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SAFE CONSENSUS CONTROL OF COOPERATIVE-COMPETITIVE MULTI-AGENT SYSTEMS VIA DIFFERENTIAL PRIVACY

Jiayue Ma and Jiangping Hu

This paper investigates a safe consensus problem for cooperative-competitive multi-agent systems using a differential privacy (DP) approach. Considering that the agents simultaneously interact cooperatively and competitively, we propose a novel DP bipartite consensus algorithm, which guarantees that the DP strategy only works on competitive pairs of agents. We then prove that the proposed algorithm can achieve the mean square bipartite consensus and (p, r)-accuracy. Furthermore, a differential privacy analysis is conducted, which shows that the performance of privacy protection is positively correlated with the number of neighbors. Thus, a practical method is established for the agents to select their own privacy levels. Finally, the simulation results are presented to demonstrate the validity of the proposed safe consensus algorithm.

Keywords: differential privacy, safe consensus, cooperative-competitive multi-agent systems, Laplace distribution, (p, r)-accuracy

Classification: 93A14, 93C10

1. INTRODUCTION

In the past two decades, distributed control of multi-agent systems (MASs) has attracted significant attention due to its wide range of applications in various fields including robotic coordination [3], sensor networks [27], and power-water network [14]. One of the fundamental research objectives of distributed controls is to guarantee consensus. Until now, some consensus problems of MASs, such as average consensus [20, 19], optimal consensus [21, 22], robust consensus [9], and bipartite consensus [11, 10], have been extensively studied from various perspectives.

It is well known that cooperative and competitive interactions exist simultaneously in many complex network systems [4]. For convenience of modeling cooperative-competitive networks, a signed graph theory was introduced in [1] and a bipartite consensus was then formulated. “Bipartite consensus” means that the agents achieve agreement with identical values but opposite signs. To deal with the effect of nonlinear unknown disturbances, Wu et al. [26] designed distributed adaptive laws to investigate a bipartite consensus for a high-order MAS by using a linearly parameterized approach. Ma et al. [16] proposed

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stochastic approximation-type protocols to achieve leader-following bipartite consensus for single-integrator MASs. Subsequently, Hu et al. [11] addressed a bipartite consensus control problem for high-order MASs with communication noise and designed a new stochastic-approximation-based control strategy to attenuate noise by using only the relative state information from neighbors. Furthermore, the distributed bipartite consensus problem of MASs subjected to both additive and multiplicative noises over time-varying random networks was handled in [5]. Recently, Peng et al. [18] proposed a model-free reinforcement-learning-based controller design method to tackle the optimal bipartite consensus problems. As discussed above, some solution strategies have been developed for bipartite consensus of MASs with communication noises, disturbances and unknown dynamics. To our knowledge, it is still scarce to study the privacy issue in multi-agent communication with competitive interactions.

In order to preserve the privacy of agents when they communication over networks, the concept of differential privacy (DP) was proposed in [6]. Then DP method has drawn considerable interest due to its remarkable advantages such as accurate formulation and verifiable privacy, which guarantees an immunity to post-processing and the irrelevance of the adversary's model. Huang et al. [12] adopted the notion of DP to an average consensus problem and proposed a distributed algorithm with decaying Laplace noises to ensure the private consensus. Thereafter, Nozari et al. [17] proposed a DP consensus algorithm with almost sure convergence. Gao et al. investigated a quantized DP consensus problem for MASs in [7] by utilizing a dynamic encoding/decoding process. In order to preserve the privacy of maximum states of individuals, Wang et al. [25] proposed a privacy-preserving mechanism that guaranteed maximum consensus by perturbing each agent's initial state drawn from the Laplace distribution. A modified push-sum DP algorithm for MASs under generally directed topology was proposed in [24] to achieve the average consensus problem. More recently, Zuo et al. [28] addressed the DP consensus problem with a signed graph, but they did not consider the relationship between privacy and the antagonistic neighbors. Furthermore, Wang el al. [23] proposed a multi-gossip Privacy-Preserving/Summation-Consistent(PPSC) mechanism for distributed computation tasks. From the literature review mentioned above, it is noted that most of the existing works related to DP control or optimization mainly focused on MASs with cooperative interactions, which forms the motivation of this paper.

Inspired by the above observations, this paper aims to develop a DP controller for a bipartite consensus problem. The main contributions of this paper are stated as follows: First, a mean square bipartite consensus problem is formulated for a MAS under the privacy protection framework. Second, a novel DP consensus algorithm is proposed to preserve the privacy of the agents' initial states. In contrast to the existing DP consensus algorithms, the privacy protection mechanism only works for competitive agents. Third, the mean square bipartite consensus is analyzed under the proposed safe controller. Furthermore, by examining the (p, r)-accuracy, a feasible strategy is established for each agent to choose his/her own privacy level based on the number of neighbors.

