KYBERNETIKA — VOLUME *58* (2022), NUMBER 3, PAGES 335–353

FIXED POINT RESULT IN CONTROLLED FUZZY METRIC SPACES WITH APPLICATION TO DYNAMIC MARKET EQUILIBRIUM

Rakesh Tiwari, Vladimir Rakočević and Shraddha Rajput

In this paper, we introduce Θ_f-type controlled fuzzy metric spaces and establish some fixed point results in this spaces. We provide suitable examples to validate our result. We also employ an application to substantiate the utility of our established result for finding the unique solution of an integral equation emerging in the dynamic market equilibrium aspects to economics.

Keywords: fixed point, fuzzy metric spaces, controlled fuzzy metric spaces, fuzzy Θ_f-contractive mapping, dynamic market equilibrium

Classification: 54H25, 47H10, A11

1. INTRODUCTION

In 1922, S. Banach [5] had given an important result of the fixed point theory. This topic has been studied, presented and generalized by many researchers in many different spaces. Firstly, the work of Bakhtin [6], Bourbaki [7] and Czerwik [8] expanded the theory of fixed points for b-metric spaces. Also, many authors proved some important fixed point theorems in b-metric spaces ([1, 2, 3]). Afterwards the controlled metric spaces introduced by Nabil Mlaiki et al. [21] which generalizes the b-metric spaces and proved some fixed point theorems.

Zadeh [28] had introduced important theoretical development in the fuzzy set theory. Fuzzy set theory is the way of defining the concept of fuzzy metric spaces was illustrated by Kramosil and Michálek [17], which can be regarded as a generalization of the statistical metric spaces. Subsequently, M. Grabiec [14] defined G-complete fuzzy metric spaces and extended the complete fuzzy metric spaces. Following Grabiec's work, George and Veeramani [10] modified the notion of M-complete fuzzy metric spaces with the help of continuous t-norms. Afterwards, this concept was extended by Nadaban [22] the context of fuzzy b-metric spaces. In the direction, Many researchers studied and generalized various fixed point results in the framework of fuzzy b-metric spaces [16, 18]. Müzeyyen Sangurlu Sezen [24] introduced controlled fuzzy metric spaces, which is a generalization of extended fuzzy b-metric spaces [18]. Furthermore, he also proved a Banach-type fixed point theorem and a new fixed point theorem for some self-mappings

DOI: 10.14736/kyb-2022-3-0335

satisfying fuzzy ψ- contraction condition and also establish some examples. Many authors introduced and generalized the numerous types of fuzzy contractive mappings ([4, 11, 12, 13, 15, 19, 20, 25, 27]) and investigated some fixed point theorems in fuzzy metric spaces.

H. Saleh Nasr et al. [23] introduced Fuzzy Θ_f- contractive mapping and established some fixed point theorems in M-complete fuzzy metric spaces and furnished an application to functional equations under Θ_f- contractive conditions.

The objective of this work is to prove a Banach type fixed point theorem in controlled fuzzy metric spaces using fuzzy Θ_f- contractive mapping, which is an extension of [23]. Our result generalizes many recent fixed point theorems in the literature ([22, 16, 18, 24]). We furnish an example to validate our result. Application is also provided to show the utility of our result to find the unique solution of an integral equation appearing in the dynamic market equilibrium aspects to economics.

2. PRELIMINARIES

Now, we begin with some basic concepts, notations and definitions. Let $\mathbb{R}$ represent the set of real numbers, $\mathbb{R}_+$ represent the set of all non-negative real numbers and $\mathbb{N}$ represent the set of natural numbers.

We start with the following definitions of a fuzzy metric space. Schweizer and Sklar introduced the continuous t- norm as follows:

Definition 2.1. (Schweizer and Sklar[26]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t- norm if for all $r_1, r_2, r_3 \in [0,1]$, the following conditions are hold:

(T-1) $r_1 * r_2 = r_2 * r_1$ and $r_1 * (r_2 * r_3) = (r_1 * r_2) * r_3$,

(T-2) $r_1 * r_2 \leq r_3 * r_4$ whenever $r_1 \leq r_3$ and $r_2 \leq r_4$,

(T-3) $r_1 * 1 = r_1$,

(T-4) $*$ is a continuous.

The most commonly used t-norms are: $r_1 *_1 r_2 = \min\{r_1, r_2\}$, $r_1 *_2 r_2 = \frac{r_1 r_2}{\max\{r_1, r_2, \lambda\}}$ for all $\lambda \in (0,1)$, $r_1 *_3 r_2 = r_1 r_2$, $r_1 *_4 r_2 = \max\{r_1 + r_2 - 1, 0\}$, the Lukasiewicz t-norms, we will denote it by $*_L$. For all $r_1 * r_2 * \cdots * r_n$ will be denoted by $\prod_{i=1}^n r_i$.

Kramosil and Michálek [17] introduced the notion of fuzzy metric space as follows:

Definition 2.2. (Kramosil and Michálek [17]) An ordered triple $(X, M, *)$ is called fuzzy metric space such that X is a nonempty set, $*$ defined a continuous t-norm and M is a fuzzy set on $X \times X \times [0, \infty)$, satisfying the following conditions, for all $x, y, z \in X, s, t > 0$,

(KM-1) $M(x, y, 0) = 0$,

(KM-2) $M(x, y, t) = 1$, for all $t > 0$ iff $x = y$,

(KM-3) $M(x, y, t) = M(y, x, t)$,

(KM-4) $(M(x, y, t) * M(y, z, s)) \leq M(x, z, t + s)$,

(KM-5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

George and Veeramani [10] modified the definition of M-complete fuzzy metric spaces due to Kramosil and Michálek and the concept as follows:

Definition 2.3. (George and Veeramani [10]) An ordered triple $(X, M, *)$ is called fuzzy metric space such that X is a nonempty set, $*$ defined a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$, satisfying the following conditions:

(FM-1) $M(x, y, t) > 0$,

(FM-2) $M(x, y, t) = 1$ if and only if $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $(M(x, y, t) * M(y, z, s)) \leq M(x, z, t + s)$,

(FM-5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is left continuous, $x, y, z \in X$ and $t, s > 0$.

George and Veeramani proved in [10] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form

$$\{B_M(x, r, t) : x \in X, 0 < r < 1, t > 0\},$$

where

$$B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\} \text{ for all } x \in X, r \in (0, 1) \text{ and } t > 0.$$

In 2017, Nădăban [22] introduced the idea of a fuzzy b-metric space to generalize the notion of a fuzzy metric spaces introduced by Kramosil and Michálek [17].

Definition 2.4. (Nădăban [22]) Let X is a non-empty set and $k \geq 1$ be a given real number and $*$ be a continuous t-norm. A fuzzy set M in $X^2 \times (0, \infty)$ is called fuzzy b-metric on X if for all $x, y, z \in X$, the following conditions hold.

(bM-1) $M(x, y, 0) = 0$,

(bM-2) $[M(x, y, t) = 1, (\forall)t > 0]$ if and only if $x = y$,

(bM-3) $M(x, y, t) = M(y, x, t), (\forall)t > 0$,

(bM-4) $M(x, z, k(t + s)) \geq M(x, y, t) * M(y, z, s), (\forall)t, s > 0$,

(bM-5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous and $\lim_{t\to\infty} M(x, y, t) = 1$.

The quadruple $(X, M, *, k)$ is said to be a fuzzy b-metric space.

