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STABILITY AND STABILIZATION OF ONE CLASS OF THREE TIME-SCALE SYSTEMS WITH DELAYS

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A singularly perturbed linear time-invariant time delay controlled system is considered. The singular perturbations are subject to the presence of two small positive multipliers for some of the derivatives in the system. These multipliers (the parameters of singular perturbations) are of different orders of the smallness. The delay in the slow state variable is non-small (of order of 1). The delays in the fast state variables are proportional to the corresponding parameters of singular perturbations. Three much simpler parameters-free subsystems are associated with the original system. It is established that the exponential stability of the unforced versions of these subsystems yields the exponential stability of the unforced version of the original system uniformly in the parameters of singular perturbations. It also is shown that the stabilization of the parameters-free subsystems by memory-free state-feedback controls yields the stabilization of the original system by a memory-free state-feedback control uniformly in the parameters of singular perturbations. Illustrative examples are presented.

Keywords: linear controlled system, time delay system, three time-scale singularly perturbed system, exponential stability, memory-free state-feedback stabilization

Classification: 93C23, 93C70, 93D15, 93D23

1. INTRODUCTION

Singularly perturbed differential systems, i. e., the systems with small positive multipliers (parameters of singular perturbations) for some of the highest order derivatives, serve as mathematical models for various real-life processes with multi-time-scale dynamics (see e. g. [6, 18, 31, 32, 36, 43] and references therein). Two classes of singularly perturbed systems are mostly studied in the literature: (I) the systems perturbed by a single small parameter (one-parameter or two time-scale systems); (II) the systems perturbed by multiple small parameters (multi-parameter systems). For the second class, three important cases of relationships between the small parameters are considered: (II_1) the ratios between the small parameters tend either to zero, or to positive infinity, i. e., the parameters are of different orders of the smallness (see e. g. [7, 8, 23, 26, 33, 34]), such singularly perturbed multi-parameter systems also are called three, four, e.t.c., time-scale ones; (II_2) the ratios between the small parameters are bounded from below and above by positive numbers, i. e., the parameters are of the same order of the smallness (see

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e. g. [27, 29, 30, 40]); (II_3) the small parameters are independent of each other (see e. g. [1, 3, 24]). Note that most of the works, devoted to analysis of singularly perturbed multi-parameter systems, deal with a delay-free version of the systems. Singularly perturbed multi-parameter time delay systems are studied much less.

In this paper, we consider one class of singularly perturbed two-parameter linear time-invariant controlled systems with state delays. We study the exponential stability of the unforced version of this system where the control function is identical zero. Also, we study the memory-free state-feedback stabilization of the original system. Stability is one of the basic properties of an uncontrolled system, as well as stabilization is one of the basic properties of a controlled system. Stability of linear time-invariant uncontrolled systems with delays, as well as stabilization of linear time-invariant controlled time delay systems, were extensively studied in the literature (see e. g. [12, 17, 21, 35, 37, 39, 41] and references therein). One can directly apply the results of these studies to a singularly perturbed system for any specified values of the small parameters. However, a stiffness of the system and its high Euclidean dimension considerably complicate such an application. Moreover, this application depends on the values of the small parameters, while in various real-life problems these values are unknown, i. e., these problems are uncertain with respect to the parameters. Thus, for singularly perturbed systems, another (than the aforementioned) conditions of their stability/stabilization, uniform (robust) with respect to the small parameters, should be derived. More precisely, these conditions should be independent of these parameters, while provide the stability/stabilization for all their sufficiently small values. Such conditions can be derived using the separation of time-scales concept (see e. g. [31]). Thus in [4, 28, 31] and references therein, the stability/stabilization conditions for standard and nonstandard single-parameter singularly perturbed systems without delays were derived. In [16, 19, 20, 25, 38] (see also references therein), the stability/stabilization analysis for various single-parameter singularly perturbed systems with delays was carried out using the separation of time-scales approach. In [2, 12, 19, 42] (see also references therein), the stability/stabilization of various single-parameter singularly perturbed systems with delays was studied using the Linear Matrix Inequality method. Stability and stabilization issues for singularly perturbed multi-parameter systems also were studied in the literature although less. Thus in [1] and references therein, various types of stability for singularly perturbed multi-parameter linear time-invariant systems without delays were studied using the spectrum analysis. In [26], a singularly perturbed multi-parameter linear time-dependent system without delays was considered. Stability analysis of this system was carried out by its diagonalization. One class of singularly perturbed three time-scale nonlinear time-invariant systems without delays was considered in [5]. Based on some approximation of the original system by three simpler systems in different intervals of the independent variable, its asymptotic stability is established. In [10], a singularly perturbed two-parameter Itô differential system without delays was analyzed. For this system, a stabilizing composite control was designed using the Linear Matrix Inequality approach. In [3], a singularly perturbed multi-parameter linear time-invariant system with a single point-wise non-small (of order of 1) delay was studied. Sufficient condition of its asymptotic stability was derived in the terms of linear matrix inequality.

The motivation of the present, rather theoretical, paper is to extend the separation of time-scales concept to analysis of the stability/stabilization for one nontrivial class of two-parameter systems with state delays of different scales. Namely, the singularly perturbed two-parameter system, considered in the present paper, is assumed to be of the three time-scale type, i. e., its two small parameters are of different orders of the smallness. The delay in the slow state variable of this system is non-small (of order of 1), while the delays in the fast state variables are proportional to the corresponding small parameters of singular perturbations. To the best of our knowledge, such a type of systems has not been considered yet in the literature, and it essentially differs from the singularly perturbed systems studied in the literature. The stability analysis of the unforced version of the original system and the stabilization analysis of this system are based on a proper extension of the separation of time-scales concept. Since the considered system is a two-parameter three time-scale one, it should be separated into more than two parameters-free subsystems (like it is done in the classical case of one-parameter singularly perturbed systems [31]). In the present paper, the considered system is separated approximately (asymptotically) into three much simpler parameters-free subsystems (purely slow, mixed slow-fast and purely fast ones). Then, the validity of the stability/stabilization of the original system uniform with respect to the small parameters is deduced from the presumed stability/stabilization of these three parameters-free subsystems. Thus, the main result of the paper consists in the reduction of the stability/stabilization analysis of the complicated two-parameter three time-scale system to the stability/stabilization analysis of the three much simpler parameter-free subsystems. Moreover, this reduction is uniform (robust) with respect to the parameters of singular perturbations for all their sufficiently small values preserving the order of the smallness.

The paper is organized of follows. In the next section, the problem formulation is given in the rigorous form. The objectives of the paper also are stated. Some preliminary results are presented in Section 3. Section 4 is devoted to the main results. In Section 5 several examples, illustrating the theoretical results of the paper, are considered. Conclusions are placed in Section 6. Sections 7–10 are devoted to technically complicated proofs of some auxiliary lemmas.

The following main notations are applied in the paper:

(1) $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space, $\|\cdot\|$ denotes the norm in this space;

(2) I_n denotes the n-dimensional identity matrix;

(3) $\mathrm{Re}\lambda$ denotes the real part of a complex number λ;

(4) $\mathrm{col}(x_1, x_2, \dots, x_k)$, where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $\dots$, $x_k \in \mathbb{R}^{n_k}$, denotes a column block-vector in which x_1 is the upper block, x_2 in the next block after x_1, and so on, x_k is the lower block;

(5) $C[a, b; \mathbb{R}^n]$ is the space of continuous functions $f(t) : [a, b] \to \mathbb{R}^n$, $\|\cdot\|_C$ denotes the uniform norm in $C[a, b; \mathbb{R}^n]$.

2. PROBLEM FORMULATION

2.1. Original system

The system under the consideration is the following:

$$\begin{aligned}\frac{dx(t)}{dt} &= A_{11}x(t) + A_{12}y_1(t) + A_{13}y_2(t) + G_1x(t-g)\\ &\quad + H_{11}y_1(t-\varepsilon_1 h_1) + H_{12}y_2(t-\varepsilon_2 h_2) + B_1u(t),\end{aligned} \tag{1}$$

$$\begin{aligned}\varepsilon_1\frac{dy_1(t)}{dt} &= A_{21}x(t) + A_{22}y_1(t) + A_{23}y_2(t) + G_2x(t-g)\\ &\quad + H_{21}y_1(t-\varepsilon_1 h_1) + H_{22}y_2(t-\varepsilon_2 h_2) + B_2u(t),\end{aligned} \tag{2}$$

$$\begin{aligned}\varepsilon_2\frac{dy_2(t)}{dt} &= A_{31}x(t) + A_{32}y_1(t) + A_{33}y_2(t) + G_3x(t-g)\\ &\quad + H_{31}y_1(t-\varepsilon_1 h_1) + H_{32}y_2(t-\varepsilon_2 h_2) + B_3u(t),\end{aligned} \tag{3}$$

where $t \geq 0$; $x(t) \in \mathbb{R}^n$, $y_k(t) \in \mathbb{R}^{m_k}$, $(k = 1,2)$, $u(t) \in \mathbb{R}^r$, ($u(t)$ is a control); $\varepsilon_k > 0$, $(k = 1,2)$ are small parameters; $g > 0$ and $h_k > 0$, $(k = 1,2)$ are given numbers independent of ε_k, $(k = 1,2)$; A_{ij}, G_i, H_{il}, B_i, $(i,j = 1,2,3;\ l = 1,2)$ are given constant matrices of corresponding dimensions.

The system (1) – (3) is a singularly perturbed time delay system with two parameters of singular perturbations ε_1 and ε_2. It is infinite-dimensional with the state variables $\big(x(t), x(t+\eta)\big)$, $\eta \in [-g,0)$, $\big(y_1(t), y_1(t+\varepsilon_1\zeta_1)\big)$, $\zeta_1 \in [-h_1,0)$ and $\big(y_2(t), y_2(t+\varepsilon_2\zeta_2)\big)$, $\zeta_2 \in [-h_2,0)$. The equation (1) and the state variable $\big(x(t), x(t+\eta)\big)$ are called a slow mode and a slow state variable of (1) – (3). The equations (2), (3) and the state variables $\big(y_1(t), y_1(t+\varepsilon_1\zeta_1)\big)$, $\big(y_2(t), y_2(t+\varepsilon_2\zeta_2)\big)$ are called fast modes and fast state variables of (1) – (3). An additional feature of (1) – (3) is that the delays in the fast state variables are proportional to the small multipliers for the derivatives in the corresponding fast modes.

In what follows, we assume that the small parameter $\varepsilon_2 > 0$ is of a higher order of smallness than the small parameter $\varepsilon_1 > 0$, i. e.,

$$\varepsilon_2/\varepsilon_1 \ll 1. \tag{4}$$

For instance, this relation between ε_1 and ε_2 is fulfilled if $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = \varepsilon^2$, where $0 < \varepsilon \ll 1$ is a small parameter.

Thus, due to (4), the fast state variable $\big(y_2(t), y_2(t+\varepsilon_2\zeta_2)\big)$ is "faster" than the fast state variable $\big(y_1(t), y_1(t+\varepsilon_1\zeta_1)\big)$. Hence, the system (1) – (3) is a three time-scale system.

Let us consider the following block vector and block matrices:

$$z(t) \triangleq \mathrm{col}\big(x(t), y_1(t), y_2(t)\big), \qquad t \geq -g, \tag{5}$$

$$A \triangleq \begin{pmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{pmatrix}, \quad G \triangleq \begin{pmatrix} G_1 & 0 & 0\\ G_2 & 0 & 0\\ G_3 & 0 & 0\end{pmatrix}, \quad H_1 \triangleq \begin{pmatrix} 0 & H_{11} & 0\\ 0 & H_{21} & 0\\ 0 & H_{31} & 0\end{pmatrix},$$

$$H_2 \stackrel{\triangle}{=} \begin{pmatrix} 0 & 0 & H_{12} \\ 0 & 0 & H_{22} \\ 0 & 0 & H_{32} \end{pmatrix}, \quad B \stackrel{\triangle}{=} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \quad E(\varepsilon_1, \varepsilon_2) \stackrel{\triangle}{=} \begin{pmatrix} I_n & 0 & 0 \\ 0 & \varepsilon_1 I_{m_1} & 0 \\ 0 & 0 & \varepsilon_2 I_{m_2} \end{pmatrix}. \tag{6}$$

Using these vector and matrices, we can rewrite the system (1) – (3) in the equivalent form as:

$$\begin{aligned} E(\varepsilon_1, \varepsilon_2)\frac{dz(t)}{dt} &= Az(t) + Gz(t-g) \\ &\quad + H_1 z(t - \varepsilon_1 h_1) + H_2 z(t - \varepsilon_2 h_2) + Bu(t), \quad t \geq 0. \end{aligned} \tag{7}$$

Let

$$0 < \varepsilon_1^* \ll 1, \qquad 0 < \varepsilon^* \ll 1 \tag{8}$$

be any given positive numbers. Let

$$\Omega(\varepsilon_1^*, \varepsilon^*) \stackrel{\triangle}{=} \{(\varepsilon_1, \varepsilon_2) : \varepsilon_1 \in (0, \varepsilon_1^*],\ \varepsilon_2/\varepsilon_1 \in (0, \varepsilon^*]\}. \tag{9}$$

Definition 2.1. The unforced system (7) ($u(t) \equiv 0$) is called exponentially stable uniformly in $(\varepsilon_1, \varepsilon_2) \in \Omega(\varepsilon_1^*, \varepsilon^*)$, if for any such pair of the parameters ε_1 and ε_2, and any function $\psi_z(\eta) \in C[-g, 0; \mathbb{R}^{n+m_1+m_2}]$ the solution $z(t)$, $t \geq 0$ of this system with the initial condition $z(\eta) = \psi_z(\eta)$, $\eta \in [-g, 0]$ satisfies the inequality

$$\|z(t)\| \leq c_z \exp(-\omega_z t)\|\psi_z(\eta)\|_C, \qquad t \geq 0,$$

where $c_z > 0$ and $\omega_z > 0$ are some constants independent of $(\varepsilon_1, \varepsilon_2) \in \Omega(\varepsilon_1^*, \varepsilon^*)$ and $\psi_z(\eta)$.

