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ON THE CONSTRUCTION OF T-NORMS (T-CONORMS) BY USING INTERIOR (CLOSURE) OPERATOR ON BOUNDED LATTICES

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Recently, the topic of construction methods for triangular norms (triangular conorms), uninorms, nullnorms, etc. has been studied widely. In this paper, we propose construction methods for triangular norms (t-norms) and triangular conorms (t-conorms) on bounded lattices by using interior and closure operators, respectively. Thus, we obtain some proposed methods given by Ertuğrul, Karaçal, Mesiar [15] and Çaylı [8] as results. Also, we give some illustrative examples. Finally, we conclude that the introduced construction methods can not be generalized by induction to a modified ordinal sum for t-norms and t-conorms on bounded lattices.

Keywords: t-norm, t-conorm, ordinal sum, bounded lattice

Classification: 03E72, 03B52

1. INTRODUCTION

Triangular norms (t-norms) and triangular conorms (t-conorms) were introduced by Schweizer and Sklar [25] in the study of probabilistic metric spaces as a special kind of associative functions defined on the unit interval $[0, 1]$. Although the t-norms and t-conorms were strictly defined on the unit interval $[0, 1]$, they were mostly studied on bounded lattices. The notion of ordinal sum of semigroups in Clifford's sense [7] was further developed by Mostert and Shields [21] and later used for introducing new t-norms and conorms on the unit interval $[0, 1]$, see [19]. Note that there is a minor difference in ordinal sum construction for t-norms (based on min operator) with those for t-conorms (based on max operator). Since Goguen' s [17] generalization of the classical fuzzy sets (with membership values from $[0, 1]$) to L-fuzzy sets (with membership values from a bounded lattice L), there is a growing interest in t-norms and t-conorms on bounded lattices, in particular in ordinal sum constructions.

In 1996, Drossos and Navara [12] studied a class of t-norms and t-conorms on any bounded lattice was generated by use of interior and closure operators, respectively. In 2006, Saminger [24] focused on ordinal sums of t-norms acting on some particular bounded lattice which is not necessarily a chain or an ordinal sum of lattices. Also, it was provided necessary and sufficient conditions for an ordinal sum operation yielding again

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a t-norm on some bounded lattice whereas the operation is determined by an arbitrary selection of subintervals as carriers for arbitrary summand t-norms. In 2012, Medina [20] presented several necessary and sufficient conditions for ensuring whether an ordinal sum on a bounded lattice of arbitrary t-norms is a t-norm. In 2015, a modification of ordinal sums of t-norms and t-conorms resulting to t-norms and t-conorms on an arbitrary bounded lattice was shown by Ertuğrul, Karaçal and Mesiar [15]. Further modifications were proposed by Aşıcı and Mesiar [3, 4], Aşıcı [2], Çaylı [8, 9] Ouyang, Zhang and Baets [22] and Dan, Hu and Qiao [10]. In 2020, a new ordinal sum construction of t-norms and t-conorms on bounded lattices based on interior and closure operators was proposed by Dvořák and Holčapek [13].

In this paper, we introduce some constructions of t-norms and t-conorms by using interior and closure operators on bounded lattices, respectively. The rest of this paper is organized as follows. In Section 2, some basic concepts and results about t-norms, t-conorms, lattices are given. In Section 3, we propose a new method to construct t-norms on bounded lattices. Using this method, in Corollary 3.12 and Corollary 3.10, we obtain the methods proposed by Ertuğrul, Karaçal and Mesiar [15] and Çaylı [8] as results, respectively. In Section 4, we propose a new method to construct t-conorms on bounded lattices. Using this method, in Corollary 4.9 and Corollary 4.11, we obtain the methods proposed by Ertuğrul, Karaçal and Mesiar [15] and Çaylı [8] as results, respectively. In Concluding Remarks, we show that the introduced construction methods can not be generalized by induction to a modified ordinal sum for t-norms and t-conorms on bounded lattices.

2. PRELIMINARIES

In this section, we present some basic facts about lattices, t-norms and t-conorms.

A lattice [6] is a partially ordered set $(L, \leq)$ in which each two element subset $\{x, y\}$ has an infimum, denoted as $x \wedge y$, and a supremum, denoted as $x \vee y$. A bounded lattice $(L, \leq, 0, 1)$ is a lattice that has the bottom and top elements written as 0 and 1, respectively.

Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, if a and b are incomparable, in this case, we use the notation $a \parallel b$. We denote the set of elements which are incomparable with a by I_a. So $I_a = \{x \in L \mid x \parallel a\}$.

Given a bounded lattice $(L, \leq, 0, 1)$ and $a, b \in L$, $a \leq b$, a subinterval $[a, b]$ of L is defined as

$[a, b] = \{x \in L \mid a \leq x \leq b\}$.

Similarly, $[a, b) = \{x \in L \mid a \leq x < b\}$, $(a, b] = \{x \in L \mid a < x \leq b\}$ and $(a, b) = \{x \in L \mid a < x < b\}$.

Definition 2.1. (Klement et al. [19], Saminger [24]) Let $(L, \leq, 0, 1)$ be a bounded lattice. A *triangular norm* T (t-norm) is a binary operation on L which is commutative, associative, increasing with respect to both variables and satisfies $T(x, 1) = x$ for all $x \in L$.

Definition 2.2. (Aşıcı and Mesiar [1], Saminger [24]) Let $(L, \leq, 0, 1)$ be a bounded lattice. A *triangular conorm* S (t-conorm) is a binary operation on L which is commutative, associative, increasing with respect to both variables and satisfies $S(x,0)=x$ for all $x \in L$.

The t-norms $T_\wedge$ and T_W on L are defined as follows, respectively:
$T_\wedge(x,y) = x \wedge y$
$T_W(x,y) = \begin{cases} x \wedge y & \text{if } 1 \in \{x,y\} , \\ 0 & \text{otherwise.} \end{cases}$
Similarly, the t-conorms $S_\vee$ and S_W on L are defined as follows, respectively:
$S_\vee(x,y) = x \vee y$
$S_W(x,y) = \begin{cases} x \vee y & \text{if } 0 \in \{x,y\} , \\ 1 & \text{otherwise.} \end{cases}$

The following definition of an ordinal sum of t-norms defined on subintervals of a bounded lattice $(L, \leq, 0, 1)$ has been extracted from [24], which generalizes the methods given in [19] on subintervals of $[0,1]$.

Definition 2.3. (Saminger [24]) Let $(L, \leq, 0, 1)$ be a bounded lattice and fix some subinterval $[a,b]$ of L. Let V be a t-norm on $[a,b]$. Then $T : L^2 \to L$ defined by

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,b]^2 , \\ x \wedge y & \text{otherwise,} \end{cases} \tag{1}$$

is an ordinal sum ($< a, b, V >$) of V on L.

Definition 2.4. (Saminger [24]) Let $(L, \leq, 0, 1)$ be a bounded lattice and fix some subinterval $[a,b]$ of L. Let W be a t-conorm on $[a,b]$. Then $S : L^2 \to L$ defined by

$$S(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in [a,b]^2 , \\ x \vee y & \text{otherwise,} \end{cases} \tag{2}$$

is an ordinal sum ($< a, b, W >$) of W on L.

