot_{\mathrm{tw}} (a_3, a_3'))\end{aligned} \tag{3.1}$$

by left-right symmetry. □

Thus we can write twist products without parentheses. The following notions have analogs in [27, § 3].

Definition 3.2.

(i) A congruence Φ of $(\mathcal{A}, \mathcal{A}_0)$ is **semiprime** if we have $\Phi_1 = \Phi$ for any congruence $\Phi_1 \supseteq \Phi$ such that $\Phi_1 \cdot_{tw} \Phi_1 \subseteq \Phi$. $(\mathcal{A}, \mathcal{A}_0)$ is a **semiprime pair** if the trivial congruence is semiprime.

In line with the standard terminology, when $\mathcal{A}$ is commutative, we write **radical congruence** instead of "semiprime congruence."

(ii) A congruence Φ of $(\mathcal{A}, \mathcal{A}_0)$ is **prime** if we have $\Phi_1 = \Phi$ or $\Phi_2 = \Phi$ for any congruences $\Phi_1, \Phi_2 \supseteq \Phi$ such that $\Phi_1 \cdot_{tw} \Phi_2 \subseteq \Phi$. $(\mathcal{A}, \mathcal{A}_0)$ is a **prime pair** if the trivial congruence is prime.

(iii) A congruence Φ of $(\mathcal{A}, \mathcal{A}_0)$ is **irreducible** if we have $\Phi_1 = \Phi$ or $\Phi_2 = \Phi$ for any congruences $\Phi_1, \Phi_2 \supseteq \Phi$ such that $\Phi_1 \cap \Phi_2 = \Phi$. $(\mathcal{A}, \mathcal{A}_0)$ is an **irreducible pair** if the trivial congruence is irreducible.

Lemma 3.3. A congruence Φ is semiprime iff $(b_1, b_1') \cdot_{\mathrm{tw}} (\mathcal{A} \times \mathcal{A}) \cdot_{\mathrm{tw}} (b_1, b_1') \subseteq \Phi$ implies $(b_1, b_1') \in \Phi$.

A congruence Φ is prime iff $(b_1, b_1') \cdot_{\mathrm{tw}} (\mathcal{A} \times \mathcal{A}) \cdot_{\mathrm{tw}} (b_2, b_2') \subseteq \Phi$ implies $(b_1, b_1') \in \Phi$ or $(b_2, b_2') \in \Phi$.

Proof. Easy consequences of Lemma 3.1, where we take Φ_i to be the congruence generated by Φ and (b_i, b_i'). $\square$

The same proofs as in Jun and Rowen [27, Proposition 3.14] can be used to prove the following, for a $\mathcal{T}$-pair $(\mathcal{A}, \mathcal{A}_0)$.

- The intersection of semiprime congruences is semiprime.
- The union of a chain of congruences is a congruence.
- A congruence is prime if and only if it is semiprime and irreducible.
- (essentially [27, Proposition 3.17]) For $\mathcal{A}$ commutative, define $\sqrt{\Phi}$ to be the set of elements (a_1, a_2) of $\mathcal{A} \times \mathcal{A}$ having a twist-power $(a_1, a_2)^{\cdot_{\mathrm{tw}} m}$ (for some $m \in \mathbb{N}$) in Φ. Then $\sqrt{\Phi}$ is a radical congruence and is the intersection of a nonempty set of prime congruences.

Proposition 3.4. [essentially [27, Proposition 3.17]] Suppose S is a multiplicative subset of $\mathcal{T}$, with $\mathbb{0} \notin S$. Then there is a prime congruence disjoint from $\hat{S} := S \times \mathbb{0} \cup \mathbb{0} \times S$.

Proof. $\hat{S}$ is a $\cdot_{\mathrm{tw}}$-multiplicative subset of $\hat{\mathcal{T}}$, disjoint from the diagonal congruence, so by Zorn's lemma there is a congruence Φ of $\mathcal{A}$ maximal with respect to being disjoint from $\hat{S}$. We claim that Φ is prime. Indeed, if $\Phi_1 \cdot_{\mathrm{tw}} \Phi_2 \subseteq \Phi$ for congruences $\Phi_1, \Phi_2 \supseteq \Phi$, then Φ_1, Φ_2 contain elements of $\hat{S}$, as does $\Phi_1 \cdot_{\mathrm{tw}} \Phi_2$, a contradiction. $\square$

Proposition 3.4 is enough to show for $\mathcal{A}$ commutative that every radical congruence is the intersection of a nonempty set of prime congruences. We can generalize this result by using a famous trick of Levitzki.

Theorem 3.5. A congruence Φ on a pair is semiprime if and only if it is the intersection of prime congruences.

Proof. $(\Leftarrow)$ is obvious. Conversely, given $(s_1, s_1') \notin \Phi$ we need to find a prime congruence containing Φ, but not containing (s_1, s_1'). Inductively we build a subset $S \subset \mathcal{A} \times \mathcal{A}$ as follows:

We start with $(s_1, s_1') \in S$. Given $(s_i, s_i') \in S$, from Lemma 3.3, there exists (a_i, a_i') such that

$$(s_{i+1}, s_{i+1}') := (s_i, s_i') \cdot_{\mathrm{tw}} (a_i, a_i') \cdot_{\mathrm{tw}} (s_i, s_i') \notin \Phi.$$

Take a congruence Φ' maximal with respect to containing Φ, but not intersecting $S = \{(s_i, s'_i) : i \in \mathbb{N}\}$. We claim that Φ' is prime. In fact, if $\Phi_1 \cdot_{\mathrm{tw}} \Phi_2 \subseteq \Phi'$ for congruences $\Phi_1, \Phi_2 \supset \Phi'$ then Φ_1, Φ_2 contain respective elements (s_i, s'_i) and (s_j, s'_j) of S from the maximality of Φ'. If $i < j$ then $\Phi_1 \cdot_{\mathrm{tw}} \Phi_2$ contains (s_{j+1}, s'_{j+1}), so Φ' contains (s_{j+1}, s'_{j+1}), a contradiction. □

Definition 3.6. Define the **Krull dimension** of $(\mathcal{A}, \mathcal{A}_0)$ to be the maximal length of a chain of prime congruences of $\mathcal{A}$ containing Diag$\mathcal{A}$.

We note that the Krull dimension in the context of congruences was first introduced and studied by Joo and Mincheva in [24].

Definition 3.7. $\mathcal{A}$ satisfies the **ACC (resp. DCC) on congruences** if every ascending (resp. descending) chain of congruences stabilizes.

We studied ACC on congruences in [27, Proposition 3.15], but it is rarer than in the classical theory, as evidenced by the following examples.

Example 3.8. $\mathbb{N}_{\max}$ does not satisfy ACC on congruences, since we can take Φ_i to be generated by $(1, 2), \dots, (1, i)$.

$\mathbb{Z}_{\max}$ satisfies the ACC on congruences but not DCC (even though it is a semifield), since we put $\Phi_i = \{(m, m + i) : m \in \mathbb{Z}\}$ (under classical addition).

$\mathbb{Z}_{\max}[\lambda]$ fails ACC since now we put Φ_i to be generated by $(\lambda, \lambda + 1), \dots, (\lambda^i, \lambda^i + 1)$. The fact that $\mathbb{B}[x]$ does not satisfy ACC was first proved in [4].

Definition 3.9. Let $(\mathcal{A}, \mathcal{A}_0)$ be a $\mathcal{T}$-semiring pair.