For the controlled system (7), let us consider the memory-free state-feedback control

$$u(t) = u[z(t)] = K_z z(t), \qquad t \geq 0, \tag{10}$$

where K_z is an $r \times (n + m_1 + m_2)$-matrix independent of ε_1 and ε_2.

Definition 2.2. The system (7) is called stabilized by a memory-free state-feedback control uniformly in $(\varepsilon_1, \varepsilon_2) \in \Omega(\varepsilon_1^*, \varepsilon^*)$, if there exists the control (10) such that the closed-loop system (7), (10) is exponentially stable uniformly in $(\varepsilon_1, \varepsilon_2) \in \Omega(\varepsilon_1^*, \varepsilon^*)$.

Remark 2.3. Since the systems (7) and (1) – (3) are equivalent to each other for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, Definitions 2.1 and 2.2 also are valid for the system (1) – (3).

Remark 2.4. For the sake of the further analysis of the original system (1) – (3) (and, therefore, (7)), we are going to introduce into the consideration three much simpler parameters-free subsystems (the purely slow, mixed slow-fast and purely fast ones). The introducing these subsystems is a generalization of the slow-fast decomposition of one-parameter singularly perturbed systems without delays (see e. g. [14, 31, 43]) and of the slow-fast decomposition of one-parameter singularly perturbed time delay systems (see e. g. [9, 18]).

2.2. Purely slow subsystem

This subsystem has the form

$$\frac{dx_s(t)}{dt} = A_{11}x_s(t) + A_{12,s}y_{1,s}(t) + A_{13,s}y_{2,s}(t) + G_1x_s(t-g) + B_1u_s(t), \quad (11)$$

$$0 = A_{21}x_s(t) + A_{22,s}y_{1,s}(t) + A_{23,s}y_{2,s}(t) + G_2x_s(t-g) + B_2u_s(t), \quad (12)$$

$$0 = A_{31}x_s(t) + A_{32,s}y_{1,s}(t) + A_{33,s}y_{2,s}(t) + G_3x_s(t-g) + B_3u_s(t), \quad (13)$$

where $x_s(t) \in \mathbb{R}^n$ and $y_{k,s}(t) \in \mathbb{R}^{m_k}$, $(k = 1,2)$; $u_s(t) \in \mathbb{R}^r$ is a control;

$$A_{lj,s} = A_{lj} + H_{lj-1}, \qquad l = 1,2,3 \qquad j = 2,3. \quad (14)$$

Remark 2.5. The subsystem (11) – (13) can be obtained from the original system (1) – (3) by setting there formally $\varepsilon_1 = 0$ and $\varepsilon_2 = 0$. Note that the purely slow subsystem (11) – (13) is a descriptor (differential-algebraic) system with a delay only in the state $x_s(\cdot)$.

Let us denote

$$\mathcal{A}_{1s} \triangleq \big(A_{12,s}\ ,\ A_{13,s}\big), \qquad \mathcal{A}_{2s} \triangleq \begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix},$$
$$\mathcal{A}_{3s} \triangleq \begin{pmatrix} A_{22,s} & A_{23,s} \\ A_{32,s} & A_{33,s} \end{pmatrix}, \qquad \mathcal{G}_{23,s} \triangleq \begin{pmatrix} G_2 \\ G_3 \end{pmatrix}, \qquad \mathcal{B}_{23,s} \triangleq \begin{pmatrix} B_2 \\ B_3 \end{pmatrix}. \quad (15)$$

If

$$\det \mathcal{A}_{3s} \neq 0, \quad (16)$$

then, using the notations (15), we can reduce the purely slow subsystem (11) – (13) to the following time delay differential equation with respect to $x_s(t)$:

$$\frac{dx_s(t)}{dt} = \bar{A}_s x_s(t) + \bar{G}_s x_s(t-g) + \bar{B}_s u_s(t), \qquad t \geq 0, \quad (17)$$

where

$$\bar{A}_s = A_{11} - \mathcal{A}_{1s}\mathcal{A}_{3s}^{-1}\mathcal{A}_{2s}, \quad \bar{G}_s = G_1 - \mathcal{A}_{1s}\mathcal{A}_{3s}^{-1}\mathcal{G}_{23,s}, \quad \bar{B}_s = B_1 - \mathcal{A}_{1s}\mathcal{A}_{3s}^{-1}\mathcal{B}_{23,s}. \quad (18)$$

The equation (17) also is called a purely slow subsystem, associated with the original system (1) – (3) (and, therefore, with (7)). This purely slow subsystem is of a lower Euclidean dimension than the original system.

2.3. Mixed slow-fast subsystem

This subsystem is the following:

$$\frac{dy_{1,sf}(\xi_1)}{d\xi_1} = A_{22}y_{1,sf}(\xi_1) + A_{23,s}y_{2,sf}(\xi_1)$$
$$+H_{21}y_{1,sf}(\xi_1 - h_1) + B_2u_{sf}(\xi_1), \quad \xi_1 \geq 0,$$

$$\begin{aligned} 0 &= A_{32}y_{1,sf}(\xi_1) + A_{33,s}y_{2,sf}(\xi_1) \\ &\quad + H_{31}y_{1,sf}(\xi_1 - h_1) + B_3 u_{sf}(\xi_1), \quad \xi_1 \geq 0, \end{aligned} \tag{19}$$

where ξ_1 is a new independent variable; the matrices $A_{23,s}$ and $A_{33,s}$ are given in (14).

Remark 2.6. The subsystem (19) can be derived from the fast modes (2) and (3) of the original system (1) – (3) by the following formal three steps' procedure. At the first step, we remove the slow state $x(\cdot)$ from the fast modes (2) and (3). Thus, we obtain the differential equations with respect to the fast states $y_k(\cdot)$, $(k = 1, 2)$, right-hand sides of which depend only on these states and the control

$$\begin{aligned} \varepsilon_1 \frac{dy_1(t)}{dt} &= A_{22}y_1(t) + A_{23}y_2(t) + H_{21}y_1(t - \varepsilon_1 h_1) + H_{22}y_2(t - \varepsilon_2 h_2) + B_2 u(t), \\ \varepsilon_2 \frac{dy_2(t)}{dt} &= A_{32}y_1(t) + A_{33}y_2(t) + H_{31}y_1(t - \varepsilon_1 h_1) + H_{32}y_2(t - \varepsilon_2 h_2) + B_3 u(t), \end{aligned} \tag{20}$$

where $t \geq 0$.

At the second step, we transform the variables in the equations (20) as: $t = \varepsilon_1\xi_1$; $y_k(\varepsilon_1\xi_1) = y_{k,sf}(\xi_1)$, $\xi_1 \geq -h_1$, $(k = 1, 2)$; $u(\varepsilon\xi_1) = u_{sf}(\xi_1)$, $\xi_1 \geq 0$. This transformation yields the system

$$\begin{aligned} \frac{dy_{1,sf}(\xi_1)}{d\xi_1} &= A_{22}y_{1,sf}(\xi_1) + A_{23}y_{2,sf}(\xi_1) \\ + H_{21}y_{1,sf}(\xi_1 - h_1) + H_{22}y_{2,sf}\big(\xi_1 - (\varepsilon_2/\varepsilon_1)h_2\big) &+ B_2 u_{sf}(\xi_1), \quad \xi_1 \geq 0, \\ (\varepsilon_2/\varepsilon_1)\frac{dy_{2,sf}(\xi_1)}{d\xi_1} &= A_{32}y_{1,sf}(\xi_1) + A_{33}y_{2,sf}(\xi_1) \\ + H_{31}y_{1,sf}(\xi_1 - h_1) + H_{32}y_{2,sf}\big(\xi_1 - (\varepsilon_2/\varepsilon_1)h_2\big) &+ B_3 u_{sf}(\zeta_1), \quad \xi_1 \geq 0. \end{aligned} \tag{21}$$

At the third step, taking into account (4), we set formally $\varepsilon_2/\varepsilon_1 = 0$ in the system (21), which yields the mixed slow-fast subsystem (19). This subsystem is a descriptor (differential-algebraic) system with a delay only in the state $y_{1,sf}(\cdot)$.

If

$$\det A_{33,s} \neq 0, \tag{22}$$

the subsystem (19) can be reduced to the following differential equation with state delay:

$$\frac{dy_{1,sf}(\xi_1)}{d\xi_1} = \bar{A}_{sf}y_{1,sf}(\xi_1) + \bar{H}_{sf}y_{1,sf}(\xi_1 - h_1) + \bar{B}_{sf}u_{sf}(\xi_1), \quad \xi_1 \geq 0, \tag{23}$$

where

$$\begin{aligned} \bar{A}_{sf} &= A_{22} - A_{23,s}(A_{33,s})^{-1}A_{32} \\ \bar{H}_{sf} &= H_{21} - A_{23,s}(A_{33,s})^{-1}H_{31}, \\ \bar{B}_{sf} &= B_2 - A_{23,s}(A_{33,s})^{-1}B_3. \end{aligned} \tag{24}$$

The system (23) also is called the mixed slow-fast subsystem of the system (1) – (3) (and, therefore, of the system (7)).

2.4. Purely fast subsystem

This subsystem has the form

$$\frac{dy_{2,f}(\xi_2)}{d\xi_2} = A_{33}y_{2,f}(\xi_2) + H_{32}y_{2,f}(\xi_2 - h_2) + B_3u_f(\xi_2), \qquad \xi_2 \geq 0, \tag{25}$$

where ξ_2 is a new independent variable.

Remark 2.7. The subsystem (25) can be obtained from the fast mode (3) of the original system (1) – (3) by the following formal two steps' procedure. At the first step, we remove the slow state $x(\cdot)$ and the fast state $y_1(\cdot)$ (the slower fast state than the fast state $y_2(\cdot)$) from the fast mode (3) (which is the faster mode than the fast mode (2)). Thus, we obtain the differential equation with respect to fast state $y_2(\cdot)$, right-hand side of which depends only on this state and the control

$$\varepsilon_2\frac{dy_2(t)}{dt} = A_{33}y_2(t) + H_{32}y_2(t - \varepsilon_2h_2) + B_3u(t), \qquad t \geq 0. \tag{26}$$

At the second step, we transform the variables in the equation (26) as: $t = \varepsilon_2\xi_2$; $y_2(\varepsilon_2\xi_2) = y_{2,f}(\xi_2)$, $\xi_2 \geq -h_2$; $u(\varepsilon\xi_2) = u_f(\xi_2)$, $\xi_2 \geq 0$. This transformation yields the purely fast subsystem (25). This subsystem is a time delay differential equation with respect to $y_{2,f}(\xi_2)$.

2.5. Objectives of the paper

The objectives of the paper are:

(I) assuming the exponential stability of the unforced purely slow subsystem, the unforced mixed slow-fast subsystem and the unforced purely fast subsystem, to establish the exponential stability of the unforced singularly perturbed system (7) (and, therefore, (1) – (3)) uniformly in $(\varepsilon_1, \varepsilon_2) \in \Omega(\varepsilon_1^*, \varepsilon^*)$ for some numbers ε_1^* and ε^* satisfying (8);

(II) assuming that the controlled purely slow subsystem, the controlled mixed slow-fast subsystem and the controlled purely fast subsystem are stabilized by memory-free state-feedback controls, to established that the controlled singularly perturbed system (7) (and, therefore, (1) – (3)) is stabilized by a memory-free state-feedback control uniformly in $(\varepsilon_1, \varepsilon_2) \in \Omega(\varepsilon_1^*, \varepsilon^*)$ for some numbers ε_1^* and ε^* satisfying (8).

Note that definitions of the exponential stability and the stabilization of the purely slow, mixed slow-fast and purely fast subsystems are presented in the next section.