However, the operation T (resp. S) given by Formula (1) (resp. Formula (2)) need not be a t-norm (resp. t-conorm), in general. Observe that condition ensuring that T (resp. S) given by (1) ((2)) is a t-norm (t-conorm) on L are given in [24].

Definition 2.5. (Everett [16]) Let $(L, \leq, 0, 1)$ be a bounded lattice. A mapping $cl : L \to L$ is said to be a closure operator if for any $x, y \in L$, it satisfies the following three conditions:

(i) $x \leq cl(x)$

(ii) $cl(x \vee y) = cl(x) \vee cl(y)$

(iii) $cl(cl(x)) = cl(x)$

Example 2.6. (Everett [16]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $b \in L$ be given. Then the mapping $cl_b : L \to L$ defined as

$$cl_b(x) = x \vee b \ (\forall x \in L)$$

is a closure operator.

Definition 2.7. (Ouyang and Zhang [22]) Let $(L, \leq, 0, 1)$ be a bounded lattice. The set of all universally comparable elements in L, denoted by $UC(L)$, be defined as

$$UC(L) = \{b \in L \mid \forall c \in L, \text{ either } b \leq c \text{ or } c \leq b \}.$$

Example 2.8. (Ouyang and Zhang [22]) Let $(L, \leq, 0, 1)$ be a complete lattice. The mapping $\Uparrow: L \to L$ defined as, for any $x \in L$,

$$\Uparrow(x) = \bigwedge\{b \in UC(L) \mid b \geq x\}.$$

is a closure operator.

Definition 2.9. (Drossos and Navara [11], Drossos [12], Ouyang and Zhang [22]) Let $(L, \leq, 0, 1)$ be a bounded lattice. A mapping $int : L \to L$ is said to be an interior operator if for any $x, y \in L$, it satisfies the following three conditions:

(i) $int(x) \leq x$

(ii) $int(x \wedge y) = int(x) \wedge int(y)$

(iii) $int(int(x)) = int(x)$.

Definition 2.10. (Ouyang and Zhang [22]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $b \in L$ be given. Then the mapping $int_b : L \to L$ defined as

$$int_b(x) = x \wedge b \ (\forall x \in L)$$

is an interior operator.

Example 2.11. (Ouyang and Zhang [22]) Let $(L, \leq, 0, 1)$ be a complete lattice. The mapping $\Downarrow: L \to L$ defined as, for any $x \in L$,

$$\Downarrow(x) = \bigvee\{b \in UC(L) \mid b \leq x\}$$

is an interior operator.

In the following, by recalled a method for generating t-norms and t-conorms on bounded lattices based on interior and closure operators, respectively.

Theorem 2.12. (Drossos [11, 12]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $int : L \to L$ and $cl : L \to L$ be an interior and a closure operators on L, respectively. Then, the functions $T : L^2 \to L$ and $S : L^2 \to L$ are, respectively, a t-norm and a t-conorm on L, where

$$T(x,y) = \begin{cases} x \wedge y & \text{if } 1 \in \{x,y\} \ , \\ int(x) \wedge int(y) & \text{otherwise,} \end{cases} \tag{3}$$

$$S(x,y) = \begin{cases} x \vee y & \text{if } 0 \in \{x,y\} \ , \\ cl(x) \vee cl(y) & \text{otherwise.} \end{cases} \tag{4}$$

Theorem 2.13. (Çaylı [8]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, \ldots, a_n\}$ be a finite chain in L such that $1 = a_0 > a_1 > a_2 > \cdots > a_n = 0$. Let $V : [a_1, 1]^2 \to [a_1, 1]$ be a t-norm. Then, the function $T_n : L^2 \to L$ defined recursively as follows is a t-norm, where $V = T_1$ and for $i \in \{2, \ldots, n\}$, the function $T_i : [a_i, 1]^2 \to [a_i, 1]$ is given by

$$T_i(x,y) = \begin{cases} T_{i-1}(x,y) & \text{if } (x,y) \in [a_{i-1}, 1)^2 \ , \\ a_i & \text{if } (x,y) \in [a_i, a_{i-1})^2 \cup [a_i, a_{i-1}) \times I_{a_{i-1}} \ , \\ & \cup I_{a_{i-1}} \times [a_i, a_{i-1}) \cup I_{a_{i-1}} \times I_{a_{i-1}} \ , \\ x \wedge y & \text{if } \ 1 \in \{x,y\} \ , \\ x \wedge y \wedge a_{i-1} & \text{otherwise .} \end{cases} \tag{5}$$

Theorem 2.14. (Ertuğrul [15]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, \ldots, a_n\}$ be a finite chain in L such that $1 = a_0 > a_1 > a_2 > \cdots > a_n = 0$. Let $V : [a_1, 1]^2 \to [a_1, 1]$ be a t-norm. Then, the function $T_n : L^2 \to L$ defined recursively as follows is a t-norm, where $V = T_1$ and for $i \in \{2, \ldots, n\}$,

$$T_i(x,y) = \begin{cases} T_{i-1}(x,y) & \text{if } (x,y) \in [a_{i-1}, 1)^2 \ , \\ x \wedge y & \text{if } x = 1 \textit{ or } y = 1 \ , \\ x \wedge y \wedge a_{i-1} & \text{otherwise .} \end{cases} \tag{6}$$

Theorem 2.15. (Çaylı [8]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, \ldots, a_n\}$ be a finite chain in L such that $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$. Let $W : [0, a_1]^2 \to [0, a_1]$ be a t-conorm. Then, the function $S_n : L^2 \to L$ defined recursively as follows is a t-conorm, where $S_1 = W$ and for $i \in \{2, \ldots, n\}$, the binary function $S_i : [0, a_i]^2 \to [0, a_i]$ is given by

$$S_i(x,y) = \begin{cases} S_{i-1}(x,y) & \text{if } (x,y) \in (0, a_{i-1}]^2 \ , \\ a_i & \text{if } (x,y) \in (a_{i-1}, a_i]^2 \cup (a_{i-1}, a_i] \times I_{a_{i-1}} \ , \\ & \cup I_{a_{i-1}} \times (a_{i-1}, a_i] \cup I_{a_{i-1}} \times I_{a_{i-1}} \ , \\ x \vee y & 0 \in \{x,y\} \ , \\ x \vee y \vee a_{i-1} & \text{otherwise .} \end{cases} \tag{7}$$

Theorem 2.16. (Ertuğrul [15]) Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, \ldots, a_n\}$ be a finite chain in L such that $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$. Let $W : [0, a_1]^2 \to$

$[0, a_1]$ be a t-conorm. Then, the function $S_n : L^2 \to L$ defined recursively as follows is a t-conorm, where $S_1 = W$ and for $i \in \{2, \ldots, n\}$,

$$S_i(x,y) = \begin{cases} S_{i-1}(x,y) & \text{if } (x,y) \in (0, a_{i-1}]^2 \ , \\ x \vee y & \text{if } x = 0 \ \textit{or} \ y = 0 \ , \\ x \vee y \vee a_{i-1} & \text{otherwise} \ . \end{cases} \tag{8}$$

3. NEW CONSTRUCTION METHOD FOR T-NORMS ON BOUNDED LATTICES BY USING INTERIOR OPERATORS

In this section, we propose a new construction method for t-norms on bounded lattices with the given t-norms by using interior operators. The main aim of this section is to present a rather effective method to construct t-norms interior operators on a bounded lattice. Using this method, in Corollary 3.10 and Corollary 3.12, we obtain the present methods given by Çaylı [8] and Ertuğrul, Karaçal, Mesiar [15], respectively.

