ON EXACT SOLUTIONS OF A CLASS OF INTERVAL BOUNDARY VALUE PROBLEMS

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In this article, we deal with the Boundary Value Problem (BVP) for linear ordinary differential equations, the coefficients and the boundary values of which are constant intervals. To solve this kind of interval BVP, we implement an approach that differs from commonly used ones. With this approach, the interval BVP is interpreted as a family of classical (real) BVPs. The set (bunch) of solutions of all these real BVPs we define to be the solution of the interval BVP. Therefore, the novelty of the proposed approach is that the solution is treated as a set of real functions, not as an interval-valued function, as usual.

It is well-known that the existence and uniqueness of the solution is a critical issue, especially in studying BVPs. We provide an existence and uniqueness result for interval BVPs under consideration. We also present a numerical method to compute the lower and upper bounds of the solution bunch. Moreover, we express the solution by an analytical formula under certain conditions. We provide numerical examples to illustrate the effectiveness of the introduced approach and the proposed method. We also demonstrate that the approach is applicable to non-linear interval BVPs.

Keywords: interval differential equations, boundary value problem, bunch of functions, linear differential equations

Classification: 34B05, 93B03, 65G40

1. INTRODUCTION

To predict or understand the behavior of a dynamic system in engineering and science problems, we create a mathematical model based on a differential equation. Often the exact values of many parameters involved in the equation are unknown, but it is possible to determine the intervals where these values lie. Thus, interval-valued differential equations arise. Interval-valued differential equations can be considered also as a particular case of set-valued differential equations. Many significant studies are devoted to set and interval differential equations [2, 6, 7, 12, 15, 17, 18, 22, 23, 29]. These studies mainly differ in the differentiability concepts that they use. The fact is that most researchers share the idea that to tackle problems with uncertainty, it is necessary to create a set-valued calculus (in particular, an interval-valued calculus). They expect that if this set-valued calculus had the same properties as the classical (real) calculus, then the ex-

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isting classical methods could be adapted to solve problems with uncertainty. Different concepts of derivative such as Huygens derivative [5], π-derivative [3], Markov-derivative [20], T-derivative [25], Hukukara derivative [13], strongly generalized derivative [4], gH-derivative [28] and dH-derivative [16] are proposed and investigated in different studies. These concepts have been employed to solve various differential equations with uncertainty [1, 6, 12, 19, 21, 24, 27, 29, 30, 31]. Undoubtedly, the above-mentioned studies enriched the theory of set-valued functions. Nevertheless, the main expectation regarding the set-valued calculus is still not met. We can indicate three essential difficulties: (i) The proposed derivatives are generally not easy to calculate and apply for problem solving (The main reason for this is that set-valued arithmetic is underdeveloped, especially for non-convex sets); (ii) The existence and uniqueness of solution cannot be guaranteed even for Initial Value Problems (IVPs); (iii) The solutions produced are not always consistent with the nature of the studied real-life problem. The main source of these difficulties is the difference operation. The Hukuhara difference is not always defined, while any other difference operation is not the opposite of addition, unlike real arithmetic. As a result, attempts to create a set-valued calculus similar to real calculus fail.

Despite the fact that interval arithmetic is relatively simple, reflections of the difficulties mentioned above are also encountered when we treat interval-valued calculus for solving interval differential equations. The main challenge in this case is as follows. To solve an interval differential equation, most studies employ the strongly generalized (Hukuhara) differentiability [28]. Unfortunately, this differentiability concept has some serious difficulties. The main difficulty is that generalized differentiability is suitable only for solutions with monotone (either increasing or decreasing) uncertainty. How to proceed in cases where the uncertainty alternately increases and decreases is an open question. Namely, to construct such a solution one should a priori anticipate the switching points, where a type of generalized derivative (say, $(1, 2)$-derivative) passes to another derivative (say, $(2, 2)$-derivative). But, up to now, there is no clear recommendation for this. Another difficulty lies in the fact that, under generalized derivative, the solution may not be unique, or may not exist at all.

Our first motivation in this article is to suggest a new approach to overcome the aforementioned difficulties. Another motivation is that the suggested approach produces results that are consistent with the nature of the real-life problems under investigation. To achieve our goals, we apply the concept of a bunch of real functions [8]. This concept allows us to dispense without using a set/interval derivative and, thus, to avoid the above-mentioned drawbacks of the generalized differentiability approach. The employed approach was applied earlier in studies [8, 10, 11] for interval IVPs and systems. The novelties of the present study compared to them are: (i) An exact solution is obtained for linear interval BVPs, for the first time; (ii) Existence and uniqueness results are formulated neatly and completely; (iii) The advantages of the proposed approach over the generalized differentiability approach are exhibited for interval BVPs; (iv) The applicability of the proposed approach to non-linear interval BVPs is demonstrated, and the difficulties encountered are described.

2. PRELIMINARIES

In this section, we provide some basic information about a bunch of functions, which is the key concept of our approach, and about boundary value problems.

2.1. Bunch of functions

Interpreting a function with interval uncertainty as an interval-valued function is a common, but not the only practice. In this article, we model such a function as a bunch of real functions (see, [8]). This concept can be briefly explained as follows. Let a set F of real functions like f_α (where $\alpha \in \Lambda$, and Λ is a set of indices) be given. If the functions are related to each other in some way, then we call this set a **bunch** of functions: $F = \{f_\alpha(\cdot) \mid \alpha \in \Lambda\}$. For a given t, we define the set $F(t) = \{f_\alpha(t) \mid f_\alpha \in F\}$ to be the **value of the bunch** F at t. That is, $F(t)$ is the set consisting of the values of all functions that constitute the bunch, at t. By definition, $F(t)$ is a subset of the set of real numbers. Geometrically, it may consist of two or more disconnected parts (intervals, or isolated points). However, in practice, $F(t)$ is usually an interval.

The following subsection contains the necessary background on boundary value problems for linear differential equations, existence and uniqueness issues, and some formulas.

2.2. BVP for second-order linear ordinary differential equations

2.2.1. Existence and uniqueness questions

The classical Boundary Value Problem (BVP) for a second-order linear differential equation is as follows:

$$\begin{cases} y'' + p(t)y' + q(t)y = f(t) \\ y(0) = a \\ y(l) = b, \end{cases} \tag{1}$$

where a, b and $l > 0$ are given real numbers; $p(t)$, $q(t)$ and $f(t)$ are given real functions. Note that in the above setting, the coefficients are not necessarily constants.

It is well-known that initial-value problems (IVPs) have unique solutions (if the functions involved are sufficiently well-behaved). But, boundary-value problems can have no solution, a unique solution, or even an infinite number of solutions. In what follows, we will refer to the lemma below to verify the existence and uniqueness of the solution (The proof can be found in [9]).

Lemma 2.1. Consider the BVP (1), where $p(t)$, $q(t)$ and $f(t)$ are continuous real functions. Let $y_1(t)$ and $y_2(t)$ be any two linearly independent solutions of the associated homogeneous equation $y'' + p(t)y' + q(t)y = 0$. BVP (1) has exactly one solution for every a and b, if and only if $y_1(0)y_2(l) - y_1(l)y_2(0) \neq 0$.

We notice that the existence and uniqueness depend on l and coefficients functions ($p(t)$ and $q(t)$), not on the right-hand side function ($f(t)$).