(i) The **$\preceq$-center** $Z((\mathcal{A}, \mathcal{A}_0))$ of $(\mathcal{A}, \mathcal{A}_0)$ is $\{z \in \mathcal{A} : yz \preceq zy, \ \forall y \in \mathcal{A}\}$.

(ii) $(\mathcal{A}, \mathcal{A}_0)$ is **commutative** if $\mathcal{A} = Z((\mathcal{A}, \mathcal{A}_0))$.

Note that $(\mathcal{A}, \mathcal{A}_0)$ commutative implies $y_1 y_2 \preceq y_2 y_1$ for all $y_1, y_2 \in \mathcal{A}$. This leads to the observation:

Lemma 3.10. Suppose $\preceq = \preceq_0$,[3] and $(\mathcal{A}, \mathcal{A}_0)$ has the property that $y + w + z = y$ for $y \in \mathcal{A}$ and $w, z \in \mathcal{A}_0$ implies $y + w = y$. Then $(\mathcal{A}, \mathcal{A}_0)$ is commutative if and only if $\mathcal{A}$ is commutative.

Proof. ($\Rightarrow$) If $\mathcal{A}$ is commutative, then clearly $(\mathcal{A}, \mathcal{A}_0)$ is commutative.

($\Leftarrow$) Take any $y_1, y_2 \in \mathcal{A}$. By hypothesis $y_1 y_2 \preceq_0 y_2 y_1$, and $y_2 y_1 \preceq_0 y_1 y_2$, so there exist $z, w \in \mathcal{A}_0$ such that

$$y_1 y_2 + z = y_2 y_1, \quad y_2 y_1 + w = y_1 y_2.$$

It follows that $y_1 y_2 = (y_1 y_2 + z) + w = y_2 y_1$, implying $y_1 y_2 = (y_1 y_2 + z) = y_2 y_1$ by hypothesis, proving that $\mathcal{A}$ is commutative. □

[3] Recall that $\preceq_0$ is a surpassing relation defined as follows: $x \preceq_0 y$ if and only if there exists $z \in \mathcal{A}_0$ such that $x = y + z$.

3.1. Module pairs

We fix a ground $\mathcal{T}$-semiring pair $(\mathcal{A}, \mathcal{A}_0)$.

Definition 3.11.

1. A **module pair** $\mathcal{M} := (\mathcal{M}, \mathcal{N})$ over $(\mathcal{A}, \mathcal{A}_0)$ is a pair of $\mathcal{A}$-modules together with a given map $\phi_{\mathcal{M},\mathcal{N}} : \mathcal{N} \to \mathcal{M}$ such that
$$\mathcal{A}_0\mathcal{M} \subseteq \phi_{\mathcal{M},\mathcal{N}}(\mathcal{N}). \tag{3.2}$$
2. A module pair $(\mathcal{M}, \mathcal{N})$ is $\mathcal{T}_\mathcal{M}$**-admissible** if there is a $\mathcal{T}$-action on $\mathcal{T}_\mathcal{M} \subseteq \mathcal{M}$, and $\mathcal{T}_\mathcal{M} \cup \{\mathbb{0}_M\}$ spans $(\mathcal{M}, +)$ with $\mathcal{T}_\mathcal{M} \cap \mathcal{N} = \emptyset$.

Intuitively we view the module pair $(\mathcal{M}, \mathcal{N})$ as $\mathcal{M}/\mathcal{N}$.

Definition 3.12. A **surpassing relation** $\preceq$ on a $\mathcal{T}_\mathcal{M}$-admissible module pair $(\mathcal{M}, \mathcal{N})$ is defined in analogy to Definition 2.4.

Given subsets $S_1, S_2 \subseteq \mathcal{M}$, we write $S_1 \preceq S_2$ if for each $s_2 \in S_2$ there is $s_1 \in S_1$ for which $s_1 \preceq s_2$.

Definition 3.13.

(i) A **homomorphism** of module pairs $F : (\mathcal{M}', \mathcal{N}') \to (\mathcal{M}, \mathcal{N})$ is a pair of module homomorphisms $f : \mathcal{M}' \to \mathcal{M}$ and $g : \mathcal{N}' \to \mathcal{N}$, such that $f(\phi_{\mathcal{M}',\mathcal{N}'}) = \phi_{\mathcal{M},\mathcal{N}}g$.

(ii) A homomorphism F is **monic** if $f^{-1}(\phi_{\mathcal{M},\mathcal{N}}(\mathcal{N})) = \phi_{\mathcal{M}',\mathcal{N}'}(\mathcal{N}')$. In this case we say that $(\mathcal{M}', \mathcal{N}')$ is a **submodule pair** of the module pair $(\mathcal{M}, \mathcal{N})$.

Example 3.14. Here are examples of some of the notions.

(i) For $\mathcal{A}$ commutative and any $a \in \mathcal{A}$ the congruence $\Phi_a(\mathcal{A})$ generated by a is $\{(a_1a, a_2a) : a \in \mathcal{A}\}$. In the image of $\mathcal{A}$ under this congruence, a is identified with $\mathbb{0}a = \mathbb{0}$, so we have the natural homomorphism sending $a \mapsto \mathbb{0}$.

(ii) For any module pair $(\mathcal{M}, \mathcal{N})$, $(a\mathcal{M}, a\mathcal{N})$ is a submodule pair, for any $a \in \mathcal{T}$.

(iii) Suppose $\mathcal{M}$ is an $\mathcal{A}$-module, and $a \in Z(\mathcal{A})$. The congruence kernel of the left multiplication homomorphism $\mathcal{M} \to a\mathcal{M}$ is $\mathcal{K} := \{(y_1, y_2) \in \mathcal{M} \times \mathcal{M} : ay_1 = ay_2\}$. $\mathcal{K}$ is an $\mathcal{A}$-module via the diagonal map, and we define $\phi_{\mathcal{K},\mathcal{M}} : K \to \mathcal{M}$ by $(y_1, y_2) \mapsto ay_1$. In this way we can identify $(\mathcal{M}, \mathcal{N})$ with $a\mathcal{M}$, and $\mathcal{K}$ is the congruence kernel. (If need be, we add on $\mathcal{A}_0\mathcal{M} \times \mathcal{A}_0\mathcal{M}$.)

(iv) For a semiring pair $(\mathcal{A}, \mathcal{A}_0)$ and an index set I, $(\mathcal{A}^{(I)}, \mathcal{A}_0^{(I)})$ is a module pair, where $\phi_{\mathcal{A}_0^{(I)},\mathcal{A}^{(I)}}$ is defined by applying $\phi_{\mathcal{A}_0,\mathcal{A}}$ componentwise, and the e_i are the usual vectors with $\mathbb{1}$ in the i position, comprising a **base** $\{e_i : i \in I\}$ of $\mathcal{A}^{(I)}$. It is uniquely quasi-negated if $(\mathcal{A}, \mathcal{A}_0)$ is uniquely quasi-negated, seen componentwise.

(v) For $\mathcal{M}$ an $\mathcal{A}$-module, take $\mathcal{N}$ to be $\mathcal{A}_0\mathcal{M}$.