Theorem 3.1. Let $(L, \leq, 0, 1)$ be a bounded lattice with $a \in L$ and $int : L \to L$ be an interior operator such that for all $x, y \in I_a$ it holds $x \wedge y \wedge a = int(x \wedge y \wedge a)$. Given a t-norm V on $[a, 1]$, then the function $T : L^2 \to L$ defined as follows is a t-norm on L where

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1)^2 \ , \\ y \wedge a & \text{if } (x,y) \in [a,1) \times I_a \ , \\ x \wedge a & \text{if } (x,y) \in I_a \times [a,1) \ , \\ x \wedge y \wedge a & \text{if } (x,y) \in I_a \times I_a \ , \\ x \wedge y & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a] \text{ or } (x = 1 \text{ or } y = 1) \ , \\ int(x) \wedge int(y) & \text{otherwise} \ . \end{cases}$$

P r o o f. It is easy to see that T is commutative and has 1 as the neutral element.

i) Monotonicity: We prove that if $x \leq y$, then $T(x,z) \leq T(y,z)$ for all $z \in L$. If $z = 1$, then we have that $T(x,z) = T(x,1) = x \leq y = T(y,1) = T(y,z)$ for all $x, y \in L$. The proof can be split into all possible cases.

1. $x \in [0, a)$

 1.1 $y \in [0, a)$

  1.1.1. $z \in [0, a)$ or $z \in I_a$

$$T(x,z) = int(x) \wedge int(z) \leq int(y) \wedge int(z) = T(y,z)$$

  1.1.2. $z \in [a, 1)$

$$T(x,z) = x \leq y = T(y,z)$$

 1.2. $y \in [a, 1)$

  1.2.1. $z \in [0, a)$

$$T(x,z) = int(x) \wedge int(z) \leq z = T(y,z)$$

1.2.2. $z \in [a, 1)$
$$T(x, z) = x \leq a \leq V(y, z) = T(y, z)$$
1.2.3. $z \in I_a$
$$T(x, z) = int(x) \wedge int(z) \leq x \wedge z \leq a \wedge z = T(y, z)$$
1.3. $y \in I_a$

1.3.1. $z \in [0, a)$
$$T(x, z) = int(x) \wedge int(z) \leq int(y) \wedge int(z) = T(y, z)$$
1.3.2. $z \in [a, 1)$
$$T(x, z) = x \leq a \wedge y = T(y, z)$$
1.3.3. $z \in I_a$
$$T(x, z) = int(x) \wedge int(z) \leq x \wedge z \leq y \wedge z \wedge a = T(y, z)$$
1.4. $y = 1$

1.4.1. $z \in [0, a)$ or $z \in I_a$
$$T(x, z) = int(x) \wedge int(z) \leq z = T(1, z)$$
1.4.2. $z \in [a, 1)$
$$T(x, z) = x \leq a \leq z = T(1, z)$$
2. $x \in [a, 1)$

2.1 $y \in [a, 1)$

2.1.1. $z \in [0, a)$
$$T(x, z) = z = T(y, z)$$
2.1.2. $z \in [a, 1)$
$$T(x, z) = V(x, z) \leq V(y, z) = T(y, z)$$
2.1.3. $z \in I_a$
$$T(x, z) = z \wedge a = T(y, z)$$
2.2 $y = 1$

2.2.1. $z \in [0, a)$
$$T(x, z) = z = T(1, z)$$
2.1.2. $z \in [a, 1)$
$$T(x, z) = V(x, z) \leq z = T(1, z)$$
2.1.3. $z \in I_a$
$$T(x, z) = z \wedge a \leq z = T(1, z)$$
3. $x \in I_a$

3.1. $y \in [a, 1)$

3.1.1. $z \in [0, a)$

$$T(x, z) = int(x) \wedge int(z) \leq z = T(y, z)$$

3.1.2. $z \in [a, 1)$

$$T(x, z) = x \wedge a \leq a \leq V(y, z) = T(y, z)$$

3.1.3. $z \in I_a$

$$T(x, z) = x \wedge z \wedge a \leq z \wedge a = T(y, z)$$

3.2. $y = 1$

3.2.1. $z \in [0, a)$

$$T(x, z) = int(x) \wedge int(z) \leq z = T(1, z)$$

3.2.2. $z \in [a, 1)$

$$T(x, z) = x \wedge a \leq a \leq z = T(1, z)$$

3.2.3. $z \in I_a$

$$T(x, z) = x \wedge z \wedge a \leq z = T(1, z)$$

4. $x = 1$.

Then, it must be $y = 1$. Clearly, monotonicity is hold.

ii) Associativity: We need to prove that $T(x, T(y, z)) = T(T(x, y), z)$ for all $x, y, z \in L$. If at least one of x, y, z in L is 1, then it is obvious. So, the proof is split into all possible cases.

1. $x \in [0, a)$

1.1 $y \in [0, a)$

1.1.1. $z \in [0, a)$ or $z \in I_a$

$$T(x, T(y, z)) = T(x, int(y) \wedge int(z)) = int(x) \wedge int(y) \wedge int(z)$$
$$= T(int(x) \wedge int(y), z) = T(T(x, y), z)$$

1.1.2. $z \in [a, 1)$

$$T(x, T(y, z)) = T(x, y) = int(x) \wedge int(y) = T(int(x) \wedge int(y), z)$$
$$= T(T(x, y), z)$$

1.2. $y \in [a, 1)$

1.2.1. $z \in [0, a)$

$$T(x, T(y, z)) = T(x, z) = T(T(x, y), z)$$

1.2.2. $z \in [a, 1)$

$$T(x, T(y, z)) = T(x, V(y, z)) = x = T(x, z) = T(T(x, y), z)$$

1.2.3. $z \in I_a$

$$T(x, T(y, z)) = T(x, z \wedge a) = int(x) \wedge int(z \wedge a) = int(x) \wedge int(z)$$
$$= T(x, z) = T(T(x, y), z)$$

1.3. $y \in I_a$

1.3.1. $z \in [0, a)$

$$T(x, T(y, z)) = T(x, int(y) \wedge int(z)) = int(x) \wedge int(y) \wedge int(z)$$
$$= T(int(x) \wedge int(y), z) = T(T(x, y), z)$$

1.3.2. $z \in [a, 1)$

$$T(x, T(y, z)) = T(x, y \wedge a) = int(x) \wedge int(y \wedge a) = int(x) \wedge int(y)$$
$$= T(int(x) \wedge int(y), z) = T(T(x, y), z)$$

1.3.3. $z \in I_a$

$$T(x, T(y, z)) = T(x, y \wedge z \wedge a) = int(x) \wedge int(y \wedge z \wedge a) = int(x) \wedge int(y) \wedge int(z)$$
$$= int(int(x) \wedge int(y)) \wedge int(z) = T(int(x) \wedge int(y), z) = T(T(x, y), z)$$