The remainder of this paper is organized as follows. Section 2 presents the necessary concepts and lemmas with problem formulation. Section 3 establishes the main results of this study. Section 4 provides a numerical simulation to illustrate the main results. Finally, section 5 concludes this paper.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1. Notations

The sets of reals, non-negative reals, positive integers, and non-negative integers are denoted by $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{N}$, and $\mathbb{Z}_{\geq 0}$, respectively. $(\mathbb{R}^n)^{\mathbb{N}}$ denotes the space of the vector-valued sequences in $\mathbb{R}^n$. $I_n \in \mathbb{R}^{n\times n}$ and $\mathbf{1}_n \in \mathbb{R}^n$ denotes the identity matrix and vector of ones, respectively. The transposes of vector v and matrix M are denoted by v^T and M^T, respectively. $\lambda\{\cdot\}$ denotes the eigenvalues. $||\cdot||$ is the Euclidean norm for a matrix. $\mathrm{sign}(\cdot)$ denotes a sign function. $\mathrm{diag}\{\cdot\}$ denotes a diagonal matrix. $\mathbb{P}\{\cdot\}$ denotes the probability of an event. $\mathbb{E}[X]$ and $\mathrm{Var}(X)$ are the mathematical expectation and variance of random variable $X \in \mathbb{R}$, respectively.

2.2. Graph theory

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be a signed digraph, where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is the set of vertices representing agents and n is the number of nodes. $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of edges, and $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the adjacency matrix of G, where $a_{ij} \neq 0 \Leftrightarrow (v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0 \Leftrightarrow (v_i, v_j) \notin \mathcal{E}$. Specifically, the interactions between agent i and j is cooperative if $a_{ij} > 0$ and competitive if $a_{ij} < 0$. It is assumed that the digraph with no self-loops, i.e., $a_{ii} = 0, i \in \{1\ldots, n\}$ and satisfies the digon sign-symmetry property $a_{ij}a_{ji} \geq 0$. Let $\mathcal{N}_i$ denote the neighbor set of the node i, that is, $\mathcal{N}_i = \{j|(v_j, v_i \in \mathcal{E})\}$. The cardinality of $\mathcal{N}_i$ is $|\mathcal{N}_i|$. A directed path from agent i to agent j is a sequence of ordered edges $\{(v_i, v_{i_1}), (v_{i_1}, v_{i_2}), \ldots, (v_{i_l}, v_j)\}$. A signed directed graph is said to contain a directed spanning tree if there at least exists one agent, which has a directed path to every other agent. The in-degree and out-degree of agent i can be separately defined as $\Delta^i_{in} = \sum_{j\in\mathcal{N}_i} |a_{ij}|$ and $\Delta^i_{out} = \sum_{j\in\mathcal{N}_i} |a_{ji}|$. A signed digraph $\mathcal{G}$ is balanced if $\Delta^i_{in} = \Delta^i_{out} = \Delta_i$ for $i \in \{1\ldots, n\}$. For a balanced digraph, we denote the greateat degree as $\Delta_{\max} = \max\{\Delta_i, i \in \mathcal{V}\}$. A signed Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ is defined as $L = D - A$, where $D = diag(\Delta^1_{in}, \ldots, \Delta^n_{in})$. Let $G_S = \{\mathcal{V}, \mathcal{E}, A_S\}$ be the corresponding unsigned graph of signed graph $\mathcal{G}$ and $L_S = D - A_S = [l_{S,ij}] \in \mathbb{R}^{n\times n}$ be the Laplician matrix of directed graph G_S.

Lemma 2.1. (Hu [9]) The Laplacian matrix L_S of the unsigned digraph G_S has at least one zero eigenvalue and all non-zero eigenvalues have positive real parts. Moreover, L_S has only one zero eigenvalue with the associated eigenvector $\mathbf{1}_n$ if and only if the unsigned digraph G_S has a spanning tree.

Lemma 2.2. (Hu [9]) Matrix $\hat{L}_S = \frac{L_S + L_S^T}{2}$ can be viewed as a Laplacian matrix if G_S is balanced.

Definition 2.3. (Hu and Wu [10]) A signed graph $\mathcal{G}$ is said to be structurally balanced if a bipartition $\mathcal{V}_1$ and $\mathcal{V}_2$ of vertices exists, where $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \cap \mathcal{V}_2 = \varnothing$ such that $a_{ij} \geq 0$ for $\forall v_i, v_j \in \mathcal{V}_h (h \in \{1, 2\})$ and $a_{ij} \leq 0$ for $\forall v_i \in \mathcal{V}_h, v_j \in \mathcal{V}_l, h \neq l (h, l \in \{1, 2\})$.