Definition 2.5. (Nădăban [22]) Let $(X, M, *, k)$ be a fuzzy b-metric space. For $x \in X,\ r \in (0,1),\ t > 0$ we define the open ball

$$B(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}.$$

Then

$$\tau_M = \{T \subset X : x \in T \ iff \ (\exists) t > 0, r \in (0,1) : B(x,r,t) \subseteq T\}$$

is a topology on X.

Mehmood et al. [18] introduced the notion of an extended fuzzy b-metric space following the approach of Grabiec [14].

Definition 2.6. (Mehmood et al. [18]) Let X be a non-empty set, $\alpha : X \times X \to [1, \infty)$, $*$ is a continuous t-norm and M_α is a fuzzy set on $X^2 \times (0, \infty)$, is called extended fuzzy b-metric on X if for all $x, y, z \in X$ and $s, t > 0$, satisfying the following conditions.

$(FbM_\alpha 1)$ $M_\alpha(x, y, 0) = 0$,

$(FbM_\alpha 2)$ $M_\alpha(x, y, t) = 1$ iff $x = y$,

$(FbM_\alpha 3)$ $M_\alpha(x, y, t) = M_\alpha(y, x, t)$,

$(FbM_\alpha 4)$ $M_\alpha(x, z, \alpha(x, z)(t + s)) \geq M_\alpha(x, y, t) * M_\alpha(y, z, s)$,

$(FbM_\alpha 5)$ $M_\alpha(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Then $(X, M_\alpha, *, \alpha(x, y))$ is an extended fuzzy b-metric space.

In [24], Sezen introduced the controlled fuzzy metric spaces, which is a generalization of extended fuzzy b-metric spaces.

Definition 2.7. (Sezen [24]) Let X be a non-empty set, $\lambda : X \times X \to [1, \infty)$, $*$ is a continuous t-norm and M_λ is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions, for all $a, c, d \in X, s, t > 0$:

$(FM\text{-}1)$ $M_\lambda(a, c, 0) = 0$,

$(FM\text{-}2)$ $M_\lambda(a, c, t) = 1$ iff $a = c$,

$(FM\text{-}3)$ $M_\lambda(a, c, t) = M_\lambda(c, a, t)$,

$(FM\text{-}4)$ $M_\lambda(a, d, t + s) \geq M_\lambda\big(a, c, \frac{t}{\lambda(a,c)}\big) * M_\lambda\big(c, d, \frac{s}{\lambda(c,d)}\big)$,

$(FM\text{-}5)$ $M_\lambda(a, c, \cdot) : [0, \infty) \to [0, 1]$ is continuous,

Then, the triple $(X, M_\lambda, *)$ is called a controlled fuzzy metric space on X.

Definition 2.8. (Gopal and Vetro [13]) Let $(X, M, *)$ be a fuzzy metric space. We say that $T : X \to X$ is α-admissible if there exists a function $\alpha : X \times X \times (0, +\infty) \to [0, +\infty)$ such that, for all $t > 0$,

$$x, y \in X, \alpha(x, y, t) \geq 1 \Rightarrow \alpha(Tx, Ty, t) \geq 1.$$

Definition 2.9. (Nasr [23]) $\Theta_f : (0,1) \to (0,1)$, such that Θ_f is non decreasing continuous and satisfying condition for each sequence $\{\beta_n\} \subset (0,1)$,

$$\lim_{n\to\infty} \Theta_f(\beta_n) = 1 \Leftrightarrow \lim_{n\to\infty} \beta_n = 1. \tag{1}$$

Definition 2.10. (Gopal [11, 12]) Let $(X, M, *)$ be a fuzzy metric space. Then

(i) A sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$, that is $\lim_{n\to+\infty} x_n = x$, if

$$\lim_{n\to+\infty} M(x_n, x, t) = 1 \text{ for all } t > 0.$$

(ii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called M-Cauchy, if for each $\epsilon \in (0,1)$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$M(x_n, x_m, t) > 1 - \epsilon \text{ for all } m, n \geq n_0.$$

(iii) A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called G-Cauchy if

$$\lim_{n\to+\infty} M(x_n, x_{n+m}, t) = 1 \text{ for each } m \in \mathbb{N} \text{ and } t > 0.$$

Now, a fuzzy metric space $(X, M, *)$ is called M-complete (G-complete) if every M-Cauchy (G-Cauchy) sequence is convergent.

Definition 2.11. (Sezen [24]) Let $(X, M_\lambda, *)$ be a controlled fuzzy metric spaces. Then

1. A sequence $\{x_n\}$ in X is said to be G-convergent to x in X, if and only if

$$\lim_{n\to\infty} M_\lambda(x_n, x, t) = 1, \text{ for any } n > 0 \text{ and for all } t > 0.$$

2. A sequence $\{x_n\}$ in X is said to be G-Cauchy sequence if and only if

$$\lim_{n\to\infty} M_\lambda(x_n, x_{n+m}, t) = 1 \text{ for any } m > 0 \text{ and for all } t > 0.$$

3. The controlled fuzzy metric space is called G-complete if every G-Cauchy sequence is convergent.

3. MAIN RESULT

In this section, we introduce some new definitions and establish a fixed point theorem in controlled fuzzy metric spaces.

Definition 3.1. Let Y be a non-empty set, $\lambda : Y \times Y \to [1, \infty)$, $*$ is a continuous t-norm and M_λ is a fuzzy set on $Y^2 \times (0, \infty)$, for all $a, c, d \in Y, s, t > 0$, $\Theta_f : [0,1] \to [0,1]$, satisfying the following conditions,

($\Theta_f FM_\lambda$-1) $\Theta_f(M_\lambda(a, c, 0)) = 0$,

($\Theta_f FM_\lambda$-2) $\Theta_f(M_\lambda(a, c, t)) = 1$ iff $a = c$,

($\Theta_f FM_\lambda$-3) $\Theta_f(M_\lambda((a,c,t)) = \Theta_f(M_\lambda(c,a,t))$,

($\Theta_f FM_\lambda$-4) $\Theta_f(M_\lambda(a,d,t+s)) \geq \Theta_f(M_\lambda\big(a,c,\frac{t}{\lambda(a,c)}\big) * M_\lambda\big(c,d,\frac{s}{\lambda(c,d)}\big)\big)$,

($\Theta_f FM_\lambda$-5) $\Theta_f(M_\lambda(a,c,\cdot)) : [0,\infty) \to [0,1]$ is continuous.

Then, the triple $(Y, \Theta_f M_\lambda, *)$ is called a Θ_f-type controlled fuzzy metric spaces on Y.

Now, we display an example to verify our definition.

Example 3.2. Let $Y = A \cup C$ where $A = (0,2)$ and $C = [2,\infty)$. Define M_λ is a fuzzy set on $Y^2 \times (0,\infty)$, as

$$M_\lambda(a,c,t) = \begin{cases} 1 & \text{if } a = c \\ e^{-\frac{3}{ct}} & \text{if } a \in A \text{ and } c \in C \\ e^{-\frac{3}{at}} & \text{if } a \in C \text{ and } c \in A \\ e^{-\frac{3}{t}} & \text{otherwise.} \end{cases}$$

With the continuous product t-norm. Taking $\Theta_f(\beta) = e^{1-\frac{1}{\beta}}$ and define $\lambda : Y \times Y \to [1,\infty)$, as

$$\lambda(a,c) = \begin{cases} 1 & \text{if } a, c \in A \\ \max\{a,c\} & \text{otherwise.} \end{cases}$$

Let us show that $(Y, \Theta_f M_\lambda, *)$ is a Θ_f - type controlled fuzzy metric space on Y. It is easy to prove conditions ($\Theta_f FM_\lambda$-1), ($\Theta_f FM_\lambda$-2) and ($\Theta_f FM_\lambda$-3). We have to examine the following cases to show that condition ($\Theta_f FM_\lambda$-4) holds.