2. $x \in [a, 1)$

2.1 $y \in [0, a)$

2.1.1. $z \in [0, a)$ or $z \in I_a$

$$T(x, T(y, z)) = T(x, int(y) \wedge int(z)) = int(y) \wedge int(z) = T(y, z) = T(T(x, y), z)$$

2.1.2. $z \in [a, 1)$

$$T(x, T(y, z)) = T(x, y) = y = T(y, z) = T(T(x, y), z)$$

2.2. $y \in [a, 1)$

2.2.1. $z \in [0, a)$

$$T(x, T(y, z)) = T(x, z) = z = T(V(x, y), z) = T(T(x, y), z)$$

2.2.2. $z \in [a, 1)$

$$T(x, T(y, z)) = T(x, V(y, z)) = V(x, V(y, z)) = V(V(x, y), z)$$
$$= T(V(x, y), z) = T(T(x, y), z)$$

2.2.3. $z \in I_a$

$$T(x, T(y, z)) = T(x, z \wedge a) = z \wedge a = T(V(x, y), z) = T(T(x, y), z)$$

2.3. $y \in I_a$

2.3.1. $z \in [0, a)$

$$T(x, T(y, z)) = T(x, int(y) \wedge int(z)) = int(y) \wedge int(z) = int(y \wedge a) \wedge int(z)$$

$$= T(y \wedge a, z) = T(T(x, y), z)$$

2.3.2. $z \in [a, 1)$

$$T(x, T(y, z)) = T(x, y \wedge a) = y \wedge a = T(y \wedge a, z) = T(T(x, y), z)$$

2.3.3. $z \in I_a$

$$T(x, T(y, z)) = T(x, y \wedge z \wedge a) = y \wedge z \wedge a = int(y \wedge z \wedge a)$$

$$= T(y \wedge a, z) = T(T(x, y), z)$$

3. $x \in I_a$

3.1 $y \in [0, a)$

3.1.1. $z \in [0, a)$ or $z \in I_a$

$$T(x, T(y, z)) = T(x, int(y) \wedge int(z)) = int(x) \wedge int(y) \wedge int(z)$$

$$= T(int(x) \wedge int(y), z) = T(T(x, y), z)$$

3.1.2. $z \in [a, 1)$

$$T(x, T(y, z)) = T(x, y) = int(x) \wedge int(y) = T(int(x) \wedge int(y), z) = T(T(x, y), z)$$

3.2. $y \in [a, 1)$

3.2.1. $z \in [0, a)$

$$T(x, T(y, z)) = T(x, z) = int(x) \wedge int(z) = int(x \wedge a) \wedge int(z)$$

$$= T(x \wedge a, z) = T(T(x, y), z)$$

3.2.2. $z \in [a, 1)$

$$T(x, T(y, z)) = T(x, V(y, z)) = x \wedge a = T(x \wedge a, z) = T(T(x, y), z)$$

3.2.3. $z \in I_a$

$$T(x, T(y, z)) = T(x, z \wedge a) = int(x) \wedge int(z \wedge a) = int(x \wedge a) \wedge int(z)$$

$$= T(x \wedge a, z) = T(T(x, y), z)$$

3.3. $y \in I_a$

3.3.1. $z \in [0, a)$

$$T(x, T(y, z)) = T(x, int(y) \wedge int(z)) = int(x) \wedge int(y) \wedge int(z)$$

$$= int(x \wedge y \wedge a) \wedge int(z) = T(x \wedge y \wedge a, z) = T(T(x, y), z)$$

3.3.2. $z \in [a, 1)$

$$T(x, T(y, z)) = T(x, y \wedge a) = int(x) \wedge int(y \wedge a) = int(x \wedge y \wedge a) = x \wedge y \wedge a$$
$$= T(x \wedge y \wedge a, z) = T(T(x, y), z)$$

3.3.3. $z \in I_a$

$$T(x, T(y, z)) = T(x, y \wedge z \wedge a) = int(x) \wedge int(y \wedge z \wedge a) = int(x \wedge y \wedge z \wedge a)$$
$$= int(x \wedge y \wedge a) \wedge int(z) = T(x \wedge y \wedge a, z) = T(T(x, y), z).$$

So, we have the fact that T is a t-norm on L. □

Proposition 3.2. Let $(L, \leq, 0, 1)$ be a bounded lattice with $a \in L$, $int : L \to L$ be an interior operator and V be a t-norm on $[a, 1]$. The function $T : L^2 \to L$ defined in Theorem 3.1 is a t-norm on L if and only if $I_a = \emptyset$ or for any $x, y \in I_a$ it holds that $x \wedge y \wedge a = int(x \wedge y \wedge a)$.

Proof. The result immediately follows from the proof of the Theorem 3.1. □

Remark 3.3. Let $(L, \leq, 0, 1)$ be a bounded lattice with $a \in L$. In Proposition 3.2, it is clear that the condition for all $x \in I_a$ it holds $x \wedge y \wedge a = int(x \wedge y \wedge a)$ can not be omitted. The following example illustrates this fact that the function $T : L^2 \to L$ defined by Theorem 3.1 is not a t-norm.

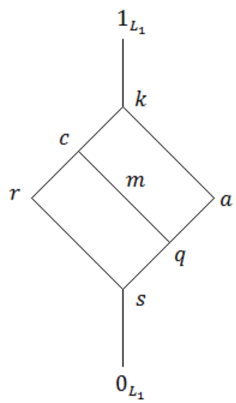

Fig. 1. The lattice L_1.

Example 3.4. Consider the lattice $(L_1 = \{0_{L_1}, s, q, a, m, r, c, k, 1_{L_1}\}, \leq, 0_{L_1}, 1_{L_1})$ in Figure 1. And we take the t-norm $V(x, y) = x \wedge y$ on $[a, 1_{L_1}]$. The interior operator $int : L_1 \to L_1$ defined by $int(0_{L_1}) = 0_{L_1}$, $int(s) = int(q) = int(r) = int(c) = int(a) = int(m) = s$, $int(k) = k$ and $int(1_{L_1}) = 1_{L_1}$. For all $x \in I_a$ it does not hold $x \wedge y \wedge a = int(x \wedge y \wedge a)$. Because, $c \wedge m \wedge a = q \neq s = int(q) = int(c \wedge m \wedge a)$. Then, the function T on L_1 defined by Table 1 is not a t-norm. Indeed, it does not satisfy the associativity. Because $T(k, T(c, m)) = T(k, q) = q \neq s = T(q, m) = T(T(k, c), m)$.

T	0_{L_1}	s	q	a	m	r	c	k	1_{L_1}
0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}	0_{L_1}
s	0_{L_1}	s	s	s	s	s	s	s	s
q	0_{L_1}	s	s	q	s	s	s	q	q
a	0_{L_1}	s	q	a	q	s	q	a	a
m	0_{L_1}	s	s	q	q	s	q	q	m
r	0_{L_1}	s	s	s	s	s	s	s	s
c	0_{L_1}	s	s	q	q	s	q	q	c
k	0_{L_1}	s	q	a	q	s	q	k	k
1_{L_1}	0_{L_1}	s	q	a	m	r	c	k	1_{L_1}

Tab. 1. The function T on L_1.