2.2.2. A representation for the solution

Consider the boundary value problem for a second-order homogeneous linear differential equation

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(0) = a \\ y(l) = b. \end{cases} \tag{2}$$

Below we create a representation for the solution, which we will use in further derivations. Let $w_1(t)$ be the solution of

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(0) = 1 \\ y(l) = 0 \end{cases} \tag{3}$$

and $w_2(t)$ be the solution of

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(0) = 0 \\ y(l) = 1. \end{cases} \tag{4}$$

Under conditions of Lemma 2.1, the functions $w_1(t)$ and $w_2(t)$ exist, and the solution to BVP (2) is

$$y(t) = a\ w_1(t) + b\ w_2(t). \tag{5}$$

Remark 2.2. The functions $w_1(t)$ and $w_2(t)$ can be computed also as follows [9]. If $y_1(t)$ and $y_2(t)$ are any two linearly independent solutions of the differential equation

$$y'' + p(t)y' + q(t)y = 0 \tag{6}$$

then

$$\begin{aligned} w_1(t) &= \tfrac{y_2(l)y_1(t) - y_1(l)y_2(t)}{y_1(0)y_2(l) - y_1(l)y_2(0)} \\ w_2(t) &= \tfrac{y_1(0)y_2(t) - y_2(0)y_1(t)}{y_1(0)y_2(l) - y_1(l)y_2(0)}. \end{aligned} \tag{7}$$

Remark 2.3. By formula (5), the solution of BVP (2) linearly depends on the boundary values a and b.

By using functions $w_1(t)$ and $w_2(t)$ we can establish a new criterion for existence and uniqueness. Based on Lemma 2.1, we can conclude the following.

Lemma 2.4. BVP (1) has exactly one solution for every a and b, if and only if both of the problems (3) and (4) have solutions, i. e., if the functions $w_1(t)$ and $w_2(t)$ exist.

2.2.3. The case of constant coefficients

In this case, i. e. when the coefficients $p(t)$ and $q(t)$ are constants, the functions $w_1(t)$ and $w_2(t)$ can be expressed by explicit formulas given below. Consider the differential equation

$$y'' + 2py' + qy = 0. \tag{8}$$

(To make further mathematical expressions a little simpler, hereinafter we take $2p$ instead of $p(t)$). The characteristic equation is

$$r^2 + 2pr + q = 0.$$

We have 3 cases for its roots depending on the sign of the discriminant $\Delta = 4\left(p^2 - q\right)$.

Case 1: Distinct real roots ($\Delta > 0$). When $\Delta > 0$, the roots are $r_1 = u - v = -p - \sqrt{p^2 - q}$ and $r_2 = u + v = -p + \sqrt{p^2 - q}$. $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$ are linear independent solutions. Then, by (7), we have:

$$w_1 = \frac{e^{r_2 l + r_1 t} - e^{r_1 l + r_2 t}}{e^{r_2 l} - e^{r_1 l}}, \quad w_2 = \frac{e^{r_2 t} - e^{r_1 t}}{e^{r_2 l} - e^{r_1 l}} \tag{9}$$

or, by using hyperbolic functions,

$$w_1 = \frac{e^{ut} \sinh v(l - t)}{\sinh vl}, \quad w_2 = \frac{e^{u(t-l)} \sinh vt}{\sinh vl}, \tag{10}$$

where $u = -p$ and $v = \sqrt{p^2 - q}$.

Case 2: Repeated real roots ($\Delta = 0$). In this case, $r_1 = r_2 = -p$, and $y_1 = e^{r_1 t}$ and $y_2 = te^{r_1 t}$. Then

$$w_1 = \frac{(l - t)e^{ut}}{l}, \quad w_2 = \frac{te^{u(t-l)}}{l}, \tag{11}$$

where $u = -p$.

Case 3: Complex roots ($\Delta < 0$). In this case, $r_1 = u + vi = -p + \sqrt{q - p^2}\, i$, $r_2 = u - vi = -p - \sqrt{q - p^2}\, i$, and $y_1 = e^{ut} \cos vt$, $y_2 = e^{ut} \sin vt$. Then

$$w_1 = \frac{e^{ut} \sin v(l - t)}{\sin vl}, \quad w_2 = \frac{e^{u(t-l)} \sin vt}{\sin vl}, \tag{12}$$

where $u = -p$ and $v = \sqrt{q - p^2}$.

When $w_1(t)$ and $w_2(t)$ are already calculated, the solution of (8) with boundary values $y(0) = a$ and $y(l) = b$ is determined by (5):

$$y(t) = a\ w_1(t) + b\ w_2(t).$$

Remark 2.5. In Cases 1 and 2, the denominators ($\sinh vl$ and l) in (10) and (11), respectively, are positive. Consequently, $w_1(t)$ and $w_2(t)$ are definite functions. Then, by Lemma 2.4, the solution of BVP (8) exists and is unique. In Case 3, the solution exists and is unique (for every a and b), if and only if $\sin vl \neq 0 \Leftrightarrow vl \neq \pi k$, $k \in Z \Leftrightarrow \frac{vl}{\pi} \notin Z$, where Z denotes the set of integer numbers.

In other words, if $p^2 - q \geq 0$, then the solution exists and it is unique regardless of the value of l (as well as the values of a and b). But, if $p^2 - q < 0$, then the value of l becomes a critical factor:

If $\frac{\sqrt{q-p^2}\,l}{\pi}$ is not an integer number, then the solution exists and is unique (for every a and b);

If $\frac{\sqrt{q-p^2}\,l}{\pi}$ is an integer number, then, depending on the values a and b, there are either an infinite number of solutions, or no solution. Namely, if $b = ae^{-pl}\cos\sqrt{q-p^2}\,l$, then there are an infinite number of solutions: $y = e^{-pt}\left(a\cos\sqrt{q-p^2}\,t + c\sin\sqrt{q-p^2}\,t\right)$, $c \in R$; otherwise, no solution exists.

3. LINEAR DIFFERENTIAL EQUATIONS WITH INTERVAL COEFFICIENTS

3.1. Formulation of the problem

In this article, we investigate an interval BVP of the following form:

$$\begin{cases} Y'' + 2PY' + QY = 0 \\ Y(0) = A \\ Y(l) = B, \end{cases} \tag{13}$$

where $P = \left[\underline{p},\ \overline{p}\right]$, $Q = \left[\underline{q},\ \overline{q}\right]$, $A = [\underline{a},\ \overline{a}]$, $B = \left[\underline{b},\ \overline{b}\right]$ are given constant intervals and $l > 0$ is a given real number.

Usually, researchers consider BVP (13) under generalized Hukuhara differentiability [13, 28] or differential inclusions approach [14]. But, the generalized differentiability approach has some essential deficiencies, indicated in Introduction. The main difficulty with differential inclusions is that there is no derivative concept behind this approach. Consequently, in the frame of this approach, developing effective solution methods is not an easy task. Therefore, our aim is to provide a new formalization of interval BVPs.

We interpret interval BVP (13) as a family of real (classical) BVPs such as

$$\begin{cases} y'' + 2py' + qy = 0 \\ y(0) = a \\ y(l) = b, \end{cases} \tag{14}$$

where $p \in P$, $q \in Q$, $a \in A$ and $b \in B$. Under the conditions of Lemma 2.1 (or 2.4), BVP (14) has a unique solution y_{pqab}, which depends on p, q, a and b.

Our solution concept for (13) is as follows.

Definition 3.1. (Solution). Let an interval BVP (13) be given. We interpret this problem as the family of all classical BVPs (14), where $p \in P$, $q \in Q$, $a \in A$ and $b \in B$. Suppose each BVP (14) has a unique solution, say $y_{pqab}(\cdot)$. Then, the bunch (set) of all these real functions $y_{pqab}(\cdot)$ we define to be the solution Y of interval BVP (13).

By Remark 2.5, we have the following criterion for existence and uniqueness.

Theorem 3.2. (Existence and uniqueness criterion). If $p^2 - q \geq 0$, or $\frac{\sqrt{q-p^2}\,l}{\pi} \notin Z$ (when $p^2 - q < 0$), for all $p \in \left[\underline{p},\ \overline{p}\right]$ and $q \in \left[\underline{q},\ \overline{q}\right]$, then, in sense of Definition 3.1, the solution Y of interval BVP (13) exists and is unique.