Lemma 2.4. (Hu and Wu [10]) The signed graph $\mathcal{G}$ is structurally balanced if and only if there exists a gauge matrix S such that $SAS = A_S$ and $SLS = L_S$ with $S = diag\{s_1, \ldots, s_n\}$ and $s_i \in \{\pm 1\}$.

Lemma 2.5. (Altafini [1]) Assume $\mathcal{G}$ is structurally balanced, undirected and connected. For the Laplacian matrix L, the property $\min_{x\neq 0} \frac{x^T L x}{x^T x} = \lambda_2(L)$ holds, where $\lambda_2(L)$ is the smallest nonzero eigenvalue of L.

Assumption 2.6. The signed graph $\mathcal{G}$ is structurally balanced and has a spanning tree.

2.3. Problem formulation

For an MAS consisting of n agents over a cooperative-competitive network, the agent dynamics are as follows:

$$x_i(k+1) = x_i(k) + u_i(k), k \in \mathbb{Z}_{\geq 0} \tag{1}$$

where $x_i(k) \in \mathbb{R}$ is the state and $u_i(k)$ is the controller.

The distributed controller has the following form:

$$u_i(k) = g(\{x_i(k), \tilde{x}_j(k), j \in \mathcal{N}_i\}), k \in \mathbb{Z}_{\geq 0} \tag{2}$$

where $\tilde{x}_j(k) = h(\{x_j(k), \eta_j(k)\})$ is the noisy state of a competitive neighbor. $\eta_j(k)$ obeys the Laplace distribution $\eta_j(k) \sim Lap(b_j(k))$, where $b_j(k) = c_j q_j^k, c_j > 0, q_j \in (0,1)$ and $g, h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions. According to [8], the DP controller with a Laplace noise is the optimal one; thus, we use it in this paper.

One objective of this study is to design a distributed controller u_i for agent i expressed as

$$\lim_{k\to\infty} \mathbb{E}[x_i(k) - s_i x^*]^2 = 0, s_i \in \{1, -1\}. \tag{3}$$

The MAS (1) achieves mean square bipartite consensus when equation (3) holds. The other objectives of this study are to guarantee (p,r)-accuracy and ϵ_i-differential privacy.

The concept of adjacency between agents and DP definitions are presented.

Definition 2.7. (Adjacency) (Nozari et al. [17]) For any given $\delta \in \mathbb{R}_{\geq 0}$, the initial state vectors $x^{(1)}(0), x^{(2)}(0) \in \mathbb{R}^n$ are δ-adjacent if one $i_0 \in \mathcal{V}$ exists, expressed as:

$$|x_i^{(2)}(0) - x_i^{(1)}(0)| \leq \begin{cases} \delta, & i = i_0 \\ 0, & i \neq i_0 \end{cases} \tag{4}$$

where δ is the adjacency distance.

Definition 2.8. (Differential privacy) (Nozari et al. [17]) Given $\delta, \epsilon \in \mathbb{R}_{\geq 0}$, the mechanism $\mathcal{M}$ is ϵ-differentially private for any pair of δ-adjacent initial states if $x^{(1)}(0), x^{(2)}(0) \in \mathbb{R}^n$ and $\mathcal{W} \in (\mathbb{R}^n)^{\mathbb{N}}$ in any set, expressed as:

$$\mathbb{P}\{\mathcal{M}(x^{(1)}(0)) \in \mathcal{W}\} \leq e^{\epsilon} \mathbb{P}\{\mathcal{M}(x^{(2)}(0)) \in \mathcal{W}\} \tag{5}$$

where ϵ is the privacy level.

Definition 2.9. (Accuracy) For $p \in [0,1]$ and $r \in \mathbb{R}_{\geq 0}$, the mechanism achieves (p,r)-accuracy, if for any initial condition $x(0)$, the state of each agent converges to a random variable x^* as $k \to \infty$ with $\mathbb{E}[x^*] = \frac{1}{N}\sum_{i=1}^{N} s_i x_i(0)$ and $\mathbb{P}\{|x^* - \mathbb{E}[x^*]| \leq r\} \geq 1 - p$, where $s_i \in \{1, -1\}$.