Remark 3.5. It can not always be easy to define the interior operator such that $int(x \wedge y \wedge a) = x \wedge y \wedge a$ for every bounded lattice. It should not be forgotten that the lattice to be taken is also important for this. In Example 3.8, if the interior operator $int : L_2 \to L_2$ is defined by $int(0_{L_2}) = 0_{L_2}$, $int(b) = int(p) = int(t) = int(a) = int(d) = int(s) = b$, $int(n) = n$ and $int(1_{L_2}) = 1_{L_2}$, then it is clear that for all $x, y \in I_a$ it holds $x \wedge y \wedge a = int(x \wedge y \wedge a)$. Similarly, readers can identify such lattices and interior operators.

Corollary 3.6. Let $(L, \leq, 0, 1)$ be a bounded lattice with $a, b \in L$ such that for all $x, y \in I_a$ it holds $x \wedge y \wedge a = x \wedge y \wedge a \wedge b$ and V be a t-norm on $[a, 1]$. Then, the function $T : L^2 \to L$ defined by

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1)^2 \ , \\ y \wedge a & \text{if } (x,y) \in [a,1) \times I_a \ , \\ x \wedge a & \text{if } (x,y) \in I_a \times [a,1) \ , \\ x \wedge y \wedge a & \text{if } (x,y) \in I_a \times I_a \ , \\ x \wedge y & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a] \text{ or } 1 \in \{x,y\} \ , \\ x \wedge y \wedge b & \text{otherwise,} \end{cases}$$

is a t-norm on L.

We give next construction methods for t-norms on complete lattices from Definition 2.9 and Definition 2.11.

Corollary 3.7. Let $(L, \leq, 0, 1)$ be a complete lattice with $a \in L$, $\Downarrow : L \to L$ be defined in Definition 2.9 such that for all $x \in I_a$ it holds $x \wedge y \wedge a = \Downarrow (x \wedge y \wedge a)$ and V be a t-norm on $[a, 1]$. Then, the binary operation $T : L^2 \to L$ defined by

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1)^2 \ , \\ y \wedge a & \text{if } (x,y) \in [a,1) \times I_a \ , \\ x \wedge a & \text{if } (x,y) \in I_a \times [a,1) \ , \\ x \wedge y \wedge a & \text{if } (x,y) \in I_a \times I_a \ , \\ x \wedge y & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a] \text{ or } 1 \in \{x,y\} \ , \\ \Downarrow (x) \wedge \Downarrow (y) & \text{otherwise,} \end{cases}$$

is a t-norm on L.

We can give an example to illustrate Corollary 3.7.

Example 3.8. Consider the complete lattice $(L_2 = \{0_{L_2}, b, d, a, p, s, t, n, 1_{L_2}\}, \leq, 0_{L_2}, 1_{L_2})$ in Figure 2. And we take the t-norm $V(x, y) = x \wedge y$ on $[a, 1_{L_2}]$. It is clear that $UC(L_2) = \{0_{L_2}, b, n, 1_{L_2}\}$. So, we obtain $\Downarrow(0_{L_2}) = 0_{L_2}$, $\Downarrow(b) = \Downarrow(p) = \Downarrow(t) = \Downarrow(a) = \Downarrow(d) = \Downarrow(s) = b$, $\Downarrow(n) = n$ and $\Downarrow(1_{L_2}) = 1_{L_2}$. Since for all $x, y \in I_a$ it holds $x \wedge y \wedge a = \Downarrow(x \wedge y \wedge a)$, L_2 satisfies the constraint of Corollary 3.7. Then the t-norm $T : L_2^2 \rightarrow L_2$ constructed via Corollary 3.7 is given by Table 2.

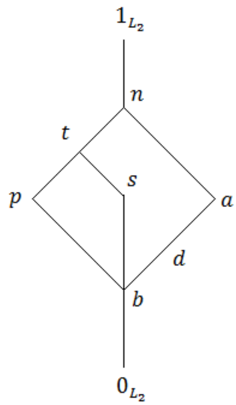

Fig. 2. The lattice L_2.

T	0_{L_2}	b	d	a	p	s	t	n	1_{L_2}
0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}
b	0_{L_2}	b	b	b	b	b	b	b	b
d	0_{L_2}	b	b	d	b	b	b	d	d
a	0_{L_2}	b	d	a	b	b	b	a	a
p	0_{L_2}	b	b	b	b	b	b	b	p
s	0_{L_2}	b	b	b	b	b	b	b	s
t	0_{L_2}	b	b	b	b	b	b	b	t
n	0_{L_2}	b	d	a	b	b	b	n	n
1_{L_2}	0_{L_2}	b	d	a	p	s	t	n	1_{L_2}

Tab. 2. The t-norm T on L_2.

Remark 3.9. If we take $b = 0$ in Corollary 3.6, then it must be $x \wedge y \wedge a = 0$. So, we obtain corresponding t-norm as follows constructed by Çaylı [8].

Corollary 3.10. Let $(L, \leq, 0, 1)$ be a bounded lattice with $a \in L \setminus \{0, 1\}$ and V is a t-norm on $[a, 1]$. Then the function $T_1 : L^2 \to L$ is a t-norm on L, where

$$T_1(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1)^2 \ , \\ 0 & \text{if } (x,y) \in [0,a)^2 \cup [0,a) \times I_a \cup I_a \times [0,a) \cup I_a \times I_a \ , \\ x \wedge y & \text{if } x = 1 \text{ or } y = 1 \ , \\ x \wedge y \wedge a & \text{otherwise.} \end{cases}$$

Remark 3.11. If we take $b = 1$ in Corollary 3.6, then we obtain corresponding t-norm as follows constructed by Ertuğrul, Karaçal and Mesiar [15].

Corollary 3.12. Let $(L, \leq, 0, 1)$ be a bounded lattice and V is a t-norm on $[a, 1]$. Then the function $T_2 : L^2 \to L$ is a t-norm on L, where

$$T_2(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1)^2 \ , \\ x \wedge y & \text{if } x = 1 \text{ } or \text{ } y = 1 \ , \\ x \wedge y \wedge a & \text{otherwise.} \end{cases}$$

Remark 3.13. It should be noted that the t-norms T_1 and T_2 in Corollary 3.10 and Corollary 3.12, respectively are different from the t-norm T in Theorem 3.1. To show that this claim, we shall consider the bounded lattice $(L_2 = \{0_{L_2}, b, d, a, p, s, t, n, 1_{L_2}\}, \leq, 0_{L_2}, 1_{L_2})$ described in Figure 2., we take the t-norm $V(x,y) = x \wedge y$ on $[a, 1_{L_2}]$ and the interior operator $int : L_2 \to L_2$ defined by $int(0_{L_2}) = 0_{L_2}$, $int(b) = int(p) = int(t) = int(a) = int(d) = int(s) = b$, $int(n) = n$ and $int(1_{L_2}) = 1_{L_2}$. According to the Table 2, Table 3 and Table 4, it is clear that the t-norms T, T_1 and T_2 different from each other.