Definition 3.1 determines the solution Y conceptually. However, the main interest is to calculate $Y(t)$, the value of Y for a given t. According to our definition, the bunch Y consists of solutions of all BVPs (14). It can be seen that the solution of (14) continuously depends on input parameters p, q, a and b. Then, the set $Y(t)$ determines an interval: $Y(t) = \left[\underline{y}(t),\ \overline{y}(t)\right]$. Thus, to finish the task we should compute the boundaries of this interval: $\underline{y}(t)$ and $\overline{y}(t)$.

3.2. Numerical algorithm

A numerical algorithm for calculation $\underline{y}(t)$ and $\overline{y}(t)$ is given in [11] and can be explained briefly as follows.

First, we focus on how to calculate the upper boundary $\overline{y}(t)$ of the solution bunch Y, for a fixed value of t. By Definition 3.1,

$$\overline{y}(t) = \max_{p\in P,\ q\in Q,\ a\in A,\ b\in B} y_{pqab}(t). \tag{15}$$

By using repeated maximums, the right-hand side of (15) can be represented as follows:

$$\max_{p\in P,\ q\in Q,\ a\in A,\ b\in B} y_{pqab}(t) = \max_{p\in P,\ q\in Q} \left(\max_{a\in A,\ b\in B} y_{pqab}(t) \right).$$

When the maximum in parentheses

$$\overline{y}_{pq}(t) = \max_{a\in A,\ b\in B} y_{pqab}(t) \tag{16}$$

is assessed, p and q are fixed. Then, we can determine the functions $w_1(t)$ and $w_2(t)$, and represent the solution as

$$y_{pqab}(t) = a\, w_1(t) + b\, w_2(t).$$

Due to the linearity of this representation we obtain:

$$\overline{y}_{pq}(t) = \max\left\{\overline{a}\, w_1(t),\ \underline{a}\, w_1(t)\right\} + \max\left\{\overline{b}\, w_2(t),\ \underline{b}\, w_2(t)\right\}. \tag{17}$$

As a result, we have:

$$\overline{y}(t) = \max_{p\in P,\ q\in Q} \overline{y}_{pq}(t). \tag{18}$$

The lower boundary $\underline{y}(t)$ of the solution bunch is evaluated similarly.

Based on (18) and (17), the following algorithm is proposed. If we use a grid of n equally spaced points for each of the intervals P and Q, then we will have n^2 pairs (p, q). For each pair (p, q), we can determine the functions $w_1(t)$ and $w_2(t)$ and compute $\overline{y}_{pq}(t)$. The maximum of these n^2 values $\overline{y}_{pq}(t)$ provides an approximation for $\overline{y}(t)$. As a result, the computational complexity of the proposed algorithm is $O(n^2)$.

Below we try to explain visually our solution concept and computational algorithm using a numerical example.

Example 3.3. Consider interval BVP (13) with $l = 5$, $P = [0.1,\ 0.3]$, $Q = [10.75,\ 11.25]$, $A = [-2.5,\ -1.5]$ and $B = [0.25,\ 0.75]$.

First, we check whether the existence and uniqueness criterion is satisfied. The first sufficient condition does not hold because $p^2 - q \le 0.3^2 - 11.25 < 0$. Therefore, to examine the second one we evaluate $\frac{\sqrt{q-p^2}\,l}{\pi}$. Since the given intervals P and Q are located on the positive semi-axis, we have $\frac{\sqrt{\underline{q}-\overline{p}^2}\,l}{\pi} \le \frac{\sqrt{q-p^2}\,l}{\pi} \le \frac{\sqrt{\overline{q}-\underline{p}^2}\,l}{\pi}$, i. e., $5.196 \le \frac{\sqrt{q-p^2}\,l}{\pi} \le 5.336$. Therefore, $\frac{\sqrt{q-p^2}\,l}{\pi}$ can not be an integer number. Consequently, by Theorem 3.2, the solution of the Example exists and it is unique.

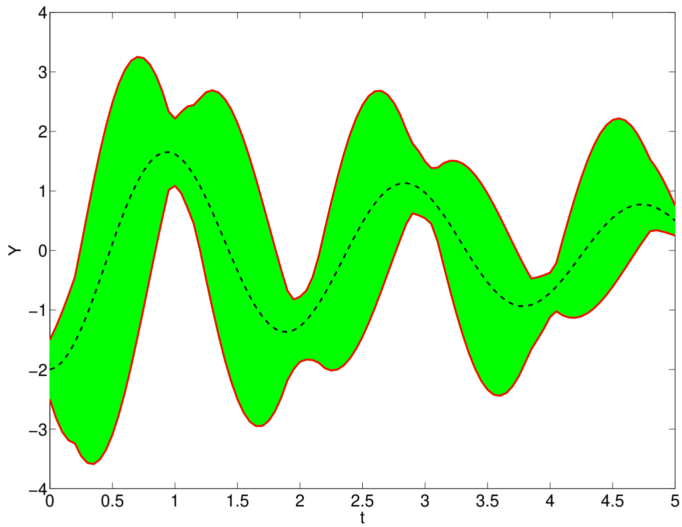

Fig. 1. The solution of Example 3.3, obtained by the proposed approach. The solution bunch forms a band in the coordinate plane. The dashed line depicts the central solution.

Now, we use the proposed algorithm to find an approximation of the solution. In calculations, we use a grid of $n = 11$ points for each of the intervals P and Q. Then, we have 121 pairs as (p, q). For each of them, we solve BVP (14) as follows. The characteristic equation of the differential equation $y'' + 2p\,y' + q\,y = 0$ is $r^2 + 2pr + q = 0$. Since $p \in [0.1,\ 0.3]$ and $q \in [10.75,\ 11.25]$, the discriminant $\Delta = 4(p^2 - q)$ is negative. Hence, the functions $w_1(t)$ and $w_2(t)$ are given by (12), and $y_{pqab}(t) = a\,w_1(t) + b\,w_2(t)$. We determine $\overline{y}_{pq}(t)$, by using formula (17). The maximum of all 121 values of $\overline{y}_{pq}(t)$ gives $\overline{y}(t)$, according to (18). We determine $\underline{y}(t)$ in similar way, and finish the computations. The resulting solution is shown in Figure 1. The dashed line depicts the "central" solution (the solution of the BVP, obtained by replacing each interval with its center, i. e., using the values $p = 0.2$, $q = 11$, $a = -2$, and $b = 0.5$ in (14)).

We wrote our code in MATLAB R2021a and implemented our numerical experiments on a PC with 64-bit MacOS operating system, MacBook Pro M1 CPU and 8 GB RAM. The actual computational time for Example 3.3 was 1.64 seconds.