3. MAIN RESULTS

3.1. Controller design

A controller is designed for agent i as follows:

$$u_i(k) = \gamma \sum_{j \in \mathcal{N}_i} a_{ij}(\tilde{x}_j(k) - \text{sign}(a_{ij})x_i(k)) \tag{6}$$

where γ is the consensus gain,

$$\tilde{x}_j(k) = x_j(k) + \frac{1 - \text{sign}(a_{ij})}{2}\eta_j(k), \eta_j(k) \sim Lap(b_j(k)). \tag{7}$$

Applying controller (6) to the MAS (1) leads to

$$x_i(k+1) = x_i(k) + \gamma \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(k) + \frac{1 - \text{sign}(a_{ij})}{2}\eta_j(k) - \text{sign}(a_{ij})x_i(k)). \tag{8}$$

Equation (8) can be rewritten in a compact form as follows:

$$x(k+1) = (I_n - \gamma L)x(k) + \frac{\gamma}{2}(A - A_S)\eta(k) \tag{9}$$

where $x = (x_1, \ldots, x_n)^T$ and $\eta = (\eta_1, \ldots, \eta_n)^T$.

Remark 3.1. In the proposed DP controllers (6) and (7), the sign function is used to describe competitive and cooperative relationships between agents. Since the graph is fixed, if $a_{ij} > 0$, $\text{sign}(a_{ij}) = 1$ and if $a_{ij} < 0$, $\text{sign}(a_{ij}) = -1$. Therefore, the Laplace noise is added to the communication link between the agents only when there is a competitive relationship between the agent i and its neighbor. This confirmed that the need for DP depends on the relationship between the agent i and its neighbors.

3.2. Bipartite consensus analysis

We now analyze the mean square bipartite consensus of the MAS under the controller (6) with (7).

Theorem 3.2. Under Assumption 2.6, the mean square bipartite consensus can be achieved for all the agents under the DP controllers (6) with (7).

Proof. Define $z(k) = Sx(k)$; then,

$$z(k+1) = (I - \gamma L_S)z(k) + \frac{\gamma}{2}S(A - A_S)\eta(k). \tag{10}$$

Let $\xi(k) = (I_n - J)z(k)$ and $J = (1/n)\mathbf{1}_n\mathbf{1}_n^T$; it takes from (10) that

$$\begin{aligned} \xi(k+1) &= (I_n - J)z(k+1) \\ &= (I_n - \gamma L_S)\xi(k) + \frac{\gamma}{2}(I_n - J)S(A - A_S)\eta(k). \end{aligned} \tag{11}$$

Let $V(k) = \xi^T(k)\xi(k)$; then

$$\begin{aligned} V(k+1) &= \xi^T(k+1)\xi(k+1) \\ &= \xi^T(k)(I_n - \gamma(L_S + L_S^T) + \gamma^2 L_S^T L_S)\xi(k) \\ &+ 2\xi^T(k)(I_n - \gamma L_S)^T(I_n - J)S(A - A_S)\eta(k) \\ &\frac{\gamma^2}{4}\eta^T(k)(A - A_S)^T S^T(I_n - J)^T(I_n - J)S(A - A_S)\eta(k). \end{aligned} \tag{12}$$

It is noted that $\xi(k)$ and $\eta(k)$ are independent of each other and $\eta(k)$ has zero mean. Taking the mathematical expectation of both sides of (12) yields

$$\begin{aligned} \mathbb{E}[V(k+1)] \leq (1 &- 2\gamma\lambda_2(\hat{L}_S) + \gamma^2||L_S||^2)\mathbb{E}[V(k)] \\ &+ \frac{\gamma^2}{4}||A - A_S||^2||I_n - J||^2\mathbb{E}[\eta^T(k)\eta(k)]. \end{aligned} \tag{13}$$

Under the assumption that $0 < \gamma < \frac{2\lambda_2(\hat{L}_S)}{||L_S||^2}$, $0 < 1 - 2\gamma\lambda_2(\hat{L}_S) + \gamma^2||L_S||^2 < 1$ holds; thus, the first term of (13) converges to 0 as $k \to \infty$. Additionally, since the elements of $\eta(k)$ are independently identically distributed (i.i.d.), for $i \neq j$, $\mathbb{E}[\eta_i(k)\eta_j(k)] = \mathbb{E}[\eta_i(k)]\mathbb{E}[\eta_j(k)] = 0$ and for any i, $\mathbb{E}[\eta_i(k)^2] = Var(\eta_i(k)) = 2c_i^2 q^{2k}$, which also converges to 0; thus, $\mathbb{E}[\eta^T(k)\eta(k)] \to 0$. We conclude that $\mathbb{E}[V(k)] \to 0$ as $k \to \infty$.