Case I. If $d = a$ or $d = c$, ($\Theta_f FM_\lambda$-4) is satisfied.

Case II. If $d \neq a$ and $d \neq c$, ($\Theta_f FM_\lambda$-4) holds when $a = c$.

Suppose that $a \neq c$. Then, we get $a \neq c \neq d$. Now, we can see that ($\Theta_f FM_\lambda$-4) is satisfied in all the cases below:

1. Let $a, c, d \in A$ and $a, c, d \in C$, choose $t = 1, s = 1$ and $a = 1, c = \frac{1}{2}, d = \frac{3}{2}$ and $\lambda(a,c) = 1,\ \lambda(a,d) = 1$, then we get $\Theta_f(e^{-1.5}) \geq \Theta_f(e^{-6})$.
2. Let $a, d \in A$ and $c \in C$, choose $t = 1, s = 1$ and $a = \frac{1}{2}, c = 3, d = \frac{3}{2}$ and $\lambda(a,c) = \max\{a,c\} = 3,\ \lambda(a,d) = \frac{3}{2}$, then we get $\Theta_f(e^{-0.5}) \geq \Theta_f(e^{-1})$.
3. Let $a, d \in C$ and $c \in A$, choose $t = 1, s = 1$ and $a = 3, c = \frac{3}{2}, d = 4$ and $\lambda(a,c) = \max\{a,c\} = 3,\ \lambda(a,d) = 1$, then we get $\Theta_f(e^{-0.5}) \geq \Theta_f(e^{-1.3})$.
4. Let $a \in C$ and $c, d \in A$, choose $t = 1, s = 1$, and $a = 4, c = \frac{1}{2}, d = \frac{3}{2}$ and $\lambda(a,c) = \max\{a,c\} = 4,\ \lambda(a,d) = 1$, then we get $\Theta_f(e^{-0.37}) \geq \Theta_f(e^{-0.93})$.
5. Let $a, c \in A$ and $d \in C$, choose $t = 1, s = 1$ and $a = \frac{1}{2}, c = \frac{3}{2}, d = 4$ and $\lambda(a,c) = 1,\ \lambda(a,d) = 1$, then we get $\Theta_f(e^{-1.5}) \geq \Theta_f(e^{-6})$.
6. Let $a, c \in C$ and $d \in A$, choose $t = 1, s = 1$, and $a = 3, c = 4, d = \frac{1}{2}$ and $\lambda(a,c) = \max\{a,c\} = 4,\ \lambda(a,d) = 3$, then we get $\Theta_f(e^{-1.5}) \geq \Theta_f(e^{-1.75})$.

7. Let $c, d \in C$ and $a \in A$, choose $t = 1, s = 1$, and $a = \frac{3}{2}, c = 3, d = 4$ and $\lambda(a, c) = \max\{a, c\} = 3,\ \lambda(a, d) = 4$, then we get $\Theta_f(e^{-0.5}) \geq \Theta_f(e^{-0.6})$.

Consequently, $(Y, \Theta_f M_\lambda, *)$ is a Θ_f type controlled fuzzy metric space on X. Also, for the same functions λ using by ($\Theta_f FM_\lambda$-4), we get

$$\Theta_f(M_\lambda(a, d, t + s)) \geq \Theta_f(M_\lambda\left(a, d, \frac{t}{\lambda(a, c)}\right) * M_\lambda\left(c, d, \frac{t}{\lambda(c, d)}\right).$$

Remark 3.3. We show by the following example, that verifies definition of Θ_f-type controlled fuzzy metric spaces and doesn't verify controlled fuzzy metric spaces.

Example 3.4. Let $Y = [1, \infty)$. Define M_λ is a fuzzy set on $Y^2 \times (0, \infty)$, as

$$M_\lambda(a, c, t) = \begin{cases} 1 & \text{if } a = c \\ te^{-(-\frac{a}{c}+1)t} & \text{if } a \leq c \\ te^{-(-\frac{c}{a}+1)t} & \text{if } c \leq a. \end{cases}$$

With the continuous product t-norm. Taking $\Theta_f(\beta) = e^{(\beta-1)e^{(1-\frac{1}{\beta})}} + \sin\frac{(\beta-1)\pi}{2} + \cos\frac{\beta\pi}{2}$, $t = 1,\ s = 2$ and define $\lambda : Y \times Y \to [1, \infty)$, as

$$\lambda(a, c) = \begin{cases} 1 & \text{if } a, c \in Y \\ \max\{a, c\} & \text{otherwise.} \end{cases}$$

Let us show that $(Y, \Theta_f M_\lambda, *)$ is a Θ_f - type controlled fuzzy metric space on Y. It is easy to prove conditions ($\Theta_f FM_\lambda$-1), ($\Theta_f FM_\lambda$-2) and ($\Theta_f FM_\lambda$-3). We have to examine the following cases to show that condition ($\Theta_f FM_\lambda$-4) holds.

Case I. If $d = a$ or $d = c$, ($\Theta_f FM_\lambda$-4) is satisfied.

Case II. If $d \neq a$ and $d \neq c$, ($\Theta_f FM_\lambda$-4) holds when $a = c$.

Suppose that $a \neq c$. Then, we get $a \neq c \neq d$. Now, we can see that ($\Theta_f FM_\lambda$-4) is satisfied in all the cases below:

1. Let $a \leq c$ and $c \leq d$ choose $t = 1, s = 2$, and $a = 1, c = 2, d = 3$ and $\lambda(a, c) = 1,\ \lambda(c, d) = 1$, then we get $\Theta_f(0.4060) \geq \Theta_f(0.620)$

$$\begin{aligned}
&\Theta_f(M_\lambda(a, d, t + s)) = \Theta_f(0.4060) \\
&= e^{(0.4060-1)e^{(1-\frac{1}{0.4060})}} + \sin\frac{(0.4060-1)\pi}{2} + \cos\frac{0.4060\pi}{2} \\
&\geq e^{(0.620-1)e^{(1-\frac{1}{0.620})}} + \sin\frac{(0.620-1)\pi}{2} + \cos\frac{0.620\pi}{2} \\
&= \Theta_f(0.620) = \Theta_f\Big(M_\lambda(a, c, \frac{t}{\lambda(a, c)}\Big) * M_\lambda\Big(c, d, \frac{s}{\lambda(c, d)}\Big),
\end{aligned}$$

2. Let $c \leq a$ and $a \leq d$ choose $t = 1, s = 2$, and $c = 2, a = 3, d = 4$ and $\lambda(a,c) = 1,\ \lambda(a,d) = 1$, then we get $\Theta_f(0.6693) \geq \Theta_f(0.86)$

$$
\begin{aligned}
&\Theta_f(M_\lambda(a,d,t+s)) \\
&= \Theta_f(0.6693) = e^{(0.6693-1)e^{(1-\frac{1}{0.6693})}} + \sin\frac{(0.6693-1)\pi}{2} + \cos\frac{0.6693\pi}{2} \\
&\geq e^{(0.86-1)e^{(1-\frac{1}{0.86})}} + \sin\frac{(0.86-1)\pi}{2} + \cos\frac{0.86\pi}{2} \\
&= \Theta_f(0.86) = \Theta_f(M_\lambda\left(a,c,\frac{t}{\lambda(a,c)}\right) * M_\lambda\left(c,d,\frac{s}{\lambda(c,d)}\right),
\end{aligned}
$$

It shows that, $(Y, \Theta_f M_\lambda, *)$ is a Θ_f-type controlled fuzzy metric spaces and condition $(FM_\lambda\text{-}4)$ is not satisfied of controlled fuzzy metric spaces, so not controlled fuzzy metric spaces.