We provide the MATLAB-code of our proposed algorithm for solving interval BVPs below.

```
function IntervalBVP
% We solve BVP with interval inputs:
% Y'' + 2 P Y' + Q Y = 0,     t in (0, l)
% Y(0) = A
% Y(l) = B
% where l = 5, P = [0.1, 0.3], Q = [10.75, 11.25],
%       A = [-2.5, -1.5], B = [0.25, 0.75]

% Input parameters
l=5;
p_min= 0.1; p_max= 0.3; q_min=10.75; q_max= 11.25;
a_min=-2.5; a_max=-1.5; b_min=0.25; b_max=0.75;

T=0:.05:l;   % Mesh on t

% Dividing the intervals P and Q into 10 parts
np=10; hp=(p_max-p_min)/np;
nq=10; hq=(q_max-q_min)/nq;

% Computing solutions of BVP for different p and q
for i=0:np
for j=0:nq
  p=p_min+i*hp;
  q=q_min+j*hq;

  [W1 W2] = w12(p,q,l,T);
  Y1=W1*a_min+W2*b_min; Y2=W1*a_max+W2*b_max;
  Y3=W1*a_min+W2*b_max; Y4=W1*a_max+W2*b_min;

  % Computing the lower and upper boundaries of solution
  if (i==0 && j==0)
    Ymin=min(min(min(Y1,Y2),Y3),Y4);
    Ymax=max(max(max(Y1,Y2),Y3),Y4);
  else
    Ymin=min(min(min(min(Y1,Y2),Y3),Y4),Ymin);
    Ymax=max(max(max(max(Y1,Y2),Y3),Y4),Ymax);
  end

end
end

% Filling the area between the lower and upper boundaries
for k=0:200
  alfa=k*0.005;
  Yw=Ymin+alfa*(Ymax-Ymin);
  h = plot(T, Yw);
  set(h, 'color', [0 1 0],'LineWidth', 2.25)
  grid off
  hold on
end
```

```
% Plotting the lower and upper boundaries of solution
h = plot(T, Ymin);
set(h, 'color', 'red', 'LineWidth', 2.25)
hold on
h = plot(T, Ymax);
set(h, 'color', 'red', 'LineWidth', 2.25)
hold on

% Plotting the central solution
p=(p_min+p_max)/2; q=(q_min+q_max)/2;
a=(a_min+a_max)/2; b=(b_min+b_max)/2;
[W1 W2] = w12(p,q,l,T);
Ycentr=W1*a+W2*b;
h = plot(T, Ycentr, '--');
set(h, 'color', 'black', 'LineWidth', 2.00)

set(gca,'FontSize',20);
xlabel('t','fontsize',20);
ylabel('Y','fontsize',20);
% ------------------------------------------------------------
% Computing the auxiliary solutions W1 and W2
function [W1 W2] = w12(p,q,l,T)
D=4*(p^2-q);
if D>0
    r1=-p-sqrt(D)/2; r2=-p+sqrt(D)/2;
    W1=(exp(r2*l+r1*T)-exp(r1*l+r2*T))/(exp(r2*l)-exp(r1*l));
    W2=(exp(r2*T)-exp(r1*T))/(exp(r2*l)-exp(r1*l));
else
  if D==0
    r1=-p;
    W1=(1-T/l).*exp(r1*T);
    W2=(T/l).*exp(r1*(T-l));
  else
    u=-p; v=sqrt(-D)/2;
    W1=exp(u*T).*sin(v*(l-T))/sin(v*l);
    W2=exp(u*(T-l)).*sin(v*T)/sin(v*l);
  end
end
% ------------------------------------------------------------
```

3.3. Sufficient conditions for analytical solution

In this subsection, we provide some sufficient conditions, under that, the solution is given by an explicit formula (A similar study, but for IVP (Initial Value Problem), was performed in [10]). For this, we will investigate how to determine the optimal values p^*, q^*, a^* and b^* such that

$$\overline{y}(t) = y_{p^*q^*a^*b^*}(t).$$

Note that, in general, the optimal values of the parameters depend on t.

Let the following condition be satisfied:

$$\underline{\text{Condition 1}}: \quad p^2 - q > 0, \text{ for all } p \in \left[\underline{p}, \overline{p}\right] \text{ and } q \in \left[\underline{q}, \overline{q}\right]$$

Then, for each BVP (14), the discriminant of the characteristic equation is positive ($\Delta > 0$), and its roots are distinct real numbers. Consequently, the functions $w_1(t)$ and $w_2(t)$ are computed by formula (10):

$$w_1 = \frac{e^{ut}\sinh v(l-t)}{\sinh vl}, \quad w_2 = \frac{e^{u(t-l)}\sinh vt}{\sinh vl}.$$

The solution is given by (5):

$$y_{pqab}(t) = a\ w_1(t) + b\ w_2(t).$$

Therefore, for each BVP (14) the solution is

$$y_{pqab}(t) = a\,\frac{e^{ut}\sinh v(l-t)}{\sinh vl} + b\,\frac{e^{u(t-l)}\sinh vt}{\sinh vl}, \tag{19}$$

where $u = -p$ and $v = \sqrt{p^2 - q}$.

Since $t > 0$, $l > 0$, $l - t > 0$ and $v > 0$, we see that w_1 and w_2 are positive functions: $w_1(t) > 0$ and $w_2(t) > 0$. Then, by (5), $y_{pqab}(t)$ attains its maximum for $a^* = \overline{a}$ and $b^* = \overline{b}$:

$$\overline{y}_{pq}(t) = \overline{a}\ w_1(t) + \overline{b}\ w_2(t).$$

Then we have to determine the optimal values of p and q, i. e., p^* and q^*. First, we will deal with q^*. For this, we need the partial derivatives of functions w_1 and w_2 on q:

$$\frac{\partial w_1}{\partial q} = \frac{e^{ut}}{2v}\frac{\sinh v(l-t)}{\sinh vl}\left(l\coth vl - (l-t)\coth v(l-t)\right)$$

$$\frac{\partial w_2}{\partial q} = \frac{e^{u(t-l)}}{2v}\frac{\sinh vt}{\sinh vl}\left(l\coth vl - t\coth vt\right).$$

Since $x\coth kx$, $(k > 0)$, is an increasing function of x, we have: $\frac{\partial w_1}{\partial q} > 0$ and $\frac{\partial w_2}{\partial q} > 0$. From this, if

<u>Condition 2</u>: $\quad \overline{a} \geq 0$ and $\overline{b} \geq 0$

is satisfied, then we have:

$$\frac{\partial \overline{y}_{pq}}{\partial q} = \overline{a}\ \frac{\partial w_1}{\partial q} + \overline{b}\ \frac{\partial w_2}{\partial q} \geq 0.$$

Therefore, $\overline{y}_{pq}$ is a non-decreasing function of q. Consequently, the maximum is attained at $q^* = \overline{q}$.

Now let us investigate how to choose the value p^*, which maximizes $y_{p\overline{q}\overline{a}\overline{b}}(t)$. We can see that

$$\frac{\partial}{\partial p}\left(\frac{\sinh vt}{\sinh vl}\right) = -\frac{p}{v}\frac{\sinh vt}{\sinh vl}\left(l\coth vl - t\coth vt\right).$$

From this, if

<u>Condition 3</u>: $\quad p < 0$

is satisfied, then $\frac{\partial}{\partial p}\left(\frac{\sinh vt}{\sinh vl}\right) > 0$. On the other hand,

$$\frac{\partial}{\partial p}\left(e^{u(t-l)}\right) = \frac{\partial}{\partial p}\left(e^{-p(t-l)}\right) = \frac{\partial}{\partial p}\left(e^{p(l-t)}\right) = (l-t)e^{p(l-t)} > 0.$$

Since $\frac{\sinh vt}{\sinh vl}$ and $e^{u(t-l)}$ are positive and increase with p, we can conclude that their product $w_2 = e^{u(t-l)}\frac{\sinh vt}{\sinh vl}$ also is positive and increases with p. Therefore,

$$\frac{\partial w_2}{\partial p} > 0.$$

To establish the conditions for w_1 be increasing with respect to p, we start with the derivative:

$$\frac{\partial w_1}{\partial p} = -\frac{p}{\sqrt{p^2-\overline{q}}}\frac{e^{ut}\sinh v(l-t)}{\sinh vl}t\left(\frac{l\coth vl-(l-t)\coth v(l-t)}{t}+\frac{\sqrt{p^2-\overline{q}}}{p}\right).$$

The factor in front of the parentheses is positive, when $p<0$. Let us investigate that under what conditions the expression in the parentheses, $\frac{l\coth vl-(l-t)\coth v(l-t)}{t}+\frac{\sqrt{p^2-\overline{q}}}{p}$, is also positive. At the beginning, we evaluate the first term. Since

$$\frac{d^2}{dx^2}\left(x\coth vx\right) = 2v\left(vx\coth vx-1\right)\left(\coth^2 vx-1\right) = \frac{2v\left(vx\coth vx-1\right)}{\sinh^2 vx} > 0,$$