3. PRELIMINARY RESULTS

3.1. Some quasi-polynomial equations: properties of the roots

The characteristic equation with respect to λ of the unforced original system (7) (and, therefore, (1) – (3)) is

$$D(\lambda, \varepsilon_1, \varepsilon_2) \stackrel{\triangle}{=} \det C(\lambda, \varepsilon_1, \varepsilon_2) = 0, \tag{27}$$

where

$$C(\lambda,\varepsilon_1,\varepsilon_2) = A + G\exp(-g\lambda) + H_1\exp(-\varepsilon_1 h_1\lambda) + H_2\exp(-\varepsilon_2 h_2\lambda) - \lambda E(\varepsilon_1,\varepsilon_2). \quad (28)$$

Similarly, the characteristic equations of the unforced purely slow subsystem (17), mixed slow-fast subsystem (23) and purely fast subsystem (25) are, respectively,

$$D_s(\lambda) \stackrel{\triangle}{=} \det C_s(\lambda) = 0, \quad C_s(\lambda) = \bar{A}_s + \bar{G}_s\exp(-g\lambda) - \lambda I_n, \quad (29)$$

$$D_{sf}(\mu) \stackrel{\triangle}{=} \det C_{sf}(\mu) = 0, \quad C_{sf}(\mu) = \bar{A}_{sf} + \bar{H}_{sf}\exp(-h_1\mu) - \mu I_{m_1}, \quad (30)$$

$$D_f(\nu) \stackrel{\triangle}{=} \det C_f(\nu) = 0, \quad C_f(\nu) = A_{33} + H_{32}\exp(-h_2\nu) - \nu I_{m_2}. \quad (31)$$

Let $\lambda_{s,k}$, $(k = 1, \ldots, k_s)$ be all distinct roots of the quasi-polynomial equation (29) satisfying the inequality

$$\mathrm{Re}\lambda_{s,k} \geq 0, \quad k = 1, \ldots, k_s. \quad (32)$$

By virtue of the results of [22], we have that

$$0 \leq k_s < +\infty. \quad (33)$$

Lemma 3.1. Let the inequality (16) hold. Let $\{\varepsilon_{1,\alpha}\}$, $\{\varepsilon_{2,\alpha}\}$, $(\alpha = 1, 2, \ldots)$ be any two sequences of real numbers and $\{\lambda_\alpha\}$, $(\alpha = 1, 2, \ldots)$ be any sequence of complex numbers such that:

(i) $\varepsilon_{l,\alpha} > 0$, $(\alpha = 1, 2, \ldots;\ l = 1, 2)$;

(ii) $\lim_{\alpha\to+\infty} \varepsilon_{l,\alpha} = 0$, $(l = 1, 2)$;

(iii) $\lim_{\alpha\to+\infty} \varepsilon_{2,\alpha}/\varepsilon_{1,\alpha} = 0$;

(iv) $\mathrm{Re}\lambda_\alpha \geq 0$, $(\alpha = 1, 2, \ldots.)$;

(v) $\lim_{\alpha\to+\infty} \varepsilon_{1,\alpha}\lambda_\alpha = 0$;

(vi) $D(\lambda_\alpha, \varepsilon_{1,\alpha}, \varepsilon_{2,\alpha}) = 0$, $(\alpha = 1, 2, \ldots)$.

Then, the sequence $\{\lambda_\alpha\}$ is bounded. Moreover, $k_s > 0$ and there exists a subsequence of $\{\lambda_\alpha\}$, which converges to one of the numbers $\lambda_{s,k}$, $(k = 1, \ldots, k_s)$.

Proof of the lemma is presented in Appendix A (see Section 7).

Let μ_p, $(p = 1, \ldots, p_{sf})$ be all distinct roots of the quasi-polynomial equations in (30) satisfying the inequality

$$\mathrm{Re}\mu_p \geq 0, \quad p = 1, \ldots, p_{sf}.$$

Similarly to (33),

$$0 \le p_{sf} < +\infty. \tag{34}$$

Consider the block-form matrix

$$E_1(\varepsilon_1) \triangleq \begin{pmatrix} \varepsilon_1 I_n & 0 & 0 \\ 0 & I_{m_1} & 0 \\ 0 & 0 & I_{m_2} \end{pmatrix}.$$

Using this matrix and the matrix $C(\lambda, \varepsilon_1, \varepsilon_2)$ (see the equation (28)), we consider the quasi-polynomial equation with respect to μ

$$\widetilde{D}(\mu, \varepsilon_1, \varepsilon_2) \triangleq \det\Big(E_1(\varepsilon_1) C(\mu/\varepsilon_1, \varepsilon_1, \varepsilon_2)\Big) = 0. \tag{35}$$

For a given pair $(\varepsilon_1 > 0, \varepsilon_2 > 0)$, μ is a root of (35) if and only if $\lambda = \mu/\varepsilon_1$ is a root of the quasi-polynomial equation (27).

Lemma 3.2. Let the inequalities (16) and (22) hold. Let $\{\varepsilon_{1,\alpha}\}$, $\{\varepsilon_{2,\alpha}\}$, $(\alpha = 1, 2, \ldots)$ be any two sequences of real numbers and $\{\mu_\alpha\}$, $(\alpha = 1, 2, \ldots)$ be any sequence of complex numbers such that:

(i) $\varepsilon_{l,\alpha} > 0$, $(\alpha = 1, 2, \ldots;\ l = 1, 2)$;

(ii) $\lim_{\alpha \to +\infty} \varepsilon_{l,\alpha} = 0$, $(l = 1, 2)$;

(iii) $\lim_{\alpha \to +\infty} \varepsilon_{2,\alpha}/\varepsilon_{1,\alpha} = 0$;

(iv) $\mathrm{Re}\mu_\alpha \ge 0$, $(\alpha = 1, 2, \ldots)$;

(v) $\lim_{\alpha \to +\infty} (\varepsilon_{2,\alpha}/\varepsilon_{1,\alpha})\mu_\alpha = 0$;

(vi) $\widetilde{D}(\mu_\alpha, \varepsilon_{1,\alpha}, \varepsilon_{2,\alpha}) = 0$, $(\alpha = 1, 2, \ldots)$.

Then:

(I) the sequence $\{\mu_\alpha\}$ is bounded;

(II) either there exists a subsequence of the sequence $\{\mu_\alpha\}$, $(\alpha = 1, 2, \ldots)$ converging to zero, or $p_{sf} > 0$ and there exists a subsequence of the sequence $\{\mu_\alpha\}$, $(\alpha = 1, 2, \ldots)$ converging to one of the numbers μ_p, $(p = 1, \ldots, p_{sf})$.

Proof of the lemma is presented in Appendix B (see Section 8).

Let ν_q, $(q = 1, \ldots, q_f)$ be all distinct roots of the quasi-polynomial equation in (31) satisfying the inequality

$$\mathrm{Re}\nu_q \ge 0, \qquad q = 1, \ldots, q_f. \tag{36}$$

Similarly to (33) and (34),

$$0 \le q_f < +\infty.$$

Consider the block-form matrix

$$E_2(\varepsilon_1,\varepsilon_2) \stackrel{\triangle}{=} \begin{pmatrix} \varepsilon_2 I_n & 0 & 0 \\ 0 & (\varepsilon_2/\varepsilon_1)I_{m_1} & 0 \\ 0 & 0 & I_{m_2} \end{pmatrix}.$$

Using this matrix, we consider the quasi-polynomial equation with respect to ν

$$\widehat{D}(\nu,\varepsilon_1,\varepsilon_2) \stackrel{\triangle}{=} \det\Big(E_2(\varepsilon_1,\varepsilon_2)C(\nu/\varepsilon_2,\varepsilon_1,\varepsilon_2)\Big) = 0. \quad (37)$$

For a given pair $(\varepsilon_1 > 0, \varepsilon_2 > 0)$, ν is a root of (37) if and only if $\lambda = \nu/\varepsilon_2$ is a root of the quasi-polynomial equation (27).

Similarly to Lemma 3.2, we obtain the following assertion.

Lemma 3.3. Let the inequality (22) hold. Let $\{\varepsilon_{1,\alpha}\}$, $\{\varepsilon_{2,\alpha}\}$, $(\alpha = 1,2,\ldots)$ be any two sequences of real numbers and $\{\nu_\alpha\}$, $(\alpha = 1,2,\ldots)$ be any sequence of complex numbers such that: (i) $\varepsilon_{l,\alpha} > 0$, $(\alpha = 1,2,\ldots;\ l = 1,2)$; (ii) $\lim_{\alpha\to+\infty}\varepsilon_{l,\alpha} = 0$, $(l = 1,2)$; (iii) $\lim_{\alpha\to+\infty}\varepsilon_{2,\alpha}/\varepsilon_{1,\alpha} = 0$; (iv) $\mathrm{Re}\nu_\alpha \geq 0$, $(\alpha = 1,2,\ldots)$; (v) $\widehat{D}(\nu_\alpha,\varepsilon_{1,\alpha},\varepsilon_{2,\alpha}) = 0$, $(\alpha = 1,2,\ldots)$. Then:

(I) the sequence $\{\nu_\alpha\}$ is bounded;

(II) either there exists a subsequence of $\{\nu_\alpha\}$, $(\alpha = 1,2,\ldots)$ converging to zero, or $q_f > 0$ and there exists a subsequence of $\{\nu_\alpha\}$, $(\alpha = 1,2,\ldots)$ converging to one of the numbers ν_q, $(q = 1,\ldots,q_f)$.

Remark 3.4. Lemmas 3.1, 3.2 and 3.3 show the following. For $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, such that $\varepsilon_2/\varepsilon_1 \to 0$, the set of roots $\lambda(\varepsilon_1,\varepsilon_2)$ with nonnegative real parts of the characteristic equation (27) of the unforced original system (7) (and, therefore, (1) – (3)) can be partitioned into three subsets. The roots of the first subset tend to the roots $\lambda_{s,k}$ of the characteristic equation (29) of the unforced purely slow subsystem (17). The products of the roots of the second subset with ε_1 tend to the roots μ_p of the characteristic equation (30) of the unforced mixed slow-fast subsystem (23). The products of the roots of the third subset with ε_2 tend to the roots ν_q of the characteristic equation (31) of the unforced purely fast subsystem (25).

Remark 3.5. Lemmas 3.1, 3.2 and 3.3 are used for a proper estimation of the roots of the characteristic equation (27) of the unforced original system (7) (and, therefore, (1) – (3)). Based on this estimation, the exponential stability of the unforced original system (7) (and, therefore, (1) – (3)), uniformly valid for all sufficiently small $(\varepsilon_1,\varepsilon_2) \in \Omega(\varepsilon_1^*,\varepsilon^*)$, is established. These results are presented in Section 4.1 (see Theorem 4.2 and its proof).

3.2. Exponential stability of the unforced purely slow, mixed slow-fast and purely fast subsystems

In this subsection, we assume that the inequalities (16) and (22) are valid. In such a case, we call the original three time-scale system (1) – (3) the standard system. Two-time scale standard systems can be found, for instance, in [14, 31] and references therein.

Definition 3.6. The unforced purely slow subsystem (17) ($u_s(t) \equiv 0$) is called exponentially stable, if for any function $\psi_s(\eta) \in C[-g, 0; \mathbb{R}^n]$ the solution $x_s(t)$, $t \geq 0$ of this subsystem with the initial condition $x_s(\eta) = \psi_s(\eta)$, $\eta \in [-g, 0]$ satisfies the inequality

$$\|x_s(t)\| \leq c_s \exp(-\omega_s t)\|\psi_s(\eta)\|_C, \qquad t \geq 0,$$

where $c_s > 0$ and $\omega_s > 0$ are some constants independent of $\psi_s(\eta)$.

Definition 3.7. The unforced mixed slow-fast subsystem (23) ($u_{sf}(\xi_1) \equiv 0$) is called exponentially stable, if for any function $\psi_{sf}(\zeta_1) \in C[-h_1, 0; \mathbb{R}^{m_1}]$ the solution $y_{1,sf}(\xi_1)$, $\xi_1 \geq 0$ of this subsystem with the initial condition $y_{1,sf}(\zeta_1) = \psi_{sf}(\zeta_1)$, $\zeta_1 \in [-h_1, 0]$ satisfies the inequality

$$\|y_{1,sf}(\xi_1)\| \leq c_{sf} \exp(-\omega_{sf}\xi_1)\|\psi_{sf}(\zeta_1)\|_C, \qquad \xi_1 \geq 0,$$

where $c_{sf} > 0$ and $\omega_{sf} > 0$ are some constants independent of $\psi_{sf}(\zeta_1)$.

Definition 3.8. The unforced purely fast subsystem (25) ($u_f(\xi_2) \equiv 0$) is called exponentially stable, if for any function $\psi_f(\zeta_2) \in C[-h_2, 0; \mathbb{R}^{m_2}]$ the solution $y_{2,f}(\xi_2)$, $\xi_2 \geq 0$ of this subsystem with the initial condition $y_{2,f}(\zeta_2) = \psi_f(\zeta_2)$, $\zeta_2 \in [-h_2, 0]$ satisfies the inequality

$$\|y_{2,f}(\xi_2)\| \leq c_f \exp(-\omega_f\xi_2)\|\psi_f(\zeta_2)\|_C, \qquad \xi_2 \geq 0,$$

where $c_f > 0$ and $\omega_f > 0$ are some constants independent of $\psi_f(\zeta_2)$.

Let $\mathcal{S}_s$, $\mathcal{S}_{sf}$ and $\mathcal{S}_f$ be the sets of all distinct roots of the quasi-polynomial equations (29), (30) and (31), respectively.

By virtue of the results of the work [22], we directly have the following three assertions.