Definition 3.5. Let $(Y, \Theta_f M_\lambda, *)$ is a Θ_f type controlled fuzzy metric space on X with $\lambda : Y \times Y \to [1, \infty)$, A mapping $S : Y \to Y$ is called a fuzzy Θ_f weak contractive with respect to $\Theta_f \in \Omega$, and Y is α- admissible. There exist $l \in (0,1)$ such that

$$M_\lambda(Sa, Sc, t) < 1 \Rightarrow \alpha(a,c)\Theta_f(M_\lambda(Sa, Sc, t)) \geq [\Theta_f(N(a,c,t))]^l \tag{2}$$

for all $a, c \in Y$ and $t > 0$, where

$$N(a,c,t) = \min\left\{M_\lambda(a,c,t), M_\lambda(a,Sa,t), M_\lambda(c,Sc,t), \frac{M_\lambda(a,Sa,t)M_\lambda(c,Sc,t)}{M_\lambda(a,c,t)}\right\}. \tag{3}$$

Theorem 3.6. Let $(Y, \Theta_f M_\lambda, *)$ is a Θ_f type controlled fuzzy metric space on Y. A mapping $S : Y \to Y$ is a fuzzy Θ_f weak contractive and Y is α- admissible, then S admits a unique fixed point.

Proof. Let a_0 is an arbitrary point in Y,

$$\alpha(a_0, Sa_0) \geq 1.$$

We define a sequence $\{a_n\}$ in Y by

$$a_{n+1} = Sa_n \quad for\ all\ n \in \mathbb{N}.$$

Obviously if, there exists $n_0 \in \mathbb{N}$ such that

$$a_{n_0} = Sa_{n_0+1},$$

then $Sa_{n_0} = a_{n_0}$ and the proof is finished. Suppose that $a_n \neq a_{n+1}$ for all $n \in \mathbb{N}$, that is, $\alpha(S_n, S_{n+1}) \geq 1$.

$$M_\lambda(Sa_{n-1}, Sa_n, t) < 1 \ \text{ for all } n \in \mathbb{N} \ \text{ and } \ s > 0.$$

Using (2), since S is a (α, Θ_f) type contraction, so for all $n \in \mathbb{N}$, we can write

$$\begin{aligned}1 > M_\lambda(Sa_{n-1}, Sa_n, t) \Rightarrow\ & \alpha(a_n, a_{n+1})\Theta_f(M_\lambda(Sa_{n-1}, Sa_n, t)) \\ & \geq [\Theta_f(N(a_{n-1}, a_n, t))]^l \\ & \geq \Big[\Theta_f\Big(\min\Big\{M_\lambda(a_{n-1}, a_n, t), M_\lambda(a_{n-1}, a_n, t), M_\lambda(a_n, a_{n+1}, t), \\ & \quad \frac{M_\lambda(a_{n-1}, a_n, t)M_\lambda(a_n, a_{n+1}, t)}{M_\lambda(a_{n-1}, a_n, t)}\Big\}\Big)\Big]^l \\ & \geq [\Theta_f(\min\{M_\lambda(a_{n-1}, a_n, t), M_\lambda(a_n, a_{n+1}, t)\})]^l.\end{aligned}$$

Thus

$$\Theta_f(M_\lambda(a_n, a_{n+1}, t) \geq [\Theta_f(\min\{M_\lambda(a_{n-1}, a_n, t), M_\lambda(a_n, a_{n+1}, t)\})]^l. \tag{4}$$

If there exists $n \in \mathbb{N}$ such that

$$\min\{M_\lambda(a_{n-1}, a_n, t), M_\lambda(a_n, a_{n+1}, t)\} = M_\lambda(a_n, a_{n+1}, t),$$

so

$$\Theta_f(M_\lambda(a_n, a_{n+1}, t) \geq [\Theta_f(M_\lambda(a_n, a_{n+1}, t))]^l > M_\lambda(a_n, a_{n+1}, t).$$

Which is contradiction. Therefore

$$\min\{M_\lambda(a_{n-1}, a_n, t), M_\lambda(a_n, a_{n+1}, t)\} = M_\lambda(a_{n-1}, a_n, t),$$

so

$$\Theta_f(M_\lambda(a_n, a_{n+1}, t) \geq [\Theta_f(M_\lambda(a_{n-1}, a_n, t))]^l > M_\lambda(a_{n-1}, a_n, t),$$

for all $n \in \mathbb{N}$. Thus (4), we get

$$\begin{aligned}\Theta_f(M_\lambda(a_n, a_{n+1}, t)) & \geq \Big[\Theta_f\big(M_\lambda\big(a_{n-1}, a_n, \frac{t}{k}\big)\big)\Big]^l \\ & \geq \Big[\Theta_f^2\big(M_\lambda\big(a_{n-2}, a_{n-1}, \frac{t}{k^2}\big)\big)\Big]^{l^2} \\ & \geq \Big[\Theta_f^3\big(M_\lambda\big(a_{n-3}, a_{n-2}, \frac{t}{k^3}\big)\big)\Big]^{l^3} \\ & \ \ \vdots \\ & \geq [\Theta_f^n(M_\lambda(a_0, a_1, \frac{t}{k^n}))]^{l^n}.\end{aligned} \tag{5}$$

Thus by (4), we have

$$\Theta_f(M_\lambda(a_n, a_{n+1}, t)) \geq \Big[\Theta_f\big(M_\lambda\big(a_0, a_1, \frac{t}{k^n}\big)\big)\Big]^{l^n} > M_\lambda\Big(a_0, a_1, \frac{t}{k^n}\Big). \tag{6}$$

Consider the triangle inequality, using the condition ($\Theta_f FM_\lambda$-4), we have

$$\begin{aligned}
&\Theta_f(M_\lambda(a_n, a_{n+m}, t)) \\
&\geq \Theta_f\Big(M_\lambda\Big(a_n, a_{n+1}, \frac{t}{2\lambda(a_n, a_{n+1})}\Big) * M_\lambda\Big(a_{n+1}, a_{n+m}, \frac{t}{2\lambda(a_{n+1}, a_{n+m})}\Big)\Big) \\
&\geq \Theta_f\Big(M_\lambda(a_n, a_{n+1}, \frac{t}{2\lambda(a_n, a_{n+1})}\Big) \\
&\quad * M_\lambda\Big(a_{n+1}, a_{n+2}, \frac{t}{(2)^2\lambda(a_{n+1}, a_{n+m})\lambda(a_{n+1}, a_{n+2})}\Big) \\
&\quad * M_\lambda\Big(a_{n+2}, a_{n+m}, \frac{t}{(2)^2\lambda(a_{n+1}, a_{n+m})\mu(a_{n+2}, a_{n+m})}\Big)
\end{aligned}$$