$y = x\coth vx$ is a concave-up function. Then, the slope of secant line, $\frac{l\coth vl-(l-t)\coth v(l-t)}{t}$, which passes through points $(l-t,(l-t)\coth v(l-t))$ and $(l, l\coth vl)$ decreases with t. Consequently, at $t\to l$ the slope is minimum:

$$\frac{l\coth vl-(l-t)\coth v(l-t)}{t} \geq \lim_{t\to l}\frac{l\coth vl-(l-t)\coth v(l-t)}{t} = \frac{l\coth vl-1/v}{l}$$

$$= \coth vl - \frac{1}{vl}.$$

One can see that $\frac{\partial}{\partial p}\left(\coth vl-\frac{1}{vl}\right) = \frac{d}{dv}\left(\coth vl-\frac{1}{vl}\right)\cdot\frac{\partial v}{\partial p} = l\left(\frac{1}{(vl)^2}-\frac{1}{\sinh^2 vl}\right)\cdot\frac{p}{\sqrt{p^2-\overline{q}}} < 0$, because of $p<0$. Therefore, $\coth vl-\frac{1}{vl}$ decreases with p. Then, we have:

$$\frac{l\coth vl-(l-t)\coth v(l-t)}{t} \geq \coth vl-\frac{1}{vl} \geq \coth\sqrt{\overline{p}^2-\overline{q}}\,l - \frac{1}{\sqrt{\overline{p}^2-\overline{q}}\,l}. \tag{20}$$

On the other hand, if

<u>Condition 4</u>: $\overline{q} > 0$

is satisfied, then $\frac{\partial}{\partial p}\left(\frac{\sqrt{p^2-\overline{q}}}{p}\right) = \frac{\overline{q}}{p^2\sqrt{p^2-\overline{q}}} > 0$. Therefore,

$$\frac{\sqrt{p^2-\overline{q}}}{p} \geq \frac{\sqrt{\underline{p}^2-\overline{q}}}{\underline{p}}. \tag{21}$$

From (20) and (21) we have:

$\frac{l \coth vl-(l-t)\coth v(l-t)}{t}+\frac{\sqrt{p^2-\overline{q}}}{p} \geq \coth \sqrt{\overline{p}^2-\overline{q}}\, l-\frac{1}{\sqrt{\overline{p}^2-\overline{q}}\, l}+\frac{\sqrt{\underline{p}^2-\overline{q}}}{\underline{p}}$.

Then, under

$$\underline{\text{Condition 5 } a}: \quad \coth \sqrt{\overline{p}^2-\overline{q}}\, l-\frac{1}{\sqrt{\overline{p}^2-\overline{q}}\, l}+\frac{\sqrt{\underline{p}^2-\overline{q}}}{\underline{p}} \geq 0$$

we get $\frac{\partial w_1}{\partial p} \geq 0$. Summing up,

$$\frac{\partial y_{\overline{p}\overline{q}\overline{a}\overline{b}}}{\partial p}=\overline{a}\, \frac{\partial w_1}{\partial p}+\overline{b}\, \frac{\partial w_2}{\partial p} \geq 0.$$

Therefore, $p^*=\overline{p}$.

We can summarize the above derivations in the form of below statement.

Proposition 3.4. Suppose that the following conditions are met:
1) $p^2-q>0$, for all $p \in [\underline{p}, \overline{p}]$ and $q \in [\underline{q}, \overline{q}]$;
$2a$) $\overline{a} \geq 0$ and $\overline{b} \geq 0$.
Then $q^*=\overline{q}$.
In addition, if
3) $\overline{p}<0$;
$4a$) $\overline{q}>0$;
$5a$) $\coth \sqrt{\overline{p}^2-\overline{q}}\, l-\frac{1}{\sqrt{\overline{p}^2-\overline{q}}\, l}+\frac{\sqrt{\underline{p}^2-\overline{q}}}{\underline{p}} \geq 0$,
then $p^*=\overline{p}$, and, consequently, $\overline{y}(t)=y_{\overline{p}\overline{q}\overline{a}\overline{b}}(t)$ (where $y_{pqab}(t)$ is determined by (19)).

Remark 3.5. If we substitute item $2a$ with the condition
$2c$) $\overline{a} \leq 0$ and $\overline{b} \leq 0$;
then we have $\overline{y}(t)=y_{\underline{p}\underline{q}\overline{a}\overline{b}}(t)$.

For the lower boundary $\underline{y}(t)$ we can obtain an analogical result:

Proposition 3.6. Suppose that the following conditions are met:
1) $p^2-q>0$, for all $p \in [\underline{p}, \overline{p}]$ and $q \in [\underline{q}, \overline{q}]$;
$2b$) $\underline{a} \geq 0$ and $\underline{b} \geq 0$.
Then $q_*=\underline{q}$.
In addition, if
3) $\overline{p}<0$;
$4b$) $\underline{q}>0$;
$5b$) $\coth \sqrt{\overline{p}^2-\underline{q}}\, l-\frac{1}{\sqrt{\overline{p}^2-\underline{q}}\, l}+\frac{\sqrt{\underline{p}^2-\underline{q}}}{\underline{p}} \geq 0$,
then $p_*=\underline{p}$, and, consequently, $\underline{y}(t)=y_{\underline{p}\underline{q}\underline{a}\underline{b}}(t)$ (where $y_{pqab}(t)$ is determined by (19)).

Remark 3.7. If we substitute item $2b$ with the condition
$2d$) $\underline{a} \leq 0$ and $\underline{b} \leq 0$;
then we have $\underline{y}(t)=y_{\overline{p}\overline{q}\underline{a}\underline{b}}(t)$.

We can combine Propositions 3.4 and 3.6 to obtain a more general result. In this, we combine the items $5a$ and $5b$ based on the following evaluations. It can be seen that $\frac{\partial}{\partial q}\left(\coth vl - \frac{1}{vl}\right) = \frac{\partial}{\partial v}\left(\coth vl - \frac{1}{vl}\right)\cdot\frac{\partial v}{\partial q} = l\left(\frac{1}{(vl)^2} - \frac{1}{\sinh^2 vl}\right)\cdot\frac{-1}{2\sqrt{p^2-q}} < 0$. Therefore, $\coth vl - \frac{1}{vl}$ decreases with q. Then, we have:

$\coth\sqrt{\overline{p}^2 - \underline{q}}\,l - \frac{1}{\sqrt{\overline{p}^2-\underline{q}}\,l} \geq \coth\sqrt{\overline{p}^2 - \overline{q}}\,l - \frac{1}{\sqrt{\overline{p}^2-\overline{q}}\,l}$.

On the other hand, $\sqrt{p^2-q}$ decreases with q. Therefore, $\frac{\sqrt{p^2-q}}{p}$ increases with q, because of $p<0$. Then, $\frac{\sqrt{\underline{p}^2-\overline{q}}}{\underline{p}} \geq \frac{\sqrt{\underline{p}^2-\underline{q}}}{\underline{p}}$.

By using the above evaluations, and by combining Propositions 3.4 and 3.6, we obtain sufficient conditions for the exact solution to be given by an explicit formula:

Lemma 3.8. Let the following conditions be satisfied:
1) $p^2 - q > 0$, for all $p \in \left[\underline{p}, \overline{p}\right]$ and $q \in \left[\underline{q}, \overline{q}\right]$;
2b) $\underline{a} \geq 0$ and $\underline{b} \geq 0$;
3) $\overline{p} < 0$;
4) $\underline{q} > 0$;
5) $\coth\sqrt{\overline{p}^2 - \overline{q}}\,l - \frac{1}{\sqrt{\overline{p}^2-\overline{q}}\,l} + \frac{\sqrt{\underline{p}^2-\underline{q}}}{\underline{p}} \geq 0$,

then $Y(t) = \left[y_{\underline{pqab}}(t),\, y_{\overline{pqab}}(t)\right]$, where $y_{pqab}(t)$ is determined by (19).

Remark 3.9. If we substitute item $2b$ with the condition
2c) $\overline{a} \leq 0$ and $\overline{b} \leq 0$;
then we have $Y(t) = \left[y_{\overline{pq}\underline{ab}}(t),\, y_{\underline{pq}\overline{ab}}(t)\right]$.

Remark 3.10. If Conditions 3 and 4 are taken into account, then Condition 1 can be reformulated as $\overline{p}^2 > \overline{q}$.