Proposition 3.9. The unforced purely slow subsystem (17) ($u_s(t) \equiv 0$) is exponentially stable if and only if the following inequality is satisfied

$$\beta_s \stackrel{\triangle}{=} \max_{\lambda \in \mathcal{S}_s} \mathrm{Re}\lambda < 0. \tag{38}$$

Proposition 3.10. The unforced mixed slow-fast subsystem (23) ($u_{sf}(\xi_1) \equiv 0$) is exponentially stable if and only if the following inequality is satisfied

$$\beta_{sf} \stackrel{\triangle}{=} \max_{\mu \in \mathcal{S}_{sf}} \mathrm{Re}\mu < 0. \tag{39}$$

Proposition 3.11. The unforced purely fast subsystem (25) ($u_f(\xi_2) \equiv 0$) is exponentially stable if and only if the following inequality is satisfied

$$\beta_f \stackrel{\triangle}{=} \max_{\nu \in \mathcal{S}_f} \mathrm{Re}\nu < 0. \tag{40}$$

3.3. Stabilization of the controlled purely slow, mixed slow-fast and purely fast subsystems

In this subsection, we assume that the inequalities (16) and (22) are not, in general, valid. In such a case, we call the original three time-scale system (1) – (3) the non-standard system. Two-time scale non-standard systems can be found, for instance, in [14, 31] and references therein. Since the inequalities (16) and (22) are not, in general, valid, then the purely slow subsystem (17) and the mixed slow-fast subsystem (23) are not, in general, exist. Therefore, to study the stabilization of the purely slow subsystem and the mixed slow-fast subsystem, we consider these subsystems in their differential-algebraic form (11) – (13) and (19), respectively.

Let us start the stabilization analysis of the purely slow, mixed slow-fast and purely fast subsystems with such an analysis of the purely fast subsystem (25). For this subsystem, let us consider the memory-free state-feedback control

$$u_f(\xi_2) = u_f[y_{2,f}(\xi_2)] = K_f y_{2,f}(\xi_2), \tag{41}$$

where K_f is an $r \times m_2$-matrix.

Definition 3.12. The purely fast subsystem (25) is called stabilized by a memory-free state-feedback control, if there exists the control (41) such that the closed-loop system (25), (41) is exponentially stable.

Let the control (41), mentioned in Definition 3.12, exists. Substituting this control into (25), we obtain the corresponding closed-loop system

$$\frac{dy_{2,f}(\xi_2)}{d\xi_2} = (A_{33} + B_3 K_f) y_{2,f}(\xi_2) + H_{32} y_{2,f}(\xi_2 - h_2), \qquad \xi_2 \geq 0. \tag{42}$$

The characteristic equation with respect to ν of the system (42) has the form

$$\det\big(A_{33} + B_3 K_f + H_{32}\exp(-h_2\nu) - \nu I_{m_2}\big) = 0. \tag{43}$$

Taking into account Definition 3.12 and applying Proposition 3.11 to the system (42), we can conclude immediately that the real parts of all the roots of the quasi-polynomial equation (43) are negative. Hence, the number $\nu = 0$ is not a root of (43), implying the following inequality:

$$\det\big(A_{33,s} + B_3 K_f\big) \neq 0, \tag{44}$$

where $A_{33,s}$ is given in (14).

Proceed to the mixed slow-fast subsystem (19). For this subsystem, we consider the memory-free state-feedback control

$$u_{sf}(\xi_1) = u_{sf}[y_{1,sf}(\xi_1), y_{2,sf}(\xi_1)] = K_{sf} y_{1,sf}(\xi_1) + K_f y_{2,sf}(\xi_1), \tag{45}$$

where K_{sf} is an $r \times m_1$-matrix; K_f is the gain-matrix in the control (41).

Definition 3.13. The mixed slow-fast subsystem (19) is called stabilized by a memory-free state-feedback control, if there exists the control (45) such that the closed-loop system (19), (45) is exponentially stable. The latter means that, for any function $\psi_{sf}(\zeta_1) \in C[-h_1, 0; \mathbb{R}^{m_1}]$, the solution $\mathrm{col}\big(y_{1,sf}(\xi_1), y_{2,sf}(\xi_1)\big)$, $\xi_1 \geq 0$ of this system with the initial condition

$$y_{1,sf}(\zeta_1) = \psi_{sf}(\zeta_1), \qquad \zeta_1 \in [-h_1, 0] \tag{46}$$

exists, is unique and satisfies the inequality

$$\big\|\mathrm{col}\big(y_{1,sf}(\xi_1), y_{2,sf}(\xi_1)\big)\big\| \leq a_{sf} \exp(-\kappa_{sf}\xi_1)\|\psi_{sf}(\zeta_1)\|_C, \qquad \xi_1 \geq 0, \tag{47}$$

where $a_{sf} > 0$ and $\kappa_{sf} > 0$ are some constants independent of $\psi_{sf}(\zeta_1)$.

Lemma 3.14. Let the control (41) stabilize the purely fast subsystem (25). Then, the control (45) stabilizes the mixed slow-fast subsystem (19) if and only if the control

$$v_{sf}(\xi_1) = v_{sf}[y_{1,sf}(\xi_1)] = K_{sf} y_{1,sf}(\xi_1) \tag{48}$$

stabilizes the following system:

$$\begin{aligned} \frac{dy_{1,sf}(\xi_1)}{d\xi_1} \;\; &= \;\; \mathcal{A}_{sf}(K_f) y_{1,sf}(\xi_1) + \mathcal{H}_{sf}(K_f) y_{1,sf}(\xi_1 - h_1) \\ &\quad + \mathcal{B}_{sf}(K_f) v_{sf}(\xi_1), \quad \xi_1 \geq 0, \end{aligned} \tag{49}$$

where

$$\begin{aligned} \mathcal{A}_{sf}(K_f) &= A_{22} - (A_{23,s} + B_2 K_f)(A_{33,s} + B_3 K_f)^{-1} A_{32}, \\ \mathcal{H}_{sf}(K_f) &= H_{21} - (A_{23,s} + B_2 K_f)(A_{33,s} + B_3 K_f)^{-1} H_{31}, \\ \mathcal{B}_{sf}(K_f) &= B_2 - (A_{23,s} + B_2 K_f)(A_{33,s} + B_3 K_f)^{-1} B_3. \end{aligned} \tag{50}$$

Proof of the lemma is presented in Appendix C (see Section 9).

Corollary 3.15. Let the control (41) stabilize the purely fast subsystem (25). Let the control (45) stabilize the mixed slow-fast subsystem (19). Then, the following inequality is valid:

$$\det\big(\mathcal{A}_{sf}(K_f) + \mathcal{B}_{sf}(K_f) K_{sf} + \mathcal{H}_{sf}(K_f)\big) \neq 0. \tag{51}$$

P r o o f. Based on Lemma 3.14, the validity of the inequality (51) is proven similarly to the validity of the inequality (44). □

Now, let us treat the purely slow subsystem (11)−(13). For this subsystem, we consider the memory-free state-feedback control

$$u_s(t) = u_s[x_s(t), y_{1,s}(t), y_{2,s}(t)] = K_s x_s(t) + K_{sf} y_{1,s}(t) + K_f y_{2,s}(t), \qquad t \geq 0, \tag{52}$$

where K_s is an $r \times n$-matrix; K_{sf} and K_f are the gain-matrices in the controls (45) and (41).

Definition 3.16. The purely slow subsystem (11) – (13) is called stabilized by a memory-free state-feedback control, if there exists the control (52) such that the closed-loop system (11) – (13), (52) is exponentially stable. The latter means that, for any function $\psi_s(\eta) \in C[-g, 0; \mathbb{R}^n]$, the solution $\mathrm{col}\big(x_s(t), y_{1,s}(t), y_{2,s}(t)\big)$, $t \geq 0$ of this system with the initial condition

$$x_s(\eta) = \psi_s(\eta), \qquad \eta \in [-g, 0] \tag{53}$$

exists, is unique and satisfies the inequality

$$\big\|\mathrm{col}\big(x_s(t), y_{1,s}(t), y_{2,s}(t)\big)\big\| \leq a_s \exp(-\kappa_s t)\|\psi_s(\eta)\|_C, \qquad t \geq 0, \tag{54}$$

where $a_s > 0$ and $\kappa_s > 0$ are some constants independent of $\psi_s(\eta)$.

Lemma 3.17. Let the controls (41) and (45) stabilize the purely fast subsystem (25) and the mixed slow-fast subsystem (19), respectively. Then, the control (52) stabilizes the purely slow subsystem (11) – (13) if and only if the control

$$v_s(t) = v_s[x_s(t)] = K_s x_s(t) \tag{55}$$

stabilizes the following system:

$$\frac{dx_s(t)}{dt} = \mathcal{A}_s(K_{sf}, K_f)x_s(t) + \mathcal{G}_s(K_{sf}, K_f)x_s(t-g) + \mathcal{B}_s(K_{sf}, K_f)v_s(t), \ t \geq 0, \tag{56}$$

where

$$\begin{aligned} \mathcal{A}_s(K_{sf}, K_f) &= A_{11} - \big(\mathcal{A}_{1s} + B_1 \cdot (K_{sf} \ , \ K_f)\big)\big(W(K_{sf}, K_f)\big)^{-1}\mathcal{A}_{2s}, \\ \mathcal{G}_s(K_{sf}, K_f) &= G_1 - \big(\mathcal{A}_{1s} + B_1 \cdot (K_{sf} \ , \ K_f)\big)\big(W(K_{sf}, K_f)\big)^{-1}\mathcal{G}_{23,s}, \\ \mathcal{B}_s(K_{sf}, K_f) &= B_1 - \big(\mathcal{A}_{1s} + B_1 \cdot (K_{sf} \ , \ K_f)\big)\big(W(K_{sf}, K_f)\big)^{-1}\mathcal{B}_{23,s}, \\ W(K_{sf}, K_f) &= \mathcal{A}_{3s} + \mathcal{B}_{23,s} \cdot (K_{sf} \ , \ K_f), \end{aligned} \tag{57}$$

and the matrices $\mathcal{A}_{1s}$, $\mathcal{A}_{2s}$, $\mathcal{A}_{3s}$, $\mathcal{G}_{23,s}$, $\mathcal{B}_{23,s}$ are given in (15).

Proof of the lemma is presented in Appendix D (see Section 10).

Remark 3.18. The gain-matrices K_f, K_{sf} and K_s, which appear in stabilizing controls (41), (48) and (55), can be derived using various methods (see e. g. [13, 18, 35, 37, 41] and references therein).

Remark 3.19. Using the stabilizing control (41) for the purely fast subsystem (25), Lemma 3.14 allows to reduce the stabilization problem for the $(m_1 + m_2)$-dimensional differential-algebraic system (19) (the mixed slow-fast subsystem) to the simpler stabilization problem for the lower dimension (m_1-dimensional) differential system (49). Similarly, using the stabilizing controls (41) and (45) for the purely fast subsystem (25) and the mixed slow-fast subsystem (19), respectively, Lemma 3.17 allows to reduce the stabilization problem for the $(n+m_1+m_2)$-dimensional differential-algebraic system (11) – (13) (the purely slow subsystem) to the simpler stabilization problem for the lower dimension (n-dimensional) differential system (56).

Remark 3.20. Lemmas 3.14, 3.17 and the corresponding gain-matrices K_f, K_{sf}, K_s are used for the design of a memory-free linear state-feedback control, which stabilizes the original singularly perturbed system (7) (and, therefore, the system (1)–(3)) uniformly in ε_1 and ε_2 for all sufficiently small values $(\varepsilon_1, \varepsilon_2) \in \Omega(\varepsilon_1^*, \varepsilon^*)$. This result is presented in Section 4.2 (see Theorem 4.4 and its proof).

4. MAIN RESULTS

4.1. Parameters-free conditions for the exponential stability of the unforced original system

First of all, let us present the parameters-dependent criterion of the exponential stability of the unforced original system (1)–(3) ($u(t) \equiv 0$). For a given pair ($\varepsilon_1 > 0$, $\varepsilon_2 > 0$), let $\mathcal{S}(\varepsilon_1, \varepsilon_2)$ be the set of all distinct roots of the quasi-polynomial equation (27). By virtue of the results of the work [22], we immediately obtain the assertion.

Proposition 4.1. For a given pair ($\varepsilon_1 > 0$, $\varepsilon_2 > 0$), the unforced original system (7) (and, therefore, the unforced system (1)–(3)) ($u(t) \equiv 0$) is exponentially stable if and only if the following inequality is satisfied

$$\beta(\varepsilon_1, \varepsilon_2) \overset{\triangle}{=} \max_{\lambda \in \mathcal{S}(\varepsilon_1, \varepsilon_2)} \mathrm{Re}\lambda < 0. \tag{58}$$

The following theorem presents the parameters-free sufficient conditions for the exponential stability of the unforced original system (1)–(3) ($u(t) \equiv 0$).

Theorem 4.2. Let the inequalities (16) and (22) be valid. Let the unforced purely slow (17), mixed slow-fast (23) and purely fast (25) subsystems be exponentially stable. Then, there exist numbers ε_1^* and ε^*, satisfying the inequalities in (8), such that for all $(\varepsilon_1, \varepsilon_2) \in \Omega(\varepsilon_1^*, \varepsilon^*)$ (see the equation (9)) the unforced original singularly perturbed system (7) (and, therefore, the unforced system (1)–(3)) is exponentially stable.