(vi) The following example from [8, Definitions 5.4, 5.6] played the main role in [8]. Suppose $(\mathcal{A}, \mathcal{A}_0)$ is a $\mathcal{T}$-semiring pair with a surpassing map $\preceq$, and $(\mathcal{M}, \mathcal{N})$ is a module over $(\mathcal{A}, \mathcal{A}_0)$. A **symmetric bilinear form** $B : \mathcal{M} \times \mathcal{N} \to \mathcal{A}$ is a symmetric map satisfying

$$B(a_1 v_1 + a_2 v_2, w) \succeq a_1 B(v_1, w) + a_2 B(v_2, w).$$

A **quadratic pair** (q, B) is a map $q : \mathcal{M} \to \mathcal{A}$ and a symmetric bilinear form $B : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$, satisfying

$$q(av) \succeq a^2 q(v), \qquad 2q(v) = B(v, v), \qquad q(v + w) \succeq q(v) + q(w) + B(v, w).$$

The pair $(\mathcal{A}, \mathcal{A}_0)$ is **Clifford** if there is a quadratic pair (q, B) on $\mathcal{A}$ for which $v_1^2 \succeq q(v_1)$ and $v_1 v_2 + v_2 v_1 \succeq B(v_1, v_2)$ for all $v_i \in \mathcal{A}$.

Remark 3.15. If Φ is a congruence on a module $\mathcal{M}$ then there is a 1:1 set-theoretic map $\Psi : \mathcal{M}/\Phi \to \Phi$, given by taking a set-theoretic retraction $\psi : \mathcal{M}/\Phi \to \mathcal{M}$ and sending $[y] \mapsto (y, y)$. Ψ will not be onto, but still is useful in estimating growth.

3.2. Bases

Definition 3.16. Let $(\mathcal{M}, \mathcal{N})$ be a module pair over $(\mathcal{A}, \mathcal{A}_0)$.

(i) A set $S \subseteq \mathcal{M}$ **($\preceq$)-spans** $\mathcal{M}$ if there are $a_i \in \mathcal{A}$ and $s_i \in S$ such that $\sum_i a_i s_i \preceq v$ for each $v \in \mathcal{M}$.

(ii) A set $S \subseteq \mathcal{M}$ is **$\preceq$-independent** if $\sum a_i s_i \preceq \sum a_i' s_i$ in $\mathcal{M}$ for $a_i, a_i' \in \mathcal{A}$ implies each $a_i \preceq a_i'$ in $\mathcal{A}$.

(iii) A **($\preceq$)-base** of $(\mathcal{M}, \mathcal{N})$ is an $\preceq$-independent set which ($\preceq$)-spans $\mathcal{M}$ over $\mathcal{N}$. A module pair with a ($\preceq$)-base is called ($\preceq$)**-free**.

An obvious example: the unit vectors $\{e_1, \dots, e_n\}$ are a base for $\mathcal{A}^{(n)}$ over $\mathcal{A}_0^{(n)}$. More generally, $(\mathcal{A}^I, \mathcal{A}_0^I)$ is free.

Remark 3.17. The universal algebraic definition of free module pair, which shall call **universally free**, is that there is a set $\{b_i : i \in I\}$ such that for every module pair $(\mathcal{M}', \mathcal{N}')$ and $\{y_i : i \in I\} \subseteq \mathcal{M}'$ there is a unique morphism $\varphi : (\mathcal{M}, \mathcal{N}) \to (\mathcal{M}', \mathcal{N}')$ sending $b_i \mapsto y_i$.

Proposition 3.18. Any universally free module pair $(\mathcal{M}, \mathcal{N})$ (over a set I) is isomorphic to $(\mathcal{A}^I, \mathcal{A}_0^I)$.

Proof. For a module pair $(\mathcal{M}', \mathcal{N}')$, define the module homomorphism φ to the free module pair $(\mathcal{M}, \mathcal{N})$ sending $\sum_i a_i e_i \to \sum_i a_i b_i$. Clearly $\mathcal{N} \to \mathcal{N}'$. The set $\{b_i : i \in I\}$ is seen to be a ($\preceq$)-base, by mapping it onto $\mathcal{A}^{(I)}$ by sending $\varphi : b_i \mapsto e_i$.

Note further that $\varphi^{-1}(\mathcal{A}_0^{(I)}) = \mathcal{N}$. Indeed, if $\sum a_i \varphi(b_i) = \varphi \sum (a_i b_i) \in \mathcal{A}_0^{(I)}$ then each $a_i \in \mathcal{A}_0$, by independence. It follows that $(\mathcal{M}, \mathcal{N}) \cong (\mathcal{A}^I, \mathcal{A}_0^I)$. ☐

This proves that the universally free module pair is ($\preceq$)-free, and unique up to the cardinality of its ($\preceq$)-base.

Remark 3.19. Given a submodule $(\mathcal{M}',\mathcal{N}')$ of $(\mathcal{M},\mathcal{N})$ and $F:(\mathcal{M}',\mathcal{N}')\to(\mathcal{M},\mathcal{N})$ with $f:\mathcal{M}'\to\mathcal{M}$ and $g:\mathcal{N}'\to\mathcal{N}$, we can define $(\mathcal{M},\mathcal{N})/(\mathcal{M}',\mathcal{N}')$ to be the pair $(\mathcal{M},\mathcal{M}')$ where $\phi_{\mathcal{M},\mathcal{N}}:\mathcal{M}'\to\mathcal{M}$ is the map f. This is a useful tool in defining "exact sequences", without congruences, namely

$$(\mathcal{K},\mathcal{K})\to(\mathcal{N},\mathcal{K})\to(\mathcal{M},\mathcal{K})\to(\mathcal{M},\mathcal{N})\to(\mathcal{M},\mathcal{M}),$$

which hints at homology, but we shall not pursue this intriguing direction in this paper.

3.3. Extensions of pairs

We assume from now on that $(\mathcal{A},\mathcal{A}_0)$ is commutative.

Definition 3.20.

(i) An **extension** of a $\mathcal{T}$-semiring pair $(\mathcal{A},\mathcal{A}_0)$ is a $\mathcal{T}_{\mathcal{W}}$-semiring pair $(\mathcal{W},\mathcal{W}_0)$ where $\mathcal{T}\subseteq\mathcal{T}_{\mathcal{W}}$. and $\mathcal{W}_0=\mathcal{A}_0\mathcal{W}$. The extension $(\mathcal{W},\mathcal{W}_0)$ is **centralizing** if $aw=wa$ for all $a\in\mathcal{T},\ w\in W$.

Note that $\mathcal{W}$ is generated by $\mathcal{A}$ and a subset $S\subset\mathcal{T}_{\mathcal{W}}$. The extension is **finitely generated** if S can be taken to be finite. Furthermore, since any nonzero element of $\mathcal{W}$ is a finite sum of elements of $\mathcal{T}_{\mathcal{W}}$, we will assume that $S\subset\mathcal{T}_{\mathcal{W}}$.

(ii) An extension of a $\mathcal{T}$-semiring pair spanned by a finite number of elements is called a **finite extension**.

(iii) A centralizing extension $(\mathcal{W},\mathcal{W}_0)$ of a $\mathcal{T}$-semifield pair $(\mathcal{A},\mathcal{A}_0)$ is called **affine** if there is a finitely generated centralizing extension $(\mathcal{W}',\mathcal{W}_0)$ of $(\mathcal{A},\mathcal{A}_0)$ with $\mathcal{W}'\preceq\mathcal{W}$.

In the natural examples, one would expect $\mathcal{A}_0=\mathcal{A}\cap\mathcal{W}_0$, but we do not see how to guarantee this.

Proposition 3.21. If a semialgebra extension $(\mathcal{W},\mathcal{W}_0)$ is a $(\preceq)$-free module over $(\mathcal{A},\mathcal{A}_0)$ with $(\preceq)$-base B and H is a sub-semialgebra of $\mathcal{A}$ over which B still $(\preceq)$-spans $\mathcal{W}$, then $\mathcal{A}\succeq H$, cf. Definition 2.4.