Under Assumption 2.6, we have $\mathbf{1}_n^T L_S = 0$. For (10),

$$\begin{aligned} \frac{1}{n}\mathbf{1}_n^T z(k+1) &= \frac{1}{n}\mathbf{1}_n^T z(k) + \frac{\gamma}{2n}\mathbf{1}_n^T S(A - A_S)\eta(k) \\ &= \frac{1}{n}\mathbf{1}_n^T z(0) + \frac{\gamma}{2n}\sum_{l=0}^{k}\sum_{i,j=1}^{n} s_i\tilde{a}_{ij}\eta_i(l) \end{aligned} \tag{14}$$

where $\tilde{a}_{ij} = a_{ij} - |a_{ij}|$. Then,

$$\frac{1}{n}\mathbf{1}_n^T z(k) = \frac{1}{n}\mathbf{1}_n^T z(0) + \frac{\gamma}{2n}\sum_{l=0}^{k-1}\sum_{i,j=1}^{n} s_i\tilde{a}_{ij}\eta_i(l) \tag{15}$$

and hence,

$$\frac{1}{n}\mathbf{1}_n^T Sx(k) = \frac{1}{n}\mathbf{1}_n^T Sx(0) + \frac{\gamma}{2n}\sum_{l=0}^{k-1}\sum_{i,j=1}^{n} s_i\tilde{a}_{ij}\eta_i(l). \tag{16}$$

We know that $\frac{1}{n}\mathbf{1}_n^T Sx(k)$ converges to x^* as time k tends to infinity in mean square, where

$$x^* = \frac{1}{n}\mathbf{1}_n^T Sx(0) + \frac{\gamma}{2n}\sum_{l=0}^{\infty}\sum_{i,j=1}^{n} s_i\tilde{a}_{ij}\eta_i(l). \tag{17}$$

Noting that $\mathbb{E}[V(k)] \to 0$ as $k \to 0$, we have $x_i(k) \to s_i x^*$ as $k \to \infty$ in mean square, which means that the mean square bipartite consensus is achieved. The proof is completed. □

3.3. Accuracy analysis

In this section, we present the computation of (p, r)-accuracy of the proposed DP control strategy.

Theorem 3.3. Considering the MAS (8) for any given initial state, the DP control strategy can achieve (p, r)-accuracy.

Proof. By the fact that $\eta_i(k)$ and $i \in \mathcal{V}$ are i.i.d.,

$$\mathbb{E}[x^*] = \mathbb{E}[\frac{1}{n}\mathbf{1}_n^T Sx(0) + \frac{\gamma}{2n}\sum_{l=0}^{\infty}\sum_{i,j=1}^{n} s_i\tilde{a}_{ij}\eta_i(l)] = \frac{1}{n}\mathbf{1}_n^T Sx(0) \tag{18}$$

and

$$\begin{aligned}\mathrm{Var}(x^*) &= \lim_{k\to\infty}\mathbb{E}[\frac{\gamma}{2n}\sum_{l=0}^{k}\sum_{i,j=1}^{n} s_i\tilde{a}_{ij}\eta_i(l)]^2 \\ &\leq \frac{\gamma^2(\tilde{a}_{\max}\Delta_{\max})^2}{4n^2}\mathbb{E}[\sum_{l=0}^{\infty}\sum_{i=1}^{n}\eta_i(l)] \\ &\leq \frac{\gamma^2(\tilde{a}_{\max}\Delta_{\max})^2}{2n^2}\sum_{l=0}^{\infty}\sum_{i=1}^{n}c_i^2 q_i^{2l} \\ &\leq \frac{\gamma^2 d^2}{2n^2}\sum_{i=1}^{n}\frac{c_i^2}{1-q_i^2}.\end{aligned} \tag{19}$$

By applying Chebyshev inequality, one has

$$\mathbb{P}\{|x^* - \mathbb{E}[x^*]| \leq r\} \geq 1 - \frac{\mathrm{var}(x^*)}{r^2}. \tag{20}$$

Furthermore, we choose $r = \frac{\sqrt{\frac{\gamma^2 d^2}{2n^2}\sum_{i=1}^{n}\frac{c_i^2}{1-q_i^2}}}{\sqrt{p}}$; then, $\mathbb{P}\{x^* - \mathbb{E}[x^*] \leq r\} \geq 1 - p$ is obtained, which implies that the proposed DP controllers can achieve (p, r)-accuracy. The proof is thus completed. □

Remark 3.4. Note that the DP control strategy can achieve $(p, \sqrt{\frac{\gamma^2 d^2}{2n^2}\sum_{i=1}^{n}\frac{c_i^2}{1-q_i^2}}/\sqrt{p})$-accuracy, where $d = \tilde{a}_{\max}\Delta_{\max}$, $\tilde{a}_{\max} = \max_{i,j}\tilde{a}_{ij}$ and $\tilde{a}_{ij} = a_{ij} - |a_{ij}|$.