P r o o f. We prove the theorem by contradiction. Namely, we suppose that its statement is wrong. Then, due to Proposition 4.1 and the assumption (4), there exist two sequences of real numbers $\{\varepsilon_{1,\alpha}\}$, $\{\varepsilon_{2,\alpha}\}$, $(\alpha = 1, 2, \ldots)$ and the sequence of complex numbers $\{\lambda_\alpha\}$, $(\alpha = 1, 2, \ldots)$ such that:

(a) $\varepsilon_{l,\alpha} > 0$, $(\alpha = 1, 2, \ldots;\ l = 1, 2)$;

(b) $\lim_{\alpha \to +\infty} \varepsilon_{l,\alpha} = 0$, $(l = 1, 2)$;

(c) $\lim_{\alpha \to +\infty} \varepsilon_{2,\alpha}/\varepsilon_{1,\alpha} = 0$;

(d) $\mathrm{Re}\lambda_\alpha \geq 0$, $(\alpha = 1, 2, \ldots)$;

(e) $\lambda_\alpha \in \mathcal{S}(\varepsilon_{1,\alpha}, \varepsilon_{2,\alpha})$, $(\alpha = 1, 2, \ldots)$.

Let us consider the sequence $\{\nu_\alpha\} = \{\varepsilon_{2,\alpha}\lambda_\alpha\}$, $(\alpha = 1, 2, \ldots)$. Due to the conditions (a) – (e) on the sequences $\{\varepsilon_{1,\alpha}\}$, $\{\varepsilon_{2,\alpha}\}$ and $\{\lambda_\alpha\}$, $(\alpha = 1, 2, \ldots)$, the sequences $\{\varepsilon_{1,\alpha}\}$, $\{\varepsilon_{2,\alpha}\}$ and $\{\nu_\alpha\}$, $(\alpha = 1, 2, \ldots)$ satisfy the conditions (i) – (v) of Lemma 3.3. Hence, the sequence $\{\nu_\alpha\}$, $(\alpha = 1, 2, \ldots)$ is bounded. Moreover, there exists a subsequence of this sequence converging to one of the numbers ν_q, $(q = 0, 1, \ldots, q_f)$, where $\nu_0 = 0$ and ν_q, $(q = 1, \ldots, q_f)$ are all distinct roots of the quasi-polynomial equation in (31) satisfying the inequality (36). However, due to the assumption on the exponential stability of the unforced purely fast subsystem (25) and Proposition 3.11, we immediately have $q_f = 0$. Therefore, the limit of the above mentioned converging subsequence of $\{\nu_\alpha\}$, $(\alpha = 1, 2, \ldots)$ is zero. For the sake of simplicity (but without loss of generality), let us assume that $\{\nu_\alpha\}$ itself is such a subsequence. Thus $\lim_{\alpha\to+\infty} \nu_\alpha = 0$. Now, we consider the sequence $\{\mu_\alpha\} = \{\varepsilon_{1,\alpha}\lambda_\alpha\}$, $(\alpha = 1, 2, \ldots)$. Hence, $\nu_\alpha = (\varepsilon_{2,\alpha}/\varepsilon_{1,\alpha})\mu_\alpha$, $(\alpha = 1, 2, \ldots)$. The latter yields that $\mathrm{Re}\mu_\alpha \geq 0$, $(\alpha = 1, 2, \ldots)$, and $\lim_{\alpha\to+\infty}(\varepsilon_{2,\alpha}/\varepsilon_{1,\alpha})\mu_\alpha = 0$. Therefore, the sequences $\{\varepsilon_{1,\alpha}\}$, $\{\varepsilon_{2,\alpha}\}$ and $\{\mu_\alpha\}$, $(\alpha = 1, 2, \ldots)$ satisfy the conditions (i) – (vi) of Lemma 3.2. Using this observation, as well as the assumption on the exponential stability of the unforced mixed slow-fast subsystem (23) and Proposition 3.10, we obtain (similarly to the analysis of the sequence $\{\nu_\alpha\}$) that $\lim_{\alpha\to+\infty} \mu_\alpha = 0$. This implies the limit equality $\lim_{\alpha\to+\infty} \varepsilon_{1,\alpha}\lambda_\alpha = 0$ meaning that the sequences $\{\varepsilon_{1,\alpha}\}$, $\{\varepsilon_{2,\alpha}\}$ and $\{\lambda_\alpha\}$, $(\alpha = 1, 2, \ldots)$ satisfy the assumptions (i) – (vi) of Lemma 3.1. By virtue of this lemma, the quasi-polynomial equation (29) has at least one root with nonnegative real part. However the latter, along with Proposition 3.9, contradicts the assumption on the exponential stability of the unforced purely slow subsystem (17). This contradiction means that the statement of the theorem is correct, which completes its proof. □

Remark 4.3. Due to Definition 2.1, Theorem 4.2 means that the unforced original singularly perturbed system (7) (and, therefore, the unforced system (1) – (3)) is exponentially stable uniformly with respect to $(\varepsilon_1, \varepsilon_2) \in \Omega(\varepsilon_1^*, \varepsilon^*)$.

4.2. Parameters-free conditions for the stabilization of the original system by a memory-free state-feedback control

Consider the control (10) with the following gain matrix:

$$K_z = (K_s \ , \ K_{sf} \ , \ K_f), \tag{59}$$

where K_s, K_{sf} and K_f are the gain-matrices in the controls (52), (45) and (41).

Theorem 4.4. Let the controls (41), (45) and (52) stabilize the purely fast subsystem (25), the mixed slow-fast subsystem (19) and the purely slow subsystem (11) – (13), respectively. Then, there exist numbers ε_1^* and ε^*, satisfying the inequalities in (8), such that the original singularly perturbed system (7) (and, therefore, the system (1) – (3)) is stabilized by the control (10), (59) uniformly in $(\varepsilon_1, \varepsilon_2) \in \Omega(\varepsilon_1^*, \varepsilon^*)$, where the domain $\Omega(\varepsilon_1^*, \varepsilon^*)$ is given by the equation (9).

P r o o f. Substitution of the control (10), (59) into the system (7) yields the closed-loop system

$$E(\varepsilon_1,\varepsilon_2)\frac{dz(t)}{dt} = [A + B\cdot(K_s\ ,\ K_{sf}\ ,\ K_f)]z(t) + Gz(t-g) \\ +H_1 z(t-\varepsilon_1 h_1) + H_2 z(t-\varepsilon_2 h_2), \quad t \geq 0. \tag{60}$$

This system is a three time-scale system with delays similar to the unforced system (7). Let us construct purely slow, mixed slow-fast and purely fast subsystems, associated with the system (60). Using the block form of the vector $z(t)$ (see the equation (5)) and the block forms of the matrices $E(\varepsilon_1,\varepsilon_2)$, A, B, G (see the equation (6)), we obtain by a routine algebra that the purely slow subsystem in the differential form, associated with the system (60), has the form

$$\frac{dx_s(t)}{dt} = [\mathcal{A}_s(K_{sf},K_f) + \mathcal{B}_s(K_{sf},K_f)K_s]x_s(t) + \mathcal{G}_s(K_{sf},K_f)x_s(t-g), \quad t \geq 0.$$

Similarly, the mixed slow-fast subsystem in the differential form and the purely fast subsystem, associated with the system (60), are

$$\frac{dy_{1,sf}(\xi_1)}{d\xi_1} = [\mathcal{A}_{sf}(K_f) + \mathcal{B}_{sf}(K_f)K_{sf}]y_{1,sf}(\xi_1) + \mathcal{H}_{sf}(K_f)y_{1,sf}(\xi_1 - h_1),\ \xi_1 \geq 0$$

and (42), respectively. Due to this observation and the results of Subsection 3.3 (see Definitions 3.12, 3.13, 3.16 and Lemmas 3.14, 3.17), the purely slow subsystem in the differential form, the mixed slow-fast subsystem in the differential form and the purely fast subsystem, associated with the system (60), are exponentially stable. Now, the application of Definition 2.1, Theorem 4.2 and Remark 4.3 to the system (60) directly yields the existence of numbers ε_1^* and ε^*, satisfying the inequalities in (8), such that this system is exponentially stable uniformly with respect to $(\varepsilon_1,\varepsilon_2)\in\Omega(\varepsilon_1^*,\varepsilon^*)$. The latter, along with Definition 2.2, means the stabilization of the original singularly perturbed system (7) (and, therefore, the system (1)–(3)) by the control (10), (59) uniformly in $(\varepsilon_1,\varepsilon_2)\in\Omega(\varepsilon_1^*,\varepsilon^*)$. This completes the proof of the theorem. □

5. EXAMPLES

5.1. Example 1: exponential stability

Consider the particular case of the unforced original system (1)–(3) ($u(t)\equiv 0$) with the following data:

$$n = 2, \quad m_1 = m_2 = 1, \quad g = 0.6, \quad h_1 = 0.8, \quad h_2 = 1,$$

$$A_{11} = \begin{pmatrix} -5 & 2 \\ 1 & -6 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad A_{13} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad A_{21} = (2\ ,\ -1),$$

$$A_{22} = -5, \quad A_{23} = -2, \quad A_{31} = (-4\ ,\ 2), \quad A_{32} = -1, \quad A_{33} = -4,$$

$$G_1 = \begin{pmatrix} 3 & 1 \\ 2 & -4 \end{pmatrix}, \quad G_2 = (-1\ ,\ 1), \quad G_3 = (1\ ,\ -1),$$

$$H_{11} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad H_{12} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}, \quad H_{21} = 4, \quad H_{22} = 2, \quad H_{31} = 1, \quad H_{32} = 3. \tag{61}$$

Using the equations (14), (15) and (61), we have by a direct calculation

$$A_{33,s} = -1, \qquad \mathcal{A}_{3s} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, the inequalities (16) and (22) are valid, meaning that the unforced original system (1)–(3) ($u(t) \equiv 0$) with the data (61) is standard.

Let us construct the purely slow, mixed slow-fast and purely fast subsystems of the unforced system (1)–(3) with the data (61). Using the results of Subsection 2.2, we obtain that the purely slow subsystem has the form

$$\frac{dx_s(t)}{dt} = \bar{A}_s x_s(t) + \bar{G}_s x_s(t - 0.6), \qquad t \geq 0, \tag{62}$$

where $x_s(t) \in \mathbb{R}^2$, $t \geq -g$, and

$$\bar{A}_s = \begin{pmatrix} -7 & 3 \\ 1 & -6 \end{pmatrix}, \qquad \bar{G}_s = \begin{pmatrix} 4 & 0 \\ 2 & -4 \end{pmatrix}. \tag{63}$$

Similarly, using the results of Subsections 2.3 and 2.4, we have the scalar mixed slow-fast subsystem and the scalar purely fast subsystem, respectively,

$$\frac{dy_{1,sf}(\xi_1)}{d\xi_1} = -5y_{1,sf}(\xi_1) + 4y_{1,sf}(\xi_1 - 0.8), \qquad \xi_1 \geq 0 \tag{64}$$

and

$$\frac{dy_{2,f}(\xi_2)}{d\xi_2} = -4y_{2,f}(\xi_2) + 3y_{2,f}(\xi_2 - 1), \qquad \xi_2 \geq 0. \tag{65}$$

Now, let us analyze the exponential stability of the above obtained subsystems. We start with the subsystem (62)–(63). Using the equations (29) and (63), we obtain the characteristic equation of this subsystem as:

$$\lambda^2 + 13\lambda + 39 - 2\exp(-0.6\lambda) - 16\exp(-1.2\lambda) = 0. \tag{66}$$

Let us show the fulfilment of the inequality (38) for the quasi-polynomial equation (66). Let us consider the domain $\mathrm{Re}\lambda \geq -0.5$ in the complex plane and estimate the functions $|\lambda^2 + 13\lambda + 39|$ and $|2\exp(-0.6\lambda) + 16\exp(-1.2\lambda)|$ in this domain. By a routine algebra we have the following:

$$\min_{\mathrm{Re}\lambda \geq -0.5} |\lambda^2 + 13\lambda + 39| = 32.75,$$

$$\max_{\mathrm{Re}\lambda \geq -0.5} |2\exp(-0.6\lambda) + 16\exp(-1.2\lambda)| < 31.8537.$$

These estimates directly yield that the equation (66) does not have roots in the domain $\mathrm{Re}\lambda \geq -0.5$. Therefore, all the roots of this equation satisfy the inequality $\mathrm{Re}\lambda < -0.5$, meaning the fulfilment of the inequality (38). Thus, by virtue of Proposition 3.9, the subsystem (62) – (63) is exponentially stable.

Proceed to the subsystem (64). Using the equation (30), we have the characteristic equation of this subsystem in the form

$$\mu + 5 - 4\exp(-0.8\mu) = 0. \tag{67}$$

Direct estimation of the expression in the left-hand side of this equation yields

$$\min_{\mathrm{Re}\mu \geq -0.2} \mathrm{Re}\big(\mu + 5 - 4\exp(-0.8\mu)\big) > 0.1059,$$

meaning that all the roots of the quasi-polynomial equation (67) satisfy the inequality $\mathrm{Re}\mu < -0.2$. Therefore, due to Proposition 3.10, the subsystem (64) is exponentially stable.

