KYBERNETIKA — VOLUME *58* (2022), NUMBER 5, PAGES 708–732

INTERVAL MULTI-LINEAR SYSTEMS FOR TENSORS IN THE MAX-PLUS ALGEBRA AND THEIR APPLICATION IN SOLVING THE JOB SHOP PROBLEM

Sedighe Khaleghzade, Mostafa Zangiabadi, Aljoša Peperko and Masoud Hajarian

In this paper, we propose the notions of the max-plus algebra of the interval tensors, which can be used for the extension of interval linear systems to interval multi-linear systems in the max-plus algebra. Some properties and basic results of interval multi-linear systems in max-plus algebra are derived. An algorithm is developed for computing a solution of the multi-linear systems in the max-plus algebra. Necessary and sufficient conditions for the interval multi-linear systems for weak solvability over max-plus algebra are obtained as well. Also, some examples are given for illustrating the obtained results. Moreover, we briefly sketch how our results can be used in the max-plus algebraic system theory for synchronized discrete event systems.

Keywords: interval tensor, max-plus algebra, multi-linear systems, weak solvability, job shop problem

Classification: 15A80, 15A06, 65G30, 15A69

1. INTRODUCTION

In the last twenty years, there has been a tremendous interest and activity in tensors, which are multi-arrays with at least $m \geq 3$ indices. Since tensors do not represent linear operators, as matrices do, the theory of tensors is more delicate than the theory of matrices. Recently, there have been many new developments in the study of tensors and their associated multi-linear system problems. In particular, results coming from the interval matrix setting when studying the max-plus algebra have shown especially attractive.

Max-plus algebra provides mathematical theory and techniques for solving nonlinear problems that can be given the form of linear problems, when arithmetical addition is replaced by the operation of maximum and arithmetical multiplication is replaced by addition. Max-plus algebra plays an important role in modeling and analysis of various types of discrete event systems, including railway networks, job shop problem, flexible manufacturing systems, intelligent transportation systems, traffic control systems and etc. [2, 4, 9, 12, 14, 19–21]. A typical application of discrete events systems are job shop problems, where an event is scheduled to meet a deadline.

DOI: 10.14736/kyb-2022-5-0708

During the last three decades the role of compact intervals as independent objects has continuously increased in numerical analysis when verifying or enclosing solutions of various mathematical problems or when proving that such problems cannot have a solution in a particular given domain. This was possible by viewing intervals as extensions of real or complex numbers, by introducing interval matrices and interval tensors. Interval tensors have been treasured for solving multi-linear systems of equations [3], and in this paper also extended to max-plus systems.

This paper is organized as follows. In Section 2, we recall some preliminary definitions and results. In addition, we give an algorithm for computing a solution of the multi-linear systems in the max-plus algebra. Necessary and sufficient conditions for weak solvability of the interval multi-linear system, together with some examples, will be discussed in Section 3. Finally, in Section 4 we give the application of max-plus algebra in the synchronized discrete event systems, more precisely in the job shop problem.

2. PRELIMINARIES AND SOME BASIC RESULTS

The following notations are used in the sequel. Vectors are written as italic lowercase letters such as $x, y, \ldots$, matrices correspond to italic capitals such as $A, B, \ldots$ and tensors are written as calligraphic capitals such as $\mathcal{A}, \mathcal{B}, \ldots$. For each nonnegative integer n, denote $[n] = \{1, \ldots, n\}$ and $\mathbb{R}$ is the set of real numbers.

The max-plus algebra $\mathbb{R}_{\max}$ is the set $\mathbb{R} \cup \{\epsilon\}$, equipped with two operations, addition ($\oplus$) and multiplication ($\otimes$), where $\epsilon = -\infty$, $x \oplus y = \max\{x, y\}$ and $x \otimes y = x + y$ for every $x, y \in \mathbb{R}_{\max}$. The algebraic structure of $\mathbb{R}_{\max}$ is an idempotent semifield, i. e., an idempotent commutative semiring where every element $x \in \mathbb{R}_{\max}$, with $x \neq \epsilon$ has an inverse under the $\otimes$ operation, denoted $-x$ (see, for instance, [4, 6, 7, 15]).

A tensor can be regarded as a higher order generalization of a matrix, which takes the form $\mathcal{A} = (a_{i_1 \cdots i_m})$, $a_{i_1 \cdots i_m} \in \mathbb{R}$. We denote the elements of an m-order tensor $\mathcal{A} \in \mathbb{R}^{n_1 \times \cdots \times n_m}$ by $a_{i_1 i_2 \cdots i_m}$ where $1 \leq i_j \leq n_j$, $1 \leq j \leq m$. If $n_1 = \cdots = n_m = n$, then it is said $\mathcal{A}$ is an m-order n-dimensional cubical tensor or for simplicity just m-order n-dimensional tensor. Obviously, a vector is a tensor of order 1 and a matrix is a tensor of order 2. The set of all tensors of size $n_1 \times n_2 \times \ldots \times n_m$ over the max-plus algebra is denoted by $\mathbb{R}_{\max}^{n_1 \times \cdots \times n_m}$. The set of all tensors of order m and dimension n over the max-plus algebra is denoted by $\mathbb{R}_{\max}^{[m,n]}$. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_{\max}^{[m,n]}$. We define its jth row tensor $\mathcal{A}_j = (a_{j i_2 \cdots i_m})$ as a tensor in $\mathbb{R}_{\max}^{[m-1,n]}$ for $j \in [n]$.

In [1], Afshin et al. extended the basic max-plus algebraic operations to tensors.

Definition 2.1. The max-plus algebra addition ($\oplus$) and multiplication ($\otimes$) are defined as follows [1, 8]:

(i) Suppose that $\mathcal{A}, \mathcal{B} \in \mathbb{R}_{\max}^{n_1 \times \cdots \times n_m}$, we have $\mathcal{A} \oplus \mathcal{B} \in \mathbb{R}_{\max}^{n_1 \times \cdots \times n_m}$ and

$$(\mathcal{A} \oplus \mathcal{B})_{i_1 \cdots i_m} = a_{i_1 \cdots i_m} \oplus b_{i_1 \cdots i_m} = \max\{a_{i_1 \cdots i_m}, b_{i_1 \cdots i_m}\}.$$

(ii) Suppose that $\mathcal{A} \in \mathbb{R}_{\max}^{n_1 \times \cdots \times n_m}$ and $x \in \mathbb{R}^{\max_{i \in \{2, \cdots, m\}} \{n_i\}}$ we have

$$(\mathcal{A} \otimes x)_i = \max_{i_2 \in [n_2], \ldots, i_m \in [n_m]} \{a_{i i_2 \cdots i_m} + x_{i_2} + \cdots + x_{i_m}\}.$$

Interval linear algebra is a mathematical field developed from classical linear algebra. The only difference is that we do not work with real numbers but with real closed intervals $x^I = [\underline{x}, \overline{x}]$, where for any $x \in x^I$ we have $\underline{x} \leq x \leq \overline{x}$ and the relation $\leq$ is always understood coordinatewise. Sometimes in applications, we do not know some parameters precisely, that is why we rather use intervals of possible values. When the components of a tensor possess interval uncertainty, we have an interval tensor. Applied problems in which there is a minimal information about the nature of the tensor coefficient uncertainty, the tensor is interval. In [3], Bozorgmanesh et al. introduced interval tensors. An interval tensor is a tensor where every element is an interval. An m-order n-dimensional cubical interval tensor is denoted by $\mathcal{A}^I = [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$, where $\underline{\mathcal{A}}$ and $\overline{\mathcal{A}}$ are real m-order n-dimensional tensors such that for any $\mathcal{A} \in \mathcal{A}^I$ we have $\underline{\mathcal{A}} \leq \mathcal{A} \leq \overline{\mathcal{A}}$. In other words, $\mathcal{A}^I$ is a tensor with coefficients formed by real closed intervals. Notice that, for $m = 2$, $\mathcal{A}^I$ is an interval matrix (see, for instance, [5, 11, 16–18]).

The interval tensor operations in max-plus algebra are defined formally in the same manner (with respect to $\oplus, \otimes$) as tensor operations in the multi-linear algebra. Let us first define a system of interval multi-linear equations or, as we abbreviate it, an interval multi-linear system. It is a set of all multi-linear systems that is defined by an interval tensor and an interval vector.

Definition 2.2. Let $\mathcal{A}^I = [\underline{\mathcal{A}}, \overline{\mathcal{A}}]$ be an interval tensor such that $\underline{\mathcal{A}}, \overline{\mathcal{A}} \in \mathbb{R}_{\max}^{[m,n]}$ and $b^I = [\underline{b}, \overline{b}]$ such that $\underline{b}, \overline{b} \in \mathbb{R}^n_{\max}$. For $x \in \mathbb{R}^n$ the notation

$$\mathcal{A}^I \otimes x = b^I, \tag{1}$$

represents max-plus interval multi-linear systems.

Interval multi-linear system (1) in max-plus algebra is the family of all multi-linear systems of the form

$$\mathcal{A} \otimes x = b, \tag{2}$$

where $\mathcal{A} \in \mathcal{A}^I$ and $b \in b^I$. Each system of the form (2) is said to be a subsystem of system (1). We say that interval multi-linear system has a constant tensor if $\underline{\mathcal{A}} = \overline{\mathcal{A}}$ and has a constant right-hand side, if $\underline{b} = \overline{b}$.