T_1	0_{L_2}	b	d	a	p	s	t	n	1_{L_2}
0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}
b	0_{L_2}	0_{L_2}	0_{L_2}	b	0_{L_2}	0_{L_2}	0_{L_2}	b	b
d	0_{L_2}	0_{L_2}	0_{L_2}	d	0_{L_2}	0_{L_2}	0_{L_2}	d	d
a	0_{L_2}	b	d	a	b	b	b	a	a
p	0_{L_2}	0_{L_2}	0_{L_2}	b	0_{L_2}	0_{L_2}	0_{L_2}	b	p
s	0_{L_2}	0_{L_2}	0_{L_2}	b	0_{L_2}	0_{L_2}	0_{L_2}	b	s
t	0_{L_2}	0_{L_2}	0_{L_2}	b	0_{L_2}	0_{L_2}	0_{L_2}	b	t
n	0_{L_2}	b	d	a	b	b	b	n	n
1_{L_2}	0_{L_2}	b	d	a	p	s	t	n	1_{L_2}

Tab. 3. The t-norm T_1 on L_2.

4. NEW CONSTRUCTION METHOD FOR T-CONORMS ON BOUNDED LATTICES BY USING CLOSURE OPERATORS

In this section, we propose new construction method for t-conorms on bounded lattices with the given t-conorms by using closure operators. The main aim of this section is to

T_2	0_{L_2}	b	d	a	p	s	t	n	1_{L_2}
0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}	0_{L_2}
b	0_{L_2}	b	b	b	b	b	b	b	b
d	0_{L_2}	b	d	d	b	b	b	d	d
a	0_{L_2}	b	d	a	b	b	b	a	a
p	0_{L_2}	b	b	b	b	b	b	b	p
s	0_{L_2}	b	b	b	b	b	b	b	s
t	0_{L_2}	b	b	b	b	b	b	b	t
n	0_{L_2}	b	d	a	b	b	b	n	n
1_{L_2}	0_{L_2}	b	d	a	p	s	t	n	1_{L_2}

Tab. 4. The t-norm T_2 on L_2.

present a rather effective method to construct t-conorms closure operators on a bounded lattice. Using this method, in Corollary 4.9 and 4.11, we obtain the methods proposed by Ertuğrul, Karaçal, Mesiar [15] and Çaylı [8], respectively.

Theorem 4.1. Let $(L, \leq, 0, 1)$ be a bounded lattice with $a \in L$ and $cl : L \to L$ be a closure operator such that for all $x \in I_a$ it holds $x \vee y \vee a = cl(x \vee y \vee a)$. Given a t-conorm W on $[0, a]$, then the function $S : L^2 \to L$ defined as follows is a t-conorm on L where

$$S(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in (0,a]^2 \ , \\ y \vee a & \text{if } (x,y) \in (0,a] \times I_a \ , \\ x \vee a & \text{if } (x,y) \in I_a \times (0,a] \ , \\ x \vee y \vee a & \text{if } (x,y) \in I_a \times I_a \ , \\ x \vee y & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a] \text{ or } 0 \in \{x,y\} \ , \\ cl(x) \vee cl(y) & \text{otherwise} \ . \end{cases}$$

Proposition 4.2. Let $(L, \leq, 0, 1)$ be a bounded lattice with $a \in L$, $cl : L \to L$ be a closure operator and W be a t-conorm on $[0, a]$. The function $S : L^2 \to L$ defined in Theorem 4.1 is a t-conorm on L if and only if $I_a = \emptyset$; otherwise, for any $x, y \in I_a$ it holds that $x \vee y \vee a = cl(x \vee y \vee a)$.

Remark 4.3. Let $(L, \leq, 0, 1)$ be a bounded lattice with $a \in L$. In Proposition 4.2, observe that the condition for all $x \in I_a$ it holds $x \vee y \vee a = cl(x \vee y \vee a)$ can not be omitted, in general. The following example illustrates this fact that the function $S : L^2 \to L$ defined by Theorem 4.1 is not a t-conorm.

Example 4.4. Consider the lattice $(L_3 = \{0_{L_3}, p, d, a, q, k, t, n, 1_{L_3}\}, \leq, 0_{L_3}, 1_{L_3})$ in Figure 3. And we take the t-conorm $W(x,y) = x \vee y$ on $[0_{L_3}, a]$. The closure operator $cl : L_3 \to L_3$ defined by $cl(0_{L_3}) = 0_{L_3}$, $cl(p) = p$, $cl(d) = cl(a) = cl(k) = cl(t) = cl(q) = cl(n) = n$, and $cl(1_{L_3}) = 1_{L_3}$. For all $x \in I_a$ it does not hold $x \vee y \vee a = cl(x \vee y \vee a)$. Because, $d \vee q \vee a = k \neq n = cl(k) = cl(d \vee q \vee a)$. Then, the function S on L_3 defined by Table 5 is not a t-conorm. Indeed, it does not satisfy the associativity. Because $S(p, S(d, q)) = S(p, k) = k \neq n = S(k, q) = S(S(p, d), q)$.

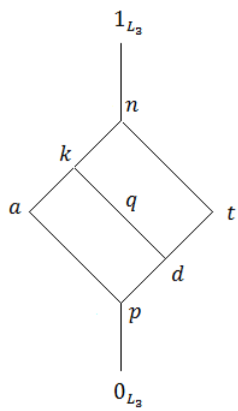

Fig. 3. The lattice L_3.

S	0_{L_3}	p	d	a	q	k	t	n	1_{L_3}
0_{L_3}	0_{L_3}	p	d	a	q	k	t	n	1_{L_3}
p	p	p	k	a	k	k	n	n	1_{L_3}
d	d	k	k	k	k	n	n	n	1_{L_3}
a	a	a	k	a	k	k	n	n	1_{L_3}
q	q	k	k	k	k	n	n	n	1_{L_3}
k	k	k	n	k	n	n	n	n	1_{L_3}
t	t	n	n	n	n	n	n	n	1_{L_3}
n	n	n	n	n	n	n	n	n	1_{L_3}
1_{L_3}	1_{L_3}	1_{L_3}	1_{L_3}	1_{L_3}	1_{L_3}	1_{L_3}	1_{L_3}	1_{L_3}	1_{L_3}

Tab. 5. The t-function S on L_3.

Corollary 4.5. Let $(L, \leq, 0, 1)$ be a bounded lattice with $a, b \in L$ such that for all $x, y \in I_a$ it holds $x \vee y \vee a = x \vee y \vee a \vee b$ and W be a t-conorm on $[0, a]$. Then, the function $S : L^2 \to L$ defined by

$$S(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in (0,a]^2 , \\ y \vee a & \text{if } (x,y) \in (0,a] \times I_a , \\ x \vee a & \text{if } (x,y) \in I_a \times (0,a] , \\ x \vee y \vee a & \text{if } (x,y) \in I_a \times I_a , \\ x \vee y & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a] \ \textit{or} \ 0 \in \{x,y\} , \\ x \vee y \vee b & \text{otherwise.} \end{cases}$$

is a t-norm on L.

We give next construction methods for t-conorms on complete lattices from Definition 2.5 and Definition 2.8.