Using the equation (31) and Proposition 3.11, one can show (quite similarly to the analysis of the subsystem (64)) the exponential stability of the subsystem (65).

Thus, the unforced system (1) – (3) with the data (61) satisfies all the conditions of Theorem 4.2. The latter implies the existence of numbers ε_1^* and ε^*, satisfying the inequalities in (8), such that the unforced system (1) – (3), (61) is exponentially stable uniformly with respect to $(\varepsilon_1, \varepsilon_2) \in \Omega(\varepsilon_1^*, \varepsilon^*)$.

5.2. Example 2: stabilization

In this example, we consider the particular case of the original system (1) – (3) with the following data:

$$\begin{gathered}
n = 2, \quad m_1 = m_2 = 1, \quad r = 1, \quad g = 1, \quad h_1 = \frac{\pi}{2}, \quad h_2 = 2, \\
A_{11} = \begin{pmatrix} 1 & 2 \\ 0.8 & -1.6 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad A_{13} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad A_{21} = (2\ ,\ -4), \\
A_{22} = 3, \quad A_{23} = -6, \quad A_{31} = (-1\ ,\ 2), \quad A_{32} = 1, \quad A_{33} = 1, \\
G_1 = \begin{pmatrix} 6 & -6 \\ -4.8 & 4.8 \end{pmatrix}, \quad G_2 = (-2\ ,\ 2), \quad G_3 = (1\ ,\ -1), \\
H_{11} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad H_{12} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad H_{21} = -4, \quad H_{22} = 1, \quad H_{31} = -1, \quad H_{32} = -1, \\
B_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad B_2 = 2, \quad B_3 = 1.
\end{gathered} \tag{68}$$

Using the equations (14), (15) and (68), we directly obtain

$$A_{33,s} = 0, \quad \mathcal{A}_{3s} = \begin{pmatrix} -1 & -5 \\ 0 & 0 \end{pmatrix}. \tag{69}$$

Thus, the inequalities (16) and (22) are not valid, meaning that the original system (1) – (3) with the data (68) is nonstandard. Let us establish the stabilization of this system by a memory-free state-feedback control uniform with respect to $(\varepsilon_1, \varepsilon_2)$. Due to Theorem 4.4 and Lemmas 3.14, 3.17, to establish such a stabilization, we should show the stabilization of the purely fast subsystem (25) by the control (41), the stabilization of the system (49) by the control (48) and the stabilization of the system (56) by the control (55).

Let us start with the purely fast subsystem (25). In the present example (see the data (68)), this subsystem becomes the following scalar system:

$$\frac{dy_{2,f}(\xi_2)}{d\xi_2} = y_{2,f}(\xi_2) - y_{2,f}(\xi_2 - 2) + u_f(\xi_2), \qquad \xi_2 \geq 0. \tag{70}$$

Note, that the unforced system (70) ($u_f(\xi_2) \equiv 0$) is not exponentially stable, because its characteristic equation has a root equals zero. To stabilize the system (70), let us choose the control (41) as:

$$u_f(\xi_2) = u_f[y_{2,f}(\xi_2)] = -5y_{2,f}(\xi_2), \qquad \xi_2 \geq 0, \tag{71}$$

i. e., $K_f = -5$. The closed-loop system (70), (71) becomes

$$\frac{dy_{2,f}(\xi_2)}{d\xi_2} = -4y_{2,f}(\xi_2) - y_{2,f}(\xi_2 - 2), \qquad \xi_2 \geq 0, \tag{72}$$

and its characteristic equation with respect to ν is

$$\nu + 4 + \exp(-2\nu) = 0. \tag{73}$$

Direct estimation of the expression in the left-hand side of this equation yields

$$\min_{\mathrm{Re}\nu \geq -0.5} \mathrm{Re}\big(\nu + 4 + \exp(-2\nu)\big) > 0.2817,$$

meaning that all the roots of the quasi-polynomial equation (73) satisfy the inequality $\mathrm{Re}\mu < -0.5$. Therefore, due to Proposition 3.11, the system (72) is exponentially stable. Thus, due to Definition 3.12, the purely fast subsystem (70) is stabilized by the memory-free control (71).

Proceed to the system (49). Using the equations (15), (50), (69) and the data of the present example (68), we directly obtain this system as the following scalar one:

$$\frac{dy_{1,sf}(\xi_1)}{d\xi_1} = -y_{1,sf}(\xi_1 - \pi/2) - v_{sf}(\xi_1), \qquad \xi_1 \geq 0. \tag{74}$$

The unforced system (74) ($v_{sf}(\xi_1) \equiv 0$) is not exponentially stable, because its characteristic equation has the purely imaginary roots $\pm i$ (here i denotes the imaginary unit). To stabilize the system (74), we choose the control (48) in the form

$$v_{sf}(\xi_1) = v_{sf}[y_{1,sf}(\xi_1)] = 4y_{1,sf}(\xi_1), \qquad \xi_1 \geq 0, \tag{75}$$

i. e., $K_{sf} = 4$. The closed-loop system (74), (75) has the form

$$\frac{dy_{1,sf}(\xi_1)}{d\xi_1} = -4y_{1,sf}(\xi_1) - y_{1,sf}(\xi_1 - \pi/2), \qquad \xi_1 \geq 0.$$

The exponential stability of this system is shown quite similarly to the exponential stability of the system (72). Thus, the system (74) is stabilized by the memory-free state-feedback control (75).

Now, let us treat the system (56). Using the equations (15), (57) and the data of the present example (68), we directly obtain the matrices of the coefficients in this system as:

$$\mathcal{A}_s(K_{sf}, K_f) = \begin{pmatrix} 7 & -10 \\ -4 & 8 \end{pmatrix}, \quad \mathcal{G}_s(K_{sf}, K_f) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{B}_s(K_{sf}, K_f) = \begin{pmatrix} -0.2 \\ 0.88 \end{pmatrix}. \tag{76}$$

Note that in the system (56), (76), the state variable $x_s(t)$ is a two-dimensional vector, while the control $v_s(t)$ is scalar.

It is verified by an immediate calculation that the roots of the characteristic equation of the unforced system (56), (76) ($v_s(t) \equiv 0$) are real positive. Therefore, the unforced system (56), (76) is not exponentially stable. To stabilize the system (56), (76), we choose the control (55) as:

$$v_s(t) = v_s[x_s(t)] = (50\ ,\ -25)x_s(t), \qquad t \geq 0, \tag{77}$$

i. e., $K_s = (50\ ,\ -25)$. The closed-loop system (56), (76), (77) is

$$\frac{dx_s(t)}{dt} = \begin{pmatrix} -3 & -5 \\ 40 & -14 \end{pmatrix} x_s(t), \qquad t \geq 0. \tag{78}$$

The eigenvalues of the matrix of the coefficients in this system are complex conjugate numbers with the real part equals -8.5. Therefore, the system (78) is exponentially stable, meaning that the system (56), (76) is stabilized by the memory-free state-feedback control (77).

Based on the gains K_f, K_{sf} and K_s of the controls (71), (75) and (77), we construct the following memory-free state-feedback control for the system (1) – (3), (68):

$$u(t) = u[x(t), y_1(t), y_2(t)] = (50\ ,\ -25)x(t) + 4y_1(t) - 5y_2(t), \qquad t \geq 0. \tag{79}$$

Taking into account the above shown stabilization of the systems (70), (74) and (56), (76) by the controls (71), (75) and (77), respectively, and using Lemmas 3.14 and 3.17, we can conclude that the system (1) – (3) with the data (68) satisfies all the conditions of Theorem 4.4. This implies the existence of numbers ε_1^* and ε^*, satisfying the inequalities in (8), such that the system (1) – (3), (68) is stabilized by the control (79) uniformly in $(\varepsilon_1, \varepsilon_2) \in \Omega(\varepsilon_1^*, \varepsilon^*)$.

5.3. Example 3: stabilization of car-following model with three scales of time

In this example, we consider the car-following model (or the vehicular traffic flow model) (see e. g. [11, 12, 18] and references therein). Here, we treat the case of four vehicles following each other in one lane, which has the geometric shape of a simple open curve (for instance, a straight line). For this shape of the lane, the car-following model can be represent as the following system of time delay differential equations (see e. g. [18]):

$$\begin{aligned} \frac{dX_{F1}(t)}{dt} &= \big(X_L(t-\eta_L) - X_{F1}(t-\eta_1)\big)/\tau_1, \\ \frac{dX_{F2}(t)}{dt} &= \big(X_{F1}(t-\eta_1) - X_{F2}(t-\eta_2)\big)/\tau_2, \\ \frac{dX_{F3}(t)}{dt} &= \big(X_{F2}(t-\eta_2) - X_{F3}(t-\eta_3)\big)/\tau_3, \end{aligned} \tag{80}$$

where $t \geq 0$; $X_L(t)$ is the current speed of the leading vehicle; $X_{F1}(t)$ is the current speed of the first following vehicle; $X_{F2}(t)$ is the current speed of the second following vehicle; $X_{F3}(t)$ is the current speed of the third following vehicle; $\tau_1 > 0$, $\tau_2 > 0$ and $\tau_3 > 0$ are the time constants of the first, second and third following vehicles; $\eta_L > 0$ is the delay in the reaction of the driver of the leading vehicle; $\eta_1 > 0$, $\eta_2 > 0$ and $\eta_3 > 0$ are the delays in the reaction of the drivers of the first, second and third following vehicles.

In the sequel of this example, we assume the following:

(i) the reaction of the driver of the leading vehicle is instantaneous, i. e., $\eta_L = 0$;

(ii) the speed of the leading vehicle $X_L(t)$ is a control $U_L(t)$ in the system (80) at the disposal of the driver of this vehicle, i. e., $X_L(t) = U_L(t)$;

(iii) $\tau_1 \ll \tau_3$, $\tau_2 \ll \tau_3$, $\tau_1 \ll \tau_2$;

(iv) $\eta_1/\tau_1 \sim O(1)$, $\eta_2/\tau_2 \sim O(1)$, $\eta_3/\tau_3 \sim O(1)$.

Based on these assumptions, we are going to make the following transformations of the independent variable and the unknown functions in the system (80):

$$t = \tau_3\theta,\ X_{F1}(\tau_3\theta) = y_2(\theta),\ X_{F2}(\tau_3\theta) = y_1(\theta),\ X_{F3}(\tau_3\theta) = x(\theta),\ U_L(\tau_3\theta) = u(\theta), \tag{81}$$

where θ is a new independent variable (the non-dimensional time); $y_1(\theta)$, $y_2(\theta)$ and $x(\theta)$ are new unknown functions; $u(\theta)$ is a new control function.

The transformation (81) converts the system (80) to the system

$$\begin{aligned} \frac{dx(\theta)}{d\theta} &= -x(\theta-g) + y_1(\theta-\varepsilon_1 h_1), \\ \varepsilon_1\frac{dy_1(\theta)}{d\theta} &= -y_1(\theta-\varepsilon_1 h_1) + y_2(\theta-\varepsilon_2 h_2), \\ \varepsilon_2\frac{dy_2(\theta)}{d\theta} &= -y_2(\theta-\varepsilon_2 h_2) + u(\theta), \end{aligned} \tag{82}$$

where $\theta \geq 0$;

$$\varepsilon_1 = \frac{\tau_2}{\tau_3}, \quad \varepsilon_2 = \frac{\tau_1}{\tau_3}, \quad g = \frac{\eta_3}{\tau_3}, \quad h_1 = \frac{\eta_2}{\tau_2}, \quad h_2 = \frac{\eta_1}{\tau_1};$$

$\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are small parameters; $\varepsilon_2 \ll \varepsilon_1$.

Thus, the system (82) is a singularly perturbed three time-scale system with delays, i. e., it is a particular case of the system (1) – (3). We are going to design a memory-free state-feedback control stabilizing the system (82) uniformly in $(\varepsilon_1, \varepsilon_2)$.

First of all, let us note that the unforced system (82) ($u(\theta) \equiv 0$) is not, in general, exponentially stable. Indeed, for $h_2 \geq \pi/2$, the characteristic equation of the third differential equation in this system has roots with nonnegative real parts.

To establish the stabilization of the system (82) and to design the stabilizing control, we use (like in the previous example) Theorem 4.4 and Lemmas 3.14, 3.17. Similarly to Example 2, we start with the purely fast subsystem (25). In the present example, this subsystem becomes the following scalar system:

$$\frac{dy_{2,f}(\xi_2)}{d\xi_2} = -y_{2,f}(\xi_2 - h_2) + u_f(\xi_2), \quad \xi_2 \geq 0. \tag{83}$$

The scalar control (41) with the gain $K_f < -1$ stabilizes this system for any $h_2 \geq 0$.

Proceed to the system (49). In the present example, this system becomes the following scalar one:

$$\frac{dy_{1,sf}(\xi_1)}{d\xi_1} = -y_{1,sf}(\xi_1 - h_1) + (1 - K_f)^{-1} v_{sf}(\xi_1), \quad \xi_1 \geq 0. \tag{84}$$

The unforced system (84) ($v_{sf}(\xi_1) \equiv 0$) is not exponentially stable for $h_1 \geq \pi/2$. Similarly to the system (83), the scalar control (48) with the gain K_{sf}, satisfying the inequality $(1 - K_f)^{-1} K_{sf} < -1$, stabilizes the system (84) for any $h_1 \geq 0$.