Remark 2.3. If $m = 2$ (namely $A^I = [\underline{A}, \overline{A}]$ is an interval matrix of order n), then

$$A^I \otimes x = b^I, \tag{3}$$

represents an interval system of linear max-plus equations (see, for instance, [5, 10, 16–18]). Therefore multi-linear system (1) is a generalization of linear system (3) in max-plus algebra.

For the study of multi-linear system (2), we have the following definitions.

Definition 2.4. For a multi-linear system $\mathcal{A} \otimes x \leq b$, we call $\bar{x}^* = (\bar{x}_1^*, \cdots, \bar{x}_n^*)^T$ a solution of $\mathcal{A} \otimes x \leq b$, if for any $i \in [n]$ and all $i_2, \ldots, i_m \in [n]$

$$\sum_{j=2}^{m} \bar{x}^*_{i_j} \leq b_i - a_{ii_2 \cdots i_m}. \tag{4}$$

Definition 2.5. Let $\mathcal{A} \in \mathbb{R}_{\max}^{[m,n]}$ and $b \in \mathbb{R}_{\max}^n$. The solution set of the multi-linear system (2) is defined as follows:

$$S(\mathcal{A}, b) = \{x \in \mathbb{R}^n \mid \mathcal{A} \otimes x = b\}.$$

Remark 2.6. Let $\bar{x}^*$ be a solution of $\mathcal{A} \otimes x \leq b$. Then by Definition 2.5, $\bar{x}^*$ is a solution of $\mathcal{A} \otimes x = b$ if and only if for any $k \in [n]$ there exist $i_2^{(k)}, \ldots, i_m^{(k)} \in [n]$ such that

$$\sum_{j=2}^{m} \bar{x}^*_{i_j^{(k)}} = b_k - a_{ki_2^{(k)} \ldots i_m^{(k)}}. \tag{5}$$

Throughout the paper we shall use the notation $x^*(\mathcal{A}, b)$ to denote a solution of the multi-linear system $\mathcal{A} \otimes x = b$ and use the notation $\bar{x}^*(\mathcal{A}, b)$ to denote a solution of the multi-linear system $\mathcal{A} \otimes x \leq b$. For notational convenience, unless otherwise stated, $x^*(\mathcal{A}, b)$ will be denoted by x^* and $\bar{x}^*(\mathcal{A}, b)$ will be denoted by $\bar{x}^*$. For $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}_{\max}^{[m,n]}$, let $r_k(\mathcal{A}) = \max_{\substack{i_2,\cdots,i_m \\ i_2 \neq \cdots \neq i_m}} a_{ki_2 \cdots i_m}$, $k \in [n]$ ($i_2 \neq \cdots \neq i_m$ means all situations excluding the situations where $i_2 = \cdots = i_m$). If in the kth row of $\mathcal{A}$ the maximum in definition of $r_k(A)$ is attained more than once, then we denote $r_{k_j}(\mathcal{A}) = a_{ki_2^{(j)} \cdots i_m^{(j)}}$, where $a_{ki_2^{(j)} \cdots i_m^{(j)}}$ is the jth entry in which the maximum in definition of $r_k(A)$ is attained.

Based on the above discussion, we give the following algorithm for solving multi-linear system $\mathcal{A} \otimes x = b$ (for more details on the following algorithm refer to Appendix A).

Algorithm 2.7. (For solving the multi-linear systems $\mathcal{A} \otimes x = b$)

Input: The tensor $\mathcal{A} \in \mathbb{R}_{\max}^{n_1 \times \cdots \times n_m}$ and $b \in \mathbb{R}_{\max}^{n_1}$.
set $j = 1$
for $k = 1, \ldots, n_1$ **do**
  set $r_{k_j}(\mathcal{A}) := a_{ki_2^{(j)} \ldots i_m^{(j)}}$
  set $\beta_j := b_k - r_{k_j}(\mathcal{A})$
  while there exist some entries $a_{ki_2 \cdots i_m}$ in kth row of $\mathcal{A}$ such that $a_{ki_2 \ldots i_m} = r_{k_j}(\mathcal{A})$ **do**
    set $j := j + 1$
    set $r_{k_j}(\mathcal{A}) := a_{ki_2 \ldots i_m}$
    set $\beta_j := b_k - r_{k_j}(\mathcal{A})$
  end while
    set $j := j + 1$
end for
set $n_{\min} = \min_{i=2,\ldots,m} \{n_i\}$ and $n_{\max} = \max_{i=2,\ldots,m} \{n_i\}$
for $k = 1, \ldots, n_1$ **do**
  for $i = 1, \ldots, n_{\min}$ **do**
    set $\alpha_i := \min_k \frac{b_k - a_{ki \cdots i}}{m-1}$
  end for
end for
Compute the following system of linear inequalities:

$$\begin{cases} \sum\limits_{t=2}^{m} x_{i_t^{(1)}} \leq \beta_1 \\ \vdots \\ \sum\limits_{t=2}^{m} x_{i_t^{(j)}} \leq \beta_j \\ x_1 \leq \alpha_1 \\ \vdots \\ x_{n_{\min}} \leq \alpha_{n_{\min}} \end{cases} \tag{6}$$

20: By solving the inequality system (6) we have $x_i \leq \gamma_i$ where $\gamma_i \in \mathbb{R}$, $i \in [n_{\max}]$
21: **for** $k = 1, \dots, n_1$
22: **if** there exists $a_{ki_2^{(k)}\dots i_m^{(k)}}$ of $\mathcal{A}$ such that $\sum_{t=2}^{m} \gamma_{i_t^{(k)}} = b_k - a_{ki_2^{(k)}\dots i_m^{(k)}}$ *and*
for any $i_2, \dots, i_m$, we have $\sum_{t=2}^{m} \gamma_{i_t} \leq b_k - a_{ki_2 \dots i_m}$,
do $x_j := \gamma_j$ where $j \in \max\{i_2, \dots, i_m\}$
23: **end if**
24: **else** the multi-linear system $\mathcal{A} \otimes x = b$ is not solvable and break
25: **end else**
26: **end for**
27: **if** $x = (x_1, \dots, x_{n_{\max}})^T = (\gamma_1, \dots, \gamma_{n_{\max}})^T$ **then**
28: $x^* = (\gamma_1, \dots, \gamma_{n_{\max}})^T$ is a solution of the multi-linear system $\mathcal{A} \otimes x = b$
29: **end if**
30: **else**
31: the multi-linear system $\mathcal{A} \otimes x = b$ is not solvable.
32: **end else**

Remark 2.8. In order to get the upper bound of solution of inequality system (6), we prepare the Algorithm 4.1 (see, Appendix B). Also we can use the Multi-Stage ABS algorithm [13].

Theorem 2.9. The computational complexity of Algorithm 2.7 is $O(n_1 \times n_2 \times \cdots \times n_m)$.

Proof. For estimation of the computational complexity realize that Algorithm 2.7 computes in Line 4, the maximum of all entries in n_1 rows of $\mathcal{A}$ and requires $n_1 \times n_2 \times \cdots \times n_m - (n_1 \times n_{\min})$ operations. The maximum number of operations in Line 6 is $n_1 \times n_2 \times \cdots \times n_m - n_1(n_{\min} + 1)$ and in Line 22 is $n_1 \times n_2 \times \cdots \times n_m$. Then the overall complexity is $O(n_1 \times n_2 \times \cdots \times n_m)$. □

In the following example we illustrate how the previous algorithm works (the implementation of Algorithm 2.7 is discussed in Appendix A).

Example 2.10. Let the entries of $\mathcal{A} \in \mathbb{R}_{\max}^{3\times 2\times 3}$ and $b \in \mathbb{R}_{\max}^{3}$ be defined as follows:

$$\begin{array}{lllllll} a_{111}=2 & a_{121}=1 & a_{112}=1 & a_{122}=-3 & a_{113}=-1 & a_{123}=3 & b_1=4,\\ a_{211}=-1 & a_{221}=0 & a_{212}=\epsilon & a_{222}=2 & a_{213}=0 & a_{223}=0 & b_2=6,\\ a_{311}=-2 & a_{321}=-1 & a_{312}=0 & a_{322}=1 & a_{313}=\epsilon & a_{323}=2 & b_3=5. \end{array}$$

Consider the multi-linear system $\mathcal{A} \otimes x = b$. Due to Lines 4 and 6 of Algorithm 2.7, $a_{123}, a_{221}, a_{213}, a_{223}$, and a_{323} are the maximum of the rows 1, 2 and 3, respectively. Due to Line 16 of the algorithm, $\alpha_1 = 1$, $\alpha_2 = 2$. Continuing the algorithm, we obtain the following system of inequalities:

$$\begin{cases} x_2 + x_3 & \leq 1 = \beta_1 \\ x_2 + x_1 & \leq 6 = \beta_2 \\ x_2 + x_3 & \leq 6 = \beta_3 \\ x_1 + x_3 & \leq 6 = \beta_4 \\ x_3 + x_2 & \leq 3 = \beta_5 \\ x_1 & \leq 1 = \alpha_1 \\ x_2 & \leq 2 = \alpha_2. \end{cases}$$

By Algorithm 4.1 (or the Multi-Stage ABS algorithm [13]), we obtain $x_1 \leq \gamma_1 = 1, x_2 \leq \gamma_2 = 2$ and $x_3 \leq \gamma_3 = -1$. Applying Line 22 of Algorithm 2.7, we have

$$\begin{array}{llll} \gamma_2 + \gamma_3 = b_1 - a_{123} & \text{and} & \sum_{t=2}^{3} \gamma_{i_t} \leq b_1 - a_{1i_2i_3} & \forall i_2, i_3, \\ \gamma_2 + \gamma_2 = b_2 - a_{222} & \text{and} & \sum_{t=2}^{3} \gamma_{i_t} \leq b_2 - a_{2i_2i_3} & \forall i_2, i_3, \\ \gamma_2 + \gamma_2 = b_3 - a_{322} & \text{and} & \sum_{t=2}^{3} \gamma_{i_t} \leq b_3 - a_{3i_2i_3} & \forall i_2, i_3, \end{array}$$

which show that the multi-linear system $\mathcal{A} \otimes x = b$ is solvable, and $x^* = (1, 2, -1)^T$ is a solution of $\mathcal{A} \otimes x = b$.