Corollary 4.6. Let $(L, \leq, 0, 1)$ be a complete lattice with $a \in L$, $\Uparrow: L \to L$ be defined in Definition 2.5 such that for all $x \in I_a$ it holds $x \vee y \vee a = \Uparrow (x \vee y \vee a)$ and W be a t-conorm on $[0, a]$. Then, the binary operation $S : L^2 \to L$ defined by

$$S(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in (0,a]^2 \ , \\ y \vee a & \text{if } (x,y) \in (0,a] \times I_a \ , \\ x \vee a & \text{if } (x,y) \in I_a \times (0,a] \ , \\ x \vee y \vee a & \text{if } (x,y) \in I_a \times I_a \ , \\ x \vee y & \text{if } (x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a] \ or \ 0 \in \{x,y\} \ , \\ \Uparrow (x) \vee \Uparrow (y) & \text{otherwise.} \end{cases}$$

is a t-conorm on L.

We can give an example to illustrate Corollary 4.6.

Example 4.7. Consider the complete lattice $(L_4 = \{0_{L_4}, k, p, r, s, a, c, t, 1_{L_4}\}, \leq, 0_{L_4}, 1_{L_4})$ in Figure 4. And we take the t-conorm $W(x, y) = x \vee y$ on $[0_{L_4}, a]$. It is clear that $UC(L_4) = \{0_{L_4}, k, t, 1_{L_4}\}$. So, we obtain $\Uparrow (0_{L_4}) = 0_{L_4}$, $\Uparrow (k) = k$, $\Uparrow (p) = \Uparrow (r) = \Uparrow (s) = \Uparrow (a) = \Uparrow (c) = \Uparrow (t) = t$, and $\Uparrow (1_{L_4}) = 1_{L_4}$. Since for all $x \in I_a$ it holds $x \vee y \vee a = \Uparrow (x \vee y \vee a)$, L_4 satisfies the constraint of Corollary 4.6. Then the t-conorm $S : L_4^2 \to L_4$ constructed via Corollary 4.6 is given by Table 6.

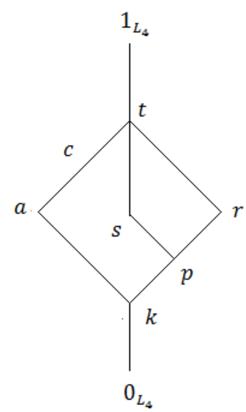

Fig. 4. The lattice L_4.

S	0_{L_4}	k	p	r	s	a	c	t	1_{L_4}
0_{L_4}	0_{L_4}	k	p	r	s	a	c	t	1_{L_4}
k	k	k	t	t	t	a	c	t	1_{L_4}
p	p	t	t	t	t	t	t	t	1_{L_4}
r	r	t	t	t	t	t	t	t	1_{L_4}
s	s	t	t	t	t	t	t	t	1_{L_4}
a	a	a	t	t	t	a	c	t	1_{L_4}
c	c	c	t	t	t	c	t	t	1_{L_4}
t	t	t	t	t	t	t	t	t	1_{L_4}
1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}

Tab. 6. The t-conorm S on L_4.

Remark 4.8. If we take $b = 0$ in Corollary 4.5, then we obtain corresponding t-conorm as follows constructed by Ertuğrul, Karaçal and Mesiar [15].

Corollary 4.9. Let $(L, \leq, 0, 1)$ be a bounded lattice and W is a t-conorm on $[0, a]$. Then the function $S_1 : L^2 \to L$ is a t-conorm on L, where

$$S_1(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in (0,a]^2 \ , \\ x \vee y & \text{if } x = 0 \ or \ y = 0 \ , \\ x \vee y \vee a & \text{otherwise.} \end{cases}$$

Remark 4.10. If we take $b = 1$ in Corollary 4.5, then it must be $x \vee y \vee a = 1$. So, we obtain corresponding t-conorm as follows constructed by Çaylı [8].

Corollary 4.11. Let $(L, \leq, 0, 1)$ be a bounded lattice and $a \in L \setminus \{0, 1\}$. If V is a t-norm on $[a, 1]$, then the function $S_2 : L^2 \to L$ is a t-norm on L, where

$$S_2(x,y) = \begin{cases} W(x,y) & \text{if } (x,y) \in (0,a]^2 \ , \\ 1 & \text{if } (x,y) \in (a,1]^2 \cup (a,1] \times I_a \cup I_a \times (a,1] \cup I_a \times I_a \ , \\ x \vee y & \text{if } x = 0 \text{ or } y = 0 \ , \\ x \vee y \vee a & \text{otherwise.} \end{cases}$$

Remark 4.12. It should be noted that the t-conorms S_1 and S_2 in Corollary 4.9 and Corollary 4.11, respectively are different from the t-conorm S in Theorem 4.1. To show that this claim, we consider the bounded lattice $(L_4 = \{0_{L_4}, k, p, r, s, a, c, t, 1_{L_4}\}, \leq, 0_{L_4}, 1_{L_4})$ in Figure 4., we take the t-conorm $W(x, y) = x \vee y$ on $[0_{L_4}, a]$ and the closure operator $cl : L_4 \to L_4$ defined by $cl(0_{L_4}) = 0_{L_4}$, $cl(k) = k$, $cl(p) = cl(r) = cl(s) = cl(a) = cl(c) = cl(t) = t$ and $cl(1_{L_4}) = 1_{L_4}$. According to the Table 6, Table 7 and Table 8, it is clear that t-conorms S, S_1 and S_2 different from each other.

S_1	0_{L_4}	k	p	r	s	a	c	t	1_{L_4}
0_{L_4}	0_{L_4}	k	p	r	s	a	c	t	1_{L_4}
k	k	k	t	t	t	a	c	t	1_{L_4}
p	p	t	t	t	t	t	t	t	1_{L_4}
r	r	t	t	t	t	t	t	t	1_{L_4}
s	s	t	t	t	t	t	t	t	1_{L_4}
a	a	a	t	t	t	a	c	t	1_{L_4}
c	c	c	t	t	t	c	t	t	1_{L_4}
t	t	t	t	t	t	t	t	t	1_{L_4}
1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}

Tab. 7. The t-conorm S_1 on L_4.

S_2	0_{L_4}	k	p	r	s	a	c	t	1_{L_4}
0_{L_4}	0_{L_4}	k	p	r	s	a	c	t	1_{L_4}
k	k	k	t	t	t	k	c	t	1_{L_4}
p	p	t	1_{L_4}	1_{L_4}	1_{L_4}	t	1_{L_4}	1_{L_4}	1_{L_4}
r	r	t	1_{L_4}	1_{L_4}	1_{L_4}	t	1_{L_4}	1_{L_4}	1_{L_4}
s	s	t	1_{L_4}	1_{L_4}	1_{L_4}	t	1_{L_4}	1_{L_4}	1_{L_4}
a	a	k	t	t	t	a	c	t	1_{L_4}
c	c	c	1_{L_4}	1_{L_4}	1_{L_4}	c	1_{L_4}	1_{L_4}	1_{L_4}
t	t	t	1_{L_4}	1_{L_4}	1_{L_4}	t	1_{L_4}	1_{L_4}	1_{L_4}
1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}	1_{L_4}

Tab. 8. The t-conorm S_2 on L_4.