Proof. For any $c\in\mathcal{A}$ write $cb_1\succeq\sum h_ib_i$. Then by definition of $(\preceq)$-base, $c\succeq h_1\in\mathcal{A}$. $\square$

3.4. Function pairs and polynomials

Some of this material is reminiscent of [27], which handled the commutative situation.

Definition 3.22. Given a set S, the **support** of a function $f:S\to\mathcal{A}$ is $\{s\in S: f(s)\neq\mathbb{0}\}$. We define $\mathcal{A}^S_{<\infty}$ to be the set of functions from S to $\mathcal{A}$ with finite support. If $\mathcal{A}$ is a $\mathcal{T}$-module, we view $\mathcal{A}^S_{<\infty}$ as a module over $\mathcal{T}_S$, defined as the set of functions $f:S\to\mathcal{A}$ whose support is a singleton $\{s\}$ with $f(s)\in\mathcal{T}\}$.

When $\mathcal{A}$ is a bimagma, we define the **convolution product** as follows:

$$f * g(s) = \sum_{uv=s} f(u)g(v) \text{ for } f, g \in \mathcal{A}^S_{<\infty}. \tag{3.3}$$

For a set of indeterminates $\Lambda = \{\lambda_i : i \in I\}$, the **polynomial magma** $\mathcal{A}[\Lambda]$ is $(\mathcal{A}^S_{<\infty})$ where $S = \mathbb{N}^I$, identifying (m_i) with $\prod \lambda_i^{m_i}$. We write $f(\lambda_1, \dots \lambda_m)$ as a typical polynomial.

Given $\{b_i : i \in I\} \subseteq \mathcal{A}$, and $\Lambda = \{\lambda_i : i \in I\}$, there is a unique homomorphism $\mathcal{A}[\Lambda] \to \mathcal{A}$ sending $\lambda_i \mapsto b_i$. We write the image of $f(\lambda_1, \dots \lambda_m)$ as $f(\mathbb{b})$, where $\mathbb{b} = \{b_1, \dots, b_m)$.

We define the **convolution product** as follows:

$$f * g(s) = \sum_{uv=s} f(u)g(v) \text{ for } f, g \in \mathcal{A}^S_{<\infty}. \tag{3.4}$$

For a set of indeterminates $\Lambda = \{\lambda_i : i \in I\}$, given $\{b_i : i \in I\} \subseteq \mathcal{A}$, and $\Lambda = \{\lambda_i : i \in I\}$, there is a unique homomorphism $\mathcal{A}[\Lambda] \to \mathcal{A}$ sending $\lambda_i \mapsto b_i$.

Proposition 3.23. If $(\mathcal{A}, \mathcal{A}_0)$ is a semiprime $\mathcal{T}$-pair then $(\mathcal{A}^S_{<\infty}, \mathcal{A}^S_{0\ <\infty})$ is a semiprime $\mathcal{T}_S$-pair.

Proof. One checks it pointwise as in [27, Proposition 4.2]. □

On the other hand, the prime analog proved in [27, Theorem 4.6] required a Vandermonde argument that relies on commutativity. One needs that two functions agreeing on "enough" points are the same.

Remark 3.24. As an illustration of Proposition 3.23, given an admissible $\mathcal{T}$-pair $(\mathcal{A}, \mathcal{A}_0)$, we have the $\mathcal{T}_\Lambda$**-polynomial pair** $(\mathcal{A}[\Lambda], \mathcal{A}_0[\Lambda])$, where $\mathcal{T}_\Lambda$ denotes the monomials with coefficients in $\mathcal{T}$.

This pair is not shallow, since $\lambda + \mathbb{1}^\circ \notin \mathcal{A}_0[\Lambda] \cup \mathcal{T}_\Lambda$. One could try to remedy this by taking $\mathcal{T}[\Lambda]$ instead of $\mathcal{T}_\Lambda$, but $\mathcal{T}[\Lambda]$ is never a monoid when $(\mathcal{A}, \mathcal{A}_0)$ satisfies Property N. For if $a + a' \in \mathcal{A}_0$ for $a, a' \in \mathcal{T}$, then $(\lambda + a)(\lambda + a') = \lambda^2 + (a + a')\lambda + aa'$.

On the other hand we could have obtained a shallow pair satisfying Property N by taking $\mathcal{A}[\Lambda]_0$ to be $\mathcal{A}_0[\Lambda] \cup \{$polynomials which are a sum of at least two monomials$\}$.

One can also define polynomials symbolically, but there are many examples in tropical mathematics of differing polynomials which agree as functions, such as $\lambda^2 + a\lambda + 4$ for all $a < 2$. This will impact on our discussion of algebraicity.

4. PAIRED VERSIONS OF CLASSICAL THEOREMS

We generalize some results from [27].

Theorem 4.1. (Artin–Tate for pairs) Suppose $\mathcal{W}$ is an affine semialgebra over $\mathcal{A}$ and has a finite $(\preceq)$-base B over a central semialgebra $\mathcal{K} \subset \mathcal{W}$. Then $\mathcal{K}$ is affine over $\mathcal{A}$.

P r o o f. As in [27, Theorem 4.26], write $\mathcal{A}[y_1, \dots, y_n] \preceq \mathcal{W}$ and

$$\sum \alpha_{ijk} b_k \preceq b_i b_j, \qquad \sum \alpha_{uk} b_k \preceq y_u, \quad 1 \leq u \leq n.$$

Let H be the sub-semialgebra generated by all the α_{ijk} and α_{uk}. Then B also ($\preceq$)-spans each y_i over H, and thus $\sum\{Hb_i : b_i \in B\}$ is a subalgebra over which B ($\preceq$)-spans $\mathcal{W}$, and thus by Proposition 3.21 $H \preceq \mathcal{K}$, so $\mathcal{K}$ is affine by definition. □

4.1. Fractions

Fractions have already been studied for monoids, cf. [31] for instance, and here we present the paired version, taking the analog from [34, §3.1].

Definition 4.2. An element $s \in \mathcal{T}$ is **left regular** if $b_1 s = b_2 s$ for $b_i \in \mathcal{A}$ implies $b_1 = b_2$,. A left regular element $s \in \mathcal{T}$ is **left $\preceq$-regular** if $b_1 s \preceq b_2 s$ for $b_i \in \mathcal{A}$ implies $b_1 \preceq b_2$. **Right $\preceq$-regular** is analogous, and **$\preceq$-regular** means left and right $\preceq$-regular.

Definition 4.3. A $\mathcal{T}$ satisfies the **(left) Ore condition** with respect to a subset S of $\preceq$-regular elements of $\mathcal{T}$ if:

- For any $b \in \mathcal{A}$ and $s \in S$ there are $b' \in \mathcal{A}$ and $s' \in S$ such that $s'b = b's$.
- If $b_1 s = b_2 s$ then there is $s' \in S$ with $s'b_1 = s'b_2$.

The main tool concerns monoids.

Lemma 4.4. Given a $\mathcal{T}$-semiring $\mathcal{A}$ with $\mathcal{T}$ satisfying the Ore condition with respect to S, one can define an equivalence relation on $\mathcal{A} \times S$, by $(b_1, s_1) \equiv (b_2, s_2)$ iff there are $a_1, a_2 \in \mathcal{T}$ for which $a_1 b_1 = a_2 b_2$ and $a_1 s_1 = a_2 s_2 \in S$.