Example 2.11. The following equation

$$x_1 + x_2 = 0$$

can be expressed as a special case of $\mathcal{A} \otimes x = b$, where $b = 0$ and $\mathcal{A} \in \mathbb{R}_{\max}^{1\times 2\times 2}$ is a tensor such that $a_{1i_2i_3} = 0$ if $(i_2, i_3) \in \{(1, 2), (2, 1)\}$ and $a_{1i_2i_3} = \epsilon$ otherwise. We check solvability of the multi-linear system $\mathcal{A} \otimes x = b$ by Algorithm 2.7.

Due to Lines 4 and 6 of the algorithm, $\max_{i_2, i_3} a_{1i_2i_3} = a_{112} = a_{121} = 0$, and Due to Line 16 of the algorithm, $\alpha_1 = \epsilon$. By (6), we have the following system of inequalities:

$$\begin{cases} x_1 + x_2 & \leq 0 \\ x_1 + x_2 & \leq 0 \\ x_1 & \leq \infty \\ x_2 & \leq \infty, \end{cases}$$

that yields $x_1 \leq \gamma_1$ and $x_2 \leq -\gamma_1$. By setting $x_1 = \gamma_1, x_2 = -\gamma_1$, and applying Line 22 of the algorithm, we have $\gamma_1 - \gamma_1 = 0$ and $2\gamma_1 \leq \infty$. This shows that $x^* = (\gamma_1, -\gamma_1)^T$ is a solution of multi-linear system $\mathcal{A} \otimes x = b$.

In this example we showed that the multi-linear system (2) is solvable even when the tensor $\mathcal{A}$ is rectangular. The following results are obtained from Algorithm 2.7.

Corollary 2.12. Let $\mathcal{A} \in \mathbb{R}_{\max}^{[m,n]}$ and $b, d \in \mathbb{R}_{\max}^n$ be such that $b \leq d$ and let $x^*(\mathcal{A}, d)$ and $\bar{x}^*(\mathcal{A}, b)$ be solutions of multi-linear systems $\mathcal{A} \otimes x = d$ and $\mathcal{A} \otimes x \leq b$, respectively. Then $\mathcal{A} \otimes \bar{x}^*(\mathcal{A}, b) \leq \mathcal{A} \otimes x^*(\mathcal{A}, d)$ and for any $k \in [n]$, there exist $i_2^{(k)}, \ldots, i_m^{(k)} \in [n]$ such that $\sum_{j=2}^{m} \bar{x}^*_{i_j^{(k)}}(\mathcal{A}, b) \leq \sum_{j=2}^{m} x^*_{i_j^{(k)}}(\mathcal{A}, d)$.

P r o o f. By Remark 2.6 for any $k \in [n]$ there exist $i_2^{(k)}, \ldots, i_m^{(k)} \in [n]$ such that $\sum_{j=2}^{m} x^*_{i_j^{(k)}}(\mathcal{A}, d) = d_k - a_{ki_2^{(k)} \ldots i_m^{(k)}}$. Therefore we have

$$\sum_{j=2}^{m} x^*_{i_j(k)}(\mathcal{A}, d) = d_k - a_{ki_2^{(k)} \ldots i_m^{(k)}} \geq b_k - a_{ki_2^{(k)} \ldots i_m^{(k)}} \geq \sum_{j=2}^{m} \bar{x}^*_{i_j^{(k)}}(\mathcal{A}, b),$$

where the last inequality follows from Eq. (4). □

Corollary 2.13. Let $\mathcal{A}, \mathcal{B} \in \mathbb{R}_{\max}^{[m,n]}$ and $d \in \mathbb{R}_{\max}^n$ be such that $\mathcal{A} \leq \mathcal{B}$ and let $x^*(\mathcal{A}, d)$ be a solution of $\mathcal{A} \otimes x = d$. If $\bar{x}^*(\mathcal{B}, d)$ is a solution of $\mathcal{B} \otimes x \leq d$, then $\mathcal{B} \otimes \bar{x}^*(\mathcal{B}, d) \leq \mathcal{A} \otimes x^*(\mathcal{A}, d)$ and for any $k \in [n]$ there exist $i_2^{(k)}, \ldots, i_m^{(k)} \in [n]$ such that

$$\sum_{j=2}^{m} \bar{x}^*_{i_j^{(k)}}(\mathcal{B}, d) \leq \sum_{j=2}^{m} x^*_{i_j^{(k)}}(\mathcal{A}, d).$$

P r o o f. By Remark 2.6 and Eq. (4), for any $k \in [n]$ there exist $i_2^{(k)}, \ldots, i_m^{(k)} \in [n]$ such that

$$\sum_{j=2}^{m} x^*_{i_j^{(k)}}(\mathcal{A}, d) = d_k - a_{ki_2^{(k)} \ldots i_m^{(k)}}, \tag{7}$$

$$\sum_{j=2}^{m} \bar{x}^*_{i_j^{(k)}}(\mathcal{B}, d) \leq d_k - b_{ki_2^{(k)} \ldots i_m^{(k)}}. \tag{8}$$

Therefore by (7) and (8) we have

$$\sum_{j=2}^{m} x^*_{i_j^{(k)}}(\mathcal{A}, d) = d_k - a_{ki_2^{(k)} \ldots i_m^{(k)}} \geq d_k - b_{ki_2^{(k)} \ldots i_m^{(k)}} \geq \sum_{j=2}^{m} \bar{x}^*_{i_j^{(k)}}(\mathcal{B}, d).$$

□

3. WEAK SOLVABILITY

Before solving a multi-linear system we might want to know whether it is actually solvable. In the first result of this section, we obtain the necessary and sufficient conditions for the interval multi-linear systems with a constant tensor to be weakly solvable. Then by definition of the canonical tensor, we obtain similar result for nonconstant tensors.

Definition 3.1. We call the system $\mathcal{A}^I \otimes x = b^I$ weakly solvable if at least one of its subsystems is solvable.

In other words, a system (1), (i. e., $\mathcal{A}^I \otimes x = b^I$) is said to be weakly solvable if *some system* (1) with data (2), (i. e., $\mathcal{A} \otimes x = b$) is solvable. Hence, the word "weakly" refers to validity of the respective property for some system in the family.

Introduction of weak properties has an obvious motivation. Assume we are to decide whether some system $\mathcal{A} \otimes x = b$ is solvable, but the exact data of this system are not directly available to us (they come from some measurements, are with rounding errors, etc.); instead, we only know that they satisfy $\mathcal{A} \in \mathcal{A}^I,\ b \in b^I$. Then we can be sure that the system $\mathcal{A}^I \otimes x = b^I$ is not solvable only if we know that the system (1) is not weakly solvable.

Characterizations of weak solvability of interval multi-linear systems are given in the following.

Proposition 3.2. An interval multi-linear system (1) with a constant tensor $\mathcal{A} = \underline{\mathcal{A}} = \overline{\mathcal{A}}$ is weakly solvable if and only if

$$\mathcal{A} \otimes x^*(\mathcal{A}, b) \geq \underline{b},$$

for some solution $x^*(\mathcal{A}, b)$ of multi-linear system (2).

Proof. Sufficiency: Since $\underline{b} \leq \mathcal{A} \otimes x^*(\mathcal{A}, b) = b \leq \overline{b}$, then $\mathcal{A} \otimes x^*(\mathcal{A}, b) \in b^I$, and the sufficiency is proved. To treat the necessity, let $\mathcal{A} \otimes x = b$ be a solvable subsystem of (1) for some $b \in b^I$. Then $\mathcal{A} \otimes x^*(\mathcal{A}, b) = b \geq \underline{b}$. □ Weak solvability of interval linear system corresponds with the existence $A \in A^I$ and $b \in b^I$. Then the aim for weak solvability is finding the matrix A and vector b, such that linear system $A \otimes x = b$ is solvable. Cechlárová et al. [5] proposed an algorithm (Canonical matrix) for finding a matrix $A \in A^I$ such that for constant vector b the subsystem $A \otimes x = b$ of interval linear system $A^I \otimes x = b$ is solvable. Similarly, a condition for weak solvability of interval multi-linear system with a nonconstant tensor can be stated formally and identically for max-plus, using a canonical tensor of an interval multi-linear system.