$$\begin{aligned}
&\Theta_f(M_\lambda\Big((a_n, a_{n+m}, t)) \qquad (7)\\
&\geq \Theta_f\Big(M_\lambda(a_n, a_{n+1}, \frac{t}{2\lambda(a_n, a_{n+1})}\Big)\Big) \\
&\quad * M_\lambda\Big(a_{n+1}, a_{n+2}, \frac{t}{(2)^2\lambda(a_{n+1}, a_{n+m})\lambda(a_{n+1}, a_{n+2})}\Big) \\
&\quad * M_\lambda\Big(a_{n+2}, a_{n+3}, \frac{t}{(2)^3\lambda(a_{n+1}, a_{n+m})\lambda(a_{n+2}, a_{n+m})\lambda(a_{n+2}, a_{n+3})}\Big) \\
&\quad * M_\lambda\Big(a_{n+3}, a_{n+m}, \frac{t}{(2)^3\lambda(a_{n+1}, a_{n+m})\lambda(a_{n+2}, a_{n+m})\lambda(a_{n+3}, a_{n+m})}\Big) \\
&\ \vdots \\
&\geq \Theta_f^n\Big(M_\lambda\Big(a_0, a_1, \frac{t}{2k^{n-1}\lambda(a_n, a_{n+1})}\Big)\Big) \\
&\quad * \Big[*_{i=n+1}^{n+m-2} \Theta_f^i\Big(M_\lambda\Big(a_0, a_1, \frac{t}{(2)^{m-1}k^{i-1}(\prod_{j=n+1}^{i} \mu(a_j, a_{n+m}))\lambda(a_i, a_{i+1})}\Big)\Big)\Big] \\
&\quad * \Big[\Theta_f^{n+m-1}\Big(M_\lambda\Big(a_0, a_1, \frac{t}{(2)^{m-1}k^{n+m-1}(\prod_{i=n+1}^{n+m-1} \mu(a_i, a_{n+m}))}\Big)\Big)\Big]. \qquad (8)
\end{aligned}$$

Θ_f is non decreasing continuous and satisfying condition for each sequence $\{\beta_n\} \subset (0,1)$

$$\lim_{n\to\infty} \Theta_f(\alpha_n) = 1 \Leftrightarrow \lim_{n\to\infty} \alpha_n = 1. \qquad (9)$$

Therefore, by taking limit as $n \to \infty$ in (7), from (6) together with (2) we have

$$\lim_{n\to\infty} \Theta_f(M_\lambda(a_n, a_{n+m}, t)) \geq (1 * 1 * \cdots * 1) = 1,$$

for all $t > 0$ and $n, m \in \mathbb{N}$. Thus, $\{a_n\}$ is a G-Cauchy sequence in X. From the completeness of $(X, M_\lambda, *)$, there exists $u \in X$ such that

$$\lim_{n\to\infty} M_\lambda(a_n, u, t) = 1, \qquad (10)$$

for all $t > 0$. Now we show that u is a fixed point of h. For any $t > 0$ and from the condition (FM_λ-4), we have

$$\begin{aligned}\Theta_f(M_\lambda(u, Su, t)) &\geq \Theta_f\Big(M_\lambda\Big(u, a_{n+1}, \frac{t}{2\lambda(a_n, a_{n+1})}\Big) * M_\lambda\Big(a_{n+1}, Su, \frac{t}{2\lambda(a_{n+1}, Su)}\Big)\Big)\\ &= \Theta_f\Big(M_\lambda\Big(u, a_{n+1}, \frac{t}{2\lambda(u, a_{n+1})}\Big)\Big) * M_\lambda\Big(Sa_n, Su, \frac{t}{2\lambda(a_{n+1}, Su)}\Big)\\ &\geq \Theta_f\Big(M_\lambda\Big(u, a_{n+1}, \frac{t}{2\lambda(u, a_{n+1})}\Big)\Big) * \Theta_f\Big(M_\lambda\Big(a_n, u, \frac{t}{2\lambda(a_{n+1}, Su)}\Big)\Big).\end{aligned} \tag{11}$$

Letting $n \to \infty$ in (11) and using (10), we get

$$\lim_{n\to\infty} \Theta_f(M_\lambda(u, Su, t)) = 1,$$

by definition of Θ_f, we have

$$\lim_{n\to\infty} \Theta_f(M_\lambda(u, Su, t)) = 1 \Leftrightarrow \lim_{n\to\infty} M_\lambda(u, Su, t) = 1,$$

for all $t > 0$, that is, $u = Su$.
Let $w \in X$ is an another fixed point of S and there exists $t > 0$ such that $u \neq w$, then it follows from (2) that

$$\begin{aligned}\Theta_f(M_\lambda(u, w, t)) &= \Theta_f(M_\lambda(Su, Sw, t))\\ &\geq \Theta_f\Big(M_\lambda\Big(u, w, \frac{t}{k}\Big)\Big)\\ &\geq \Theta_f\Big(M_\lambda\Big(u, w, \frac{t}{k^2}\Big)\Big)\\ &\ \ \vdots\\ &> \Theta_f\Big(M_\lambda\Big(u, w, \frac{t}{k^n}\Big)\Big),\end{aligned} \tag{12}$$

for all $n \in \mathbb{N}$. By taking limit as $n \to \infty$ in (11), $M_\lambda(u, w, t) = 1$ for all $t > 0$, that is, $u = w$. This completes the proof. □

Remark 3.7. Putting $\Theta_f(\beta) = \beta, \quad \alpha(a, c) = 1$ and

$$\begin{aligned}N(a, c, t) &= \min\Big\{M_\lambda(a, c, t), M_\lambda(a, Sa, t), M_\lambda(c, Sc, t), \frac{M_\lambda(a, Sa, t)M_\lambda(c, Sc, t)}{M_\lambda(a, c, t)}\Big\}\\ &= M_\lambda(a, c, t)\end{aligned}$$

in Theorem 3.6, we obtain the following result.

Corollary 3.8. Let $(Y, \Theta_f M_\lambda, *)$ is a Θ_f-type controlled fuzzy metric space on Y. If $S : Y \to Y$ is a mapping such that for all $a, c \in Y$, and $\lambda : Y \times Y \to [1, \infty)$, $t > 0$ for some $l \in (0, 1)$

$$M_\lambda(Sa, Sc, t) \geq [M_\lambda(a, c, t)]^l,$$

then S admits a unique fixed point.

Remark 3.9. By taking $l = 1$, in corollary 3.8, we infer the Theorem 2 in [24].

We furnish an example to validate our main result.

Example 3.10. Let $Y = A \cup C$ where $A = \{2^{2^n} : n \in \mathbb{N}\}$ and $C = \{2\}$. Define $M_\lambda : Y \times Y \times [0, \infty) \to [0, 1]$ as

$$M_\lambda(a, c, t) = \begin{cases} 1 & \text{if } a = c \\ \frac{t}{t+\frac{1}{a}} & \text{if } a \in A \text{ and } c \in C \\ \frac{t}{t+\frac{1}{c}} & \text{if } a \in C \text{ and } c \in A \\ (\frac{t}{t+1})^{\frac{1}{3}} & \text{otherwise.} \end{cases}$$

With the continuous product t-norm. Define $\lambda : Y \times Y \to [1, \infty)$, as

$$\lambda(a, c) = \begin{cases} 1 & \text{if } a, c \in A \\ \max\{a, c\} & \text{otherwise.} \end{cases}$$

Clearly $(Y, \Theta_f FM_\lambda, *)$ is a Θ_f type controlled fuzzy metric space and we take $\Theta_f(\beta) = \beta, \quad \beta \in (0, 1]$, $\alpha(a, c) \geq 1$. Consider $S : Y \to Y$ by

$$S(a) = \begin{cases} \sqrt{2} & \text{if } a \in A \\ 2^{2^{n+1}} & \text{if } a \in C. \end{cases}$$

Now let us show that $(Y, \Theta_f FM_\lambda, *)$ is a Θ_f controlled fuzzy metric space. It is easy to prove conditions $(\Theta_f FM_\lambda$-1), $(\Theta_f FM_\lambda$-2) and $(\Theta_f FM_\lambda$-3). We have to examine the following cases to show that condition $(\Theta_f FM_\lambda$-4) holds.