P r o o f. Reflexivity and symmetry are clear. For transitivity we need a sublema:

> If $(s_1, b_1) \sim (s_2, b_2)$ and $c_1 s_1 = c_2 s_2 \in S$ for $c_i \in \mathcal{T}$, then there is $a \in \mathcal{A}$ such that $ac_1 s_1 = ac_2 s_2 \in S$ and $ac_1 b_1 = ac_2 b_2$.
>
> P r o o f of sublemma: Take $a_i \in \mathcal{T}$ with $a_1 b_1 = a_2 b_2$ and $a_1 s_1 = a_2 s_2 \in S$. Take $s \in S$, $y \in \mathcal{A}$, such that $ya_1 s_1 = sc_1 s_1$. The Ore condition gives $s_1' \in S$ with $s_1' y a_1 = s_1' s c_1 \in \mathcal{T}$. Then
>
> $$s_1' y c_2 s_2 = s_1' y c_1 s_1 = s_1' y a_1 s_1 = s_1' y a_2 s_2.$$
>
> Hence there is s_2' for which $s_2' s_1' y c_2 = s_2' s_1' y a_2$, and we take $a = s_2' s_1' y$ and check that $ac_1 s_1 = ac_2 s_2 \in S$ and $ac_1 b_1 = ac_2 b_2$ as desired.

Transitivity now follows easily as in [34, p. 350]. □

When $\mathcal{T}$ satisfies the (left) Ore condition with respect to S, take the equivalence of the lemma, and write $S^{-1}\mathcal{A}$ for $\{s^{-1}b := [(b, s)] : b \in \mathcal{A}, s \in S\}$, $S^{-1}\mathcal{A}_0$ for $\{s^{-1}b := [(b, s)] : b \in \mathcal{A}_0, s \in S\}$, and $S^{-1}\mathcal{T}$ for $\{s^{-1}a := [(a, s)] : a \in \mathcal{T}, s \in S\}$.

Theorem 4.5. With the above notation,

(i) Any two elements $s_1^{-1}b_1$ and $s_2^{-1}b_2$ can be written with a common denominator s where we have $s = s's_1 = b's_2 \in S$ for suitable $s' \in S$ and $b' \in \mathcal{A}$.

(ii) $S^{-1}\mathcal{A}$ has well-defined addition given by $s^{-1}b_1 + s^{-1}b_2 = s^{-1}(b_1 + b_2)$ and multiplication $s^{-1}b_1 s^{-1}b_2 = s's^{-1}b'b_2$ where $b's_1 = s'b_1$ from Definition 4.3.

(iii) $(S^{-1}\mathcal{A}, S^{-1}\mathcal{A}_0)$ is an $S^{-1}\mathcal{T}$-pair. When $\mathcal{T}$ is multiplicatively cancellative, the $\mathcal{T}^{-1}\mathcal{T}$-pair $(\mathcal{T}^{-1}\mathcal{A}, \mathcal{T}^{-1}\mathcal{A}_0)$ is a $\mathcal{T}^{-1}\mathcal{T}$-semifield pair.

Proof. (i) $s^{-1}(s'b_1) \equiv s_1^{-1}b_1$ and $s^{-1}b'b_2 \equiv s_2^{-1}b_2$.

(ii) is as in [34, p. 351], using the same argument for well-definedness of multiplication using the trick used in the proof of (i) to avoid the use of negation. (The action is well-defined since S is regular.)

(iii) Straightforward, by the definition of $\preceq$-regularity. □

We call $(\mathcal{T}^{-1}\mathcal{A}, \mathcal{T}^{-1}\mathcal{A}_0)$ of (iv) the **pair of fractions of the pair** $(\mathcal{A}, \mathcal{A}_0)$. In all of our applications, the $\preceq$-regular set S will be central, and thus automatically Ore.

4.2. Integral extensions

Definition 4.6.

(i) Suppose $(\mathcal{W}, \mathcal{W}_0)$ is a centralizing extension of a commutative $\mathcal{T}$-semiring pair $(\mathcal{A}, \mathcal{A}_0)$. An element $y \in \mathcal{W}$ is **integral over** $(\mathcal{A}, \mathcal{A}_0)$, if there are $a_0, \dots, a_{n-1} \in \mathcal{A}$ such that $\sum_{i=0}^{n-1} a_i y^i \preceq y^n$. The minimal such n is called the **degree** of y. An element $y \in \mathcal{W}$ is $\mathcal{T}$**-integral** if each $a_i \in \mathcal{T}$.

(ii) $(\mathcal{W}, \mathcal{W}_0)$ is an **integral extension** of $(\mathcal{A}, \mathcal{A}_0)$ if each element of $\mathcal{W}$ is integral over $(\mathcal{A}, \mathcal{A}_0)$.

Remark 4.7. It follows at once that if y is integral over $(\mathcal{A}, \mathcal{A}_0)$ then $(\mathcal{A}[y], \mathcal{A}_0[y])$ is a finite centralizing extension of $(\mathcal{A}, \mathcal{A}_0)$, spanned by $\mathbb{1}, y, \dots, y^{n-1}$.

We need the following property, to provide the converse of Remark 4.7.

Definition 4.8.

(i) $(\mathcal{A}, \mathcal{A}_0)$ satisfies **reversibility** for a if $b + a \succeq \mathbb{0}$ implies $b \succeq a$, $\forall b \in \mathcal{A}$. In this case, we say that a is **reversible**.

(ii) We say that a is **power-reversible** if a^n is reversible for each n.

(iii) $(\mathcal{A}, \mathcal{A}_0)$ satisfies **tangible reversibility** if it satisfies reversibility for each $a \in \mathcal{T}$.

Definition 4.9. Suppose that $(\mathcal{A}, \mathcal{A}_0)$ has a negation map $(-)$.

(i) $(\mathcal{A}, \mathcal{A}_0)$ satisfies $(-)$**reversibility** for a if $b(-)a \succeq \mathbb{0}$ implies $b \succeq a$, $\forall b \in \mathcal{A}$. In this case, we shall say that a is $(-)$**-reversible**.

(ii) We say that a is $(-)$**-power-reversible** if a^n is $(-)$-reversible for each n.

(iii) $(\mathcal{A}, \mathcal{A}_0)$ satisfies **tangible** $(-)$**-reversibility** if it satisfies $(-)$-reversibility for each $a \in \mathcal{T}$.

Lemma 4.10. Tangible reversibility holds in the following settings: supertropical system, hypersystem, and symmetricized system.

Tangible $(-)$-reversibility holds in the following settings: classical system, supertropical system, and hypersystem.

Proof. Suppose $b(-)a \succeq \mathbb{0}$.

The classical case is obvious, since then $b - a = 0$ implies $a = b$.

For the supertropical, $(-) = +$; if $b \neq a$ and $b \in \mathcal{A}^\circ$ then $b + a = b$.

For a hypersystem built on a hypergroup H, $a \in H$ and $a' - a = 0$ for some element $a' \in b$, so $a = a' \in b$.

For a symmetricized system, let $b = (b_1, b_2)$ and $a = (a, 0)$. Then $b + a = (b_2, b_1) + (a, 0) = (b_2 + a, b_1)$, so $b_1 = b_2 + a$ and $(a, 0) + (b_1, b_2) = (b_1, b_2)$. □

Clearly tangible $(-)$-reversibility implies power $(-)$-reversibility for each $a \in \mathcal{T}$. In this case we can define the **negated determinant** $\det(A)$ of a matrix $A = (c_{ij})$ to be the following:

$$\det(A) := \sum_{\pi \in S_n} ((-)\mathbb{1})^{\operatorname{sgn}(\pi)} c_{1,\pi(1)} c_{2,\pi(2)} \cdots c_{n,\pi(n)}. \tag{4.1}$$

Proposition 4.11. Suppose $(\mathcal{A}[y], \mathcal{A}_0[y])$ is a finite centralizing extension of a commutative pair $(\mathcal{A}, \mathcal{A}_0)$, and is reversible or $(-)$-reversible. Then y is integral over $(\mathcal{A}, \mathcal{A}_0)$.