3.4. Privacy analysis

In this section, we refer to the initial state of an individual agent $x_i(0)$ as the private data and denote the private dataset of this system as $P = \{x_i(0), i \in \mathcal{V}\}$. Then, we denote the sequence of transmitted states $W = \{\tilde{x}_i(k), i \in \mathcal{V}, k = 0, 1, 2, \dots\}$ as the observation of the system and the sequence of the trajectory as $q(P, W) = \{x_i(k), i \in \mathcal{V}, k = 0, 1, 2, \dots\}$.

Theorem 3.5. The proposed DP control strategy preserves ϵ_i-differential privacy for agent $i \in \mathcal{V}$ with

$$\epsilon_i = \begin{cases} 0, & a_{ij} > 0 \\ \dfrac{\delta(1-\gamma\Delta_i)}{c_i(q_i - 1 + \gamma\Delta_i)}, \ q_i \in (1-\gamma\Delta_i, 1), & a_{ij} < 0. \end{cases} \tag{21}$$

Proof. Suppose there are two private datasets, $P^{(1)} = \{x_i^{(1)}(0), i \in \mathcal{V}\}$ and $P^{(2)} = \{x_i^{(2)}(0), i \in \mathcal{V}\}$, and a set of observation $\mathcal{W}$. Given the controller (6), assume $R^{(1)} = \{q(P^{(1)}, W) : W \in \mathcal{W}\}$ and $R^{(2)} = \{q(P^{(2)}, W) : W \in \mathcal{W}\}$ to be the set of possible trajectories in the observation set $\mathcal{W}$. Meanwhile, let $f(P^{(1)}, q(P^{(1)}, W))$ and $f(P^{(2)}, q(P^{(2)}, W))$ be the probability density function of the trajectories. Due to the fact that the observations $W = \{\tilde{x}_j(k), j \in \mathcal{V}\}$ for $P^{(1)}$ and $P^{(2)}$ are the same, based on equation (8), we have

$$x_i^{(1)}(k+1) = (1-\gamma\Delta_i)x_i^{(1)}(k) + \gamma \sum_{j\in\mathcal{N}_i} a_{ij}\tilde{x}_j(k) \tag{22}$$

and

$$x_i^{(2)}(k+1) = (1-\gamma\Delta_i)x_i^{(2)}(k) + \gamma \sum_{j\in\mathcal{N}_i} a_{ij}\tilde{x}_j(k). \tag{23}$$

Therefore,

$$x_i^{(2)}(k+1) - x_i^{(1)}(k+1) = (1-\gamma\Delta_i)(x_i^{(2)}(k) - x_i^{(1)}(k)) \tag{24}$$

which obtains

$$x_i^{(2)}(k) - x_i^{(1)}(k) = (1-\gamma\Delta_i)^k(x_i^{(2)}(0) - x_i^{(1)}(0)). \tag{25}$$

Based on (1)-(2), for a dataset P, given an initial state $x_i(0)$, the observation $W = \{\tilde{x}_i(0), \tilde{x}_i(1), \cdots, i \in \mathcal{V}\}$ is uniquely defined by the noise sequence $\{\eta_i(k), i \in \mathcal{V}, k = 0, 1, 2, \dots\}$. According to (7), the probability density function is

$$\begin{aligned} f(D, q(P,W)) &= \prod_{i=1}^{n}\prod_{l=0}^{k} f(P, q(P,W)_i(l)) \\ &= \prod_{i=1}^{n}\prod_{l=0}^{k} \mathcal{L}(\tilde{x}_i(l) - q(P,W)_i(l); b_i(l)). \end{aligned} \tag{26}$$

For a pair of private datasets, since they have the same observation, there exists a bijection $u(\cdot) : R^{(1)} \to R^{(2)}$ such that for $q(P^{(1)}, W) \in R^{(1)}$, $q(P^{(2)}, W) \in R^{(2)}$, it has

$u(q(P^{(1)}, W)) = q(P^{(2)}, W)$. Using the bijection $u(\cdot)$, we have

$$
\begin{aligned}
\frac{f(P^{(1)}, q(P^{(1)}, W))}{f(P^{(2)}, q(P^{(2)}, W))} &= \frac{\prod_{i=1}^{n}\prod_{l=0}^{k} \mathcal{L}(\tilde{x}_i(l) - q(P^{(1)}, W)_i(l); b_i(l))}{\prod_{i=1}^{n}\prod_{l=0}^{k} \mathcal{L}(\tilde{x}_i(l) - u(q(P^{(1)}, W))_i(l); b_i(l))} \\
&\leq \prod_{i=1}^{n}\prod_{l=0}^{k} e^{\frac{|(\tilde{x}_i(l) - q(P^{(1)}, W)_i(l)) - (\tilde{x}_i(l) - u(q(P^{(1)} W))_i(l))|}{b_i(l)}} \\
&\leq e^{\sum_{l=0}^{k} \frac{|x_{i_0}^{(2)}(l) - x_{i_0}^{(1)}(l)|}{c_{i_0} q_{i_0}^{l}}} \leq e^{\sum_{l=0}^{k} \frac{\delta(1-\gamma\Delta_{i_0})^{l}}{c_{i_0} q_{i_0}^{l}}}.
\end{aligned}
\tag{27}
$$