Case I. If $a = c$ then we have $Sa = Sc$. In this case:

$$\Theta_f(M_\lambda(Sa, Sc, t)) = 1 = \Theta_f(N_\lambda(a, c, t))^{\frac{1}{2}}.$$

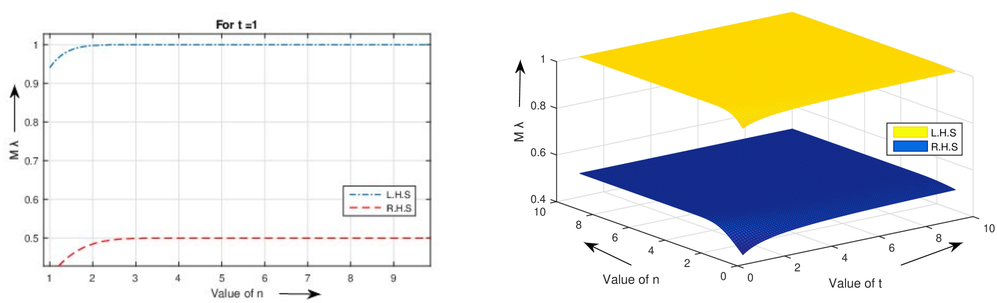

Fig. 1. Variation of $L.H.S = \Theta_f(M_\lambda(Sa, Sc, t))$ with $R.H.S = \Theta_f[N(a, c, t)]^l$ of Example 3.10 case-II on $2D$ and $3D$ view, for: (a) $\Theta_f(M_\lambda(Sa, Sc, t))$ vs $\Theta_f[N(a, c, t)]^l$ at $t, n \in (1, 10)$.

Case II. Let $a \in A$ and $c \in C$, then we have $Sa \in A$ and $Sc \in C$. In this case:
Since

$$\begin{aligned}
\Theta_f(M_\lambda(Sa, Sc, t)) &= \Theta_f\Big(\frac{t}{\big(t + \frac{1}{S(a)}\big)}\Big) \\
&= \Theta_f\Big(\frac{t}{(t + \frac{1}{2^{2^{n+1}}})}\Big) \\
&\geq \Theta_f\Big[\frac{t}{(t + \frac{1}{2^{2^n}})}\Big]^{\frac{1}{2}} \\
&= \Theta_f\big[N(a, c, t)\big]^l. \qquad (13)
\end{aligned}$$

Table 1 and 2 show the variation between $\Theta_f(M_\lambda(Sa, Sc, t))$ and $\Theta_f[N(a, c, t)]^l$ as a function of n with relative to t. This table justifies inequality (13), which observed in both the curves for the value of t is a higher than 50 as a function of n.

Value of t	Value of n	$\Theta_f(M_\lambda(Sa, Sc, t))$	$\Theta_f[N(a, c, t)]^l$
1	1	0.9412	0.4000
	2	0.9961	0.4706
	5	1	0.5000
	10	1	0.5000
50	1	0.9988	0.4975
	2	0.9999	0.4994
	20	1.0000	0.5000
	50	1.0000	0.5000

Tab. 1. Variation of $\Theta_f(M_\lambda(Sa, Sc, t))$ with $\Theta_f[N(a, c, t)]^l$ of inequality (13), as a function of n with fixed value of $t = 1$ and $t = 50$.

Value of n	Value of t	$\Theta_f(M_\lambda(Sa, Sc, t))$	$\Theta_f[N(a, c, t)]^l$
1	1	0.9412	0.4000
	2	0.9697	0.4444
	50	0.9988	0.4975
	100	0.9994	0.4988
50	1	1.000	0.5000
	2	1.000	0.5000
	50	1.000	0.5000
	100	1.000	0.5000

Tab. 2. Variation of $\Theta_f(M_\lambda(Sa, Sc, t))$ with $\Theta_f[N(a, c, t)]^l$ of inequality (13) as a function of n with fixed value of $n = 1$ and $n = 50$.

Case III. Let $a \in C$ and $c \in A$, then we have $Sa \in C$ and $Sc \in A$. In this case:

$$\begin{aligned}\Theta_f(M_\lambda(Sa,Sc,t)) &= \Theta_f\Big(\frac{t}{(t+\frac{1}{S(c)})}\Big)\\ &= \Theta_f\Big(\frac{t}{(t+\frac{1}{2^{2n+1}})}\Big)\\ &\geq \Theta_f\Big[\frac{t}{(t+\frac{1}{2^{2n}})}\Big]^{\frac{1}{2}}\\ &= \Theta_f[N(a,c,t)]^l.\end{aligned}$$

Case IV. For other states of $a,\ c$ and similarly $Sa,\ Sc$, we have,

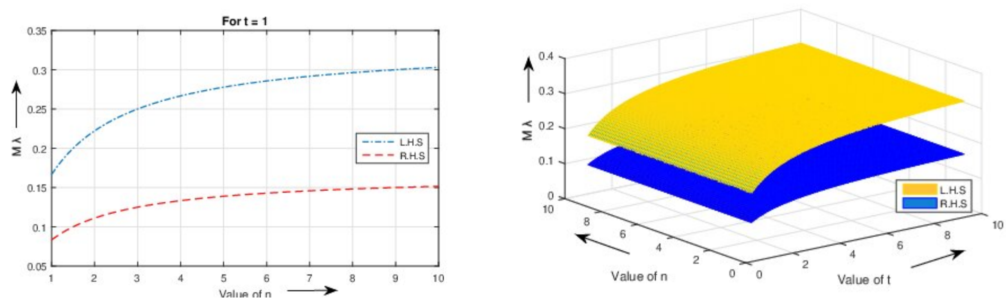

Fig. 2. Variation of $L.H.S = \Theta_f(M_\lambda(Sa,Sc,t))$ with $R.H.S. = \Theta_f[N(a,c,t)]^l$ of Example 3.8, case-IV on $2D$ and $3D$ view, for: (a) $\Theta_f(M_\lambda(Sa,Sc,t))$ vs $\Theta_f[N(a,c,t)]^l$ at $t,n \in (1,10)$.

$$\begin{aligned}\Theta_f(M_\lambda(Sa,Sc,t)) &= \Theta_f(\frac{t}{t+1})^{\frac{1}{3}}\\ &\geq \Theta_f[\frac{t}{t+1}]^{\frac{1}{6}}\\ &= \Theta_f[N(a,c,t)]^l.\end{aligned}$$

Therefore, all the conditions of Theorem 3.6 hold and S has a unique fixed point $y=1$.

Remark 3.11. Putting $\Theta_f(\beta) = e^{\beta-1}$, $\alpha(a,c)=1$,

$$\begin{aligned}N(a,c,t) &= \min\Big\{M_\lambda(a,c,t), M_\lambda(a,Sa,t), M_\lambda(c,Sc,t), \frac{M_\lambda(a,Sa,t)M_\lambda(c,Sc,t)}{M_\lambda(a,c,t)}\Big\}\\ &= M_\lambda(a,c,t)\end{aligned}$$

and $\alpha(a,c)=1$ in Theorem 3.6, we obtain the following result.