Proof. Write $\mathcal{A}[y] \succeq \sum_{i=0}^n \mathcal{A} y_i$. Then, for each $1 \le i \le n$, $y y_i \succeq \sum c_{ij} y_i$ for suitable $c_{ij} \in \mathcal{A}$, implying the matrix

$$A = \begin{pmatrix} c_{11}(-)y & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22}(-)y & \dots & c_{2n} \\ & & \ddots & \\ c_{n1} & c_{n2} & \dots & c_{nn}(-)y \end{pmatrix}$$

satisfies $Av \succeq \mathbb{0}$, for v the column vector $(y_1, \dots y_n)$, and letting $\operatorname{adj}(A)$ denote the (negated) adjoint matrix as in [36, Definition 8.2], we have $\det(A)v = \operatorname{adj}(A)Av \succeq \mathbb{0}$. Hence $\det(A) y_i \succeq \mathbb{0}$ for each i, and thus

$$\det(A)\mathcal{A}[y] \succeq \sum_i \det(A)\mathcal{A} y_i \succeq \mathbb{0},$$

implying $\det(A) \succeq \mathbb{0}$. But opening $\det(A)$ and reversing y^n if necessary gives an integral relation for y over $(\mathcal{A}, \mathcal{A}_0)$. □

Definition 4.12. Suppose $(\mathcal{W}, \mathcal{W}_0)$ is an extension of $(\mathcal{A}, \mathcal{A}_0)$. An element $y \in \mathcal{W}$ is **algebraic over** $(\mathcal{A}, \mathcal{A}_0)$ if there are $a_i \in \mathcal{A}$ such that $\sum_{i=0}^{n} a_i y^i \in \mathcal{W}_0$. The minimal such n is called the **degree** of y. The a_i are called the **coefficients** of y.

Unfortunately, an algebraic element over a $\mathcal{T}$-semifield pair need not be $\mathcal{T}$-integral. Here is a special case where $\mathcal{T}$-integrality holds.

Proposition 4.13.

(i) Every power-reversible or $(-)$-power-reversible algebraic element over a $\mathcal{T}$-semifield pair, with leading coefficient in $\mathcal{T}$, is $\mathcal{T}$-integral.

(ii) If $s = \sum b_i a^i \in \mathcal{T}$ and the pair $(\mathcal{A}, \mathcal{A}_0)$ is shallow, then we can take all the coefficients b_i to be in $\mathcal{T}_{\mathbb{0}}$.

Proof. (i) This is clear as we can divide through by the leading coefficient, and then apply reversibility to the leading term.

(ii) Write $s = s_1 + s_2$, where $s_1 = \sum b_i a^i$ with $b_i \in \mathcal{T}$ and $s_2 = \sum b_i' a^i$ with $b_i' \in \mathcal{A}_0$. Then $s \succeq s_1$, implying $s = s_1$ since s is tangible, cf. Lemma 2.6. $\square$

4.3. Hilbert Nullstellensatz

Definition 4.14. [Another version of algebraicity]

(i) A polynomial f is **tangible** if $f = \sum a_i \lambda^i$, with $a_i \in \mathcal{T}$.

(ii) $y \in \mathcal{W}$ is **transcendental** if for any polynomials f_1, f_2, if $f_1(y) \succeq f_2(y)$ then $f_1(b) \succeq f_2(b)$ for all $b \in \mathcal{A}$; $y \in \mathcal{W}$ is **congruence algebraic** over the pair $(\mathcal{A}, \mathcal{A}_0)$ if y is not transcendental.

Remark 4.15. If $(\mathcal{A}, \mathcal{A}_0)$ is shallow and $\preceq$ -nondegenerate, then every tangible polynomial takes on a tangible value. Indeed for all $a \in \mathcal{T}$, if $f(a) \notin \mathcal{T}$ then $f(a) \in \mathcal{A}_0$, so $f \in \mathcal{A}_0[\Lambda]$.

This can be coupled with Lemma 2.6.

Proposition 4.16. Suppose a shallow semiring pair $(\mathcal{A}, \mathcal{A}_0)$ is $\preceq_0$ -nondegenerate, and $(\mathcal{W}, \mathcal{W}_0)$ is a centralizing extension of $(\mathcal{A}, \mathcal{A}_0)$, with $y \in \mathcal{T}_{\mathcal{W}}$ such that $(\mathcal{A}[y], \mathcal{A}_0[y])$ is tangibly separating. Let $\mathcal{T}' = \{ay^i : a \in \mathcal{T}, i \in \mathbb{N}\}$. Let $(\mathcal{K}, \mathcal{K}_0)$ be the $\mathcal{T}'$-semifield of fractions of $(\mathcal{A}[y], \mathcal{A}_0[y])$. If $(\mathcal{K}, \mathcal{K}_0)$ is affine, then y is congruence algebraic over the pair $(\mathcal{A}, \mathcal{A}_0)$.

Proof. Write $(\mathcal{K}, \mathcal{K}_0) \succeq (\mathcal{A}[a_1, \dots, a_m], \mathcal{K}_0)$, with each $a_i \in \mathcal{T}_{\mathcal{W}}$. Write each $a_i = s^{-1} f_i(y)$, $1 \le i \le t$, where $s \in \mathcal{T}_{\mathcal{W}}$, cf. Theorem 4.5(2). Note that $\deg g > 1$, or else $\mathcal{K} = \mathcal{A}[y]$ and we are done. Write $s = g(y)$, where by Proposition 4.13 we may assume that $g \in \mathcal{A}[\lambda]$ has coefficients in $\mathcal{T}$. By $\preceq$-nondegeneracy we have $g(a)$ tangible for some $a \in \mathcal{T}'$. Since $(\mathcal{A}[y], \mathcal{A}_0[y])$ is tangibly separating, we have $a' \in \mathcal{T}'$ such that $y + a' \in \mathcal{T}'$ and $a + a' \in \mathcal{K}_0$. Hence $(y + a')^{-1} \in \mathcal{K} = {\mathcal{T}'}^{-1}\mathcal{A}[y]$ so $(y + a')^{-1} \succeq f(y)$ for some

$f \in \mathcal{A}[\lambda]$. Then $(y+a')^{-1} \succeq s^{-k}f(y) = g(y)^{-k}f(y)$, so $g(y)^k \succeq (y+a')f(y)$. If y were transcendental then $g(a)^k \succeq (a+a')f(a)$, i.e., $g(a)^k \in \mathcal{A}_0$, contrary to $g(a)$ tangible. Hence y is congruence algebraic. □

Theorem 4.16 could be viewed as a "baby Nullstellensatz," and we would like the conclusion that y is integral over the pair $(\mathcal{A}, \mathcal{A}_0)$. But there are counterexamples.

Example 4.17. (i) Take any pair $(\mathcal{A}, \mathcal{A}_0)$ and adjoin an additively absorbing element ∞ to $\mathcal{T}$, in the sense that $\infty^n + a = \infty^n$ for all $a \in \mathcal{T}$ and all n. Then for any tangibly monic polynomial f of degree n, $f(\infty) = \infty^n$, implying the $\mathcal{T}$-semifield pair of fractions of $(\mathcal{A}[\infty], \mathcal{A}_0[\infty])$ is affine (generated by ∞^{-1}).