Remark 3.11. If Conditions 1 and $2b$ are satisfied, then $q_* = \underline{q}$ and $q^* = \overline{q}$. (In the case, when Conditions 1 and $2c$ are satisfied, we have $q_* = \overline{q}$ and $q^* = \underline{q}$). Thus, we only need to compute p^* (that maximizes (19), where $a = a^* = \overline{a}$, $b = b^* = \overline{b}$, $q = q^* = \overline{q}$) and p_*. Therefore, if Conditions 1 and $2b$ are satisfied, then the computational complexity of the introduced algorithm reduces to $O(n)$.

Below we give an example to justify Lemma 3.8.

Example 3.12. Consider interval BVP

$$\left\{\begin{array}{l} Y'' + 2PY' + QY = 0 \\ \quad Y(0) = A \\ \quad Y(l) = B \end{array}\right. \tag{22}$$

with $l = 5$, $P = [-5.75, -5.25]$, $Q = [3.0, 4.0]$, $A = [1.5, 2.5]$ and $B = [9.0, 11.0]$.

All conditions of Lemma 3.8 are satisfied:
1) $p^2 - q \geq \overline{p}^2 - \overline{q} = (-5.25)^2 - 4 = 23.563 > 0 \quad \checkmark$;

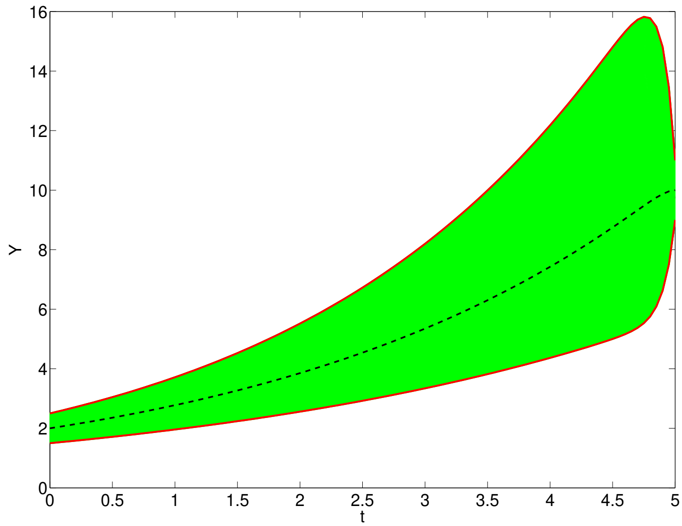

Fig. 2. The solution of Example 3.12 by the proposed method. The dashed line depicts the central solution.

2b) $\underline{a} = 1.5 \geq 0$ and $\underline{b} = 4.0 \geq 0 \quad \checkmark$;
3) $\overline{p} = -5.25 < 0 \quad \checkmark$;
4) $\underline{q} = 3.0 > 0 \quad \checkmark$;
5) $\coth \sqrt{\overline{p}^2 - \overline{q}}\, l - \frac{1}{\sqrt{\overline{p}^2 - \overline{q}}\, l} + \frac{\sqrt{\underline{p}^2 - \underline{q}}}{\underline{p}} =$

$$\coth\left(\sqrt{(-5.25)^2 - 4} \cdot 5\right) - \frac{1}{\sqrt{(-5.25)^2 - 4} \cdot 5} + \frac{\sqrt{(-5.75)^2 - 3}}{-5.75} = 5.2452 \times 10^{-3} \geq 0 \quad \checkmark.$$

Then, $a^* = \overline{a}$, $b^* = \overline{b}$, $p^* = \overline{p}$, $q^* = \overline{q}$; $a_* = \underline{a}$, $b_* = \underline{b}$, $p_* = \underline{p}$, $q_* = \underline{q}$, and $Y(t) = \left[\underline{y}(t), \overline{y}(t)\right] = \left[y_{\underline{p}\underline{q}\underline{a}\underline{b}}(t), y_{\overline{p}\overline{q}\overline{a}\overline{b}}(t)\right]$, where $y_{pqab}(t)$ is determined by (19). The numerical solution, which is depicted in Figure 2, justifies this derivation.

At the end of the section, we underline once again that Lemma 3.8 provides only some necessary conditions. Therefore, it covers only a restricted class of interval BVPs. We saw above that the BVP in Example 3.12 belongs to this class. At the same time, we see that BVP from Example 3.3 does not satisfy the conditions of Lemma 3.8 (for example, the first condition), and its solution is not represented by the explicit formula given in the lemma.

4. COMPARISON WITH THE GENERALIZED DIFFERENTIABILITY APPROACH

In this section, we demonstrate the advantages of the proposed approach. We compare it with the strongly generalized (Hukuhara) differentiability approach that is commonly used in solving differential equations with uncertainties. First, we briefly give the main definitions and theorems. More detailed information can be found, for example, in [12, 28].

Definition 4.1. Let F be an interval-valued function defined on $(a,\ b)$ and let $t_0 \in (a,\ b)$. We say that F is strongly generalized (Hukuhara) differentiable at t_0 if there exists an interval $F'(t_0)$, such that, for all $h > 0$ sufficiently small, there exist the involved Hukuhara differences and either

$$(i) \quad \lim_{h \searrow 0^+} \frac{1}{h}\left(F(t_0+h) \ominus F(t_0)\right) = \lim_{h \searrow 0^+} \frac{1}{h}\left(F(t_0) \ominus F(t_0-h)\right) = F'(t_0),$$

or

$$(ii) \quad \lim_{h \searrow 0^+} \frac{1}{-h}\left(F(t_0) \ominus F(t_0+h)\right) = \lim_{h \searrow 0^+} \frac{1}{-h}\left(F(t_0-h) \ominus F(t_0)\right) = F'(t_0).$$

We call a function to be differentiable in the first form ((i)-differentiable, or 1-differentiable) if it is strongly generalized differentiable as in the case (i) of the definition above, etc.

We say that F is strongly generalized differentiable on the interval $(a,\ b)$ if F is strongly generalized differentiable at each point $t_0 \in (a,\ b)$.

It is difficult to find the derivative of a function based on the above definition. The following theorem allows us to compute the derivative more practically [12, 28].

Theorem 4.2. Let F be an interval-valued function defined on $(a,\ b)$ such that $F(t) = [f(t),\ g(t)]$. Also, let F be strongly generalized differentiable at $t_0 \in (a,\ b)$. Then
(a) if F is differentiable in the first form then f and g are differentiable at t_0 and $F'(t_0) = [f'(t_0),\ g'(t_0)]$,
(b) if F is differentiable in the second form then f and g are differentiable at t_0 and $F'(t_0) = [g'(t_0),\ f'(t_0)]$.

We define second-order strongly generalized differentiability as follows. We say that F is $(i,\ j)$-differentiable if F is differentiable in the ith form and F' is differentiable in the jth form. For example, if F is 2-differentiable, but F' is 1-differentiable, then F is $(2,\ 1)$-differentiable. There are 4 types of generalized second-order differentiability: $(1,\ 1)$ or $(1,\ 2)$ or $(2,\ 1)$ or $(2,\ 2)$-differentiability.

One can establish an analogue of Theorem 4.2 for the second-order derivative [12, 28].

Theorem 4.3. Let F be an interval-valued function defined on $(a,\ b)$ such that $F(t) = [f(t),\ g(t)]$. Also, let F be second-order strongly generalized differentiable at $t_0 \in (a,\ b)$. Then
(a) if F is $(1,1)$-differentiable then $F'(t_0) = [f'(t_0),\ g'(t_0)]$ and $F''(t_0) = [f''(t_0),\ g''(t_0)]$,
(b) if F is $(1,2)$-differentiable then $F'(t_0) = [f'(t_0),\ g'(t_0)]$ and $F''(t_0) = [g''(t_0),\ f''(t_0)]$,
(c) if F is $(2,1)$-differentiable then $F'(t_0) = [g'(t_0),\ f'(t_0)]$ and $F''(t_0) = [g''(t_0),\ f''(t_0)]$,
(d) if F is $(2,2)$-differentiable then $F'(t_0) = [g'(t_0),\ f'(t_0)]$ and $F''(t_0) = [f''(t_0),\ g''(t_0)]$.