Corollary 3.12. Let $(Y, \Theta_f M_\lambda, *)$ is a Θ_f-type controlled fuzzy metric space on Y. If $S : Y \to Y$ is a mapping such that for all $a, c \in Y$, $t > 0$. Define $\lambda : Y \times Y \to [1, \infty)$, for some $l \in (0, 1)$.

$$e^{M_\lambda(Sa,Sc,t)-1} \geq [e^{M_\lambda(a,c,t)-1}]^l,$$

then S admits a unique fixed point.

Similarly, by choosing different values of $\Theta_f(\beta)$, we can obtain several results in the literature.

4. APPLICATION

We also provide an application to substantiate the utility of our established result to find the unique solution of an integral equation appearing in the dynamic market equilibrium aspects to economics.

Supply Q_s and demand Q_d are influenced by current prices and price trends (i. e., whether prices are rising or falling and whether they are rising or falling at an increasing or decreasing rate) in many markets [9]. The economist, therefore, needs to know the current price $P(t)$, the first derivative $\frac{\mathrm{d}P(t)}{\mathrm{d}t}$, and the second derivative $\frac{\mathrm{d}^2P(t)}{\mathrm{d}t^2}$. Assume

$$Q_s = g_1 + u_1 P(t) + e_1 \frac{\mathrm{d}P(t)}{\mathrm{d}t} + c_1 \frac{\mathrm{d}^2P(t)}{\mathrm{d}t^2}$$
$$Q_d = g_2 + u_2 P(t) + e_2 \frac{\mathrm{d}P(t)}{\mathrm{d}t} + c_2 \frac{\mathrm{d}^2P(t)}{\mathrm{d}t^2}.$$

g_1, g_2, u_1, u_2, e_1 and e_2 are constants. Comment on the dynamic stability of the market, if price clears the market at each point in time. In equilibrium, $Q_s = Q_d$. Therefore,

$$g_1 + u_1 P(t) + e_1 \frac{\mathrm{d}P(t)}{\mathrm{d}t} + c_1 \frac{\mathrm{d}^2P(t)}{\mathrm{d}t^2} = g_2 + u_2 P(t) + e_2 \frac{\mathrm{d}P(t)}{\mathrm{d}t} + c_2 \frac{\mathrm{d}^2P(t)}{\mathrm{d}t^2}$$

since

$$(c_1 - c_2)\frac{\mathrm{d}^2P(t)}{\mathrm{d}t^2} + (e_1 - e_2)d\frac{\mathrm{d}P(t)}{\mathrm{d}t} + (u_1 - u_2)P(t) = -(g_1 - g_2).$$

Letting $c = c_1 - c_2$, $e = e_1 - e_2$, $u = u_1 - u_2$ and $g = g_1 - g_2$ in above, we have

$$c\frac{\mathrm{d}^2P(t)}{\mathrm{d}t^2} + e\frac{\mathrm{d}P(t)}{\mathrm{d}t} + uP(t) = -g,$$

dividing through by c, $P(t)$ is governed by the following initial value problem

$$\begin{cases} P'' + \frac{e}{c}P' + \frac{u}{c}P(t) = -\frac{g}{c} \\ P(0) = 0 \\ P'(0) = 0, \end{cases} \tag{14}$$

where $\frac{e^2}{c} = \frac{4u}{c}$ and $\frac{u}{e} = \mu$ is a continuous function. It is easy to show that the problem (14) is equivalent to the integral equation:

$$P(t) = \int_0^T \zeta(t,r) F(t,r,P(r))\,\mathrm{d}r, \tag{15}$$

where $\zeta(t,r)$ is Green's function given by

$$\zeta(t,r) = \begin{cases} re^{\frac{\mu}{2}(t-r)} & \text{if } 0 \le r \le t \le T \\ te^{\frac{\mu}{2}(r-t)} & \text{if } 0 \le t \le r \le T. \end{cases} \tag{16}$$

In this section, by using Corollary 3.8, we will show the existence of a solution to the integral equation:

$$P(t) = \int_0^T G(t,r,P(r))\,\mathrm{d}r. \tag{17}$$

Let $X = C([0,T])$ be the set of real continuous functions defined on $[0,T]$. For $t > 0$, we define

$$M_\lambda(a,c,t) = \sup_{t\in[0,T]} \frac{\min\{a,c\}+t}{\max\{a,c\}+t} \tag{18}$$

for all $a, c \in Y$, with the continuous t-norm $*$ such that $t_1 * t_2 = t_1 t_2$. Taking $\Theta_f(\beta) = \beta$ and define $\lambda : Y \times Y \to [1,\infty)$, as

$$\lambda(a,c) = \begin{cases} 1 & \text{if } a, c \in A \\ \max\{a,c\} & \text{otherwise.} \end{cases}$$

It is easy to prove that $(Y, M_\lambda, *)$ is a controlled fuzzy metric spaces. Consider the mapping $S : Y \to Y$ defined by

$$SP(t) = \int_0^T G(t,r,P(r))\,\mathrm{d}r. \tag{19}$$

Theorem 4.1. Consider equation (17) and suppose that

1. There exist a continuous function $\zeta : [0,T] \times [0,T] \to \mathbb{R}^+$ such that $\sup_{t\in[0,T]} \int_0^T \zeta(t,r)\,\mathrm{d}r \ge 1$
2. $\max\{G(t,r,a(r)) - G(t,r,c(r))\} \ge \zeta(t,r) \max\{a(r),c(r)\}$ and $\min\{G(t,r,a(r)) - G(t,r,c(r))\} \ge \zeta(t,r) \min\{a(r),c(r)\}$
3. $\max\{a(r),c(r)\} + t \ge (\max\{a(r),c(r)\}+t)^l$ and $\min\{a(r),c(r)\} + t \ge (\min\{a(r),c(r)\}+t)^l$,

for all $l \in (0,1)$. Then, the integral equation (17) has a unique solution.

P r o o f. For $a, c \in Y$, by using of assumptions (1)(3), we have

$$\begin{aligned}
M_\lambda(Sa, Sc, t) &= \sup_{t\in[0,T]} \frac{\min\{\int_0^T G(t,r,a(r))\,\mathrm{d}r, \int_0^T G(t,r,c(r))\,\mathrm{d}r.)\} + t}{\max\{\int_0^T G(t,r,a(r))\,\mathrm{d}r, \int_0^T G(t,r,c(r))\,\mathrm{d}r.)\} + t} \\
&= \sup_{t\in[0,T]} \frac{\int_0^T \min\{G(t,r,a(r), G(t,r,c(r))\}\,\mathrm{d}r + t}{\int_0^T \max\{G(t,r,a(r))\,\mathrm{d}r, G(t,r,c(r))\}\,\mathrm{d}r + t} \\
&\geq \sup_{t\in[0,T]} \frac{\int_0^T \zeta(t,r) min\{a(r), c(r)\}\,\mathrm{d}r + t}{\int_0^T \zeta(t,r)\max\{a(r), c(r)\}\,\mathrm{d}r + t} \\
&\geq \sup_{t\in[0,T]} \frac{\min\{a(r), c(r)\}\int_0^T \zeta(t,r)\,\mathrm{d}r + t}{\max\{a(r), c(r))\}\int_0^T \zeta(t,r)\,\mathrm{d}r + t} \\
&\geq \frac{(\min\{a(r), c(r)\} + t)^l}{(\max\{a(r), c(r))\} + t)^l} \\
&\geq (\frac{\min\{a(r), c(r)\} + t}{\max\{a(r), c(r))\} + t})^l \\
&\geq (M_\lambda(a, c, t))^l.
\end{aligned}$$