For a given $x^*(\underline{\mathcal{A}}, b) \in S(\underline{\mathcal{A}}, b)$ the canonical tensor $\mathcal{A}(b) \in \mathcal{A}^I$ is defined by the following algorithm (for more details on the following algorithm refer to Appendix C).

Algorithm 3.3. (For finding the canonical tensor)

1: **Input:** The tensors $\underline{\mathcal{A}}, \overline{\mathcal{A}} \in \mathbb{R}_{\max}^{[m,n]}$ and $b \in \mathbb{R}^n_{\max}$.
2: Suppose that $i_2, i_3, \ldots, i_m \in [n]$.
3: **for** $j' = 1, \cdots, n$ and j' is at least one of $i_2, i_3, \ldots, i_m$ **do**

$$a_{j'}(b) = \max_{k,\, i_2,\dots,i_m} \left(\frac{\underline{a}_{ki_2\cdots i_m} - b_k + \sum\limits_{j=2,\, i_j \neq j'}^{m} x^*_{i_j}(\underline{\mathcal{A}}, b)}{l} \right);$$

where l is the number of indices i_j such that $i_j = j'$.

end for

for $k = 1, \cdots, n$ **do**

for $i_2, i_3, \dots, i_m = 1, \dots, n$ **do**

if $\overline{a}_{ki_2\dots i_m} \geq \sum\limits_{j=2}^{m} a_{i_j}(b) + b_k$ **then**

$$a_{ki_2\dots i_m}(b) = \sum_{j=2}^{m} a_{i_j}(b) + b_k$$

end if

else

$$a_{ki_2\dots i_m}(b) = \overline{a}_{ki_2\dots i_m}$$

end else

end for

end for

Now, we give some properties of the canonical tensor according to Algorithm 3.3.

Proposition 3.4. Let $\mathcal{A}^I \otimes x = b$ be any interval multi-linear system with a constant right-hand side such that $\mathcal{A}^I = [\underline{\mathcal{A}}, \overline{\mathcal{A}}] \in \mathbb{R}^{[m,n]}_{\max}$, and $x^*(\underline{\mathcal{A}}, b) \in S(\underline{\mathcal{A}}, b)$. Then

(a) $x^*_{j'}(\underline{\mathcal{A}}, b) = -a_{j'}(b)$ *for each* $j' \in [n]$,

(b) $$\max_{k,\, i_2,\dots,i_m} \left(\frac{a_{ki_2\dots i_m}(b) - b_k + \sum\limits_{j=2,\, i_j \neq j'}^{m} x^*_{j'}(\underline{\mathcal{A}}, b)}{l} \right) = a_{j'}(b) \quad \textit{for each } j' \in [n], \text{ where}$$
l is the number of indices i_j such that $i_j = j'$.

(c) If $x^*(\mathcal{A}(b), b) \in S(\mathcal{A}(b), b)$, then $\mathcal{A}(b) \otimes x^*(\mathcal{A}(b), b) = \underline{\mathcal{A}} \otimes x^*(\underline{\mathcal{A}}, b)$ and for each $k \in [n]$ there exist $i_2^{(k)}, \dots, i_m^{(k)}$ such that

$$\sum_{j=2}^{m} x^*_{i_j^{(k)}}(\mathcal{A}(b), b) = -\sum_{j=2}^{m} a_{i_j^{(k)}}(b).$$

(d) For any solvable subsystem $\mathcal{A} \otimes x = b$, if $x^*(\mathcal{A}(b), b) \in S(\mathcal{A}(b), b)$, then for each $k \in [n]$ there exist $i_2^{(k)}, \dots, i_m^{(k)}$ such that

$$\sum_{j=2}^{m} x^*_{i_j^{(k)}}(\mathcal{A}, b) \leq \sum_{j=2}^{m} x^*_{i_j^{(k)}}(\mathcal{A}(b), b).$$

Proof. (a) Since $S(\underline{\mathcal{A}}, b) \neq \varnothing$, then by Remark 2.6 for any $k \in [n]$ there exist $i_2^{(k)}, \dots, i_m^{(k)} \in [n]$

$$\sum_{j=2}^{m} x^*_{i_j^{(k)}}(\underline{\mathcal{A}}, b) = b_k - \underline{a}_{k i_2^{(k)} \dots i_m^{(k)}},$$

which shows that $\underline{a}_{k i_2^{(k)} \dots i_m^{(k)}} + \sum_{j=2}^{m} x^*_{i_j^{(k)}}(\underline{\mathcal{A}}, b) = b_k$. Therefore

$$-x^*_{j'}(\underline{\mathcal{A}}, b) = \left(\frac{\underline{a}_{k i_2^{(k)} \dots i_m^{(k)}} - b_k + \sum_{j=2,\, i_j^{(k)} \neq j'}^{m} x^*_{i_j^{(k)}}(\underline{\mathcal{A}}, b)}{l} \right)$$

$$= \max_{k,\, i_2, \dots, i_m} \left(\frac{\underline{a}_{k i_2 \dots i_m} - b_k + \sum_{j=2,\, i_j \neq j'}^{m} x^*_{i_j}(\underline{\mathcal{A}}, b)}{l} \right) = a_{j'}(b),$$

where l is the number of indices i_j such that $i_j = j'$.
(b) Due to Line 4 of the Algorithm 3.3, we have

$$a_{j'}(b) = \max_{k,\, i_2, \dots, i_m} \left(\frac{\underline{a}_{k i_2 \dots i_m} - b_k + \sum_{j=2,\, i_j \neq j'}^{m} x^*_{i_j}(\underline{\mathcal{A}}, b)}{l} \right)$$

$$\leq \max_{k,\, i_2, \dots, i_m} \left(\frac{\overline{a}_{k i_2 \dots i_m} - b_k + \sum_{j=2,\, i_j \neq j'}^{m} x^*_{i_j}(\underline{\mathcal{A}}, b)}{l} \right). \tag{9}$$

Let us assume that the condition (b) is not satisfied, so we have two cases:
(I) If

$$\max_{k,\, i_2, \dots, i_m} \left(\frac{a_{k i_2 \dots i_m}(b) - b_k + \sum_{j=2,\, i_j \neq j'}^{m} x^*_{i_j}(\underline{\mathcal{A}}, b)}{l} \right) < a_{j'}(b),$$

then $a_{j'}(b) > \dfrac{a_{k i_2 \dots i_m}(b) - b_k + \sum_{j=2,\, i_j \neq j'}^{m} x^*_{i_j}(\underline{\mathcal{A}}, b)}{l}$. Due to Lines 7-13 of the Algorithm 3.3, this is possible only if

$$a_{k i_2 \dots i_m}(b) = \overline{a}_{k i_2 \dots i_m} < l\, a_{j'}(b) + b_k - \sum_{j=2,\, i_j \neq j'}^{m} x^*_{i_j}(\underline{\mathcal{A}}, b).$$

Therefore

$$a_{j'}(b) > \frac{\overline{a}_{ki_2\ldots i_m} - b_k + \sum_{j=2,\, i_j\neq j'}^{m} x^*_{i_j}(\underline{\mathcal{A}}, b)}{l},$$

which is impossible by Eq. (9).
(II) If

$$\max_{k,\, i_2,\ldots,i_m}\left(\frac{a_{ki_2\ldots i_m}(b) - b_k + \sum_{j=2,\, i_j\neq j'}^{m} x^*_{i_j}(\underline{\mathcal{A}}, b)}{l}\right) > a_{j'}(b),$$

then similar to (I), we have

$$a_{ki_2\ldots i_m}(b) > l\, a_{j'}(b) + b_k - \sum_{j=2, i_j\neq j'}^{m} x^*_{i_j}(\underline{\mathcal{A}}, b).$$

Due to Lines 9-10 of the Algorithm 3.3, this is possible only if

$$a_{ki_2\ldots i_m}(b) = l\, a_{j'}(b) + b_k - \sum_{j=2, i_j\neq j'}^{m} x^*_{i_j}(\underline{\mathcal{A}}, b),$$

that is a contradiction and the proof is complete.