Now, let us deal with the system (56). In the present example, we obtain this system as the following scalar one:

$$\frac{dx_s(\theta)}{d\theta} = -x_s(\theta - g) + \big(1 - (1 - K_f)^{-1} K_{sf}\big)^{-1} (1 - K_f)^{-1} v_s(\theta), \quad \theta \geq 0. \tag{85}$$

The unforced system (85) ($v_s(\theta) \equiv 0$) is not exponentially stable for $g \geq \pi/2$. However, the scalar control (55) with the gain K_s, satisfying the inequality $\big(1 - (1 - K_f)^{-1} K_{sf}\big)^{-1} (1 - K_f)^{-1} K_s < -1$, stabilizes the system (85) for any $g \geq 0$.

Based on the gains K_f, K_{sf} and K_s of the stabilizing controls for the systems (83), (84) and (85), we construct the following memory-free state-feedback control for the system (82):

$$u(\theta) = u[x(\theta), y_1(\theta), y_2(\theta)] = K_s x(\theta) + K_{sf} y_1(\theta) + K_f y_2(\theta), \quad \theta \geq 0. \tag{86}$$

Thus, by virtue of Lemmas 3.14, 3.17 and Theorem 4.4, there exist numbers ε_1^* and ε^*, satisfying the inequalities in (8), such that the system (82) is stabilized by the control (86) uniformly in $(\varepsilon_1, \varepsilon_2) \in \Omega(\varepsilon_1^*, \varepsilon^*)$.

6. CONCLUSIONS

In this paper, the three-time scale singularly perturbed linear time-invariant differential system with point-wise state delays was considered. The three-time scale nature of the system is due to the presence of two small positive multipliers (the parameters of the singular perturbations) for part of its derivatives, where one of these parameters is of a higher order of the smallness then the other. The delay in the slow state variable is of order of 1 (the non-small delay). The delay in each of the fast state variables is small of order of the corresponding singular perturbations' parameter. This system significantly differs from the singularly perturbed systems studied in the literature. To the best of our knowledge, such a type of singularly perturbed systems has not been considered yet in the literature. The exponential stability of the unforced version of original system and the memory-free state-feedback stabilization of its controlled version, uniform with respect to the parameters of singular perturbations, were studied. This study is based on the asymptotic replacing the three time-scale original system with three much simpler parameters-free subsystems: the purely slow, mixed slow-fast and purely fast ones. The purely slow and mixed slow-fast subsystems are descriptor (differential-algebraic) systems with state delays, while the purely fast subsystem is a differential system with state delay. The purely fast subsystem is of a lower Euclidean space dimension than the original system. Subject to some additional assumptions, the purely slow and mixed slow-fast subsystems can be reduced to differential systems with state delays, and these systems also are of lower Euclidean space dimensions than the original system. By spectrum analysis of the unforced version of the original system and its purely slow subsystem in the differential form, the mixed slow-fast subsystem in the differential form and the purely fast subsystem, it was established that the exponential stability of these subsystems yields the exponential stability of the original system (the unforced version) uniformly (robustly) with respect to the parameters of singular perturbations. Based on this result, it was established the following. If the purely fast subsystem, as well as the mixed slow-fast and purely slow subsystems in the differential-algebraic form, are memory-free state-feedback stabilized, then the original system also is memory-free state-feedback stabilized uniformly (robustly) with respect to the parameters of singular perturbations. Using the stabilizing controls of the purely fast, mixed slow-fast and purely slow subsystems, the stabilizing control of the original system was designed. Due to the results of the paper, the stability/stabilization analysis of the complicated parameters-dependent system is reduced to the stability/stabilization analysis of several much simpler parameter-free subsystems.

Completing this section, we would like to mention several issues of the paper's topic, which are interesting ones for future investigations. These issues are the following: (a) stability/stabilization analysis of two-parameters singularly perturbed systems with multiple point-wise delays and distributed delays; (b) obtaining estimates of the small positive numbers ε_1^* and ε^*, mentioned in Theorems 4.2, 4.4; (c) asymptotic solution of an initial-value problem for the original unforced system, considered in the paper; (d) asymptotic solution of a linear-quadratic optimal control problem with the dynamics, described by the original system of the paper.

7. APPENDIX A: PROOF OF LEMMA 3.1

Let us start with the proof of the boundedness of the sequence $\{\lambda_\alpha\}$. Assume the opposite. In this case, there exists a subsequence of $\{\lambda_\alpha\}$, such that absolute values of its elements tend to $+\infty$ as $\alpha \to +\infty$. For the sake of simplicity (but without loss of generality), we assume that $\{\lambda_\alpha\}$ itself has such a behavior, i. e., $\lim_{\alpha\to+\infty} |\lambda_\alpha| = +\infty$.

Let us partition the matrix $C(\lambda, \varepsilon_1, \varepsilon_2)$ into blocks as:

$$C(\lambda, \varepsilon_1, \varepsilon_2) = \begin{pmatrix} C_1(\lambda) & C_2(\lambda, \varepsilon_1, \varepsilon_2) \\ C_3(\lambda) & C_4(\lambda, \varepsilon_1, \varepsilon_2) \end{pmatrix}, \tag{87}$$

where

$$\begin{aligned} C_1(\lambda) &= A_{11} + G_1 \exp(-g\lambda) - \lambda I_n, \\ C_2(\lambda, \varepsilon_1, \varepsilon_2) &= (A_{12}\ ,\ A_{13}) + (H_{11}\ ,\ 0) \exp(-\varepsilon_1 h_1 \lambda) + (0\ ,\ H_{12}) \exp(-\varepsilon_2 h_2 \lambda), \\ C_3(\lambda) &= \begin{pmatrix} A_{21} \\ A_{31} \end{pmatrix} + \begin{pmatrix} G_2 \\ G_3 \end{pmatrix} \exp(-g\lambda), \\ C_4(\lambda, \varepsilon_1, \varepsilon_2) &= \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} + \begin{pmatrix} H_{21} & 0 \\ H_{31} & 0 \end{pmatrix} \exp(-\varepsilon_1 h_1 \lambda) \\ &+ \begin{pmatrix} 0 & H_{22} \\ 0 & H_{32} \end{pmatrix} \exp(-\varepsilon_2 h_2 \lambda) - \lambda \begin{pmatrix} \varepsilon_1 I_{m_1} & 0 \\ 0 & \varepsilon_2 I_{m_2} \end{pmatrix}. \end{aligned} \tag{88}$$

Due to (14) – (15) and the conditions (iii), (v) on the sequences $\{\varepsilon_{1,\alpha}\}$, $\{\varepsilon_{2,\alpha}\}$, $\{\lambda_\alpha\}$, $(\alpha = 1, 2, \ldots)$, we have

$$\lim_{\alpha\to+\infty} C_4(\lambda_\alpha, \varepsilon_{1,\alpha}, \varepsilon_{2,\alpha}) = \mathcal{A}_{3s}. \tag{89}$$

The latter, along with the inequality (16), yields the inequality

$$\det C_4(\lambda_\alpha, \varepsilon_{1,\alpha}, \varepsilon_{2,\alpha}) \neq 0 \tag{90}$$

for all sufficiently large α.

Now, using the equations (27), (87), the inequality (90) and the formula for the determinant of a block matrix (see [15]), we obtain the following equality for all sufficiently large α:

$$\begin{aligned} D(\lambda_\alpha, \varepsilon_{1,\alpha}, \varepsilon_{2,\alpha}) &= \det \Big(C_1(\lambda_\alpha) \\ -C_2(\lambda_\alpha, \varepsilon_{1,\alpha}, \varepsilon_{2,\alpha}) \big(C_4(\lambda_\alpha, \varepsilon_{1,\alpha}, \varepsilon_{2,\alpha}) \big)^{-1} & C_3(\lambda_\alpha) \Big) \det C_4(\lambda_\alpha, \varepsilon_{1,\alpha}, \varepsilon_{2,\alpha}). \end{aligned}$$

Thus, for all sufficiently large α, the condition (vi) on the sequences $\{\varepsilon_{1,\alpha}\}$, $\{\varepsilon_{2,\alpha}\}$, $\{\lambda_\alpha\}$ can be rewritten in the form

$$\det \Big(C_1(\lambda_\alpha) - C_2(\lambda_\alpha, \varepsilon_{1,\alpha}, \varepsilon_{2,\alpha}) \big(C_4(\lambda_\alpha, \varepsilon_{1,\alpha}, \varepsilon_{2,\alpha}) \big)^{-1} C_3(\lambda_\alpha) \Big) = 0. \tag{91}$$

Using the above made assumption that $\lim_{\alpha\to+\infty}|\lambda_\alpha| = +\infty$, let us divide the equality (91) by λ_α^n. Thus, we obtain for all sufficiently large α:

$$\frac{1}{\lambda_\alpha^n}\det\Big(C_1(\lambda_\alpha) - C_2(\lambda_\alpha,\varepsilon_{1,\alpha},\varepsilon_{2,\alpha})\big(C_4(\lambda_\alpha,\varepsilon_{1,\alpha},\varepsilon_{2,\alpha})\big)^{-1}C_3(\lambda_\alpha)\Big) = 0. \tag{92}$$

Using the expression for $C_1(\lambda)$ (see the equation (88)), the equality (92) can be rewritten as:

$$\det\bigg(\frac{1}{\lambda_\alpha}\Big(A_{11} + G_1\exp(-g\lambda_\alpha)\Big) - I_n \\ - \frac{1}{\lambda_\alpha}\Big(C_2(\lambda_\alpha,\varepsilon_{1,\alpha},\varepsilon_{2,\alpha})\big(C_4(\lambda_\alpha,\varepsilon_{1,\alpha},\varepsilon_{2,\alpha})\big)^{-1}C_3(\lambda_\alpha)\Big)\bigg) = 0.$$

Now, calculating the limit of this equality for $\alpha \to +\infty$ and taking into account the assumption that $\lim_{\alpha\to+\infty}|\lambda_\alpha| = +\infty$, as well as the expressions for $C_2(\lambda,\varepsilon_1,\varepsilon_2)$ and $C_3(\lambda)$ (see the equation (88)), the equation (89), and the conditions (iii)−(v) on the sequences $\{\varepsilon_{1,\alpha}\}$, $\{\varepsilon_{2,\alpha}\}$, $\{\lambda_\alpha\}$, $(\alpha = 1,2,\dots)$, we obtain the contradiction $(-1)^n = 0$. This contradiction means that the above made assumption on the unboundedness of the sequence $\{\lambda_\alpha\}$, $(\alpha = 1,2,\dots)$ is wrong, implying the boundedness of this sequence. Thus, the first statement of the lemma is proven.

Proceed to the proof of the second statement. Since the sequence $\{\lambda_\alpha\}$, $(\alpha = 1,2,\dots)$ is bounded, then there exists a convergent subsequence of this sequence. For the sake of simplicity (but without loss of generality), we assume that the sequence $\{\lambda_\alpha\}$, $(\alpha = 1,2,\dots)$ itself is such a subsequence. Let us denote $\bar\lambda \stackrel{\triangle}{=} \lim_{\alpha\to+\infty}\lambda_\alpha$. By virtue of the condition (iv) on the sequence $\{\lambda_\alpha\}$, $(\alpha = 1,2,\dots)$, we have that $\mathrm{Re}\bar\lambda \ge 0$. Now, calculating the limit of the equality (91) for $\alpha \to +\infty$ and using the equations (14)−(15), (18), (29) and the inequality (16), we obtain $D_s(\bar\lambda) = 0$. The latter means that $\bar\lambda$ is a root of the quasi-polynomial equation (29) satisfying the inequality (32), which proves the second statement of the lemma. Thus, the lemma is proven. □

8. APPENDIX B: PROOF OF LEMMA 3.2

First of all, we represent the matrix $E_1(\varepsilon_1)C(\mu/\varepsilon_1,\varepsilon_1,\varepsilon_2)$ in the block form as:

$$E_1(\varepsilon_1)C(\mu/\varepsilon_1,\varepsilon_1,\varepsilon_2) = \begin{pmatrix} F_1(\mu,\varepsilon_1,\varepsilon_2) & F_2(\mu,\varepsilon_1,\varepsilon_2) \\ F_3(\mu,\varepsilon_1,\varepsilon_2) & F_4(\mu,\varepsilon_1,\varepsilon_2)\end{pmatrix}, \tag{93}$$

where

$$F_1(\mu,\varepsilon_1) = \begin{pmatrix} \varepsilon_1 A_{11} & \varepsilon_1 A_{12} \\ A_{21} & A_{22}\end{pmatrix} + \begin{pmatrix} \varepsilon_1 G_1 & 0 \\ G_2 & 0\end{pmatrix}\exp(-g\mu/\varepsilon_1) \\ + \begin{pmatrix} 0 & \varepsilon_1 H_{11} \\ 0 & H_{21}\end{pmatrix}\exp(-h_1\mu) - \mu\begin{pmatrix} I_n & 0 \\ 0 & I_{m_1}\end{pmatrix}, \tag{94}$$

$$F_2(\mu,\varepsilon_1,\varepsilon_2) = \begin{pmatrix} \varepsilon_1 A_{13} \\ A_{23}\end{pmatrix} + \begin{pmatrix} \varepsilon_1 H_{12} \\ H_{22}\end{pmatrix}\exp\big(-(\varepsilon_2/\varepsilon_1)h_2\mu\big), \tag{95}$$

$$F_3(\mu,\varepsilon_1) = (A_{31}\ ,\ A_{32}) + (G_3\ ,\ 0)\exp(-g\mu/\varepsilon_1) + (0\ ,\ H_{31})\exp(-h_1\mu), \tag{96}$$

$$F_4(\mu,\varepsilon_1,\varepsilon_2) = A_{33} + H_{32}\exp\big(-(\varepsilon_2/\varepsilon_1)h_2\mu\big) - (\varepsilon_2/\varepsilon_1)\mu I_{m_2}. \tag{97}$$

Using the condition (v) on the sequences $\{\varepsilon_{1,\alpha}\}$, $\{\varepsilon_{2,\alpha}\}$, $\{\mu_\alpha\}$, $(\alpha = 1,2,\ldots)$, as well as the equation (14), we obtain

$$\lim_{\alpha\to+\infty} F_4(\mu_\alpha,\varepsilon_{1,\alpha},\varepsilon_{2,\alpha}) = A_{33,s}, \tag{98}$$

which, along with the inequality (22), yields

$$\det F_4(\mu_\alpha,\varepsilon_{1,\alpha},\varepsilon_{2,\alpha}) \neq 0 \tag{99}$$

for all sufficiently large α.