Therefore all the conditions of Corollary 3.8 are satisfied. As a result, the mapping S has a unique fixed point $y \in Y$, which is a solution of the integral equation (17). □

CONCLUSIONS

In this article, motivated and inspired by the work of Müzeyyen Sangurlu Sezen [24] and H. Saleh Nasr et al. [23], we generalize the controlled fuzzy metric spaces using fuzzy Θ_f- contractive mapping. We obtain a fixed point result by using generalized contractive conditions in the framework of controlled fuzzy metric spaces. Our investigations and results obtained were supported by suitable examples. We also provide an application of our result to the existence of a solution to an integral equation, which determine dynamic market equilibrium aspects to economics problems. This work provides a new path for researchers in the concerned field.

ACKNOWLEDGEMENT

The authors are very grateful to the anonymous referee, for his precise remarks to improve the paper.

(Received April 29, 2022)

R E F E R E N C E S

[1] H. Aydi, M. Bota, E. Karapinar, and S.Moradi: A common fixed point for weak ϕ-contractions on b-metric spaces. Fixed Point Theory *13* (2012), 337–346.

[2] H. Afshari, M. Atapour, and H. Aydi : Generalized $\alpha - \psi -$Geraghty multivalued mappings on b-metric spaces endowed with a graph. J. Appl. Eng. Math. *7* (2017), 248–260.

[3] H. Aydi, R.Banković, I.Mitrović, and M. Nazam: Nemytzki–Edelstein–Meir–Keeler type results in b-metric spaces. Discret. Dyn. Nat. Soc. (2018), 4745764. DOI:10.1155/2018/4745764

[4] N. Alharbi, H. Aydi, A. Felhi, C. Ozel, and S. Sahmim: α-Contractive mappings on rectangular b-metric spaces and an application to integral equations. J. Math. Anal. *9* (2018), 47–60.

[5] S. Banach: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrals. Fund. Math. *3* (1922), 133–181.DOI:10.4064/fm-3-1-133-181

[6] I. A. Bakhtin: The contraction mapping principle in almost metric spaces. Funct. Anal. *30* (1989), 26–37. DOI:10.1039/ap9892600037

[7] M. Boriceanu, A. Petrusel. and I. A. Rus: Fixed point theorems for some multivalued generalized contraction in b-metric spaces. Int. J. Math. Statist. *6* (2010), 65–76.

[8] S. Czerwik: Contraction mappings in b-metric spaces. Acta Math. Inform. Univ. Ostrava *1* (1993), 5–11.`http://dml.cz/dmlcz/120469.`

[9] T. E. Dowling: Introduction to Mathematical Economics. Schaum's Outline Series, 2001.

[10] A. George and P. Veeramani: On some results in fuzzy metric spaces. Fuzzy Sets Systems *64* (1994), 395–399.DOI:10.1016/0165-0114(94)90162-7

[11] D. Gopal: Contributions to fixed point theory of fuzzy contractive mappings. Adv. Metric Fixed Point Theory Appl. (2021), 241–282. DOI:10.1007/978-981-33-6647-3_11

[12] D. Gopal and T.Došenović: Fixed point theory for fuzzy contractive mappings. Metric Struct. Fixed Point Theory (2021), 199–244. DOI:10.4324/9781003139607-6

[13] D. Gopal and C. Vetro: Some new fixed point theorems in fuzzy metric spaces. Iranian J. Fuzzy Systems *11*(2014), 3, 95–107.DOI:10.4324/9781003139607-6

[14] M. Grabiec: Fixed points in fuzzy metric spaces. Fuzzy Sets Systems *27* (1988), 385–389.DOI:10.1016/0165-0114(88)90064-4

[15] Y. Hao and H. Guan: On some common fixed point results for weakly contraction mappings with application. J. Funct. Spaces *2021* (2021), 5573983.DOI:10.1155/2021/5573983

[16] J. K. Kim: Common fixed point theorems for non-compatible self-mappings in b-fuzzy metric spaces. J. Comput. Anal. Appl. *22* (2017), 336–345.

[17] I. Kramosil and J. Michálek: Fuzzy metric and statistical metric spaces. Kybernetika *11* (1975), 326–334.`http://dml.cz/dmlcz/125556.`

[18] F. Mehmood, R. Ali, C. Ionescu, and T. Kamran: Extended fuzzy b-metric spaces. J. Math. Anal. *8* (2017), 124–131.`http://www.ilirias.com.`

[19] S. Melliani and A. Moussaoui: Fixed point theorem using a new class of fuzzy contractive mappings. J. Univer. Math. *1* (2018), 2, 148–154.

[20] D. Mihet: Fuzzy ψ-contractive mappings in non-archimedean fuzzy metric spaces. Fuzzy Sets Systems *159* (2008), 6, 739–744.DOI:10.1016/j.fss.2007.07.006

[21] N. Mlaiki, H. Aydi, N. Souayah, and T. Abdeljawad: Controlled metric type spaces and the related contraction principle. Math. Molecul. Divers. Preservat. Int. *6* (2018), 1–7.DOI:10.3390/math6100194

[22] S. Nădăban: Fuzzy b-metric spaces. Int. J. Comput. Commun. Control *11* (2016), 273–281.DOI:10.15837/ijccc.2016.2.2443

[23] H. S. Nasr, M. Imdad, I. Khan and M. Hasanuzzaman: Fuzzy Θ_f-contractive mappings and their fixed points with applications. J. Intell. Fuzzy Systems (2020), 1–10.DOI:10.3233/jifs-200319

[24] M. S. Sezen: Controlled fuzzy metric spaces and some related fixed point results. Numer. Part. Different. Equations (2020), 1–11.DOI:10.1002/num.22541

[25] S. Shukla, D. Gopal and W. Sintunavarat: A new class of fuzzy contractive mappings and fixed point theorems. Fuzzy Sets Systems *350* (2018), 85–94.DOI:10.1016/j.fss.2018.02.010

[26] B. Schweizer and A. Sklar: Statistical metric spaces. Paciffc J. Math. *10* (1960), 313–334. DOI:10.2140/pjm.1960.10.313

[27] D. Wardowski: Fuzzy contractive mappings and fixed points in fuzzy metric spaces. Fuzzy Sets Systems *222* (2013), 108–114.DOI:10.1016/j.fss.2013.01.012

[28] A. L. Zadeh: Fuzzy sets. Inform. Control *8* (1965), 338–353.DOI:10.1016/ S0019-9958(65)90241-X

Rakesh Tiwari, Department of Mathematics, Government V. Y. T. Post-Graduate Autonomous College, Durg 491001, Chhattisgarh. India.
e-mail: rtiwari@govtsciencecollegedurg.ac.in

Vladimir Rakočević, University of NIŠ, Faculty of Sciences and Mathematics, Višegradska 33, 18000 NIŠ. Serbia.
e-mail: vrakoc@sbb.rs

Shraddha Rajput, Department of Mathematics, Government V. Y. T. Post-Graduate Autonomous College, Durg 491001, Chhattisgarh. India.
e-mail: shraddhasss112@gmail.com