(ii) Call $\mathcal{A}$ **weakly bipotent** if $a, a' \in \mathcal{T}$ implies $a + a' \in \{a, a'\}$ or $a^2 = (a')^2$. (This is implied by $(-)$-bipotence in systems, cf. [36], and often is the case in tropical mathematics.) Then, taking $a' = a^2$, either $a^2 + a = a^2$ or $a^2 + a = a$ or $a^2 = a^4$. But we get the reverse for a^{-1} instead of a. Thus for any such $\mathcal{T}$ not satisfying the identity $x^2 = x^4$, any invertible $a \in \mathcal{T}$ is $\mathcal{T}$-congruence algebraic, but not necessarily $\mathcal{T}$-integral.

These examples weaken the next result.

Theorem 4.18. Suppose $(\mathcal{W}, \mathcal{W}_0)$ is a commutative affine $\mathcal{T}$-semifield pair over a shallow, $\preceq$-nondegenerate semifield pair $(\mathcal{A}, \mathcal{A}_0)$, and also having the property that $\mathcal{T}$-congruence algebraic implies $\mathcal{T}$-integral. Then $(\mathcal{W}, \mathcal{W}_0)$ is $\mathcal{T}$-integral.

Proof. This can be easily proved by induction on n, where $\mathcal{W} = \mathcal{A}[y_1, \dots, y_n]$.

Write $\mathcal{W} \preceq \mathcal{A}[y_1]([y_2, \dots, y_n])$, and let $(\mathcal{K}, \mathcal{K}_0)$ be the $\mathcal{T}$-semifield of fractions of the pair $(\mathcal{A}[y_1], \mathcal{A}[y_1] \cap \mathcal{W}_0)$. By Theorem 4.1, $(\mathcal{K}, \mathcal{K}_0)$ is affine. Hence, by Theorem 4.16, $\mathcal{K} \preceq \mathcal{A}[y_1]$, and is finite over $(\mathcal{A}, \mathcal{A}_0)$. But by induction, $(\mathcal{W}, \mathcal{W}_0)$ is integral over $(\mathcal{K}, \mathcal{K}_0)$, and thus over $(\mathcal{A}, \mathcal{A}_0)$. □

Unfortunately, from the standpoint of tropical mathematics, the property that $\mathcal{T}$-algebraic implies $\mathcal{T}$-integral is not compatible with the tropical viewpoint, by Example 4.17. But it is all we have.

5. GROWTH IN SEMIALGEBRAS

Growth in algebraic structures has been an active area of study in the last 30 years, cf. [30], mostly for groups and algebras, although recently growth in semigroups and other algebraic structures has been investigated in [6, 18, 39]. Some of the basic properties carry over to semialgebras, as we review here.

Definition 5.1. Let f be a function from the set of algebraic pairs to natural numbers. We say that f is **sub-additive** (resp. **sub-multiplicative**) if the following holds: for any pairs of objects (A, B) and (B, C),

$$f(A, C) \leq f(A, B) + f(B, C) \quad (\text{resp. } f(A, C) \leq f(A, B)f(B, C));$$

when equality holds we say that f is **additive** (resp. **multiplicative**).

5.1. Growth in a pair

Let $[\mathcal{M} : \mathcal{N}]$ denote the minimum number of elements need to generate $\mathcal{M}$ over $\mathcal{N}$, called the **rank**. The rank is sub-multiplicative: If $\mathcal{N} \subseteq \mathcal{N}' \subseteq \mathcal{M}$, then $[\mathcal{M} : \mathcal{N}] \leq [\mathcal{M} : \mathcal{N}'][\mathcal{N}' : \mathcal{N}]$. In situations where equality holds (such as in the classical situation for modules over Artinian rings), one can develop dimension theories.

In this subsection we assume that $(\mathcal{W}, \mathcal{W}_0)$ is an affine centralizing extension over a $\mathcal{T}$-semifield pair $(\mathcal{A}, \mathcal{A}_0)$. In the non-relative case we take $\mathcal{W}_0 = 0$.

Now, we introduce a notion of the growth rate. We take a generating set $\{a_1, \dots, a_{t'}\}$ of $\mathcal{W}_0$ over $\mathcal{A}$ (which is empty when $\mathcal{W}_0 = 0$), which we extend to a generating set $\{a_1, \dots, a_t\}$ of $\mathcal{W}$ over $\mathcal{A}$; we put $V' = \sum_{i=1}^{t'} \mathcal{A}a_i$ and $V = \sum_{i=1}^{t} \mathcal{A}a_i$. Note that $V = V'$ when $\mathcal{W}_0 = 0$. We have the filtration $\mathcal{W}_k = \sum_{i=1}^{k} V^k$ (which is just V^k if $1 \in V$) of $\mathcal{W}$, $1 \leq k < \infty$, and $\mathcal{W}_{0k} = \sum_{i=1}^{k} {V'}^k$. We define the following numbers:

$$d_k := [\mathcal{W}_k : \mathcal{W}_{k-1} + \mathcal{W}_{0k}]. \tag{5.1}$$

Definition 5.2. The **growth rate** of $\mathcal{W}$ (with respect to $\mathcal{W}_0$) is the sequence $\{d_1, d_2, \dots\}$, where d_k is as defined in (5.1).

If we start with a different set of generators then we may get a different growth rate $\{d'_1, d'_2, \dots\}$, but they are **equivalent** in the sense that there are numbers m_1, m_2 such that $d'_k \leq m_1 d_{m_2 k}$ and $d_k \leq m_2 d'_{m_1 k}$ for all k. Since $\mathcal{T}_{\mathbb{0}}$ $(\preceq)$-spans $\mathcal{W}$ we may take the generating set in $\mathcal{T}_{\mathbb{0}}$.

Remark 5.3. The basic definitions do not involve subtraction, and thus many classical results go over directly to semialgebras, and then to semialgebra pairs, almost word for word. So we shall quote proofs of standard results.

Example 5.4. The basic examples are analogous to the ones found in any standard algebra text, such as [35, Chapter 17].

1. The **free semialgebra** over a commutative semiring is the monoid semialgebra over the word monoid $X = \{x_1, \dots, x_t\}$ in t letters. The words in X are a base, so d_k is the number of words of length k, which is t^k. Obviously this exponential growth rate is the largest possible growth rate for a semialgebra.

2. When $\mathcal{A}$ is finitely spanned over $\mathcal{A}_0$, the d_k are bounded. If $\mathcal{A}$ is not finitely spanned over $\mathcal{A}_0$, then $d_k \geq k$.

3. The **free commutative semialgebra** is the polynomial semialgebra $\mathcal{A}[\lambda_1, \dots \lambda_t]$. When $1 \in \mathcal{A}$, $d_k = \binom{k+t}{t} \leq k^t$, which has polynomial growth.

Here is a semialgebraic analog of a theorem of Jategaonkar (that any domain of subexponential growth is Ore). We say that a $\mathcal{T}$-pair $(\mathcal{A}, \mathcal{A}_0)$ is a **$\mathcal{T}$-semidomain pair** if every element of $\mathcal{T}$ is $\preceq$-regular.

Proposition 5.5. Any $\mathcal{T}$-semidomain pair $(\mathcal{A}, \mathcal{A}_0)$ with subexponential growth has the property that for any $a_1, a_2 \in \mathcal{T}$ there are $b_1, b_2 \in \mathcal{A} \setminus \mathcal{A}_0$ such that $b_1 a_1 + b_2 a_2 \in \mathcal{A}_0$.