Integrating both sides over $R^{(1)}$ and letting $k \to \infty$, we have

$$
\mathbb{P}[\mathcal{M}(x^{(1)}(0)) \in \mathcal{W}] \leq e^{\frac{\delta(1-\gamma\Delta_{i_0})}{c_{i_0}(q_{i_0} - 1 + \gamma\Delta_{i_0})}} \mathbb{P}[\mathcal{M}(x^{(2)}(0)) \in \mathcal{W}] \tag{28}
$$

which establishes ϵ_{i_0}-differential privacy for agent i_0. It is worth noting that the agent i_0 can be any agent in the entire network. The proof is thus completed. □

Remark 3.6. Theorem 3.5 guarantees that each agent can determine its own privacy level based on the number of neighbors it has. From (21), it can be seen that the privacy level ϵ_i is related to Δ_i, which means that ϵ_i can be small when the agent i has a large number of neighbors. It is well known that a smaller ϵ_i provides a stronger privacy guarantee, then the proposed DP control strategy can provide better privacy protection.

4. SIMULATION RESULTS

In this section, a numerical example is given to demonstrate the theoretical results. We consider a structurally balanced network, which is illustrated in Figure 1, where the red dash lines and blue solid lines denote the competitive and cooperative relationships, respectively. The distribution of the Laplace noise $\eta_i(k)$ is $\mathcal{L}(\mu, b_i(k)) = \frac{1}{2b_i(k)} e^{-\frac{|x-\mu|}{b_i(k)}}$,

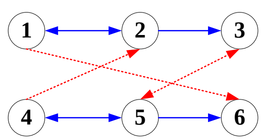

Fig. 1. Structurally balanced signed graph $\mathcal{G}$ containing a spanning tree.

where $\mu = 0$ and $b_i(k) = c_i q_i^k, q_i \in (1 - \gamma\Delta_i, 1)$. By setting $c_i = 0.2, q_i = 0.95$, and $x(0) = [-3, 3, -4, 3, 0.5, 2]$ and calculating the corresponding parameter to obtain $0 < \gamma < 0.2969$, we choose $\gamma = 0.18$. The evolution of the states of the six agents is shown in Figure 2. It is clear that the agents 1, 2, and 3 converge to one group and

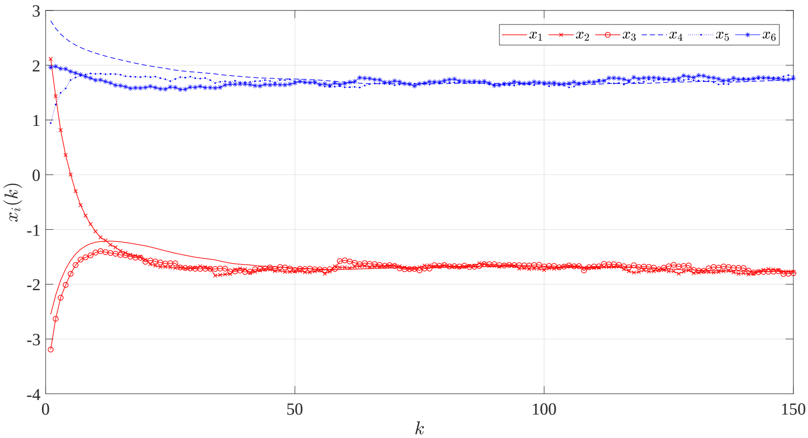

Fig. 2. Evolution of the states of the agents.

the agents 4, 5, and 6 tend to another, which implies that the six agents can reach a mean square bipartite consensus. Furthermore, the evolution of the mathematical expectations of the consensus errors of all the agents is given in Figure 3. Figure 4 shows the histogram of convergence points for 10^4 times under the proposed DP controllers and the red lines indicate the theoretical x^*, which shows that the existence of the noise results in inaccurate bipartite consensus and the consensus values are near the average initial value of the agents in most cases. The relationship between the privacy level ϵ_i and the parameter set (Δ_i, γ) is shown in Figure 5. It is observed that ϵ_i decreases as Δ_i or γ increases.