5. CONCLUDING REMARKS

In this paper, we have proposed the constructions of t-norms and t-conorms on bounded lattices with the interior and closure operators. The main aim of this paper is to present a rather effective method to construct t-norms and t-conorms interior and closure operators on a bounded lattice. Also, using these methods, in Corollary 3.12 and Corollary 4.9, we obtain the methods proposed by Ertuğrul, Karaçal and Mesiar [15]. Also, in Corollary 3.10 and Corollary 4.11, we obtain the methods proposed by Çaylı [8]. From [8] and [15], we have known that new t-norms and t-conorms on bounded lattices can be obtained using recursion in Theorem 2.13, Theorem 2.14 and Theorem 2.15, Theorem 2.16, respectively. Unlike them, in this paper, the new construction methods defined in Theorem 3.1 and Theorem 4.1 can not be generalized by induction to a modified ordinal sum for t-norms and t-conorms on arbitrary bounded lattice, respectively. We can explain this condition as follows:

First, comparing with the methods in Theorem 2.13 and Theorem 2.14. Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, \ldots, a_n\}$ be a finite chain in L such that

$1 = a_0 > a_1 > a_2 > \cdots > a_n = 0$. Let $x \wedge y \wedge a_i = int(x \wedge y \wedge a_i)$ for all $x \in I_{a_i}$, let $V : [a_1, 1]^2 \to [a_1, 1]$ be a t-norm and $int : L \to L$ be an interior operator. It should be noted that our construction method in Theorem 3.1 can not be obtained using recursion. Because, we can not obtain the binary operation $T_i : [a_i, 1]^2 \to [a_i, 1]$ as follows, where $T_1 = V$ and for $i \in \{2, \ldots, n\}$,

$$T_i(x,y) = \begin{cases} T_{i-1}(x,y) & \text{if } (x,y) \in [a_{i-1}, 1)^2 \ , \\ y \wedge a_{i-1} & \text{if } (x,y) \in [a_{i-1}, 1) \times I_{a_{i-1}} \ , \\ x \wedge a_{i-1} & \text{if } (x,y) \in I_{a_{i-1}} \times [a_{i-1}, 1) \ , \\ x \wedge y \wedge a_{i-1} & \text{if } (x,y) \in I_{a_{i-1}} \times I_{a_{i-1}}, \\ x \wedge y & \text{if } x = 1 \textit{ or } y = 1 \ , \\ int(x) \wedge int(y) & \text{otherwise.} \end{cases} \tag{9}$$

To illustrate this claim we consider the lattice $(L_5 = \{0_5, a_4, m, n, a_3, a_2, a_1, 1_{L_5}\}, \leq, 0_{L_5}, 1_{L_5})$ described in Figure 5 with the finite chain $0_{L_5} < a_4 < a_3 < a_2 < a_1 < 1_{L_5}$ in L_5. Then, the interior operator $int : L_5 \to L_5$ defined by $int(0_{L_5}) = 0_{L_5}$, $int(a_4) = int(a_3) = int(a_2) = int(a_1) = int(m) = int(n) = a_4$, $int(1_{L_5}) = 1_{L_5}$. It is clear that $x \wedge y \wedge a_i = int(x \wedge y \wedge a_i)$ for all $x, y \in I_{a_i}$. Define the t-norm $V : [a_1, 1_{L_5}]^2 \to [a_1, 1_{L_5}]$ by $V = T_\wedge$. Since $int(a_1) \wedge int(a_2) = a_4 \notin [a_2, 1_{L_5}]$, we can not obtain the binary operation T_2 on $[a_2, 1_{L_5}]$. Since $int(a_3) \wedge int(a_2) = a_4 \notin [a_3, 1_{L_5}]$, we can not obtain the binary operation T_3 on $[a_3, 1_{L_5}]$.

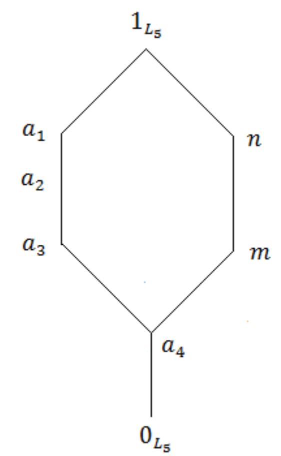

Fig. 5. The lattice L_5-

Secondly, comparing with the methods in Theorem 2.15 and Theorem 2.16. Let $(L, \leq, 0, 1)$ be a bounded lattice and $\{a_0, a_1, a_2, \ldots, a_n\}$ be a finite chain in L such that $0 = a_0 < a_1 < a_2 < \cdots < a_n = 1$. Let $x \vee y \vee a_i = cl(x \vee y \vee a_i)$ for all $x, y \in I_{a_i}$, let $W : [0, a_1]^2 \to [0, a_1]$ be a t-conorm and $cl : L^2 \to L$ be an interior operator. It should be noted that our construction method in Theorem 4.1 can not be obtained using recursion. Because we can not obtain the binary operation $S_i : [0, a_i]^2 \to [0, a_i]$ as follows, where $S_1 = W$ and for $i \in \{2, \ldots, n\}$,

$$S_i(x,y) = \begin{cases} S_{i-1}(x,y) & \text{if } (x,y) \in (0, a_{i-1}]^2 \ , \\ y \vee a_{i-1} & \text{if } (x,y) \in (0, a_{i-1}] \times I_{a_{i-1}} \ , \\ x \vee a_{i-1} & \text{if } (x,y) \in I_{a_{i-1}} \times (0, a_{i-1}] \ , \\ x \vee y \vee a_{i-1} & \text{if } (x,y) \in I_{a_{i-1}} \times I_{a_{i-1}} \ , \\ x \vee y & \text{if } (x,y) \in [0, a_{i-1}] \times [a_{i-1}, 1] \cup [a_{i-1}, 1] \times [0, a_{i-1}] \text{ or } 0 \in \{x, y\}, \\ cl(x) \vee cl(y) & \text{otherwise.} \end{cases} \tag{10}$$

To illustrate this claim we consider the lattice $(L_6 = \{0_{L_6}, a_1, a_2, a_3, p, k, a_4, 1_{L_6}\}, \leq, 0_{L_6}, 1_{L_6})$ described in Figure 6 with the finite chain $0_{L_6} < a_1 < a_2 < a_3 < a_4 < 1_{L_6}$ in L_6. Then, the closure operator $cl : L_6 \to L_6$ defined by $cl(0_{L_6}) = 0_{L_6}$, $cl(p) = cl(k) = cl(a_1) = cl(a_2) = cl(a_3) = cl(a_4) = a_4$, $cl(1_{L_6}) = 1_{L_6}$. It is clear that $x \vee y \vee a_i = cl(x \vee y \vee a_i)$ for all $x, y \in I_{a_i}$. Define the t-conorm $W : [0_{L_6}, a_1]^2 \to [0_{L_6}, a_1]$ by $W = S_\vee$. Since $int(a_1) \vee int(a_2) = a_4 \notin [0_{L_6}, a_2]$, we can not obtain the binary operation S_2 on $[0_{L_6}, a_2]$. Since $int(a_3) \vee int(a_2) = a_4 \notin [0_{L_6}, a_3]$, we can not obtain the binary operation S_3 on $[0_{L_6}, a_3]$.

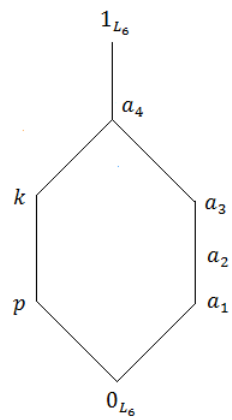

Fig. 6. The lattice L_6.

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