(c) Since the Canonical tensor $\mathcal{A}(b)$ belongs to interval tensor $\mathcal{A}^I$, i.e., $\underline{\mathcal{A}} \leq \mathcal{A}(b) \leq \overline{\mathcal{A}}$ and by Corollary 2.13 for each $k \in [n]$, there exist $i_2^{(k)}, \ldots, i_m^{(k)} \in [n]$ such that

$$\sum_{j=2}^{m} a_{i_j^{(k)}}(b) = -\sum_{j=2}^{m} x^*_{i_j^{(k)}}(\underline{\mathcal{A}}, b) \leq -\sum_{j=2}^{m} x^*_{i_j^{(k)}}(\mathcal{A}(b), b). \tag{10}$$

Since $S(\mathcal{A}(b), b) \neq \varnothing$, then by Remark 2.6 for any $k' \in [n]$ there exist $i_2^{(k')}, \ldots, i_m^{(k')} \in [n]$ such that

$$\sum_{j=2}^{m} x^*_{i_j^{(k')}}(\mathcal{A}(b), b) = b_k - a_{ki_2^{(k')}\ldots i_m^{(k')}}(b).$$

Then by (b) for any $j' \in [n]$, we have

$$
\begin{aligned}
a_{j'}(b) &= \max_{k,\, i_2,\dots,i_m} \left(\frac{a_{ki_2\dots i_m}(b) - b_k + \sum\limits_{j=2,\, i_j \neq j'}^{m} x^*_{i_j}(\underline{\mathcal{A}}, b)}{l} \right) \\
&\geq \max_{k,\, i_2,\dots,i_m} \left(\frac{a_{ki_2\dots i_m}(b) - b_k + \sum\limits_{j=2,\, i_j \neq j'}^{m} x^*_{i_j}(\mathcal{A}(b), b)}{l} \right) \\
&= \frac{-b_k + a_{ki_2^{(k')}\dots i_m^{(k')}}(b) + \sum\limits_{j=2,\, i_j^{(k')} \neq j'}^{m} x^*_{i_j^{(k')}}(\mathcal{A}(b), b)}{l} \\
&= -x^*_{j'}(\mathcal{A}(b), b),
\end{aligned}
$$

then

$$\sum_{j=2}^{m} a_{i_j^{(k)}}(b) \geq -\sum_{j=2}^{m} x^*_{i_j^{(k)}}(\mathcal{A}(b), b). \tag{11}$$

Then by Eqs. (10) and (11) for each $k \in [n]$ there exist $i_2^{(k)}, \dots, i_m^{(k)} \in [n]$ such that $\sum\limits_{j=2}^{m} x^*_{i_j^{(k)}}(\mathcal{A}(b), b) = -\sum\limits_{j=2}^{m} a_{i_j^{(k)}}(b)$.

(d) According to parts (a) and (c), for each $k \in [n]$ there exist $i_2^{(k)}, \dots, i_m^{(k)} \in [n]$ such that

$$\sum_{j=2}^{m} x^*_{i_j^{(k)}}(\mathcal{A}(b), b) = -\sum_{j=2}^{m} a_{i_j^{(k)}}(b) = \sum_{j=2}^{m} x^*_{i_j^{(k)}}(\underline{\mathcal{A}}, b).$$

Therefore by Corollary 2.13, one obtains $\sum\limits_{j=2}^{m} x^*_{i_j^{(k)}}(\mathcal{A}(b), b) \geq \sum\limits_{j=2}^{m} x^*_{i_j^{(k)}}(\mathcal{A}, b)$.

□

Corollary 3.5. An interval multi-linear system (1) with a constant right-hand side b is weakly solvable if subsystem $\mathcal{A}(b) \otimes x = b$ is solvable.

P r o o f. The result trivially follows from $\mathcal{A}(b) \in \mathcal{A}^I$. □

Theorem 3.6. Let $S(\mathcal{A}(\bar{b}), \bar{b}) \neq \varnothing$, where $\mathcal{A}(\bar{b}) \in \mathcal{A}^I$ and $\bar{b} \in b^I$. Then an interval multi-linear system (1) is weakly solvable and it holds

$$\mathcal{A}(\bar{b}) \otimes x^*(\mathcal{A}(\bar{b}), \bar{b}) \geq \underline{b}, \tag{12}$$

for any solution $x^*(\mathcal{A}(\bar{b}), \bar{b})$ of multi-linear system $\mathcal{A}(\bar{b}) \otimes x = \bar{b}$.

Proof. Since $S(\mathcal{A}(\bar{b}), \bar{b}) \neq \varnothing$, then the solvable subsystem is obtained with $\mathcal{A} = \mathcal{A}(\bar{b})$ and $b = \mathcal{A}(\bar{b}) \otimes x^*(\mathcal{A}(\bar{b}), \bar{b})$. It follows from $x^*(\mathcal{A}(\bar{b}), \bar{b}) \in S(\mathcal{A}(\bar{b}), \bar{b})$ that

$$\mathcal{A}(\bar{b}) \otimes x^*(\mathcal{A}(\bar{b}), \bar{b}) = \bar{b} \geq \underline{b}.$$

□

3.1. Examples

In this subsection we illustrate the main result of this section by some examples.

Example 3.7. Let us consider an interval tensor $\mathcal{A}^I$ given by the corresponding interval for each entry in the form

$$\mathcal{A}^I(:,:,1) = \begin{pmatrix} [2,6] & [3,8] & [4,6] \\ [1,4] & [6,6] & [5,7] \\ [4,5] & [3,7] & [6,7] \end{pmatrix},$$
$$\mathcal{A}^I(:,:,2) = \begin{pmatrix} [3,5] & [2,7] & [4,4] \\ [2,3] & [1,5] & [4,6] \\ [5,8] & [2,4] & [1,3] \end{pmatrix},$$
$$\mathcal{A}^I(:,:,3) = \begin{pmatrix} [6,8] & [3,5] & [5,9] \\ [3,5] & [6,6] & [6,7] \\ [2,6] & [4,5] & [2,3] \end{pmatrix}.$$

Consider the interval multi-linear system $\mathcal{A}^I \otimes x = b^I$, where

$$b^I = \begin{pmatrix} [7,10] \\ [5,9] \\ [8,11] \end{pmatrix}.$$

By Algorithm 2.7 (see Appendix A), we consider the following system of inequalities:

$$\begin{cases} x_1 + x_3 & \leq 4 = \beta_1 \\ x_2 + x_3 & \leq 3 = \beta_2 \\ x_2 + x_1 & \leq 3 = \beta_2 \\ x_3 + x_1 & \leq 5 = \beta_3 \\ x_1 & \leq 3.5 = \alpha_1 \\ x_2 & \leq 4 = \alpha_2 \\ x_3 & \leq 1.5 = \alpha_3. \end{cases}$$

By solving the above inequality system (Algorithm 4.1) and Line 22 of Algorithm 2.7, we have

$$x^*(\underline{\mathcal{A}}, \bar{b}) = (1.5, 1.5, 1.5)^T.$$

Then we obtain $a_{j'}(\bar{b}) = (-1.5, -1.5, -1.5)^T$ by applying Algorithm 3.3 (for more details, see Appendix C). Therefore the Canonical tensor $\mathcal{A}(\bar{b})$ is in the following form:

$$\mathcal{A}(\bar{b})(:,:,1) = \begin{pmatrix} 6 & 7 & 6 \\ 4 & 6 & 6 \\ 5 & 7 & 7 \end{pmatrix},$$
$$\mathcal{A}(\bar{b})(:,:,2) = \begin{pmatrix} 5 & 7 & 4 \\ 3 & 5 & 6 \\ 8 & 4 & 3 \end{pmatrix},$$
$$\mathcal{A}(\bar{b})(:,:,3) = \begin{pmatrix} 7 & 5 & 7 \\ 5 & 6 & 6 \\ 6 & 5 & 3 \end{pmatrix}.$$

Similarly, we get $x^*(\mathcal{A}(\bar{b}), \bar{b}) = (1.5, 1.5, 1.5)^T$, which in this case equals $x^*(\underline{\mathcal{A}}, \bar{b})$ and the system $\mathcal{A}^I \otimes x = b^I$ is weakly solvable.

Example 3.8. Consider the interval multi-linear system $\mathcal{A}^I \otimes x = b^I$, where

$$\mathcal{A}^I(:,:,1) = \begin{pmatrix} [2,6] & [3,8] & [6.5,7] \\ [1,4] & [6,6] & [5,7] \\ [4,5] & [3,7] & [7.5,8] \end{pmatrix},$$
$$\mathcal{A}^I(:,:,2) = \begin{pmatrix} [1,5] & [2,6] & [3,4] \\ [2,3] & [4,6] & [3,5] \\ [1,6] & [4,7] & [2,8.5] \end{pmatrix},$$
$$\mathcal{A}^I(:,:,3) = \begin{pmatrix} [2,4] & [3,6] & [1,4] \\ [2,4] & [5,6] & [7,8] \\ [3,7] & [2,6] & [1,7] \end{pmatrix},$$
$$b^I = \begin{pmatrix} [4,7] \\ [2,6] \\ [5,8] \end{pmatrix}.$$

Algorithm 2.7 gives the following system of inequalities:

$$\begin{cases} x_1 + x_3 & \leq 0.5 = \beta_1 \\ x_2 + x_1 & \leq 0 = \beta_2 \\ x_3 + x_1 & \leq 1 = \beta_3 \\ x_1 & \leq 2 = \alpha_1 \\ x_2 & \leq 1 = \alpha_2 \\ x_3 & \leq -0.5 = \alpha_3. \end{cases}$$

By solving the above inequality system (Algorithm 4.1) and Line 22 of Algorithm 2.7, we have

$$x^*(\underline{\mathcal{A}}, \bar{b}) = (1, -1, -0.5)^T.$$

By Algorithm 3.3, one can obtain the Canonical tensor $\mathcal{A}(\bar{b})$ with entries as follows:

$$\mathcal{A}(\bar{b})(:,:,1) = \begin{pmatrix} 5 & 7 & 6.5 \\ 4 & 6 & 5.5 \\ 5 & 7 & 7.5 \end{pmatrix},$$

$$\mathcal{A}(\bar{b})(:,:,2) = \begin{pmatrix} 5 & 6 & 4 \\ 3 & 6 & 5 \\ 6 & 7 & 8.5 \end{pmatrix},$$

$$\mathcal{A}(\bar{b})(:,:,3) = \begin{pmatrix} 4 & 6 & 4 \\ 4 & 6 & 7 \\ 7 & 6 & 7 \end{pmatrix}.$$

Similarly, we get $x^*(\mathcal{A}(\bar{b}), \bar{b}) = (0, 0, -0.5)^T$ and $\underline{b} \leq \mathcal{A}(\bar{b}) \otimes x^*(\mathcal{A}(\bar{b}), \bar{b}) = (7, 6, 7)^T \leq \bar{b}$. Therefore $x^*(\underline{\mathcal{A}}, \bar{b})$ is a solution of the multi-linear system $\underline{\mathcal{A}} \otimes x = \bar{b}$ and $x^*(\mathcal{A}(\bar{b}), \bar{b})$ is a solution of the multi-linear system $\mathcal{A}(\bar{b}) \otimes x = \bar{b}$, so the interval multi-linear system $\mathcal{A}^I \otimes x = b^I$ is weakly solvable.