As it was stated in the Introduction, the purpose of this article is to overcome the known shortcomings of the generalized differentiability approach. To justify that this goal has been achieved we use Example 3.12. In the beginning, we show some circumstances that openly reveal the shortcomings of the generalized differentiability approach.

1) Under generalized differentiability, BVP (22) has not a solution. This statement can be explained as follows. Since the boundary values are proper intervals (that is, intervals with non-zero widths), the solution is proper too. Then, in (22), the left-hand side (which is a sum of intervals) is not identically zero, while the right-hand side is. Therefore, (22) cannot have any solution under the generalized differentiability approach.

2) If we move, for example, the second term of the equation from the left-hand side to the right-hand side, that is, if we reformulate the problem as

$$\begin{cases} Y'' + QY = KY' \\ Y(0) = A \\ Y(l) = B \end{cases} \tag{23}$$

(where $l = 5$, $Q = [3.0, 4.0]$, $K = -2P = [10.5, 11.5]$, $A = [1.5, 2.5]$ and $B = [9.0, 11.0]$), then it can potentially have solutions.

3) Let us find a solution to the problem (23) under generalized differentiability. Since the differential equation under consideration is of the second order, the solution function $Y(t) = \left[\underline{y}(t), \overline{y}(t)\right]$ can have one of the four derivatives: $(1,1)$, $(1,2)$, $(2,1)$, or $(2,2)$-derivative. For each of these 4 cases, to find a solution, first, (a) we should constitute the corresponding classical problem (for $\underline{y}(t)$ and $\overline{y}(t)$) and solve it, then, (b) we should check whether the obtained solution is valid (i. e., whether $\underline{y}(t) \le \overline{y}(t)$ and whether the derivative of $Y(t)$ is of the type under consideration).

Geometrically, the width of a $(1,1)$-differentiable function increases (as time goes) at a rate that increases in time. $(1,2)$-differentiable function also expands, but the rate of the expansion decreases. $(2,1)$ and $(2,2)$-differentiable functions narrow with time, respectively, with increasing and decreasing rates.

Since, in our problem, the interval B (the value at the right boundary) is wider than the interval A, the solution can be either $(1,1)$ or $(1,2)$-differentiable. For simplicity of further derivations we rename $x(t) := \underline{y}(t)$ and $z(t) := \overline{y}(t)$. Then, $Y(t) = [x(t), z(t)]$.

First, let us look for $(1,1)$-solution of (23). In this case, $Y'(t) = [x'(t), z'(t)]$ and $Y''(t) = [x''(t), z''(t)]$. All coefficients of the differential equation (23) are positive. Then, we obtain the following $(1,1)$-problem:

$$\begin{cases} z'' + 4z = 11.5z' \\ x'' + 3x = 10.5x' \\ z(0) = 2.5 \\ z(5) = 11 \\ x(0) = 1.5 \\ x(5) = 9. \end{cases}$$

Since the equations and boundary values for $z(t)$ and $x(t)$ are independent, we determine them separately:

$$z(t) = 2.5e^{0.35904t} - 2.6020 \cdot 10^{-24} e^{11.141t}$$
$$x(t) = 1.5e^{0.29394t} + 1.7058 \cdot 10^{-22} e^{10.206t}.$$

We depict the result in Figure 3 (Note that although it is close to the solution in Figure 2, it is not the same). We see that the condition $\underline{y}(t) \leq \overline{y}(t)$ is satisfied everywhere. Now, we have to check whether the obtained function is $(1,1)$-differentiable or not? But, it can be seen from Figure 3 that the width of function does not increase on the entire time domain $[0,\, 5]$. More precisely, the solution is valid only on the interval $[0,\, 4.39]$; on the reminder interval $[4.39,\, 5]$ it is not valid. Therefore, BVP (23) has not a global $(1,1)$-differentiable solution.

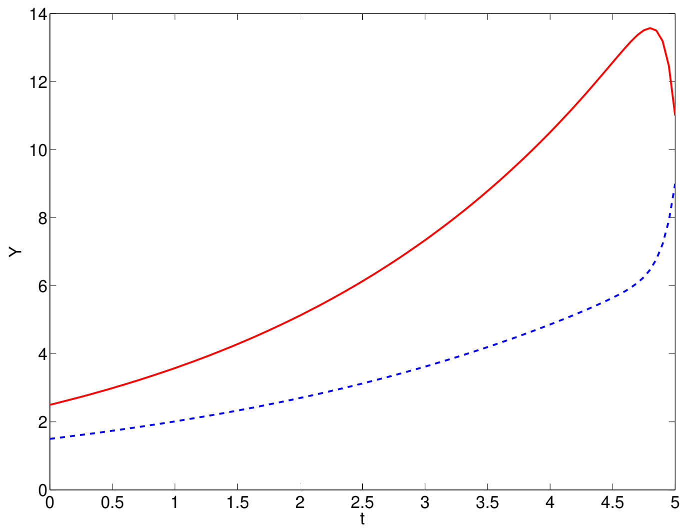

Fig. 3. The solution of $(1,1)$-problem for (23).

Second, let us consider for $(1,2)$-solution to (23). In this case, $Y'(t) = [x'(t),\, z'(t)]$ and $Y''(t) = [z''(t),\, x''(t)]$. We have the following $(1,2)$-problem:

$$\begin{cases} x'' + 4z = 11.5z' \\ z'' + 3x = 10.5x' \\ z(0) = 2.5 \\ z(5) = 11 \\ x(0) = 1.5 \\ x(5) = 9. \end{cases}$$

The solution of this system is (see Figure 4)

$$\begin{aligned} z\,(t) &= \;\; 2.1145e^{0.34976t} - 0.19297e^{0.28485t} - 2.4753\cdot 10^{-24}e^{10.662t} + 0.57850e^{-11.297t} \\ x\,(t) &= 0.38464e^{0.34976t} + \;\; 1.7224e^{0.28485t} - 2.5827\cdot 10^{-24}e^{10.662t} - 0.60705e^{-11.297t}. \end{aligned}$$

Since the lower boundary exceeds the upper one on some interval, the function $Y(t) = [x(t),\, z(t)]$ is not a proper solution. Therefore, an $(1,2)$-differentiable solution also does not exist.

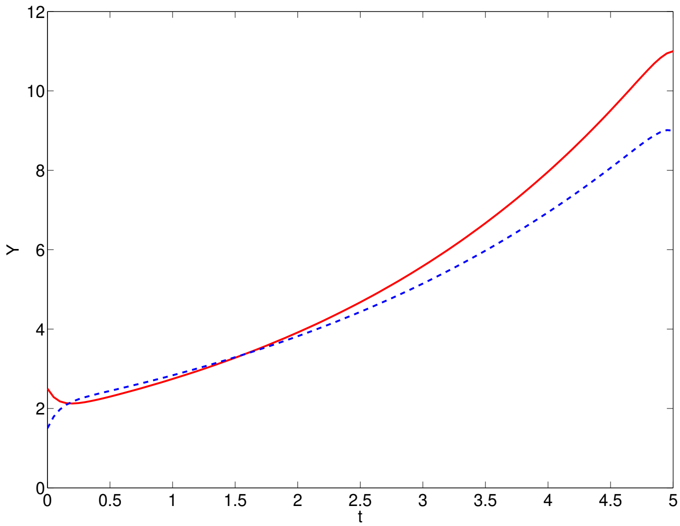

Fig. 4. The solution of $(1, 2)$-problem for (23).

Summing up, BVP (23) has not a global $(1, 1)$, or $(1, 2)$, or $(2, 1)$, or $(2, 2)$ generalized differentiable solution.