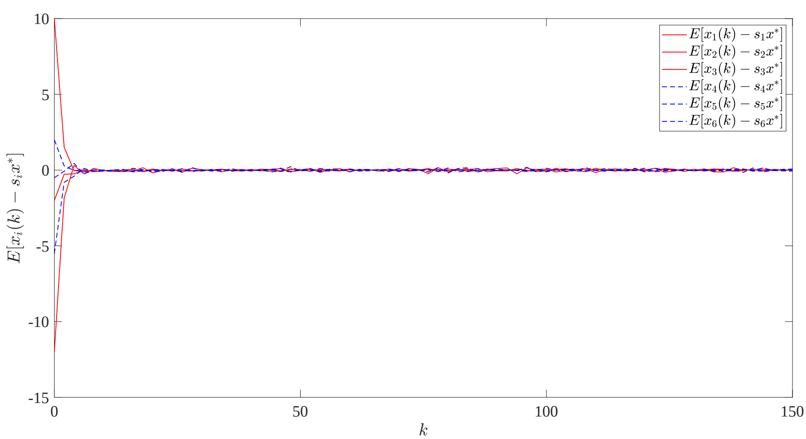

Fig. 3. Evolution of the mathematical expectations of the consensus errors $x_i(k) - s_ix^*$.

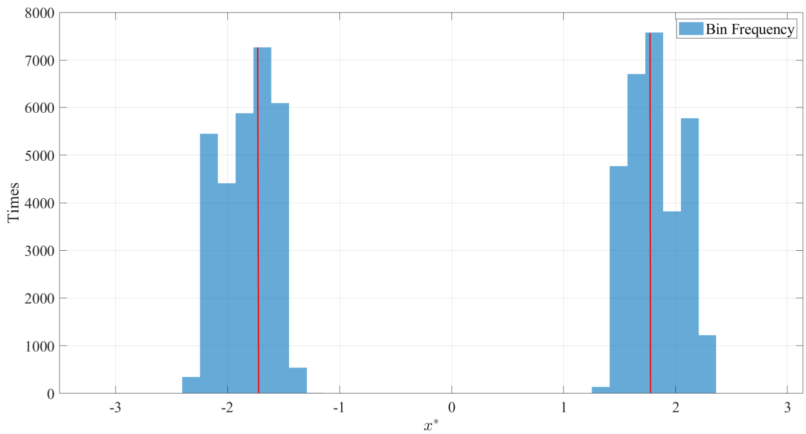

Fig. 4. Histogram of convergence point x^*.

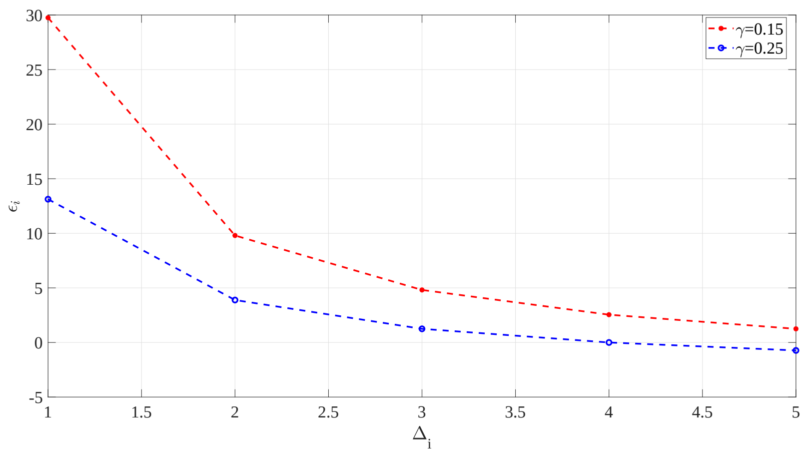

Fig. 5. Privacy level ϵ_i with respect to the parameter set (Δ_i, γ).

5. CONCLUSIONS

This study investigated a DP consensus problem for cooperative-competitive MASs. We first developed a novel DP bipartite consensus algorithm by adding Laplace noise, which was based on the relationship of the agents with its neighbors. Then, we proved that the agents reached a mean square bipartite consensus using our algorithm. We also characterized the accuracy and DP properties, where the choice of privacy level was more practical and relevant to the number of neighbor agents. Finally, a simulation was presented to illustrate the results. Many attractive subjects, such as the extension of the results to switching topologies and distributed DP filtering for MASs, still need to be investigated further.

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Jiayue Ma, School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu 611731, P. R. China.
e-mail: jiayuem@std.uestc.edu.cn

Jiangping Hu, School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu 611731, P. R. China; Yangtze Delta Region Institute (Huzhou), University of Electronic Science and Technology of China, Huzhou 313001, P. R. China.
e-mail: hujp@uestc.edu.cn