4. USING MAX-PLUS ALGEBRA TO SOLVE THE JOB SHOP PROBLEM

Max-plus algebra is an effective tool for modeling discrete event systems, especially synchronized discrete event systems (like a job shop manufacturing system). Synchronized discrete event problem is a problem in which an event is scheduled to meet a deadline. The events run simultaneously and the completion of the lengthiest event has to compulsorily happen exactly at the deadline.

Job shops are a special category of synchronized discrete event systems. In a job shop, the flow of resources through the jobs is not identical. This means that each job might not require the machines in the same order for processing. Also, all the machines may not be required by all jobs. In a job shop scheduling problem, the number of schedules generated is equal to the number of machines in the system, because each machine can have a distinct schedule (see, for instance, [19]).

In this section, we considered the max-plus algebra to job-shop problem in synchronized discrete event system as follows [2, Example 3.5]:

Six shops which are within the same market but are located at some meters from each other were studied. The six shops find out that customers start buying at 7:00 a.m. They all decided to open their shops for customers at exactly 7:00 a.m. Since the shops want to meet that deadline, the sale representatives (reps) for each product for each shop are to start restocking before the set time. This will enable the shops to serve their customers on time and other consumers to make more profit because of the competitions. The six shops A, B, C, D, E and F sell six different beverage products, Vigul milk(V), Peak milk(P), Coastal milk(C), Dano milk (D), Cowbell milk (CB) and Three Crown milk (TC).

The shops work six days within the week, that is from Monday to Saturday. For the shops to avoid losses, each product has one Representative. The time available to the sale representatives to restock the shops depend on when the shops are opened to them before the set time 8 a.m. The time available for reps and the time each rep spent on each product was taken on each of the six days for each of the six shops. Suppose we only coordinate the events of a single deadline, then the latest start time can be obtained via the difference between the finish time and individual event duration times. If we are to take shop A for example, when the shop is opened to the reps for V, P, C, D, CB, and TC, the time each rep took was 20 min, 25 min, 30 min, 35 min, 30 min and 35 min, respectively. Where they were to finish within 45 min. Obtaining the difference implies that the latest starting time for each event is 25 min, 20 min, 15 min, 10 min, 15 min and 10 min, respectively. After considering the events of all the six shops, we will get a multiple deadline. When we consider the case where we have six shops, each shop will have different time available to the reps for their respective products. This will depend on the size of the shop, quantity of products available to the reps to restock, time the reps report at work, and also the time the shops are opened to the reps to start restock.

Below are the tables for shops A - F for various data taken for the respective shops in the selected week.

Day	V	P	C	D	CB	TC	Time available
MON	17	15	20	25	15	19	30
TUE	16	18	27	28	18	20	30
WED	25	17	25	22	23	25	30
THUR	23	25	30	33	27	37	40
FRI	40	45	40	37	30	35	50
SAT	35	35	30	32	35	35	40

Tab. 1. Shop A.

Day	V	P	C	D	CB	TC	Time available
MON	33	35	35	32	32	42	50
TUE	37	38	29	29	29	49	55
WED	38	45	43	35	35	50	60
THUR	42	40	47	43	43	55	60
FRI	25	25	26	28	28	30	35
SAT	35	42	35	38	38	40	45

Tab. 2. Shop B.

Day	V	P	C	D	CB	TC	Time available
MON	28	20	25	25	25	25	30
TUE	30	30	35	35	38	35	40
WED	32	33	30	28	30	32	35
THUR	40	40	43	42	40	36	45
FRI	31	32	30	35	33	40	50
SAT	40	40	30	35	35	30	40

Tab. 3. Shop C.

Day	V	P	C	D	CB	TC	Time available
MON	32	33	47	50	40	48	50
TUE	40	42	43	35	38	37	40
WED	30	31	35	40	42	43	45
THUR	20	29	30	30	33	32	35
FRI	30	37	30	29	30	35	45
SAT	45	40	38	35	25	20	40

Tab. 4. Shop D.

Day	V	P	C	D	CB	TC	Time available
MON	20	25	30	35	30	35	45
TUE	25	30	45	47	45	45	50
WED	30	20	25	33	35	25	40
THUR	25	35	40	30	43	30	55
FRI	30	32	40	43	33	30	45
SAT	35	40	45	30	45	40	60

Tab. 5. Shop E.

Day	V	P	C	D	CB	TC	Time available
MON	15	30	18	15	33	20	40
TUE	23	25	28	30	33	30	40
WED	30	32	29	23	30	33	35
THUR	40	27	41	35	35	40	45
FRI	35	35	43	40	35	48	50
SAT	26	40	50	30	35	50	55

Tab. 6. Shop F.

We consider a tensor $\mathcal{A}_{ijk} \in \mathbb{R}_{\max}^{[3,6]}$ such that the index i shows the reps for V, P, C, D, CB and TC when the shops is opened and the index j shows the six shops A, B, C, D, E and F. The index k shows the shops work six days within the week, that is from Monday to Saturday. Also we consider a vector $b \in \mathbb{R}_{\max}^{6}$ such that b_l is a maximum of each time available corresponding to row l of each of six shops, for example for Monday we have $b_1 = \max\{30, 50, 30, 50, 45, 40\} = 50$. Then we have a tensor $\mathcal{A}$ and time available corresponding to each days as follows.

$$\mathcal{A}_{MON}=\mathcal{A}(:,:,1)=\begin{array}{c} \text{shop} \\ V \\ P \\ C \\ D \\ CB \\ TC \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 17 & 33 & 28 & 32 & 20 & 15 \\ 15 & 35 & 20 & 33 & 25 & 30 \\ 20 & 35 & 25 & 47 & 30 & 18 \\ 25 & 32 & 25 & \boxed{50} & 35 & 15 \\ 15 & 32 & 25 & \boxed{40} & 30 & 33 \\ 19 & 42 & 25 & 48 & 35 & 20 \end{pmatrix} \end{array}, \quad \begin{array}{c} \text{Time available} \\ \begin{pmatrix} 30 \\ 50 \\ 30 \\ 50 \\ 45 \\ 40 \end{pmatrix} \end{array},$$

$$\mathcal{A}_{TUE}=\mathcal{A}(:,:,2)=\begin{array}{c} \text{shop} \\ V \\ P \\ C \\ D \\ CB \\ TC \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 16 & 37 & 30 & 40 & 25 & 23 \\ 18 & 38 & 30 & 42 & 30 & 25 \\ 27 & 29 & 35 & 43 & 45 & 28 \\ 28 & 29 & 35 & 35 & 47 & 30 \\ 18 & 29 & 38 & 38 & \boxed{45} & 33 \\ 20 & 49 & 35 & 37 & 45 & 30 \end{pmatrix} \end{array}, \quad \begin{array}{c} \text{Time available} \\ \begin{pmatrix} 30 \\ 55 \\ 40 \\ 40 \\ 50 \\ 40 \end{pmatrix} \end{array},$$

$$\mathcal{A}_{WED}=\mathcal{A}(:,:,3)=\begin{array}{c} \text{shop} \\ V \\ P \\ C \\ D \\ CB \\ TC \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 25 & 38 & 32 & 30 & 30 & 30 \\ 17 & \boxed{45} & 33 & 31 & 20 & 32 \\ 25 & 43 & 30 & 35 & 25 & 29 \\ 22 & 35 & 28 & 40 & 33 & 23 \\ 23 & 35 & 30 & 42 & 35 & 30 \\ 25 & \boxed{50} & 32 & 43 & 25 & 33 \end{pmatrix} \end{array}, \quad \begin{array}{c} \text{Time available} \\ \begin{pmatrix} 30 \\ 60 \\ 35 \\ 45 \\ 40 \\ 35 \end{pmatrix} \end{array},$$

$$\mathcal{A}_{THUR}=\mathcal{A}(:,:,4)=\begin{array}{c} \text{shop} \\ V \\ P \\ C \\ D \\ CB \\ TC \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 23 & 42 & 40 & 20 & 25 & 40 \\ 25 & 40 & 40 & 29 & 35 & 27 \\ 30 & 47 & 43 & 30 & 40 & 41 \\ 33 & 43 & 42 & 30 & 30 & 35 \\ 27 & 43 & 40 & 33 & 43 & 35 \\ 37 & \boxed{55} & 36 & 32 & 30 & 40 \end{pmatrix} \end{array}, \quad \begin{array}{c} \text{Time available} \\ \begin{pmatrix} 40 \\ 60 \\ 35 \\ 35 \\ 55 \\ 45 \end{pmatrix} \end{array},$$