4) Above, we were looking for a solution to (23) that has one of the 4 generalized derivatives globally (over the entire time domain). We were convinced that there is no such solution. To defeat such deficiencies, some researchers suggest to consider switching between types of generalized differentiability as time goes. For example, up to the first switching point, the solution can be a $(1, 1)$-differentiable function, then it switches to $(2, 1)$-differentiability, and so on. This suggestion leads to some questions such as how to choose the switching points, how to determine the type of differentiability at the beginning, and what type of differentiability to switch to next time. These questions still remain unanswered.

Above, based on Example 3.12, we indicated four shortcomings of the generalized derivative approach: 1) A class of problems has not a solution under generalized differentiability even when the corresponding real (classical) problems have unique solutions; 2) If we transfer a term from one side of the equation to another, the solution can change; 3) The generalized differentiability approach does not guarantee existence and uniqueness of the solution: a problem can have several number of solutions, or no solution; 4) To include a broader class of solutions, one have to alternate the types of the generalized derivative. However, a general strategy on how to accomplish this process does not exist at present.

Our proposed approach is exempt from the above shortcomings. In particular, in its frame, BVP (22) from Example 3.12 has a unique solution (Figure 2). The solution of BVP (23) is the same. In general, the proposed approach has the same properties as the classical approaches in the theory of differential equations. Namely, under the proposed approach, 1) An interval BVP has a solution, whenever the corresponding real (classical) problems have solutions; 2) The solution does not depend on the representation of the problem (The solution is not changed, if any term is moved from one side of the equation to the other side); 3) The solution exists and is unique (under the usual conditions); 4) The solution is obtained in a natural way by using methods of real calculus, therefore, no special strategy is required for choice of switching points.

5. APPLICABILITY OF THE METHOD TO THE NON-LINEAR CASE

Above, we have considered the case when the problem is linear. In this case, the solution can be found using well-known methods, and the analysis of the problem is rather easy. Therefore, the non-linear case is of particular interest. Below, we demonstrate the applicability of the proposed method to non-linear problems and discuss the difficulties that arise.

Example 5.1. Consider the following non-linear interval BVP:

$$\begin{cases} Y'' + 2Pt\,(Y' + Q)^2 = 0 \\ Y(0) = A \\ Y(1) = B \end{cases} \tag{24}$$

with $P = \left[\frac{11}{4},\ \frac{13}{4}\right]$, $Q = \left[-\frac{1}{3},\ \frac{1}{3}\right]$, $A = \left[-\frac{5}{2},\ -\frac{3}{2}\right]$ and $B = \left[\frac{2}{5},\ \frac{3}{5}\right]$.

We interpret this interval BVP as the set of real BVPs

$$\begin{cases} y'' + 2pt\,(y' + q)^2 = 0 \\ y(0) = a \\ y(1) = b, \end{cases} \tag{25}$$

where $p \in P$, $q \in Q$, $a \in A$ and $b \in B$.

The differential equation does not contain y implicitly. The substitution $z = y'$ brings it to a first-order equation [26]:

$$z' + 2pt\,(z + q)^2 = 0.$$

The general solution of this separable equation is

$$z = \frac{1}{p}\frac{1}{t^2 + C} - q, \qquad C \in R.$$

It can be divided into two subsets depending on the sign of the arbitrary constant C:

$z = \frac{1}{p}\frac{1}{t^2+s^2} - q$ and $z = \frac{1}{p}\frac{1}{t^2-s^2} - q$, where $s \in R^+$.

In the end, it can be established that the first subset occurs when $p\,(b - a + q) > 0$; the second set corresponds to the case of $p\,(b - a + q) < 0$. The first condition holds for our input parameters. Therefore, $z = \frac{1}{p}\frac{1}{t^2+s^2} - q$. Since $y = \int z(t)dt$, we have:

$$y = \frac{1}{ps}\arctan\frac{t}{s} - qt + c, \qquad s \in R^+, \quad c \in R.$$

If to put $k = \frac{1}{s}$, we obtain:

$$y = \frac{k}{p} \arctan kt - qt + c, \qquad k \in R^{+}, \quad c \in R.$$

The first boundary condition implies that $c = a$. By the second condition, k must satisfy the algebraic equation

$$k \arctan k = p\,(b - a + q)\,. \tag{26}$$

Since $k \arctan k$ is an increasing function of k on $(0,\ \infty)$, there is a unique positive root k.

Summing up, the solution of real BVP (25) is

$$y = \frac{k}{p} \arctan kt - qt + a,$$

where k is the root of (26).

Above, we have interpreted the given interval BVP (24) as the set of all real BVPs (25). Therefore, its solution is the following bunch of functions:

$Y = \{y(\cdot) \mid$

$y(t) = \frac{k}{p} \arctan kt - qt + a;\ p \in P,\ q \in Q,\ a \in A,\ b \in B;\ k \arctan k = p\,(b - a + q)\Big\}$

We plot the solution in Figure 5. The dashed line represents the solution of the associated real BVP.

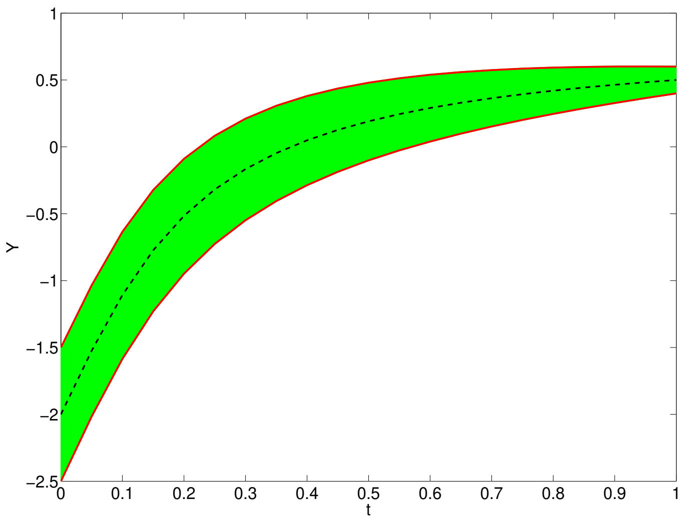

Fig. 5. The solution of non-linear interval BVP (24), obtained by the proposed approach.

We can make the following observations from the above example. The given non-linear interval BVP has 4 interval parameters: P, Q, A and B. If each of them is approximated by n points, we have n^4 number of different combinations of parameters, such as $(p,\ q,\ a,\ b)$. Therefore, to build the solution bunch, we need to solve n^4 real BVPs. Consequently, the computational complexity of the proposed method is n^4 for a non-linear problem with 4 interval inputs. The complexity in the linear case is n^2 under the same conditions. Thus, our first observation is that the complexity in the non-linear case is 2 orders of magnitude higher than in the linear case.

Another observation concerns solving related real BVPs. In the linear case, we can use known explicit formulas. But, in the non-linear case, we have to treat special techniques or numerical methods [26].

6. CONCLUSION

In this article, for a linear differential equation with interval coefficients and interval boundary values, we introduced a new approach that differs significantly from the ones described in the literature. This approach considers the solution as a set of real functions. The superiority of the introduced approach over the others is that the existence and uniqueness issues are analogical to the classical (real) case. Another advantage is that the approach does not use an interval-valued derivative. Consequently, there is no need for a strategy to choose the switching points, as required in the generalized differentiability approach. As a result, the solution is found naturally by using only the means of real calculus. We proposed a numerical algorithm to compute this solution. The algorithm requires solving n^2 classical BVPs, i.e. its complexity is $O(n^2)$. To find the ways to improve the algorithm, and deeply understand the problem, we provided a theoretical investigation. In particular, we derived a sufficient condition under which the solution is given analytically.

In future studies, the results obtained can be generalized to non-homogeneous interval differential equations, as well as to differential equations, whose coefficients are interval functions (rather than constant intervals). The approach can be useful also for solving fuzzy differential equations, formulated in terms of α-cuts.

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