Using the equations (35), (93), the inequality (99) and the formula for the determinant of a block matrix (see [15]), we can rewrite (for all sufficiently large α) the condition (vi) on the sequences $\{\varepsilon_{1,\alpha}\}$, $\{\varepsilon_{2,\alpha}\}$, $\{\mu_\alpha\}$ in the following form:

$$\det\Big(F_1(\mu_\alpha,\varepsilon_{1,\alpha}) - F_2(\mu_\alpha,\varepsilon_{1,\alpha},\varepsilon_{2,\alpha})\big(F_4(\mu_\alpha,\varepsilon_{1,\alpha},\varepsilon_{2,\alpha})\big)^{-1}F_3(\mu_\alpha,\varepsilon_{1,\alpha})\Big) = 0. \tag{100}$$

Now, based on the equations (93) – (97), (98), (100) and the inequality (99), let us prove the statement (I) of the lemma. Assume the opposite. In this case, there exists a subsequence of $\{\mu_\alpha\}$, such that absolute values of its elements tend to $+\infty$ as $\alpha\to+\infty$. For the sake of simplicity (but without loss of generality), we assume that $\{\mu_\alpha\}$ itself has such a behavior, i. e., $\lim_{\alpha\to+\infty}|\mu_\alpha| = +\infty$. Dividing the equality (100) by $\mu_\alpha^{n+m_1}$ and using the expression for $F_1(\mu,\varepsilon_1)$ (see the equation (94)), we obtain the following equality for all sufficiently large α:

$$\begin{aligned}\frac{1}{\mu_\alpha^{n+m_1}}\det\Big(F_1(\mu_\alpha,\varepsilon_{1,\alpha}) - F_2(\mu_\alpha,\varepsilon_{1,\alpha},\varepsilon_{2,\alpha})\big(F_4(\mu_\alpha,\varepsilon_{1,\alpha},\varepsilon_{2,\alpha})\big)^{-1}F_3(\mu_\alpha,\varepsilon_{1,\alpha})\Big)\\ = \det\bigg(\frac{1}{\mu_\alpha}\widetilde{F}_1(\mu_\alpha,\varepsilon_{1,\alpha}) - I_{n+m_1}\\ -\frac{1}{\mu_\alpha}F_2(\mu_\alpha,\varepsilon_{1,\alpha},\varepsilon_{2,\alpha})\big(F_4(\mu_\alpha,\varepsilon_{1,\alpha},\varepsilon_{2,\alpha})\big)^{-1}F_3(\mu_\alpha,\varepsilon_{1,\alpha})\bigg) = 0,\end{aligned} \tag{101}$$

where

$$\begin{aligned}\widetilde{F}_1(\mu_\alpha,\varepsilon_{1,\alpha}) = \begin{pmatrix}\varepsilon_{1,\alpha}A_{11} & \varepsilon_{1,\alpha}A_{12}\\ A_{21} & A_{22}\end{pmatrix} + \begin{pmatrix}\varepsilon_{1,\alpha}G_1 & 0\\ G_2 & 0\end{pmatrix}\exp(-g\mu_\alpha/\varepsilon_{1,\alpha})\\ + \begin{pmatrix}0 & \varepsilon_{1,\alpha}H_{11}\\ 0 & H_{21}\end{pmatrix}\exp(-h_1\mu_\alpha).\end{aligned} \tag{102}$$

Calculating the limit of the equality (101) for $\alpha\to+\infty$, and taking into account the equations (95), (96), (98), (102), the inequality (99) and the conditions (ii), (iv), (v) on the sequences $\{\varepsilon_{1,\alpha}\}$, $\{\varepsilon_{2,\alpha}\}$, $\{\mu_\alpha\}$, $(\alpha = 1, 2, \ldots)$, we obtain the contradiction $(-1)^{n+m_1} = 0$. This contradiction means that the above made assumption on the unboundedness of the sequence $\{\mu_\alpha\}$, $(\alpha = 1, 2, \ldots)$ is wrong. Thus, this sequence is bounded. Using this conclusion, proceed to the proof of the second statement of the lemma. Since $\{\mu_\alpha\}$ is bounded, then there exist its convergent subsequence. For the sake of simplicity (but without loss of generality), we suppose that $\{\mu_\alpha\}$ itself is convergent, i. e., $\lim_{\alpha\to+\infty} \mu_\alpha = \bar{\mu}$. Moreover, by virtue of the condition (iv), $\mathrm{Re}\bar{\mu} \geq 0$. Furthermore, calculating the limit of the equality (100) for $\alpha \to +\infty$, and using the equations (24), (30), (94) – (97), (98), the inequality (99) and the conditions (ii) – (v), we obtain by a routine algebra

$$(-1)^n \bar{\mu}^n \big(\det C_{sf}(\bar{\mu})\big) = 0,$$

which directly yields the second statement of the lemma. Thus, the lemma is proven.□

9. APPENDIX C: PROOF OF LEMMA 3.14

Necessity. First of all, let us note the following. Since the control (41) stabilizes the purely fast subsystem (25), then the inequality (44) is valid. This means the feasibility of the system (49). Now, let us assume that the control (45) stabilizes the mixed slow-fast subsystem (19). The closed-loop system (19), (45) has the form

$$\begin{aligned}\frac{dy_{1,sf}(\xi_1)}{d\xi} &= (A_{22} + B_2K_{sf})y_{1,sf}(\xi_1) + (A_{23,s} + B_2K_f)y_{2,sf}(\xi_1)\\ &\qquad + H_{21}y_{1,sf}(\xi_1 - h_1), \quad \xi_1 \geq 0,\\ 0 &= (A_{32} + B_3K_{sf})y_{1,sf}(\xi_1) + (A_{33,s} + B_3K_f)y_{2,sf}(\xi_1)\\ &\qquad + H_{31}y_{1,sf}(\xi_1 - h_1), \quad \xi_1 \geq 0.\end{aligned} \tag{103}$$

Since the inequality (44) is valid, the initial-value problem (103), (46) can be transformed to an equivalent problem consisting of the algebraic expression for $y_{2,sf}(\xi_1)$

$$y_{2,sf}(\xi_1) = -(A_{33,s} + B_3K_f)^{-1}[(A_{32} + B_3K_{sf})y_{1,sf}(\xi_1) + H_{31}y_{1,sf}(\xi_1 - h_1)],\ \xi_1 \geq 0,$$

and the differential equation with respect to $y_{1,sf}(\xi_1)$

$$\frac{dy_{1,sf}(\xi_1)}{d\xi_1} = [\mathcal{A}_{sf}(K_f) + \mathcal{B}_{sf}(K_f)K_{sf}]y_{1,sf}(\xi_1) + \mathcal{H}_{sf}(K_f)y_{1,sf}(\xi_1 - h_1),\ \xi_1 \geq 0 \tag{104}$$

with the initial condition (46).

Due to the equivalence of the above mentioned problems and Definition 3.13, we have that the solution $y_{1,sf}(\xi_1)$, $\xi_1 \geq 0$ of the differential equation (104) with the initial condition (46) satisfies the inequality

$$\|y_{1,sf}(\xi_1)\| \leq a_{1,sf}\exp(-\kappa_{sf}\xi_1)\|\psi_{sf}(\zeta_1)\|_C, \quad \xi_1 \geq 0 \tag{105}$$

with some number $a_{1,sf} > 0$ independent of $\psi_{sf}(\zeta_1)$.

From the other hand, the differential equation (104) can be obtain from the system (49) by substituting there the state-feedback control (48). The latter, along with the inequality (105), means that the control (48) stabilizes the system (49), which completes the proof of the necessity.

Sufficiency. The sufficiency is proven similarly to the necessity.

Thus, the lemma is proven. □

10. APPENDIX D: PROOF OF LEMMA 3.17

Necessity. First of all, let us show that the matrix $W(K_{sf},K_f)$ is invertible. This matrix can be represented in the block form as:

$$W(K_{sf},K_f) = \begin{pmatrix} A_{22,s}+B_2K_{sf} & A_{23,s}+B_2K_f \\ A_{32,s}+B_3K_{sf} & A_{33,s}+B_3K_f \end{pmatrix}.$$

Using this representation of $W(K_{sf},K_f)$, as well as the formula for the determinant of a block matrix (see [15]) and the equations (15), (50), we obtain that:

$$\det W(K_{sf},K_f) = \det\big(\mathcal{A}_{sf}(K_f)+\mathcal{B}_{sf}(K_f)K_{sf}+\mathcal{H}_{sf}(K_f)\big)\det\big(A_{33,s}+B_3K_f\big).$$

Thus, by virtue of (44) and (51), $\det W(K_{sf},K_f)\neq 0$, meaning the invertibility of the matrix $W(K_{sf},K_f)$. Hence, the system (56) is feasible. Now, let us assume that the control (52) stabilizes the purely slow subsystem (11) – (13). The closed-loop system (11) – (13), (52) is

$$\begin{aligned} \frac{dx_s(t)}{dt} &= (A_{11}+B_1K_s)x_s(t)+(A_{12,s}+B_1K_{sf})y_{1,s}(t) \\ &\quad +(A_{13,s}+B_1K_f)y_{2,s}(t)+G_1x_s(t-g), \quad t\geq 0, \\ 0 &= (A_{21}+B_2K_s)x_s(t)+(A_{22,s}+B_2K_{sf})y_{1,s}(t) \\ &\quad +(A_{23,s}+B_2K_f)y_{2,s}(t)+G_2x_s(t-g), \quad t\geq 0, \\ 0 &= (A_{31}+B_3K_s)x_s(t)+(A_{32,s}+B_3K_{sf})y_{1,s}(t) \\ &\quad +(A_{33,s}+B_3K_f)y_{2,s}(t)+G_3x_s(t-g), \quad t\geq 0. \end{aligned} \tag{106}$$

Since the matrix $W(K_{sf},K_f)$ is invertible, the initial-value problem (106), (53) can be transformed to an equivalent problem consisting of the algebraic expression for the vector $\mathrm{col}\big(y_{1,s}(t),y_{2,s}(t)\big)$

$$\mathrm{col}\big(y_{1,s}(t),y_{2,s}(t)\big) = -\big(W(K_{sf},K_f)\big)^{-1}\big((\mathcal{A}_{2s}+\mathcal{B}_{23,s}K_s)x_s(t)+\mathcal{G}_{23,s}x_s(t-g)\big),\ t\geq 0,$$

and the differential equation with respect to $x_s(t)$

$$\frac{dx_s(t)}{dt} = [\mathcal{A}_s(K_{sf},K_f)+\mathcal{B}_s(K_{sf},K_f)K_s]x_s(t)+\mathcal{G}_s(K_{sf},K_f)x_s(t-g), \quad t\geq 0 \tag{107}$$

with the initial condition (53).

Due to the equivalence of the above mentioned problems and Definition 3.16, we have that the solution $x_s(t)$, $t\geq 0$ of the initial-value problem (107), (53) satisfies the inequality

$$\|x_s(t)\| \leq a_{1,s}\exp(-\kappa_s t)\|\psi_s(\eta)\|_C, \quad t\geq 0 \tag{108}$$

with some number $a_{1,s} > 0$ independent of $\psi_s(\eta)$.

From the other hand, the differential equation (107) can be obtain from the system (56) by substituting there the state-feedback control (55). The latter, along with the inequality (108), means that the control (55) stabilizes the system (56), which completes the proof of the necessity.

Sufficiency. The sufficiency is proven similarly to the necessity.

Thus, the lemma is proven. □

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