$$\mathcal{A}_{FRI}=\mathcal{A}(:,:,5)=\begin{array}{c} \text{shop} \\ V \\ P \\ C \\ D \\ CB \\ TC \end{array}\begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} \boxed{40} & 25 & 31 & 30 & 30 & 35 \\ \boxed{45} & 25 & 32 & 37 & 32 & 35 \\ 40 & 26 & 30 & 30 & 40 & 43 \\ 37 & 28 & 35 & 29 & 43 & 40 \\ 30 & 28 & 33 & 30 & 33 & 35 \\ 35 & 30 & 40 & 35 & 30 & 48 \end{pmatrix} \end{array}, \quad \begin{array}{c} \text{Time available} \\ \begin{pmatrix} 40 \\ 60 \\ 35 \\ 35 \\ 55 \\ 45 \end{pmatrix} \end{array},$$

$$\mathcal{A}_{SAT}=\mathcal{A}(:,:,6)=\begin{matrix} \text{shop} \\ V \\ P \\ C \\ D \\ CB \\ TC \end{matrix}\begin{matrix} \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{pmatrix} \boxed{35} & 35 & \boxed{40} & \boxed{45} & 35 & 26 \\ 35 & 42 & 40 & 40 & 40 & 40 \\ 30 & 35 & 30 & 38 & 45 & \boxed{50} \\ 32 & 38 & 35 & 35 & 30 & 30 \\ \boxed{35} & 38 & 35 & 25 & \boxed{45} & 35 \\ 35 & 40 & 30 & 20 & 40 & \boxed{50} \end{pmatrix} \end{matrix}, \quad \begin{matrix} \text{Time available} \\ \begin{pmatrix} 40 \\ 45 \\ 40 \\ 40 \\ 60 \\ 55 \end{pmatrix} \end{matrix}.$$

Consider the events completed at 7 a.m. when the shops are opened to customers. We need to find the latest starting times for the various product Vigul milk, Peak milk. Coastal milk, Dano milk, Cowbell milk, and Three crown milk.

The problem is formulated as a multi-linear system $\mathcal{A}\otimes x=b$, where the entries of $\mathcal{A}$ and b are given as shown in the above with $\mathcal{A}(:,:,1)=\mathcal{A}_{MON}$, $\mathcal{A}(:,:,2)=\mathcal{A}_{TUE}$, $\mathcal{A}(:,:,3)=\mathcal{A}_{WED}$, $\mathcal{A}(:,:,4)=\mathcal{A}_{THUR}$, $\mathcal{A}(:,:,5)=\mathcal{A}_{FRI}$, $\mathcal{A}(:,:,6)=\mathcal{A}_{SAT}$, and $b=[50,55,60,60,50,60]^T$.

Now we use Algorithm 2.7 for calculating a solution of the multi-linear system $\mathcal{A}\otimes x=b$, as follows:

Due to Lines 4 and 6 of Algorithm 2.7, $a_{146}=45$, $a_{223}=a_{215}=45$, $a_{341}=a_{324}=47$, $a_{441}=50$, $a_{552}=45$, and $a_{624}=55$ are the maximum of the rows 1, 2, 3, 4 , 5 and 6, respectively. Due to Line 16 of the algorithm, $\alpha_1=17.5$, $\alpha_2=5.5$, $\alpha_3=11$, $\alpha_4=8.5$, $\alpha_5=8.5$, and $\alpha_6=5$. We have the following inequality system

$$\begin{cases} x_4+x_6 & \leq 5=\beta_1 \\ x_2+x_3 & \leq 10=\beta_2 \\ x_1+x_5 & \leq 10=\beta_3 \\ x_1+x_4 & \leq 13=\beta_4 \\ x_2+x_4 & \leq 13=\beta_5 \\ x_1+x_4 & \leq 10=\beta_6 \\ x_2+x_5 & \leq 5=\beta_7 \\ x_2+x_4 & \leq 5=\beta_8 \\ x_1 & \leq 17.5=\alpha_1 \\ x_2 & \leq 5.5=\alpha_2 \\ x_3 & \leq 11=\alpha_3 \\ x_4 & \leq 8.5=\alpha_4 \\ x_5 & \leq 8.5=\alpha_5 \\ x_6 & \leq 5=\alpha_6. \end{cases}.$$

By solving the above inequality we have $x_1\leq 10=\gamma_1$, $x_2\leq 5=\gamma_2$, $x_3\leq 5=\gamma_3$, $x_4\leq 0=\gamma_4$, $x_5\leq 0=\gamma_5$, $x_6\leq 5=\gamma_6$. By checking conditions in Line 22 of Algorithm 2.7, $x^*=(10,5,5,0,0,5)^T$ is a solution of multi-linear system $\mathcal{A}\otimes x=b$.

As observed in the above, some entries of the tensor $\mathcal{A}$ are marked with a $\square$ symbol, to indicate that these entries are used in solving multi-linear system $\mathcal{A}\otimes x=b$. In other words, these marked entries are satisfied in the multi-linear system. For example, consider the following equation:

$$(\mathcal{A} \otimes x)_1 = \max_{j,k \in [6]} (a_{1jk} + x_j + x_k) = b_1. \tag{13}$$

By substituting x^* in Eq. (13), we see that only entries $a_{115} = 40,\ a_{116} = 35,\ a_{136} = 40$ and $a_{146} = 45$ of $\mathcal{A}(1, :, :)$, where marked with a $\square$ symbol, satisfy in Eq. (13).

Now, these marked entries show that the delay time on Saturday is less than for the other days, because $\mathcal{A}(:, :, 6) = \mathcal{A}_{SAT}$ has the most marked entries. After Saturday the delay time on Monday, Wednesday and Friday are the same because on each of these days there are two marked entries. Similarly, shop A has the least delay time among other shops and Vigul milk has the least delay time among other products.

Notice that if the problem is formulated as a max-plus matrix equation $A \otimes x = b$, then matrix A and vector b formed from the averages to show the preparation before a shop is opened to customers and the averages of the time available to the shop, respectively (see, for instance, [2, Example 3.5]). However, in multi-linear system $\mathcal{A} \otimes x = b$, all entries of $\mathcal{A}$ and b have been used. Also in the matrix case, we can only consider the time delays for products V, P, C, D, CB, TC and shops A-F, while in the tensor case, the time delays for the days of the week were also considered.

In this section the focus is on third-order tensors. However, our approach naturally generalizes to higher-order tensors in a similar manner.

CONCLUSIONS

The max-plus algebra system of interval matrices has been studied widely in the literature. In this paper, we proposed the interval multi-linear systems for tensors in the max-plus algebra. We have developed an algorithm for computing the multi-linear system $\mathcal{A} \otimes x = b$ and used for the job shop problem in a synchronized discrete events system. The weak solvability of interval multi-linear systems in max-plus algebra have been studied. The results obtained have shown that the method is very efficient for solving multi-linear max-plus systems and is also applicable to job shop problems.

Appendix A. Implementation of Algorithm 2.7

We derive an alternative expression for implementation of Algorithm 2.7 which provides step by step procedure.

Step 1: We find the maximum entries in each row of $\mathcal{A} \in \mathbb{R}_{\max}^{n_1 \times \cdots \times n_m}$. Notice that, it is possible the maximum in some row of $\mathcal{A}$ is attained more than once, then before we find the maximum in the next row, the condition (Lines 6-10) will be checked for other maximum in the same row. We also computed β_j for each maximum (Lines 5 and 8).
Step 2: We compute the $\alpha_i,\ i \in [n_{\min}]$ and inserted in the linear inequality system (6) of Algorithm 2.7.
Step 3: In this step, we solve the linear inequality system (6) by using the Multi-Stage ABS algorithm [13] or Algorithm 4.1 in the Appendix B. Therefore, we have $x_1 \le \gamma_1, \ldots, x_{n_{\max}} \le \gamma_{n_{\max}}$, where $\gamma_i \in \mathbb{R}$, $i \in [n_{\max}]$.
Step 4: Finally, if two conditions of Line 22 are satisfied, then $(x_1^*, \ldots, x_{n_{\max}}^*) =$

$(\gamma_1, \ldots, \gamma_{n_{\max}})$ is a solution of the multi-linear system (2). Otherwise the multi-linear system (2) is not solvable.

Appendix B. An algorithm for solving the system of linear inequalities (6)

In [13] Guo and Liu propose the Multi-Stage ABS algorithm, for solving the system of linear inequalities. In the following, we suggest a new algorithm for solving the system of linear inequalities (6) and we will illustrate this by an example.

Let the number of inequalities in the inequality system (6) be q, where $q \in \mathbb{N}$.