Proof. Suppose $a_1, a_2 \in \mathcal{T}$. Then $f(a_1, a_2) \in \mathcal{A}_0$ for some polynomial $f = g\lambda_1 + h\lambda_2 \in \mathcal{T}[\lambda_1, \lambda_2]$. Take such f of minimal total degree. Then $\deg g, \deg h < \deg f$, so by hypothesis $g(a_1, a_2), h(a_1, a_2) \notin \mathcal{A}_0$, and thus $g(a_1, a_2)a_1, h(a_1, a_2)a_2 \notin \mathcal{A}_0$. Thus we can take $b_1 = g(a_1, a_2)$ and $b_2 = h(a_1, a_2)$. □

The conclusion of the proposition could be interpreted as saying that $(\mathcal{A}, \mathcal{A}_0)$ satisfies the left Ore condition with respect to $\mathcal{T}$.

Definition 5.6. We define the **Hilbert Series** of $(\mathcal{W}, \mathcal{W}_0)$ to be $\sum_{k=1}^{\infty} d_k \lambda^k \in \mathbb{N}[\![\lambda]\!]$.[4]

This is completely analogous to the standard algebraic situation, and can be viewed in terms of the **graded pair** $\oplus(\mathcal{W}_k, \mathcal{A}_{k-1} + \mathcal{A}_{0k})$. Thus growth, Gelfand-Kirillov dimension and Hilbert Series of semialgebras are closely related to semigroups, and Example 2.3 is relevant. Shneerson [39], Smoktunowicz [38], and Greenfeld [18] have interesting semigroup examples of varied growth.

Question 5.7. Suppose $\mathcal{A}$ is commutative and $\mathbb{N}$-graded. Is the Hilbert series of a finitely spanned $\mathcal{A}$-module $\mathcal{M}$ (over $\mathcal{A}_0$) rational?

Lemma 5.8. When $\mathcal{A}_0 = 0$, the Hilbert Series of the polynomial semialgebra $\mathcal{A}[\lambda_1, \dots, \lambda_t]$ is $\frac{1}{(1-\lambda)^t}$.

Proof. Just as in the classical case, cf. [35, Example 17.39]. □

Remark 5.9. The referee has suggested that the Hilbert series of a tensor product is the product of the Hilbert series.

Define the **Gelfand–Kirillov** dimension GK-dim$(\mathcal{A}, \mathcal{A}_0)$ of $(\mathcal{A}, \mathcal{A}_0)$ to be

$$\limsup \log_k [\mathcal{A}_k : \mathcal{A}_{0k}].$$

Lemma 5.10.
1. GK-dim$(M_n(\mathcal{A}), \mathcal{A}) = 0$. More generally, GK-dim$(\mathcal{A}, \mathcal{A}_0) = 0$ whenever $\mathcal{A}$ is a finite extension of $\mathcal{A}_0$.
2. GK-dim$(\mathcal{A}[\lambda_1, \dots, \lambda_n], \mathcal{A}[\lambda_1, \dots, \lambda_n]_0) =$ GK-dim$(\mathcal{A}, \mathcal{A}_0) + n$, for any n.

Proof. (1) The dimensions are bounded. (2) The same proof as for the classical case, cf. [35, Example 17.47], for example. □

Question 5.11. For $(\mathcal{A}, \mathcal{A}_0)$ commutative, is GK-dim$(\mathcal{A}, \mathcal{A}_0)$ always an integer, and does it equal the Krull dimension?

[4] See [12] for an alternative definition.

6. APPENDIX A: ROOTS OF A POLYNOMIAL

Since algebra often serves as a tool for geometry, we use this appendix to lay out the geometric concepts arising from pairs.

Definition 6.1. A $\mathcal{A}_0$**-root** of a polynomial $f(\lambda_1, \dots, \lambda_m)$ is a tuple $(b_1, \dots, b_m) \in \mathcal{A}^{(m)}$ such that $f(b_1, \dots, b_m) \in \mathcal{A}_0$.

This definition is natural, consistent with [21, 22]. Then for $Z \subset \mathcal{A}^{(n)}$ one would take $\Lambda = \{\lambda_1, \dots, \lambda_n\}$ and $\mathcal{I}(Z) = \{f \in \mathcal{A}[\Lambda] : f(z) \in \mathcal{A}_0, \ \forall z \in Z\}$.

The difficulty in tying this in to algebra is that $\mathcal{I}(Z)$ is an ideal of $\mathcal{A}[\Lambda]$, not a congruence.

Accordingly we take instead points of $\mathcal{A}^{(n)} \times \mathcal{A}^{(n)}$ and for $(z_1, z_2) \in \mathcal{A}^{(n)} \times \mathcal{A}^{(n)}$ we define the **twist substitution**

$$(f_1, f_2)_{\mathrm{tw}}(z_1, z_2) = (f_1(z_1) + f_2(z_2), (f_1(z_2) + f_2(z_1)).$$

Lemma 6.2. $((f_1, f_2) \cdot_{\mathrm{tw}} (f_3, f_4))_{\mathrm{tw}}(z_1, z_2) = (f_1, f_2)_{\mathrm{tw}}((f_3, f_4)_{\mathrm{tw}}(z_1, z_2))$

P r o o f. By distributivity we may assume that our polynomials are monomials $f_i = \Lambda^{k_i}$ for $1 \le i \le 4$. Then

$$\begin{aligned} ((f_1, f_2) \cdot_{\mathrm{tw}} (f_3, f_4))_{\mathrm{tw}}(z_1, z_2) &= (\Lambda^{k_1k_3+k_2k_4}, (\Lambda^{k_1k_4+k_2k_3})(z_1, z_2) \\ &= z_1^{k_1k_3+k_2k_4} + z_2^{k_1k_4+k_2k_3}, z_1^{k_1k_4+k_2k_3} + z_2^{k_1k_3+k_2k_4} \\ &= (\Lambda^{k_1}, \Lambda^{k_2})_{\mathrm{tw}}(z_1^{k_3} + z_2^{k_4}, z_1^{k_4} + z_2^{k_3}) \\ &(f_1, f_2)_{\mathrm{tw}}((f_3, f_4)_{\mathrm{tw}}(z_1, z_2)). \end{aligned} \tag{6.1}$$

□

Definition 6.3. The **congruence** $\widehat{\mathcal{I}}(S)$ of $S \subseteq \mathcal{A}^{(n)} \times \mathcal{A}^{(n)}$ then is

$$\{(f_1, f_2) \in \mathcal{A}[\Lambda] \times \mathcal{A}[\Lambda] : (f_1, f_2)(z_1, z_2) \in \mathcal{A}_0 \times \mathcal{A}_0, \forall (z_1, z_2) \in S\}.$$

The congruence of a point is called a **geometric congruence**.

Lemma 6.2 implies that $\widehat{\mathcal{I}}(S)$ is a radical congruence, clearly the intersection of a nonempty set of geometric congruences.

For example for a system with unique negation, if $S = \{(a, \mathbb{0}, \dots, \mathbb{0}), (\mathbb{0}))\}$ for $a \in \mathcal{T}$, then one has $\widehat{\mathcal{I}}(S) = (\lambda_1(-)a + \mathcal{A}_0[\Lambda], \mathcal{A}_0[\Lambda])$, a prime congruence.

This opens the door to the Zariski topology on pairs, but pursuing this path is out of the scope of this paper.

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Jaiung Jun, Department of Mathematics, State University of New York at New Paltz, New Paltz, NY 12561. U. S. A.
e-mail: jujun0915@gmail.com

Kalina Mincheva, Department of Mathematics, Tulane University, New Orleans, LA 70118. U. S. A.
e-mail: kmincheva@tulane.edu

Louis Rowen, Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900. Israel.
e-mail: rowen@math.biu.ac.il