Algorithm 4.1. (Solving the system of linear inequalities (6))

1: **Input:** Linear inequality system (6)
2: $l = 1$
3: **set** $\alpha_s := \min_i \alpha_i \neq \epsilon$, $r_l := s$ and $\gamma_{r_l} = x_{r_l} := \alpha_s$
4: **for** $j = 1, \ldots, q$ **do**
5:     **if** $x_{i_t^{(j)}} = x_{r_l}$ and $x_{i_t^{(j)}}$ is not a single variable in some inequality of inequality system (6) **then**
6:         insert x_{r_l} in part1 of inequality system (6)
7:     **end if**
8:     **else** break to Line 11
9:     **end else**
10: **end for**
11: **if** $\bigcup_l r_l \neq [n_{\max}]$ i.e., there exists $f \in [n_{\max}]$ and $f \notin \bigcup_l r_l$ **then**
12:     **if** x_f is a single variable in some inequality of inequality system (6), which $x_f \leq \theta_r$ ($\theta_r \in \mathbb{R}$ and $r \in q$) **then**
13:         **set** $x_f := \min\left\{\min_r\{\theta_r\}, \alpha_f\right\}$
14:         $l := l + 1$
15:         **set** $\gamma_{r_l} = x_{r_l} := x_f$ and go to Line 4
16:     **end if**
17:     **else**
18:         **set** $\alpha_s := \min_{i \neq \bigcup_l r_l} \{\alpha_i\}$
19:         **set** $l := l + 1$, $r_l := s$ and $\gamma_{r_l} = x_{r_l} := \alpha_s$
20:         go to Line 4
21:     **end else**
22: **end if**
23: **else return** $\gamma_1, \ldots, \gamma_{n_{\max}}$
24: **end else**

Let us illustrate this algorithm by an example.

Consider the linear inequality system in Example 2.10 as follows:

$$\begin{cases} x_2 + x_3 & \leq 1 = \beta_1 \\ x_2 + x_1 & \leq 6 = \beta_2 \\ x_2 + x_3 & \leq 6 = \beta_3 \\ x_1 + x_3 & \leq 6 = \beta_4 \\ x_3 + x_2 & \leq 3 = \beta_5 \\ x_1 & \leq 1 = \alpha_1 \\ x_2 & \leq 2 = \alpha_2. \end{cases} \tag{14}$$

In order to get the upper bound of solution of inequality system, we implement Algorithm 4.1 in the following way:

Step 1: $\min_i \alpha_i \neq \epsilon = \min\{1, 2\} = 1 = \alpha_1$, then set $i_1 = 1$, $r_1 = 1$ and $\gamma_1 = x_1 = 1$.
Step 2: We inserted $x_1 = 1$ in inequality system (14) and we get

$$\begin{cases} x_2 + x_3 & \leq 1 \\ x_2 & \leq 5 \\ x_2 + x_3 & \leq 6 \\ x_3 & \leq 5 \\ x_3 + x_2 & \leq 3 \\ x_1 & \leq 1 = \alpha_1 \\ x_2 & \leq 2 = \alpha_2. \end{cases} \tag{15}$$

Step 3: Because $\bigcup_l r_l = \{1\} \neq [3]$, we set $f = 2$. Since the second inequality of (15) has a single variable, then $x_2 = \min\{5, 2 = \alpha_2\} = 2$. Set $l = 2$, $r_2 = f = 2$ and we go to line 6 of Algorithm 4.1, that give $\gamma_2 = 2$. We insert $x_2 = 2$ in inequality system (15) and we have

$$\begin{cases} x_3 & \leq -1 \\ x_3 & \leq 4 \\ x_3 & \leq 5 \\ x_3 & \leq 1 \\ x_1 & \leq 1 = \alpha_1 \\ x_2 & \leq 2 = \alpha_2. \end{cases} \tag{16}$$

Step 4: Because $\bigcup_l r_l = \{1, 2\} \neq [3]$, we set $f = 3$. Since inequalites 1,2,3 and 4 of (16) have a single variable, then $x_3 = \min\{5, 4, 1, -1\} = -1$.

Step 5: Finally, we set $r_3 = 3$, $\gamma_3 = -1$ and because $\bigcup_l r_l = \{1, 2, 3\} = [3]$, then the algorithm is finished and returns $\gamma_1 = 1, \gamma_2 = 2, \gamma_3 = -1$.

Appendix C. Implementation of Algorithm 3.3

In this section we consider step by step, how Algorithm 3.3 can obtain the canonical tensor in Example 3.7.
Consider the tensor $\underline{\mathcal{A}} \in \mathcal{A}^I$ and vector $\bar{b} \in b^I$ given in Example 3.7 as follows:

$$\underline{\mathcal{A}}(:,:,1) = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 6 & 5 \\ 4 & 3 & 6 \end{pmatrix},$$
$$\underline{\mathcal{A}}(:,:,2) = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 1 & 4 \\ 5 & 2 & 1 \end{pmatrix},$$
$$\underline{\mathcal{A}}(:,:,3) = \begin{pmatrix} 6 & 3 & 5 \\ 3 & 6 & 6 \\ 2 & 4 & 2 \end{pmatrix}, \qquad \bar{b} = \begin{pmatrix} 10 \\ 9 \\ 11 \end{pmatrix}.$$

Step 1: We calculated $x^*(\underline{\mathcal{A}}, \bar{b})$ by Algorithm 2.7, that gives $x^*(\underline{\mathcal{A}}, \bar{b}) = (1.5, 1.5, 1.5)^T$.
Step 2: We obtained $a_{j'}(\bar{b})$ for $j' \in [3]$ (by Line 4 of Algorithm 3.3). For example for $j' = 1$ we have the following:

$$a_1(\bar{b}) = \max_{k,\, i_2, i_3} \left(\frac{\underline{a}_{ki_2i_3} - \bar{b}_k + \sum\limits_{j=2,\, i_j \neq 1}^{3} x^*_{i_j}(\underline{\mathcal{A}}, \bar{b})}{l} \right) =$$
$$\max\left\{ \frac{\underline{a}_{111} - \bar{b}_1}{2}, \frac{\underline{a}_{211} - \bar{b}_2}{2}, \frac{\underline{a}_{311} - \bar{b}_3}{2}, \underline{a}_{112} - \bar{b}_1 + x^*_2(\underline{\mathcal{A}}, \bar{b}), \underline{a}_{212} - \bar{b}_2 + x^*_2(\underline{\mathcal{A}}, \bar{b}), \right.$$
$$\underline{a}_{312} - \bar{b}_3 + x^*_2(\underline{\mathcal{A}}, \bar{b}), \underline{a}_{113} - \bar{b}_1 + x^*_3(\underline{\mathcal{A}}, \bar{b}), \underline{a}_{213} - \bar{b}_2 + x^*_3(\underline{\mathcal{A}}, \bar{b}), \underline{a}_{313} - \bar{b}_3 + x^*_3(\underline{\mathcal{A}}, \bar{b}),$$
$$\underline{a}_{121} - \bar{b}_1 + x^*_2(\underline{\mathcal{A}}, \bar{b}), \underline{a}_{221} - \bar{b}_2 + x^*_2(\underline{\mathcal{A}}, \bar{b}), \underline{a}_{321} - \bar{b}_3 + x^*_2(\underline{\mathcal{A}}, \bar{b}), \underline{a}_{131} - \bar{b}_1 + x^*_3(\underline{\mathcal{A}}, \bar{b}),$$
$$\left. \underline{a}_{231} - \bar{b}_2 + x^*_3(\underline{\mathcal{A}}, \bar{b}), \underline{a}_{331} - \bar{b}_3 + x^*_3(\underline{\mathcal{A}}, \bar{b}) \right\} = \underline{a}_{221} - \bar{b}_2 + x^*_2(\underline{\mathcal{A}}, \bar{b}) = -1.5.$$

Similarly, we obtained $a_2(\bar{b}) = -1.5$ and $a_3(\bar{b}) = -1.5$.
Step 3: Finally, we determined all entries of $\mathcal{A}(\bar{b})$ by Lines 7-15 of Algorithm 3.3. For $a_{111}(\bar{b})$, we have the following

$$2a_1(\bar{b}) + b_1 = 7 \geq \bar{a}_{111} = 6,$$

then by Line 13 of Algorithm 3.3

$$a_{111}(\bar{b}) = \bar{a}_{111} = 6.$$

Similarly, we determined the other entries of tensor $\mathcal{A}(\bar{b})$ (see, Example 3.7).

ACKNOWLEDGMENTS

The authors would like to thank the editor, Prof. Sergey Sergeev and the reviewers for their helpful comments and corrections. We also thank Prof. Stephane Gaubert for his helpful comments. The third author acknowledges a partial support of the Slovenian Research Agency (grants P1-0222, J1-8133, J1-8155, J2-2512 and N1-0071).

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Sedighe Khaleghzade, Department of Mathematics, University of Hormozgan, P. O. Box 3995, Bandar Abbas. Iran.
e-mail: saharkh547@gmail.com

Mostafa Zangiabadi, Department of Mathematics, University of Hormozgan, P. O. Box 3995, Bandar Abbas. Iran.
e-mail: zangiabadi1@gmail.com

Aljoša Peperko, Faculty of Mechanical Engineering, University of Ljubljana, Aškerčeva 6, SI-1000 Ljubljana, Slovenia, and Institute of Mathematics, Physics and Mechanics, Jadranska 19, SI-1000 Ljubljana. Slovenia.
e-mail: Aljosa.Peperko@fs.uni-lj.si

Masoud Hajarian, Department of Mathematics, Faculty of Mathematical Sciences, Shahid Beheshti University, General Campus, Evin, Tehran 19839. Iran.
e-mail: m-hajarian@sbu.ac.ir